Weakly half-factorial sets in finite abelian groups

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Abstract

We investigate weakly half-factorial sets in finite abelian groups, a concept introduced by J. Śliwa to study half-factorial sets. We fully characterize weakly half-factorial sets in a given group, and determine the maximum cardinality of such a set. This leads to several new results on half-factorial sets; in particular we solve a problem of W. Narkiewicz in some special cases. We also study the arithmetical consequences of weakly-half-factoriality in terms of factorization lengths in block monoids.

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1 Introduction

A monoid H, a commutative cancellative semigroup with identity element, is called atomic if every non-unit $a \in H$ has a factorization $a = u_1 \cdot \ldots \cdot u_l$ with irreducible elements (atoms) $u_1, \ldots, u_l \in H$; l is called the length of such a factorization. An atomic monoid is called half-factorial if for each $a \in H$ all factorizations of a (into atoms) have the same length. Of course, if H is factorial, that is there is an essentially unique factorization for each $a \in H$, then it is half-factorial. The investigation of half-factorial monoids and domains is one of the central topics in the theory of non-unique factorization (cf. the surveys by S.T. Chapman [3, 4], J. Coykendall [3, 6], M. Freeze [4], and W.W. Smith [4]).

It is well known that if H is a Krull monoid, for instance the multiplicative monoid of a Dedekind domain, then the properties of sets of lengths, in particular whether H is half-factorial or not, just depend on the class group G of H and the set $G_0 \subset G$ of classes containing prime divisors. More precisely, H is a half-factorial monoid if and only if the block monoid $\mathcal{B}(G_0)$, the monoid of zero-sum sequences in G_0, is a half-factorial monoid; then the set $G_0 \subset G$ is called a half-factorial set (cf. Preliminaries for further details). Thus, instead of studying the problem of half-factoriality for Krull monoids directly, one can instead study half-factorial sets, which is a problem on zero-sum sequences in abelian groups. Half-factorial sets have been studied in a variety of papers, see for instance those of P. Erdős and A. Zaks [7] on splittable sets (that is, half-factorial sets in cyclic groups), of S.T. Chapman and W.W. Smith [5] where among others a generalization of half-factorial sets, different from the one investigated here, is considered, of W. Gao and A. Geroldinger [9] for various results and an overview of the subject, and of A. Geroldinger and R. Göbel [16] on half-factorial sets in infinite abelian groups.

One of the main problems related to half-factorial sets is to determine the value of $\mu(G)$, the maximal cardinality of a half-factorial set $G_0 \subset G$, for a given finite abelian group G. This problem was first posed by W. Narkiewicz [25, P 1142] and is motivated by factorization problems in rings of algebraic integers. For the same number theoretic applications it is also desirable to know the structure of the half-factorial sets with maximal cardinality (cf. [12, 28]). This problem, in general, is wide open; an answer is known for cyclic groups, where the order is a product of at most two prime powers, for elementary p-groups, and in some other special cases (see [31, 32, 34, 19, 26, 27, 29]).

A fundamental tool in the study of half-factorial sets, for finite (resp. torsion) abelian groups, is a characterization of half-factorial sets in terms of the cross number of minimal zero-sum sequences (cf. Definition 2.1.1) due to L. Skula [31] and A. Zaks [34]. Starting from this characterization, J. Śliwa [33] introduced the more general concept of a weakly half-factorial set (which he defined as C_0 sets) as a further tool in the investigation of half-factorial sets and in particular of $\mu(G)$ (cf. Definition 2.1.2), and characterized these sets in terms of characters of G (cf. Lemma 3.1). Using this characterization, the maximal cardinality of a weakly half-factorial set was determined for all groups of the form C_n^r with integers n and r; and this way the best general upper bound for $\mu(G)$ so far was obtained (see [9]).

Here, we first determine, for an arbitrary finite abelian group G, the structure of (inclusion-maximal) weakly half-factorial subsets of G and in particular the maximal cardinality of a weakly half-factorial set in G (see Theorem 3.2); as a consequence we obtain an improved upper bound for $\mu(G)$ (see Corollary 4.1). Then in Section 4, we apply the results on weakly half-factorial sets to obtain results on half-factorial sets. In particular, we determine $\mu(G)$ (and the structure of half-factorial sets with maximal cardinality) for some new classes of groups (see Theorem 4.5 and Proposition 4.8). Finally in Section 5, we study the "arithmetic" of weakly half-factorial sets, more specifically the system of sets of lengths of block monoids over (maximal) weakly half-factorial sets. These results show that the arithmetic properties of weakly half-factorial sets are seemingly quite diverse, depending on the underlying group G. Thus, it could be difficult (or perhaps impossible) to give a (meaningful) arithmetic characterization of a weakly half-factorial set.

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2 Preliminaries

We recall some notation, in particular for sequences and block monoids; for further details we refer to the surveys [2, 22] in [1], or [9]. We denote by $\mathbb{N}$ the positive integers and by $\mathbb{N}_0$ the non-negative integers. For integers $a, b \in \mathbb{Z}$, let $[a, b] = \{z \in \mathbb{Z} : a \le z \le b\}$.

Let G be an, additively written, finite abelian group. Then $\exp(G)$ denotes its exponent, $\mathsf{r}(G)$ its rank, $\mathsf{r}^*(G)$ its total rank, and $\mathsf{r}_p(G)$, for $p \in \mathbb{P}$, its p-rank. For $g \in G$ we let $\operatorname{ord} g$ denote the order of g and we put $G[d] = \{g \in G\colon \operatorname{ord} g \mid d\}$ for $d \in \mathbb{N}$. A subset, respectively its elements,

$G_0 \subset G$ is called independent if $\sum_{g\in G} m_g g = 0$ with $m_g \in \mathbb{Z}$ implies $m_g g = 0$ for each $g \in G_0$. We refer to an independent generating set of G as basis.

We denote by $\widehat{G}$ the group of characters of G; we use the shorthand notation $e(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$, and consider characters as homomorphisms from G to $\{z \in \mathbb{C} : |z| = 1\}$. For $n \in \mathbb{N}$, let C_n denote a cyclic group of order n.

For $G_0 \subset G$, let $\mathcal{F}(G_0)$ denote the, muliplicatively written, free abelian monoid with basis G_0 (equivalently, the multisets in G_0). An element

$$S = \prod_{i=1}^{l} g_i = \prod_{g\in G_0} g^{v_g} \in \mathcal{F}(G_0)$$

is called a sequence in G_0. The divisors (in $\mathcal{F}(G_0)$) of a sequence S are called subsequences of S; for $T \mid S$, we denote by $T^{-1}S$ the co-divisor of T in S. Further, $|S| = l$ denotes the length of S, $\sigma(S) = \sum_{i=1}^{l} g_i \in G$ the sum, and $\mathsf{k}(S) = \sum_{i=1}^{l} \frac{1}{\operatorname{ord} g_i}$ the cross number. For $g \in G_0$ let $\mathsf{v}_g(S) = v_g$ denote the multiplicity of g in S and $\operatorname{supp}(S) = \{g \in G_0 : \mathsf{v}_g(S) \neq 0\} \subset G_0$ the support of S. The identity element of $\mathcal{F}(G_0)$, the empty sequence, is simply denoted by 1; it has length 0, sum 0, and cross number 0.

A sequence $S \in \mathcal{F}(G_0)$ is called a zero-sum sequence (in G_0), if $\sigma(S) = 0$; and a sequence is called zero-sumfree if it has no proper non-trivial zero-sum subsequence. The set of all zero-sum sequences $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is a submonoid and is called the block monoid over G_0. It is an atomic monoid; its atoms are the minimal zero-sum sequences, that is non-empty zero-sum sequences such that no proper, non-empty subsequence has sum 0. The set of atoms is denoted by $\mathcal{A}(G_0) \subset \mathcal{B}(G_0)$. The following definition summarizes the central notions of this paper.

Definition 2.1. Let G be a finite abelian group.

