ON ZERO-SUM SUBSEQUENCES IN FINITE ABELIAN GROUPS

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Abstract

Let G be a finite abelian group and $k \in \mathbb{N}$ with $k \nmid \exp(G)$. Then $\mathsf{E}_k(G)$ denotes the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ with $|S| \geq l$ has a zero-sum subsequence T with $k \nmid |T|$. In this paper we prove that if $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ is a p-group, $k \in \mathbb{N}$ with $k \nmid \exp(G)$ and $\gcd(p, k) = 1$, then

$$\mathsf{E}_k(G) = \left\lfloor \frac{k}{k-1} \sum_{i=1}^{r} (n_i - 1) \right\rfloor + 1.$$

1. Introduction and Main Result

Let G be an additively written, finite abelian group and let $\exp(G)$ denote its exponent. We will consider sequences in G and recall some terminology. Let $\mathcal{F}(G)$ be the multiplicatively written, free abelian monoid with basis G and let $S = \prod_{i=1}^{l} g_i \in \mathcal{F}(G)$ be a *sequence* in G. We denote by $|S| = l \in \mathbb{N}_0$ the *length* of S and by $\sigma(S) = \sum_{i=1}^{l} g_i \in G$ the *sum* of S. We call the sequence S a *zero-sum sequence*, if $\sigma(S) = 0$. If $\emptyset \neq I \subset \{1, \ldots, l\}$, then we call $T = \prod_{i \in I} g_i \in \mathcal{F}(G)$ a *subsequence* of S. If $\sigma(T) = 0$, we call T a *zero-sum subsequence* of S.

In 1961 P. Erdős, A. Ginzburg and A. Ziv (cf. [4]) proved that in case G is a cyclic group, $2|G| - 1$ is the smallest integer l such that every sequence $S \in \mathcal{F}(G)$ with $|S| \geq l$ has a zero-sum subsequence with $|T| = |G|$.

This was a starting point to study subsequences of given sequences, that have sum zero and satisfy some given additional property. Each of the following problems had its own motivation and its own history.

Problem: Determine the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ with $|S| \geq l$ has a zero-sum subsequence T such that

1. $|T| = |G|$ (cf. [2, 5, 13]).

2. $|T| = \exp(G)$ (cf. [1, 14, 15]).
3. $1 \le |T| \le \exp(G)$ (cf. [9]).
4. T is a product of k zero-sum subsequences (for given $k \in \mathbb{N}$) (cf. [12]).

Recently W. D. Gao studied Problem 2 in a series of papers (cf. [6, 7, 8]). To do so he introduced the following invariant.

Definition 1.1. Let G be a finite abelian group and $k \in \mathbb{N}$ with $k \nmid \exp(G)$. Then $\mathsf{E}_k(G)$ denotes the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ with $|S| \ge l$ has a zero-sum subsequence T with $k \nmid |T|$.

W. D. Gao showed how the invariant is related with Problem 2 and he determined $\mathsf{E}_2(G)$ in case G is a p-group with odd p or G is a cyclic group of odd order (cf. [8]). In this paper we determine $\mathsf{E}_k(G)$ in case G is a p-group, $k \in \mathbb{N}$ with $k \nmid \exp(G)$ and $\gcd(p, k) = 1$.

For some real number $x \in \mathbb{R}$ let $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \le x\}$ and for some $n \in \mathbb{N}$ let C_n denote the cyclic group with n elements.

The aim of the paper is to prove the following result:

Theorem 1.2. *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ be a p-group, $k \in \mathbb{N}$ with $k \nmid \exp(G)$ and $\gcd(p, k) = 1$. Then*

$$\mathsf{E}_k(G) = \left\lfloor \frac{k}{k-1} \sum_{i=1}^{r} (n_i - 1) \right\rfloor + 1.$$

2. Proof of the Main Result

Throughout, let G denote a finite abelian group and let $k \in \mathbb{N}$ with $k \nmid \exp(G)$. If $|G| > 1$, then there are uniquely determined $n_1, \ldots, n_r \in \mathbb{N}$ with $1 < n_1 | \ldots | n_r$ and

$$G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}.$$

If $|G| = 1$, we set $r = n_r = 1$.

Let $\mathsf{D}(G)$ denote *Davenport's constant*, which is defined as the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ with $|S| \ge l$ contains a zero-sum subsequence. Furthermore, let $\mathsf{s}(G)$ denote the invariant arising from Problem 2, i.e. the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ with $|S| \ge l$ has a zero-sum subsequence T such that $|T| = \exp(G)$. We start with a simple lemma showing relations between $\mathsf{D}(G)$, $\mathsf{s}(G)$ and $\mathsf{E}_k(G)$

Lemma 2.1. *1.* $\mathsf{D}(G) \le \mathsf{E}_k(G) \le \mathsf{s}(G)$.