1. A set $G_0 \subset G$ is called half-factorial if $\mathsf{k}(A) = 1$ for every atom $A \in \mathcal{A}(G_0)$. We put

$$\mu(G) = \max\{|G_0| : G_0 \subset G \text{ half-factorial}\}.$$

2. A set $G_0 \subset G$ is called weakly half-factorial (whf) if $\mathsf{k}(A) \in \mathbb{N}$ for every atom $A \in \mathcal{A}(G_0)$. We put

$$\mu_0(G) = \max\{|G_0| : G_0 \subset G \text{ whf}\}.$$

Clearly, every half-factorial set is whf and $\mu(G) \leq \mu_0(G)$. The above definition of a half-factorial set emphasizes the relation between half-factorial and whf sets. The more common definition is the arithmetical one: a set G_0 is called half-factorial if $\mathcal{B}(G_0)$ is a half-factorial monoid. Both are equivalent by the characterization result of L. Skula and A. Zaks. (As mentioned in the Introduction, the characterization result is valid for all torsion groups and thus Definition 2.1 would remain meaningful for torsion groups; the arithmetical definition is meaningful for any abelian group.)

For H an atomic monoid and a non-unit $a \in H$ let

$$\mathsf{L}_H(a) = \{l \in \mathbb{N} : a \text{ has a factorization into atoms of length } l\}$$

denote the set of lengths of a; if $a \in H$ is a unit, then set $\mathsf{L}_H(a) = \{0\}$. Further, $\mathcal{L}(H) = \{\mathsf{L}_H(a) : a \in H\}$ is called the system of sets of lengths of H. For $L = \{l_1, l_2, l_3, \dots\} \subset \mathbb{N}_0$ with $l_1 < l_2 < l_3, \dots$, let $\Delta(L) = \{l_2 - l_1, l_3 - l_2, \dots\}$ denote the set of (successive) distances; and $\Delta(H) = \bigcup \big\{\Delta(L) : L \in \mathcal{L}(H)\big\}$ denotes the set of distances of H. Having this notation at hand we can rephrase the informal statement, given in the Introduction, on lengths of factorizations in Krull monoids. For H a Krull monoid $\mathsf{L}_H(a) = \mathsf{L}_{\mathcal{B}(G_0)}(B)$, where G_0 is the set of classes containing prime divisors and B is the block obtained by mapping each prime divisor in the factorization of a to its class. In particular, $\mathcal{L}(H) = \mathcal{L}(\mathcal{B}(G_0))$. We use these notations mainly for block monoids and usually omit the subscript H; we write $\mathcal{L}(G_0)$ and $\Delta(G_0)$ instead of $\mathcal{L}(\mathcal{B}(G_0))$ and $\Delta(\mathcal{B}(G_0))$, respectively. Note that G_0 is half-factorial if and only if $\mathcal{L}(G_0) = \{\{n\} : n \in \mathbb{N}_0\}$, which is equivalent to $\Delta(G_0) = \emptyset$.

We also use some specific shorthand notation. Let G be a finite abelian group. For a character $\chi \in \widehat{G}$, let

$$U_\chi = \big\{g \in G : \chi(g) = e\big(\tfrac{1}{\operatorname{ord} g}\big)\big\}$$

and

$$m(\chi) = \max\{\operatorname{ord} g : g \in G, \operatorname{ord} \chi(g) = \operatorname{ord} g\}.$$

For $G = H \oplus \langle u \rangle$ with $\operatorname{ord} u = \exp(G)$, possibly $|H| = 1$, and $m \mid \exp(G)$, let

$$U_{u,H}(m) = \bigcup_{d \mid m} \big(\tfrac{\exp(G)}{d} u + H[d]\big).$$

We frequently write $U(m)$ instead of $U_{u,H}(m)$ if the choice of u and H is clear from the context.

3 Structure of weakly half-factorial sets

In this section, we determine the structure of (inclusion-maximal) weakly half-factorial sets (see Theorem 3.2). In particular, we derive an explicit formula for $\mu_0(G)$. It turns out that there exists, up to automorphisms of the group, a unique whf set with maximal cardinality, and every whf set corresponds to a certain subset of this "universal" whf subset. Not every inclusion-maximal whf set has maximal cardinality (cf. Proposition 3.4). Yet, we show (see Corollary 3.3) that every inclusion-maximal whf set is a whf set with maximal cardinality in the subgroup it generates; the analogous statement for half-factorial sets is in general not true (cf. [9, Corollary 6.5]).

The key tool in investigations on whf sets is the following lemma that gives a characterization of whf sets in terms of characters of the group. It is due to J. Śliwa [33, Lemma 1]; in [9, Lemma 4.1] it is stated for bounded groups, and using essentially the same arguments it can be seen that it holds for torsion groups as well.

Lemma 3.1 (J. Śliwa)**.** *Let G be a finite abelian group. A subset $G_0 \subset G$ is whf if and only if there exists a character $\chi \in \widehat{G}$ such that $G_0 \subset U_\chi$.*

Now, we state our main result on whf sets.

Theorem 3.2. *Let $G = \oplus_{i=1}^r \langle e_i \rangle$ with $\operatorname{ord} e_i = n_i$, $i = 1, \dots, r$ and $n_1 \mid \dots \mid n_r$. Let $H = \langle e_1, \dots, e_{r-1} \rangle$.*

1. *A set $G_0 \subset G$ is an inclusion-maximal whf set if and only if there exists a character $\chi \in \widehat{G}$ such that $G_0 = U_\chi$ and $|G| \bigm| m(\chi)^k$ for some k.*

2. *For each $\chi \in \widehat{G}$, we have $\langle U_\chi \rangle = G[m(\chi)]$. There exists an automorphism f_χ of $\langle U_\chi \rangle$ such that $f_\chi(U_\chi) = U_{e_r, H}(m(\chi))$.*

3. *We have*
$$\mu_0(G) = \sum_{d \mid n_r} \prod_{i=1}^{r-1} \gcd(n_i, d).$$

4. *For $\chi \in \widehat{G}$ the following are equivalent:*
   (a) $|U_\chi| = \mu_0(G)$.
   (b) $\langle U_\chi \rangle = G$.
   (c) $m(\chi) = \exp(G)$.

Proof. We start with the case when G is a non-zero p-group; and we show 1. and 2. in this case. (Then we establish 1.–4. in the general case.) We begin by showing
$$\langle U_\chi \rangle = G[m(\chi)]. \tag{1}$$

It is clear that $U_\chi \subset G[m(\chi)]$, hence $\langle U_\chi \rangle \subset G[m(\chi)]$. To obtain the converse inclusion we first show that $g \in \langle U_\chi \rangle$ for all $g \in G$ such that $\operatorname{ord} g \mid m(\chi)$ and $\operatorname{ord} \chi(g) = \operatorname{ord} g$. Indeed, then we have $\chi(g) = e\big(\frac{a}{\operatorname{ord} g}\big)$ for some a, $p \nmid a$, so $\chi(bg) = e\big(\frac{1}{\operatorname{ord} g}\big) = e\big(\frac{1}{\operatorname{ord} bg}\big)$ for b such that $ab \equiv 1 \ (\operatorname{ord} g)$. Hence $bg \in U_\chi$ and $g = abg \in \langle U_\chi \rangle$. Now we suppose that $g \in G$, $\operatorname{ord} g \mid m(\chi)$, and $\operatorname{ord} \chi(g) < \operatorname{ord} g$. We take any $g' \in G$ such that $\operatorname{ord} \chi(g') = \operatorname{ord} g' = m(\chi)$. We have $g' \in \langle U_\chi \rangle$ by the previous argument, and also $g'' = \frac{m(\chi)}{\operatorname{ord} g} g' \in \langle U_\chi \rangle$, $\operatorname{ord} g'' = \operatorname{ord} g$. Moreover $\operatorname{ord} \chi(g'') = \frac{\operatorname{ord} g}{m(\chi)} \operatorname{ord} \chi(g') = \operatorname{ord} g > \operatorname{ord} \chi(g)$, so

$$\operatorname{ord} \chi(g'' - g) = \operatorname{ord} \chi(g'') = \operatorname{ord} g \geq \operatorname{ord}(g'' - g).$$

Consequently, also $g'' - g \in \langle U_\chi \rangle$ and $g \in \langle U_\chi \rangle$. We have shown (1). Note that the condition $|G| \big| m(\chi)^k$ for some k is, for p-groups, equivalent to $m(\chi) \neq 1$. Hence the 'only if'-part of 1. follows from Lemma 3.1 and the fact that for $m(\chi) = 1$ we have $U_\chi = \{0\}$, which is not inclusion-maximal. Conversely, suppose $\chi \in \widehat{G}$ and $p \mid m(\chi)$. To prove that U_χ is inclusion-maximal suppose $\chi' \in \widehat{G}$ and $U_\chi \subset U_{\chi'}$. Then $\langle U_\chi \rangle \subset \langle U_{\chi'} \rangle$. We have $\chi'|_{\langle U_\chi \rangle} = \chi|_{\langle U_\chi \rangle}$, because $\chi'(g) = \chi(g) = e\big(\frac{1}{\operatorname{ord} g}\big)$ for all $g \in U_\chi$. Take $g \in G$ such that $\operatorname{ord} g > m(\chi)$. We have $\operatorname{ord} \chi(g) < \operatorname{ord} g$. Since $\frac{\operatorname{ord} g}{m(\chi)} g \in \langle U_\chi \rangle$, we have $\chi'\big(\frac{\operatorname{ord} g}{m(\chi)} g\big) = \chi\big(\frac{\operatorname{ord} g}{m(\chi)} g\big)$, so $\operatorname{ord} \chi'\big(\frac{\operatorname{ord} g}{m(\chi)} g\big) < m(\chi)$ and $\operatorname{ord} \chi'(g) < \operatorname{ord} g$. Hence $m(\chi') \leq m(\chi)$, $\langle U_{\chi'} \rangle = \langle U_\chi \rangle$, and $U_{\chi'} = U_\chi$ as required. We have shown 1. To construct the automorphism f_χ, we take some $x \in U_\chi$ with $\operatorname{ord} x = m(\chi)$; such an element exists since $\langle U_\chi \rangle$ contains elements of order $m(\chi)$. Let $H' = \ker \chi \cap \langle U_\chi \rangle$. We have $\langle U_\chi \rangle = H' \oplus \langle x \rangle$ and

$$U_\chi = \bigcup_{d \mid m(\chi)} \big(\tfrac{m(\chi)}{d} x + H'[d]\big).$$

On the other hand, $\langle U_\chi \rangle = (H \cap \langle U_\chi \rangle) \oplus (\langle e_r \rangle \cap \langle U_\chi \rangle)$, and $\langle e_r \rangle \cap \langle U_\chi \rangle = \big\langle \frac{n_r}{m(\chi)} e_r \big\rangle \cong \langle x \rangle$, so $H' \cong H \cap \langle U_\chi \rangle$. Therefore f_χ can be defined by $x \mapsto \frac{n_r}{m(\chi)} e_r$ and any isomorphism of H' and $H \cap \langle U_\chi \rangle$. We have completed the proof of 1. and 2. for p-groups.

Now, let G be an arbitrary finite abelian group, and let $G = \oplus_{j=1}^s G_{p_j}$ be its decomposition to non-zero p-groups. We can assume $s \geq 1$, as all the assertions hold trivially for $G \cong \{0\}$. We show that every whf set associated to a character of G is an image of a cartesian product of whf sets in G_{p_j}, $j = 1, \ldots, s$, under a permutation of G. Let $\gamma : G \to G$,

$$\gamma(g_1 + \ldots + g_s) = \sum_{j=1}^{s} g_j \prod_{i \neq j} \operatorname{ord} g_i, \qquad g_j \in G_{p_j},\ j = 1, \ldots, s.$$

Note that the mapping $g \mapsto \gamma(g)$ preserves the order of g and of its coordinates. Hence

$$\gamma^{\varphi(|G|)}(g) = \sum_{j=1}^{s} g_j \left(\prod_{i \neq j} \operatorname{ord} g_i \right)^{\varphi(|G|)}$$

and, since $\varphi(\operatorname{ord} g_j) \mid \varphi(|G|)$, we have $\gamma^{\varphi(|G|)}(g) = \sum_{j=1}^{s} g_j = g$, so $\gamma^{\varphi(|G|)} = \mathrm{id}_G$, and γ is a permutation.

Let $\chi \in \widehat{G}$, $\chi_j = \chi|_{G_{p_j}}$ $(j = 1, \dots, s)$, and $g \in G$, $g = g_1 + \ldots + g_s$, $g_j \in G_{p_j}$ $(j = 1, \dots, s)$, as above. Suppose $\chi(g_j) = e\big(\frac{a_j}{\operatorname{ord} g_j}\big)$, $a_j \in \mathbb{Z}$, $j = 1, \dots, s$. We have $g \in U_\chi$ if and only if

$$\sum_{j=1}^{s} a_j \prod_{i \neq j} \operatorname{ord} g_i \equiv 1 \pmod{\prod_{j=1}^{s} \operatorname{ord} g_j},$$

which is equivalent to

$$a_j \prod_{i \neq j} \operatorname{ord} g_i \equiv 1 \pmod{\operatorname{ord} g_j}, \qquad \text{for each } j = 1, \dots, s,$$

further to

$$g_j \prod_{i \neq j} \operatorname{ord} g_i \in U_{\chi_j}, \qquad \text{for each } j = 1, \dots, s,$$

and thus to $\gamma(g) \in U_{\chi_1} + \ldots + U_{\chi_s}$. Therefore

$$U_\chi = \gamma^{-1}(U_{\chi_1} + \ldots + U_{\chi_s}).$$

Hence U_χ is maximal if and only if each U_{χ_j} is maximal, that is if $p_j \mid m(\chi_j)$ for each $j = 1, \dots, s$. Since $\operatorname{ord}(g_1 + \ldots + g_s) = \prod_{j=1}^{s} \operatorname{ord} g_j$ and $\operatorname{ord} \chi(g_1 + \ldots + g_s) = \prod_{j=1}^{s} \operatorname{ord} \chi_j(g_j)$, we have $m(\chi) = \prod_{j=1}^{s} m(\chi_j)$, and U_χ is maximal if and only if $p_j \mid m(\chi)$ for each $j = 1, \dots, s$. This implies 1. in view of Lemma 3.1.

To show $\langle U_\chi \rangle = G[m(\chi)]$ we note that

$$U_\chi = \gamma^{-1}(U_{\chi_1} + \ldots + U_{\chi_s}) \subset G[m(\chi)],$$

because γ is order-preserving. On the other hand $U_{\chi_j} \subset U_\chi$, $j = 1, \dots, s$, so

$$G[m(\chi)] = \langle U_{\chi_1} \rangle + \cdots + \langle U_{\chi_s} \rangle \subset \langle U_\chi \rangle$$

and $\langle U_\chi \rangle$ is as claimed. We construct the required automorphism. Let $\chi' \in \widehat{\langle U_\chi \rangle}$ be defined by

$$\chi'(h + a e_r) = e\big(\tfrac{a}{n_r}\big), \text{ where } h \in H \cap \langle U_\chi \rangle \text{ and } a \in \tfrac{n}{m(\chi)}\mathbb{Z}.$$

We have $m(\chi') = m(\chi)$ and

$$U_{\chi'} = U_{\frac{n}{m(\chi)}e_r, H\cap\langle U_\chi\rangle}(m(\chi)) = U_{e_r,H}(m(\chi)).$$

Further, let $\chi'_j = \chi'|_{G_{p_j}\cap\langle U_\chi\rangle}$ for $j = 1, \ldots, s$. Then we have, $m(\chi'_j) = \gcd(m(\chi'), |G_{p_j}|) = m(\chi_j)$, $\langle U_{\chi_j}\rangle = \langle U_{\chi'_j}\rangle$, and

$$U_{\chi'_j} = U_{e_r,H\cap G_{p_j}}(m(\chi_j)), \qquad j = 1, \ldots, s.$$

By the result for p-groups applied for every j, there exist automorphisms f_j of $\langle U_{\chi_j}\rangle$ such that $f_j(U_{\chi_j}) = U_{\chi'_j}$; that is $\chi_j|_{\langle U_{\chi_j}\rangle} = \chi'_j \circ f_j$. Let $f_\chi : \langle U_\chi\rangle \to \langle U_\chi\rangle$ be the automorphism induced by $f_1, \ldots, f_s$. Then $\chi|_{\langle U_\chi\rangle} = \chi' \circ f$, since the restrictions are equal for every p-component. We obtain $f_\chi(U_\chi) = U_{\chi'} = U_{e_r,H}(m(\chi))$ and the proof of 2. is complete.