2. If $\mathsf{D}(G) < k$, *then* $\mathsf{D}(G) = \mathsf{E}_k(G)$.

Proof. The inequality $\mathsf{D}(G) \le \mathsf{E}_k(G)$ holds by definition. The inequaltiy $\mathsf{E}_k(G) \le \mathsf{s}(G)$ holds, since $k \nmid \exp(G)$ and therefore a zero-sum subsequence of length $\exp(G)$ is as well a zero-sum subsequence of length not divisible by k.
To prove $\mathsf{D}(G) = \mathsf{E}_k(G)$, in case $\mathsf{D}(G) < k$, it suffices to prove $\mathsf{E}_k(G) \le \mathsf{D}(G)$. Let $T \in \mathcal{F}(G)$ with $|T| = \mathsf{D}(G)$. By definition T has a zero-sum subsequence Z. Since $|Z| \le |T| < k$, we have $k \nmid |T|$. Therefore every $S \in \mathcal{F}(G)$ with $|S| \ge \mathsf{D}(G)$, has a zero-sum subsequence of length not divisible by k. This implies $\mathsf{E}_k(G) \le \mathsf{D}(G)$. □

In various problems involving zero-sum sequences it has turned out to be useful to reformulate the original problem into an equivalent one involving zerofree sequences (as usual, we call a sequence *zerofree*, if it has no zero-sum subsequence). This procedure proved successful in all investigations on the generalized Davenport's constant (cf. Problem 4 of the Introduction) and in all investigations on the cross number of sequences (cf. [10, 11]). Although the above reformulation of the given problem is quite simple in many cases, we regard this as a key idea which we are going to apply for investigating $\mathsf{E}_k(G)$.

We need some further notations. Let $\mathsf{d}(G)$ denote the largest integer $l \in \mathbb{N}$ such that there exists a sequence $S \in \mathcal{F}(G)$ which is zerofree and has length l. It is well known that

$$\mathsf{D}(G) = \mathsf{d}(G) + 1 \qquad \text{and} \qquad \sum_{i=1}^{r} (n_i - 1) \le \mathsf{d}(G).$$

Definition 2.2. Let $\mathsf{e}_k(G)$ denote the largest integer $l \in \mathbb{N}$ such that there exists a sequence $S \in \mathcal{F}(G)$ with $|S| = l$ and $k \mid |T|$ for all zero-sum subsequences T of S.

The following will show that there are relations among $\mathsf{E}_k(G)$ and $\mathsf{e}_k(G)$, which are similar to those among $\mathsf{D}(G)$ and $\mathsf{d}(G)$.

Lemma 2.3.

$$\mathsf{E}_k(G) = \mathsf{e}_k(G) + 1.$$

Proof. By definition, $\mathsf{e}_k(G) < \mathsf{E}_k(G)$. Indeed, there exists a sequence $S \in \mathcal{F}(G)$ of length $\mathsf{e}_k(G)$ such that k divides the lengths of all zero-sum subsequences of S. On the other hand, the maximality of $\mathsf{e}_k(G)$ implies that every sequence with length greater $\mathsf{e}_k(G)$ has a zero-sum subsequence with length not divisible by k. Therefore $\mathsf{E}_k(G) \le \mathsf{e}_k(G) + 1$, and the equality follows. □

Lemma 2.4.

$$\left\lfloor \frac{k}{k-1} \sum_{i=1}^{r} (n_i - 1) \right\rfloor \le \mathsf{e}_k(G).$$

Proof. The proof is done by construction of a sequence of length $\lfloor \frac{k}{k-1} \sum_{i=1}^{r}(n_i-1) \rfloor$ such that k divides the length of every zero-sum subsequence. Let $e_1, \ldots, e_r \in G$ such that

$$G = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle$$

and $\mathrm{ord}(e_i) = n_i$ for all $i \in \{1, \ldots, r\}$.

If $S' = \prod_{i=1}^{r} (-e_i)^{(n_i-1)}$, then S' is zerofree and it remains to construct a sequence S'' of length $\lfloor \frac{\sum_{i=1}^{r}(n_i-1)}{k-1} \rfloor$ such that $k \mid |Z|$ for every zero-sum subsequence Z of $S'S''$. We consider the sequence $T = \prod_{i=1}^{r} e_i^{(n_i-1)}$, which is zerofree and we write it as a product of sequences $B_1, \ldots, B_l$ of length $k-1$ and a rest R of length less than $k-1$:

$$T = (\prod_{i=1}^{l} B_i) R$$

with $|B_i| = k-1$ for all $i \in \{1, \ldots, l\}$ and $0 \le |R| < k-1$. We define