We have shown that every set U_χ, $\chi \in \widehat{G}$, is an image of $U_{e_r,H}(m(\chi))$ under subgroup automorphism. We have $m(\chi) \mid n_r$ for all χ and we can find a $\chi \in \widehat{G}$ such that $m(\chi) = n_r$. Hence $U_{e_r,H}(n_r)$ is a whf set with maximum cardinality and an easy calculation leads to 3. Moreover, if $m(\chi) \neq n_r$, then $U_{e_r,H}(m(\chi))$ is a proper subset of $U_{e_r,H}(n_r)$, because the subgroup it generates is a proper subset of G. Thus we obtain 4. $\square$

Corollary 3.3. *If a set G_0 is inclusion-maximal whf, then $|G_0| = \mu_0(\langle G_0\rangle)$.*

Proof. By Theorem 3.2.1 we can find $\chi \in \widehat{\langle G_0\rangle}$ such that $G_0 = U_\chi$. Then the assertion follows from points 2. and 4. of the theorem. $\square$

To make the classification of whf sets complete we determine the possible values of $m(\chi)$ for $\chi \in G$.

Proposition 3.4. *Let G be a finite abelian group and let $m \in \mathbb{N}$. Then there exists a character $\chi \in \widehat{G}$ such that $m = m(\chi)$ if and only if $G \cong C_m \oplus H$ for some subgroup $H \subset G$. Specifically, if $G = \oplus_{p\in\mathcal{P}} \oplus_{i=1}^{r_p} C_{p^{k(p,i)}}$ for some finite set $\mathcal{P} \subset \mathbb{P}$ and r_p, $k(p,i) \in \mathbb{N}$, then the set of possible $m(\chi)$ equals $\prod_{p\in\mathcal{P}}\{1, p^{k(p,1)}, \ldots, p^{k(p,r_p)}\}$.*

Proof. If $G = \langle u\rangle \oplus H$ with $\operatorname{ord} u = m$, then the character defined by $\chi(au + h) = e(\frac{a}{m})$, for $a \in \mathbb{Z}$ and $h \in H$, fulfills $m(\chi) = m$. This proves the 'if'-part, and it remains to show the 'only if'-part.

Let $\chi \in \widehat{G}$. Since $m(\chi) = \prod_{p\in\mathcal{P}} m(\chi_p)$ where $\mathcal{P}$ is the set of primes dividing $|G|$ and χ_p the restriction of χ to the p-component (cf. the proof of Theorem 3.2), it suffices to prove the result for p-groups.

Further, if G is a p-group, say $G = \oplus_{i=1}^r C_{p^{k(i)}}$, and χ_i denotes the restriction of χ to $C_{p^{k(i)}}$ for $i = 1, \ldots, r$, then it is easy to see that $m(\chi) =$

$\max\{m(\chi_i)\colon i = 1, \dots, r\}$. Indeed, for all $g = \sum_{i=1}^r g_i$ with $g_i \in C_{p^{k_i}}$, we have $\operatorname{ord} g = \max\{\operatorname{ord} g_i\colon i = 1, \dots, r\}$ and $\operatorname{ord} \chi(g) = \max\{\operatorname{ord} \chi_i(g_i)\colon i = 1, \dots, r\}$. Thus it remains to consider the problem for cyclic groups of prime power order $G = C_{p^k}$. Let $\chi \in \widehat{C_{p^k}}$. If $\ker \chi = \{0\}$, then clearly $m(\chi) = p^k$. Otherwise, for every $g \in C_{p^k}$ we have either $g \in \ker \chi$ or $\operatorname{ord} \chi(g) = \frac{\operatorname{ord} g}{|\ker \chi|} < \operatorname{ord} g$, so $m(\chi) = 1$. $\square$

4 Some results on half-factorial sets and $\mu(G)$

As mentioned in the Introduction, the notion of weakly half-factorial sets has been introduced in order to investigate half-factorial sets and in particular the invariant $\mu(G)$. In this section we apply our result on weakly half-factorial sets for this purpose.

As an immediate consequence of Theorem 3.2 we obtain an upper bound for $\mu(G)$ valid for any finite abelian group, which improves the estimate obtained in [9, Corollary 4.4].

Corollary 4.1. *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $n_1 \mid \cdots \mid n_r$. Then*

$$\mu(G) \leq \sum_{d \mid n} \prod_{i=1}^{r-1} \gcd(n_i, d). \tag{2}$$

Proof. Obvious, by Theorem 3.2 and the inequality $\mu(G) \leq \mu_0(G)$. $\square$

In the rest of this section we are mainly concerned with the question of equality in (2). We exhibit certain classes of groups for which equality holds (see Theorem 4.2 and also Proposition 4.8.1), and conversely we derive conditions a group must fulfill when equality holds (see Theorem 4.5).

We start with the former problem and determine the value of $\mu(G)$ for certain types of groups. We point out that the first of the following results is known (for $n = 0$ see [31, 32, 34], for $m = n = 1$ see [19], and for the general case see [27]). We include it to emphasize that the present approach gives a unified proof. Also note that the second result was studied in some special cases (for $m = 1$ and $n = 0$ see [19] and for $n = 0$ see [29]).

Theorem 4.2. *Let p and q be distinct primes and $m, n \in \mathbb{N}_0$.*

1. $\mu(C_{p^m q^n}) = (m + 1)(n + 1)$.
2. $\mu(C_{p^m} \oplus C_{p^{m+n}}) = np^m + \frac{p^{m+1}-1}{p-1}$.
3. $\mu(C_2 \oplus C_{2p^n}) = 3(n + 1)$ *for $p \neq 2$.*

In order to prove the theorem, we recall the definition of the cross number of a group. It was initially considered by U. Krause [23]. For G a finite abelian group

$$\mathsf{K}(G) = \max\{\mathsf{k}(A) : A \in \mathcal{A}(G)\}$$

is called the cross number of G; moreover

$$\mathsf{k}(G) = \max\{\mathsf{k}(A) : A \in \mathcal{F}(G) \text{ zero-sumfree}\}$$

is called the little cross number of G. The value of the cross number of a group is in general unknown. Yet, the following lower bound was established in [24]. Let $G = C_{q_1} \oplus \cdots \oplus C_{q_r}$ with prime powers $q_1, \dots, q_r$, then

$$\mathsf{K}(G) \geq \frac{1}{\exp(G)} + \sum_{i=1}^{r} \frac{q_i - 1}{q_i}. \tag{3}$$

For some classes of groups it is known that equality holds in (3), for instance for p-groups (see [15]); and no example is known where equality does not hold. Moreover, $\mathsf{k}(G) \leq \mathsf{K}(G) - \frac{1}{\exp(G)}$ and equality holds for all G for which equality holds in (3). In Proposition 4.4 and the proof of Theorem 5.4 we will recall further results on the (little) cross number.

The following observation will be crucial in the proof of Theorem 4.2.

Lemma 4.3. *Let G be a finite abelian group. If $\mathsf{K}(G) < 2$, then every weakly half-factorial set is half-factorial. In particular, $\mu(G) = \mu_0(G)$.*

Proof. Obvious, by the definitions (resp. the characterization of half-factorial sets). □

We will later provide a counterexample to the converse of this result. By combining (3) with a result of A. Geroldinger and R. Schneider [20], it is straightforward to determine all groups with $\mathsf{K}(G) < 2$.

Proposition 4.4. *Let G be a finite abelian group. Then $\mathsf{K}(G) < 2$ if and only if*

- *$G \cong C_{p^m} \oplus C_{q^n}$ with p and q prime (not necessarily distinct) and $m, n \in \mathbb{N}_0$, or*
- *$G \cong C_2 \oplus C_{2p^n}$ with p prime and $n \in \mathbb{N}_0$.*

Proof. We note that $\mathsf{K}(G) = 1$ if $|G| = 1$, and assume that $|G| \geq 2$. Let $r = \mathsf{r}^*(G)$ and $q_1 \leq \cdots \leq q_r$ be prime powers such that $G \cong \oplus_{i=1}^r C_{q_i}$. Suppose $\mathsf{K}(G) < 2$. By (3) it follows that $r \leq 3$. And, if $r = 3$, then $q_1 = q_2 = 2$ and $\exp(G) > q_3$. This proves the 'only if'-part. To obtain the 'if'-part, we note that by [20, Theorem 2] for the groups under consideration equality holds in (3). □

With the preparatory results at hand, the theorem follows easily.

Proof of Theorem 4.2. Let G be any of the groups mentioned in the theorem. By Proposition 4.4 we have $\mathsf{K}(G) < 2$ and by Lemma 4.3 this implies $\mu(G) = \mu_0(G)$. Finally, Theorem 3.2 yields the explicit value for $\mu_0(G)$. $\square$

In view of what we mentioned in the Introduction, we point out that this approach and Theorem 3.2 also yields the structure of half-factorial sets (with maximal cardinality) for these groups. By Proposition 4.4 it is clear that there exist no further groups for which one can obtain the value of $\mu(G)$ by this argument.

Next, we address the converse problem and derive necessary conditions for a group G to satisfy $\mu(G) = \mu_0(G)$. We point out that by Theorem 3.2, for any finite abelian group, the condition $\mu(G) = \mu_0(G)$ is equivalent to the formally stronger condition that every weakly half-factorial set is half-factorial.

Theorem 4.5. *Let $G \neq \{0\}$ be a finite abelian group for which $\mu(G) = \mu_0(G)$. Then*

$$G \cong C_{p^m} \oplus C_n$$

with p prime and $m, n \in \mathbb{N}$. Moreover, if $p \mid n$, then for every $k \mid n$ with $p \nmid k$ the congruence $k \equiv 1 \pmod{p^m}$ holds.

Before we turn to the proof of this theorem we contrast it with Theorem 4.2 and discuss the problem of completely classifying groups with $\mu(G) = \mu_0(G)$.

On the one hand, Theorem 4.5 yields that if $\mu(G) = \mu_0(G)$, then G is cyclic or of rank two and has to fulfill additional conditions, and by Theorem 4.2 we know that for some of these groups the equality $\mu(G) = \mu_0(G)$ actually holds.

On the other hand, it was already shown by A. Zaks [34] that there exist cyclic groups for which $\mu(G) < \mu_0(G)$. Indeed, already the minimal cyclic group (in terms of cardinality and cross number) not covered by Theorem 4.2, which is $G = C_{30}$, satisfies $\mu(G) < \mu_0(G)$; that is, the converse of Theorem 4.5 cannot hold in general. However, it is not the case that all cyclic groups with cross number greater or equal to 2, equivalently with order divisible by three distinct primes, satisfy $\mu(G) < \mu_0(G)$, and the converse of Theorem 4.2 cannot hold either. For example, it is know that $\mu(C_{6p}) = 8$ if p is a prime congruent to 1 mod 6 (see [27, Proposition 7.5]). This and other results, obtained in [27], indicate that a complete answer to the problem $\mu(G) = \mu_0(G)$ could be quite complicated. There, the problem was investigated for cyclic groups; and it turned out that a criterion cannot depend just on the number and multiplicities of prime divisors of $|G|$ or some threshold-value for $\mathsf{K}(G)$, but needs to take into account at least congruence relations among the (prime) divisors of $|G|$.

Here, we investigate the problem for certain groups of rank two. We obtain a result (cf. Proposition 4.8) that suggests that the 'moreover'-condition in Theorem 4.5 is natural; and in addition it yields the value of $\mu(G)$ for a further class of groups.

The proof of Theorem 4.5 is mainly based on the following two auxiliary results.

Lemma 4.6. *Let G be a finite abelian group and G' a subgroup of G. If $\mu(G') < \mu_0(G')$, then $\mu(G) < \mu_0(G)$.*

Proof. Suppose $\mu(G') < \mu_0(G')$. Let $U \subset G$ and $U' \subset G'$ be whf sets with maximal cardinality $\mu_0(G)$ and $\mu_0(G')$, respectively. By assumption we have $|U'| > \mu(G')$ and thus U' is not half-factorial. Let $U'' \subset G$ be an inclusion-maximal whf subset of G containing U'. By Theorem 3.2, assertion 1., $U'' = U_\chi$ for some character $\chi \in \widehat{G}$ and by assertion 2. of that theorem there exists an automorphism f of $\langle U'' \rangle$ such that $f(U'') \subset U$, hence $f(U') \subset U$. Thus, U is not half-factorial. □

Proposition 4.7. *Let p and q be distinct primes, $k, m, n \in \mathbb{N}$ with $m \le n$ and $G = C_{p^m} \oplus C_{p^n q^k}$. If $q \not\equiv 1 \pmod{p^n}$, then $\mu(G) < \mu_0(G)$.*

Proof. Suppose $q \not\equiv 1 \pmod{p^n}$. Using Lemma 4.6 we easily reduce the problem to the case $k = m = 1$ and $q \equiv 1 \pmod{p^{n-1}}$, since it suffices to prove $\mu(G') < \mu_0(G')$ for a subgroup $G' \subset G$ isomorphic to $C_p \oplus C_{p^{n'}q}$ with n' such that $q \not\equiv 1 \pmod{p^{n'}}$ but $q \equiv 1 \pmod{p^{n'-1}}$. Let $e_1, e_2 \in G$ with $\operatorname{ord} e_1 = p$ and $\operatorname{ord} e_2 = p^n q$ a basis of G. Further, let $a \in [1, q-1]$ such that $ap^n \equiv 1 \pmod{q}$ and $b \in [1, p^n - 1]$ such that $bq \equiv 1 \pmod{p^n}$. Since $q \equiv 1 \pmod{p^{n-1}}$, we have $b = 1 + jp^{n-1}$ with $j \in [1, p-1]$.

By Theorem 3.2 we know that the set

$$G_0 = \{e_2 + e_1, qe_2, p^n e_2, p^{n-1}qe_2 + e_1, p^{n-1}qe_2\}$$

is contained in a maximal whf set, and we show that G_0 is not half-factorial. Let $S = (e_2 + e_1)(qe_2)^{p^{n-1}-1}(p^n e_2)^{q-a}$. Then $\sigma(S) \in \langle e_1, p^{n-1}qe_2 \rangle$ and no proper non-empty subsequence of S has sum in $\langle e_1, p^{n-1}qe_2 \rangle$; more precisely $\sigma(S) = (j+1)p^{n-1}qe_2 + e_1$. We set

$$F = (p^{n-1}qe_2 + e_1)^{p-1}(p^{n-1}qe_2)^{p-j}.$$

Then, $\sigma(SF)$ is a zero-sum sequence and in fact an atom. Since $\mathsf{k}(SF) > \mathsf{k}(F) \ge 1$, it follows that G_0 is not half-factorial. □

Proof of Theorem 4.5. Let $U \subset G$ be a whf subset with maximal cardinality $\mu_0(G)$. By the condition $\mu_0(G) = \mu(G)$ and Theorem 3.2 we know that U is half-factorial.