$$S'' = \prod_{i=1}^{l} \sigma(B_i).$$

It follows that S'' is zerofree and has length $|S''| = l = \lfloor \frac{|T|}{k-1} \rfloor = \lfloor \frac{\sum_{i=1}^{r}(n_i-1)}{k-1} \rfloor$. Therefore $|S'S''| = \lfloor \frac{k}{k-1} \sum_{i=1}^{r}(n_i-1) \rfloor$, and it remains to show that k divides the length of every zero-sum subsequence. Let Z denote an arbitrary zero-sum subsequence of $S'S''$. Since S' and S'' are both zerofree, Z can be written as $Z'Z''$ with subsequences Z' of S' and Z'' of S''. Every element z'' of Z'' can be written in the form $z'' = \sum_{j=1}^{k-1} e_{i_j}$ with suitable $i_j \in \{1, \ldots, r\}$. Since Z is a zero-sum sequence, we get $\prod_{j=1}^{k-1}(-e_{i_j}) | Z'$. The zero-sum sequence $z''(\prod_{j=1}^{k-1}(-e_{i_j}))$ is of length k and Z can be written as a product of sequences of this form. Therefore k divides $|Z|$. □

Lemma 2.5. *If $G = G_1 \oplus G_2$, then*

$$\mathsf{e}_k(G_1) + \mathsf{e}_k(G_2) \le \mathsf{e}_k(G).$$

Proof. Since $\exp(G) = \mathrm{lcm}(\exp(G_1), \exp(G_2))$ and $k \nmid \exp(G)$, it follows that $k \nmid \exp(G_1)$ and $k \nmid \exp(G_2)$. Therefore $\mathsf{e}_k(G_1)$ and $\mathsf{e}_k(G_2)$ are well-defined. For $i \in \{1, 2\}$ let $S_i \in \mathcal{F}(G_i)$ be a sequence with $|S_i| = \mathsf{e}_k(G_i)$ such that for every zero-sum subsequence T_i of S_i, k divides $|T_i|$. We define $S = S_1 S_2 \in \mathcal{F}(G)$. For every zero-sum subsequence T of S, there exist $T_i \in \mathcal{F}(G_i)$ for $i \in \{1, 2\}$ such that $T = T_1 T_2$. Since T has sum zero, the sequences T_1 and T_2 have sum zero too. Due to the definition of S_1 and S_2, we have

$k \mid |T_1|$ and $k \mid |T_2|$. Therefore $k \mid |T_1| + |T_2| = |T|$ and S is a sequence in G of length $\mathsf{e}_k(G_1) + \mathsf{e}_k(G_2)$, for which every zero-sum subsequence has a length divisible by k. By definition of $\mathsf{e}_k(G)$, we have

$$\mathsf{e}_k(G_1) + \mathsf{e}_k(G_2) = |S_1| + |S_2| = |S| \le \mathsf{e}_k(G).$$

□

For the proof of Theorem 1.2 we need two results on p-groups. The first result has been proved independently by D. Kruyswijk and J. E. Olson (cf. [3, 16])

Theorem 2.6. *If G is a p-group, then* $\mathsf{d}(G) = \sum_{i=1}^r (n_i - 1)$.

Theorem 2.6 implies that for two p-groups G and H

$$\mathsf{d}(G) + \mathsf{d}(H) = \mathsf{d}(G \oplus H).$$

The second result is due to W. D. Gao. For convenience we repeat its short proof.

Lemma 2.7. *[8] Let G be a p-group. Then there exists a p-group H such that $D(G \oplus H)$ is a power of p.*

Proof. Let $G = \bigoplus_{i=1}^r C_{p^{m_i}}$ with $m_i \in \mathbb{N}$ and $M = \prod_{i=1}^r m_i$. Then G is a direct summand of

$$\bar{G} = C_{p^M}^{p^M - r + 1} \oplus \bigoplus_{i=1}^r C_{p^{m_i}}^{\frac{p^M - 1}{p^{m_i} - 1}}$$

and by Theorem 2.6

$$\begin{aligned} \mathsf{D}(\bar{G}) =& 1 + \mathsf{d}(\bar{G}) = 1 + (p^M + 1 - r)(p^M - 1) + \sum_{i=1}^r \frac{p^M - 1}{p^{m_i} - 1}(p^{m_i} - 1) = \\ & 1 + (p^M + 1)(p^M - 1) = p^{2M}. \end{aligned}$$

□

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.3 and Lemma 2.4 it suffices to prove

$$\mathsf{e}_k(G) \le \left\lfloor \frac{k}{k-1} \sum_{i=1}^r (n_i - 1) \right\rfloor.$$

The proof is done in three steps. In the first and the second step the proof is given for special groups. In the third step the general case is proved, by using the result for the groups of special type.