First, we assert that for every prime p the p-rank of G is at most two. Assume to the contrary that we have $\mathsf{r}_p(G) > 2$ for some p. Then there exist three independent elements $e_1, e_2, e_3 \in G$, each of which has order p. We consider the subgroup $G' = \langle e_1, e_2, e_3 \rangle \cong C_p^3$. We know

$$\mu(G') \leq 1 + 3\frac{p}{2} < 1 + p^2 = \mu_0(G');$$

the first inequality by [19] (cf. Introduction) and the equality (for instance) by Theorem 3.2. By Lemma 4.6 this contradicts $\mu_0(G) = \mu(G)$. Thus, we obtain that $\mathsf{r}_p(G) \leq 2$ for each prime p.

Now, assume there exist distinct primes p and q such that $\mathsf{r}_p(G) = \mathsf{r}_q(G) = 2$. Then, there exist subgroups $G_1 \cong C_p \oplus C_{pq}$ and $G_2 \cong C_q \oplus C_{pq}$ of G. Assume without restriction $q < p$. Then $q \not\equiv 1 \pmod{p}$ and by Proposition 4.7 we have $\mu(G_1) < \mu_0(G_1)$, again a contradiction by Lemma 4.6. Thus, there exists at most one prime p for which actually $\mathsf{r}_p(G) = 2$; and therefore $G \cong C_{p^m} \oplus C_n$.

To obtain the 'moreover'-statement, it remains to show that if $p \mid n$, then $k \equiv 1 \pmod{p^m}$ for every $k \mid n$ with $p \nmid k$. Clearly, it suffices to show this for every prime $q \neq p$ with $q \mid n$. Let $q \neq p$ be a prime with $q \mid n$. Then $pq \mid n$ and thus there exists a subgroup $H \cong C_{p^m} \oplus C_p \oplus C_q$ of G. By Lemma 4.6 we have $\mu(H) = \mu_0(H)$ and thus by Proposition 4.7 we get $q \equiv 1 \pmod{p^m}$. $\square$

We end this section with a result on $\mu(C_p \oplus C_{pq})$.

Proposition 4.8. *Let p and q be distinct primes and $G = C_p \oplus C_{pq}$.*

1. *If $q \equiv 1 \pmod{p}$, then $\mu(G) = \mu_0(G) = 2p + 2$.*
2. *If $q \not\equiv 1 \pmod{p}$, then $\mu(G) = \mu_0(G) - 1 = 2p + 1$.*

Proof. Let $e_1, e_2 \in G$ with $\operatorname{ord} e_1 = p$ and $\operatorname{ord} e_2 = pq$ a basis of G. By Theorem 3.2 we have $\mu_0(G) = 2p + 2$ and

$$U = U_{e_2, \langle e_1 \rangle}(pq) = (e_2 + \langle e_1 \rangle) \cup (qe_2 + \langle e_1 \rangle) \cup \{pe_2, 0\}$$

is whf with maximal cardinality. We note that $(qe_2 + \langle e_1 \rangle) \cup \{0\}$ is half-factorial, which can be seen easily since it is a subset of the group $qG \cong C_p^2$. Moreover, since $G = \langle pe_2 \rangle \oplus qG$, the set $(qe_2 + \langle e_1 \rangle) \cup \{pe_2, 0\}$ is half-factorial as well.

First, we show that $G_0 = U \setminus \{pe_2\}$ is half-factorial. Let $A \in \mathcal{A}(G_0)$. We have to show that $\mathsf{k}(A) = 1$. If $\operatorname{ord} g < pq$ for every $g \in \operatorname{supp}(A)$, then $\operatorname{supp}(A) \subset (qe_2 + \langle e_1 \rangle) \cup \{0\}$. Consequently, $\operatorname{supp}(A)$ is half-factorial and we have $\mathsf{k}(A) = 1$.

Let $S \mid A$ be the subsequence consisting of those elements that have order pq, that is those from $e_2 + \langle e_1 \rangle$, and assume that $S \neq 1$. We note that $\sigma(S) \in qG$ and thus $|S| = lq$ for some $l \in \mathbb{N}$. We factor S into l subsequences $S_1, \dots, S_l$ each of length q, and set $h_i = \sigma(S_i) \in qe_2 + \langle e_1 \rangle$ for $i = 1, \dots, l$. We consider the sequence $A' = (\prod_{i=1}^{l} h_i)(S^{-1}A) \in \mathcal{F}(qe_2 + \langle e_1 \rangle)$. The sequence A' is a zero-sum sequence and in fact it is even an atom, since every factorization of A' would yield a factorization of A by replacing h_i with S_i. Since $\mathsf{k}(S_i) = \frac{q}{pq} = \frac{1}{p} = \mathsf{k}(h_i)$, we have $\mathsf{k}(A) = \mathsf{k}(A') = 1$. Thus, G_0 is half-factorial and $\mu(G) \geq \mu_0(G) - 1$.

By Proposition 4.7 we know that if $q \not\equiv 1 \pmod p$, then $\mu(G) < \mu_0(G)$ and 2. follows. It remains to prove 1.

Suppose $q \equiv 1 \pmod p$. We need to show that U is half-factorial. Let $A \in \mathcal{A}(U)$, and we again assert that $\mathsf{k}(A) = 1$. If $\operatorname{supp}(A) \cap (e_2 + \langle e_1 \rangle) = \emptyset$, then this is obvious. Thus, suppose this is not the case and let again $S \mid A$ be the subsequence consisting of the elements from $e_2 + \langle e_1 \rangle$. We may assume that $|S| \leq q - 1$; by the same reasoning as above we could replace subsequences of S of length q by their sum, an element of $qe_2 + \langle e_1 \rangle$. Let $v = \mathsf{v}_{pe_2}(A)$. Since A is an atom, we have $v \in [0, q-1]$; moreover, since $g_0 = \sigma(S) + vpe_2 \in qG$, we have

$$|S| + vp \equiv 0 \pmod q,$$

hence $|S| + vp = kq$ for some $k \in [1, p-1]$. We write $|S| = sp + r$ with $r \in [1, p]$ and $s \in \mathbb{N}_0$. Then, since $q \equiv 1 \pmod p$, we have $k = r$ and

$$v = r\frac{q-1}{p} - s.$$

Therefore $\mathsf{k}(S(pe_2)^v) = \frac{r}{p}$.

We set $F = (pe_2)^{-v} S^{-1} A \in \mathcal{F}(qe_2 + \langle e_1 \rangle)$ and have to show that $\mathsf{k}(F) = \frac{p-r}{p}$. Since U is whf, we know that $\mathsf{k}(A) \in \mathbb{N}$ and therefore $p\mathsf{k}(F) \equiv p - r \pmod p$. Since F has to be zero-sum free, we get $\mathsf{k}(F) < \mathsf{K}(\langle qe_2 + \langle e_1 \rangle \rangle) = \frac{2p-1}{p}$ (cf. the discussion after (3)). Thus, it remains to show that $\mathsf{k}(F) \neq \frac{2p-r}{p}$, that is $|F| \neq 2p - r$, for $r \geq 2$. We assume to the contrary that $r \geq 2$ and $|F| = 2p - r$. Let $S_1 = \prod_{i=1}^{r-1} g_i \mid S$ be some proper subsequence of length $r - 1$. Since $q \equiv 1 \pmod p$, the homomorphism from G to qG defined via multiplication with q is the projection from $G = \langle pe_2 \rangle \oplus qG$ to qG; in particular, it is identity on qG. We denote by $\overline{S_1} = \prod_{i=1}^{r-1}(qg_i) \in \mathcal{F}(qG)$ the projection of S_1.

The sequence $\overline{S_1}F$ has length $2p-1$ and thus a zero-sum subsequence, say, $\overline{S_2}F_1$ with $S_2 \mid S_1$ and $F_1 \mid F$, where again $\overline{S_2}$ denotes the projection to qG. Since F is zero-sum free, we have $S_2 \neq 1$. We observe that $\sigma(S_2F_1) \in \langle pe_2 \rangle$.

Since $v \geq (r-1)\frac{q-1}{p}$ and $|S_2| \in [1, r-1]$, we get that

$$S_2(pe_2)^{|S_2|\frac{q-1}{p}}F_1$$

is a proper subsequence of A. This sequence is a zero-sum sequence, since $\sigma(S_2(pe_2)^{|S_2|\frac{q-1}{p}}) = \sigma(\overline{S_2})$; which contradicts that A is an atom. Thus, $|F| \neq 2p - r$ and $\mathsf{k}(A) = 1$. $\square$

5 Arithmetic of weakly half-factorial sets

In this section, we present some results on the arithmetic of block monoids over weakly half-factorial sets; more specifically we study the systems of sets of lengths of these monoids. To this end, we investigate a certain subset of $\Delta(G)$, which we denote by $\Delta_0^*(G)$ (cf. below). It is to a certain extent a "measure" for the complexity of these systems.

It is well known (see [11, Satz 1] and [8, 17] for generalizations and developments) that $\mathcal{L}(G_0)$ consists of almost arithmetical multiprogressions bounded by some constant M depending only on G_0. Roughly, this means that every $L \in \mathcal{L}(G_0)$ is, up to at most $2M$ exceptional elements, equal to $y + \mathcal{D} + d \cdot [0, l]$ for $y, l \in \mathbb{N}_0$, $d \in [1, M]$, and $\{0, d\} \subset \mathcal{D} \subset [0, d]$; the integer d is equal to $\min \Delta(G_0')$ for some $G_0' \subset G_0$ and is called a difference of L. Conversely, for every (non-half-factorial) $G_0' \subset G_0$ there exist (arbitrarily large) sets in $\mathcal{L}(G_0)$ with difference $\min \Delta(G_0')$. Thus, the values $\min \Delta(G_0')$ for $G_0' \subset G_0$ are important invariants when studying $\mathcal{L}(G_0)$.

These minimal distances were initially considered by A. Geroldinger [13, 14] and in [10] the following notation has been introduced: For G a finite abelian group, let

$$\Delta^*(G) = \{\min \Delta(G_0) : G_0 \subset G,\ \Delta(G_0) \neq \emptyset\}.$$

Here, we are only interested in block monoids over weakly half-factorial sets and thus consider the set

$$\Delta_0^*(G) = \{\min \Delta(G_0) : G_0 \subset G \text{ whf},\ \Delta(G_0) \neq \emptyset\}.$$

We note that by Theorem 3.2, for $U \subset G$ a whf subset with maximal cardinality, $\Delta_0^*(G) = \{\min \Delta(G_0) : G_0 \subset U,\ \Delta(G_0) \neq \emptyset\}$.

Clearly, we have $\Delta_0^*(G) \subset \Delta^*(G)$, and $\Delta_0^*(G) = \emptyset$ if and only if $\mu(G) = \mu_0(G)$. For $G' \subset G$ a subgroup, we have $\Delta_0^*(G') \subset \Delta_0^*(G)$. We freely make use of the fact that $\min \Delta(G_0) = \gcd \Delta(G_0)$ for any $G_0 \subset G$ (see [11, Proposition 4]).

For $\Delta^*(G)$ various results are known. Among others, it is known that $[1, \mathsf{r}(G) - 1] \subset \Delta^*(G)$ and that $\operatorname{ord}(g) - 2 \in \Delta^*(G)$ for each $g \in G$ with $\operatorname{ord} g \geq 3$ (see [10, Proposition 5.2]). And, several bounds for $\max \Delta^*(G)$ are known (see [10, 18, 30] and cf. below). For certain types of groups, for instance elementary p-groups, these bounds are known to be sharp. In the remainder of this section, we obtain similar results for $\Delta_0^*(G)$.

It follows from our remarks above that $\Delta^*(G)$ can be arbitrarily large (in terms of cardinality) while $\Delta_0^*(G) = \emptyset$; for instance consider cyclic groups of order p^k with large k. Yet, we show that it is also possible that $\max \Delta^*(G) = \max \Delta_0^*(G)$, even if $\max \Delta^*(G)$ is large (cf. Theorem 5.6).

Given that results of the preceding sections show that although for some groups all whf sets are half-factorial, for other groups, for instance elementary p-groups with large rank, $\mu_0(G)$ is much larger than $\mu(G)$, it is not surprising that both the absolute and the relative size of $\Delta_0^*(G)$, compared with $\Delta^*(G)$, depends heavily on the group G.

First, we give a "lower bound" for $\Delta_0^*(G)$, that is we determine various elements that are contained in $\Delta_0^*(G)$. Then, we obtain upper bounds for $\max \Delta_0^*(G)$. Moreover, we determine $\Delta_0^*(G)$ for elementary 2-groups (cf. Theorem 5.6).

Proposition 5.1. *Let G be a finite abelian group and let $r_p = \mathsf{r}_p(G)$ for some prime p.*

1. $p \cdot [1, \lfloor \frac{r_p - 1}{p} \rfloor] \subset \Delta_0^*(G)$.
2. *If p is odd, then* $[1, \lfloor \frac{r_p - 1}{2} \rfloor] \subset \Delta_0^*(G)$.

An essential part of the proof of this result is carried out in the following lemma. There, we compute $\min \Delta(G_0)$ for certain (not necessarily whf) sets $G_0 \subset G$. We consider more general sets than those that were considered in the proofs of similar results for $\Delta^*(G)$ (cf. below). This generality is needed because of the additional condition that the sets, to be considered in the proof of Proposition 5.1, have to be whf.

Lemma 5.2. *Let $\{e_1, \dots, e_t\}$ be independent elements with $\operatorname{ord} e_i = n > 2$ for $i = 1, \dots, t$. Further, let $s \in [1, t]$, $g_s = \sum_{i=1}^s e_i - \sum_{i=s+1}^t e_i$, and $G_s = \{g_s, e_1, \dots, e_t\}$. Then*

$$\min \Delta(G_s) = \gcd(s - 1, t - 1).$$

Proof. Set $g = g_s$. We consider the sequences $U_i = e_i^n$ for each i, and $W_j = g^j \prod_{i=1}^s e_i^{n-j} \prod_{i=s+1}^t e_i^j$ for $j = 1, \dots, n-1$, and $W_n = g^n$; we note that $\mathcal{A}(G_s) = \{U_1, \dots, U_t\} \cup \{W_1, \dots, W_n\}$. Among these atoms we have the following relations. For $a, b \in [1, n-1]$ we have

$$W_a W_b = \begin{cases} W_{a+b} \prod_{i=1}^s e_i^n & \text{if } a+b < n \\ W_n \prod_{i=1}^t e_i^n & \text{if } a+b = n \\ W_{a+b-n} W_n \prod_{i=s+1}^t e_i^n & \text{if } a+b > n \end{cases}.$$

These relations yield the distances $s-1$, $t-1$, and $t-s$, respectively. Clearly, $\gcd(s-1, t-1, s-t) = \gcd(s-1, t-1)$ and thus $\min \Delta(G_s) \mid \gcd(s-1, t-1)$. Conversely, we observe that for each block $B \in \mathcal{B}(G_s)$ any two factorizations of B can be connected by a chain of factorizations such that any two consecutive factorizations can be obtained from each other by applying one of the relations given above. Therefore we have $\gcd(s-1, t-1) \mid \min \Delta(G_s)$ and equality holds. □

The case $s = t$ occurred already in the proof of [10, Proposition 5.2]. The case $s = 0$, which is not covered by this lemma, was considered in [4, Example 4.11]; it yields $|t+1-n|$ as minimal distance. These two results hold for $n = 2$ as well and yield the minimal distance $t-1$. In the following lemma, which we will need in the proof of Theorem 5.6, we consider the minimal distance of more general subsets of elementary 2-groups. It contains the above mentioned result, for $n = 2$, as the special case $I = J$.