1. Suppose that there exists some $n \in \mathbb{N}$ such that $\mathsf{d}(G) = (k-1)(p^n - 1)$.
   Let $S \in \mathcal{F}(G)$ with $|S| = \left\lfloor \frac{k}{k-1} \sum_{i=1}^r (n_i - 1) \right\rfloor + 1$. We shall prove that S possesses a zero-sum subsequence T such that $k \nmid |T|$. This implies that

$$\mathsf{e}_k(G) = \mathsf{E}_k(G) - 1 \le |S| - 1 \le \left\lfloor \frac{k}{k-1} \sum_{i=1}^r (n_i - 1) \right\rfloor .$$

   We consider the map

$$\zeta : \begin{cases} G \to G \oplus C_{p^n} \\ g \mapsto g + e \end{cases}$$

   where $G \oplus C_{p^n} = G \oplus \langle e \rangle$. For $W = \prod_{i=1}^l g_i \in \mathcal{F}(G)$ we set $\zeta(W) = \prod_{i=1}^l \zeta(g_i) \in \mathcal{F}(G \oplus C_{p^n})$. By Theorem 2.6 we get

$$|S| = \frac{k}{k-1}\mathsf{d}(G) + 1 = \mathsf{d}(G) + 1 + (p^n - 1) = k(p^n - 1) + 1 < kp^n$$

   and

$$\mathsf{d}(G \oplus C_{p^n}) = \mathsf{d}(G) + (p^n - 1) = k(p^n - 1) < |S| = |\zeta(S)|.$$

   Therefore, again by Theorem 2.6, there exists a subsequence T of S such that $\sigma(\zeta(T)) = 0$. By construction $\sigma(T) = 0$, $p^n \mid |T|$ and, since $|T| \le |S| < kp^n$, we have $k \nmid |T|$.

2. Suppose that $(k-1)$ divides $\mathsf{d}(G)$.
   By Lemma 2.7 there exists a p-group H and an integer $n \in \mathbb{N}$ such that $\mathsf{d}(G \oplus H) = p^n - 1$. If $H' = G^{k-2} \oplus H^{k-1}$, then $\mathsf{d}(G \oplus H') = \mathsf{d}((G \oplus H)^{k-1}) = (k-1)\mathsf{d}(G \oplus H) = (k-1)(p^n - 1)$. Since $\mathsf{d}(H') = \mathsf{d}((G \oplus H)^{k-1}) - \mathsf{d}(G)$, we also get $(k-1)|\mathsf{d}(H')$. From the previous step we obtain

$$\begin{aligned} &\frac{k}{k-1}\mathsf{d}((G \oplus H)^{k-1}) = \frac{k}{k-1}\mathsf{d}(G \oplus H') = \frac{k}{k-1}\mathsf{d}(G) + \frac{k}{k-1}\mathsf{d}(H') \le \\ &\mathsf{e}_k(G) + \mathsf{e}_k(H') \le \mathsf{e}_k(G \oplus H') = \mathsf{e}_k((G \oplus H)^{k-1}) = \frac{k}{k-1}\mathsf{d}((G \oplus H)^{k-1}), \end{aligned}$$

   where the first inequality holds by Lemma 2.4, Theorem 2.6 and the fact that $(k-1)|\mathsf{d}(G)$ and $(k-1)|\mathsf{d}(H')$. In this chain of inequalities equality holds and therefore

$$\mathsf{e}_k(G) = \frac{k}{k-1}\mathsf{d}(G) = \left\lfloor \frac{k}{k-1} \sum_{i=1}^r (n_i - 1) \right\rfloor .$$

3. Assume to the contrary that $\left\lfloor \frac{k}{k-1} \sum_{i=1}^r (n_i - 1) \right\rfloor + 1 \le \mathsf{e}_k(G)$.
   Since $k-1$ divides $(k-1)\mathsf{d}(G) = \mathsf{d}(G^{k-1})$, we obtain by the previous step and

Lemma 2.5:

$$k\,\mathsf{d}(G) =$$
$$(k-1)(\frac{k}{k-1}\mathsf{d}(G)-1)+(k-1) < (k-1)\left\lfloor\frac{k}{k-1}\mathsf{d}(G)\right\rfloor+(k-1) =$$
$$(k-1)(\left\lfloor\frac{k}{k-1}\mathsf{d}(G)\right\rfloor+1) = (k-1)(\left\lfloor\frac{k}{k-1}\sum_{i=1}^{r}(n_i-1)\right\rfloor+1) \leq$$
$$(k-1)\mathsf{e}_k(G) \leq \mathsf{e}_k(G^{k-1}) = k\,\mathsf{d}(G),$$

a contradiction. Therefore we have $\mathsf{e}_k(G) \leq \left\lfloor\frac{k}{k-1}\sum_{i=1}^{r}(n_i-1)\right\rfloor$.

□

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