Lemma 5.3. *Let $\{e_1, \dots, e_t\}$ be independent elements with $\operatorname{ord} e_i = 2$ for $i = 1, \dots, t$. Let $I, J \subset [1, t]$ with $|I|, |J| \ge 2$ and $I \cap J \ne \emptyset$. Further, let $g_I = \sum_{i \in I} e_i$, $g_J = \sum_{j \in J} e_j$, and $G_0 = \{g_I, g_J, e_1, \dots, e_t\}$. Then*

$$\min \Delta(G_0) = \gcd(|I|-1, |J|-1, |I \cap J|-1).$$

Proof. Let $A_I = g_I \prod_{i \in I} e_i$, $A_J = g_J \prod_{j \in J} e_j$ and $A_{I,J} = g_I g_J \prod_{i \in I \triangle J} e_i$, where $\triangle$ denotes the symmetric difference. Then $\mathcal{A}(G_0) = \{A_I, A_J, A_{I,J}\} \cup \{g^2 \colon g \in G_0\}$. We have the following relations $A_I^2 = g_I^2 \prod_{i \in I} e_i^2$, $A_J^2 = g_J^2 \prod_{j \in J} e_j^2$, $A_{I,J}^2 = g_I^2 g_J^2 \prod_{i \in I \triangle J} e_i^2$, and $A_I A_J = A_{I,J} \prod_{i \in I \cap J} e_i^2$, yielding the distances $|I|-1$, $|J|-1$, $|I \triangle J|$, and $|I \cap J|-1$, respectively. Thus, $\min \Delta(G_0)$ divides the greatest common divisor of these distances, which is $\gcd(|I|-1, |J|-1, |I \cap J|-1)$, since $|I \triangle J| = (|I|-1) + (|J|-1) - 2(|I \cap J|-1)$. As in the proof of Lemma 5.2, we can argue that in fact $\min \Delta(G_0) = \gcd(|I|-1, |J|-1, |I \cap J|-1)$. □

Proof of Proposition 5.1. We consider a maximal whf set $U = U_{u,H}(\exp(G))$, with suitable u and H, as defined in the Preliminaries. We start with the following assertion: Let $\{e_1, \ldots, e_t\} \subset U$ independent with $\operatorname{ord} e_1 = \cdots = \operatorname{ord} e_t = n$, $s \in [1, t]$, $\sum_{i=1}^{s} e_i - \sum_{i=s+1}^{t} e_i$, and $G_s = \{g_s, e_1, \ldots, e_t\}$. Then $G_s \subset U$ if and only if $2s - 1 \equiv t \pmod n$.

Let $\chi \in \widehat{G}$ such that $U_\chi = U$. Then $\chi(e_i) = e(\frac{1}{n})$ for $i = 1, \ldots, t$ and thus $\chi(g_s) = e(\frac{s-(t-s)}{n})$. Since $\operatorname{ord} g_s = n$, we have $g_s \in U$ if and only if $e(\frac{2s-t}{n}) = e(\frac{1}{n})$, which proves the assertion.

By Theorem 3.2 it follows that there exist independent $e_1, \ldots, e_{r_p} \in U$ with $\operatorname{ord} e_i = p$. We assume $r_p \geq 3$, since otherwise the result follows trivially.

1. Let $t \in [2, r_p]$ such that $t \equiv 1 \pmod p$, that is $t = 1 + jp$ for some $j \in [1, \lfloor \frac{r_p - 1}{p} \rfloor]$. Let $\{e_1, \ldots, e_t\} \subset U$ independent with $\operatorname{ord} e_1 = \cdots = \operatorname{ord} e_t = p$. We set $g_t = \sum_{i=1}^{t} e_i$ and $G_t = \{g_t, e_1, \ldots, e_t\}$. Then $G_t \subset U$ and $\min \Delta(G_t) = t - 1$, by Lemmas 5.2 and 5.3 (for $p = 2$).

2. Suppose p is odd and let $t \in [3, r_p]$ be an odd integer. Further, let $\{e_1, \ldots, e_t\} \subset U$ independent with $\operatorname{ord} e_1 = \cdots = \operatorname{ord} e_t = p$. We set $s = \frac{t+1}{2}$ and $g_s = \sum_{i=1}^{s} e_i - \sum_{i=s+1}^{t} e_i$, and $G_s = \{g_s, e_1, \ldots, e_t\}$. Then, by our assertion $G_s \subset U$ and by Lemma 5.2 we have $\min \Delta(G_s) = \gcd(s-1, t-1) = \frac{t-1}{2}$. Thus, $\{\frac{t-1}{2} : t \in [3, r_p], 2 \nmid t\} \subset \Delta_0^*(G)$, and the result follows. $\square$

Next, we establish an upper bound for $\max \Delta_0^*(G)$ valid for any finite abelian group.

Theorem 5.4. *Let G be a finite abelian group. Then*

$$\max \Delta_0^*(G) \leq 2\mathsf{k}(G) - 1.$$

In particular, $\max \Delta_0^*(G) \leq 2 \log |G| - 1$.

In the proof of this theorem the following lemma is essential.

Lemma 5.5. *Let G be a finite abelian group and let $G_0 \subset G$ a non-half-factorial subset. If $\mathsf{k}(A) \geq 1$ for each $A \in \mathcal{A}(G_0)$, then*

$$\min \Delta(G_0) \leq 2\mathsf{k}(G) - 1.$$

Proof. Suppose $\mathsf{k}(A) \geq 1$ for each $A \in \mathcal{A}(G_0)$. We assume without restriction that the set G_0 is minimal non-half-factorial, that is each proper subset is half-factorial. (Otherwise, we could consider a proper non-half-factorial subset $G_0' \subset G_0$; then $\Delta(G_0') \subset \Delta(G_0)$ and consequently $\min \Delta(G_0) \leq \min \Delta(G_0')$.)

Since the set is not half-factorial, there exists an atom with cross number greater than 1; let $W \in \mathcal{A}(G_0)$ with maximal cross number. Since G_0 is minimal non-half-factorial, we have $\operatorname{supp}(W) = G_0$. We note that for each block $B \in \mathcal{B}(G_0)$ we have $\min \mathsf{L}(B) \geq \frac{\mathsf{k}(B)}{\mathsf{k}(W)}$ and $\max \mathsf{L}(B) \leq \mathsf{k}(B)$. Let k

be the minimum of all integers l with $|\mathsf{L}(W^l)| \geq 2$; by a standard argument, considering for instance $W^{\exp(G)}$ (also cf. the argument below), such a k exists and indeed $k \in [2, \exp(G)]$.

For each $l \in \mathbb{N}$ we have $l = \min \mathsf{L}(W^l)$. Thus, we have $\min \mathsf{L}(W^k) = k$ and we estimate $\max \mathsf{L}(W^k)$, which can be done by estimating $\mathsf{k}(W^k)$. Let $g \in G_0$, $v = \mathsf{v}_g(W^{k-1})$, and $F = g^{-v}W^{k-1}$. We assert that $v < \operatorname{ord} g$ and F is zero-sumfree. Assume to the contrary that $v \geq \operatorname{ord} g$. Then $g^{\operatorname{ord} g} \mid W^{k-1}$, and since $\mathsf{k}(g^{-\operatorname{ord} g}W^{k-1}) = \mathsf{k}(W^{k-1}) - 1 > (k-2)\mathsf{k}(W)$, we have $\max \mathsf{L}(W^{k-1}) \geq 1 + \max \mathsf{L}(g^{-\operatorname{ord} g}W^{k-1}) > 1 + (k-2)$, which contradicts $|\mathsf{L}(W^{k-1})| = 1$. Similarly, if F is not zero-sumfree, then there exists an atom $A \mid F$ and, since $\operatorname{supp}(A) \subset G_0 \setminus \{g\}$ is half-factorial, we have $\mathsf{k}(A) = 1$. This yields again a factorization of W^{k-1} with length larger than $k-1$. Thus, we conclude

$$\begin{aligned} \mathsf{k}(W^k) &= \mathsf{k}(W^{k-1}) + \mathsf{k}(W) = \frac{v}{\operatorname{ord} g} + \mathsf{k}(F) + \mathsf{k}(W) \\ &\leq \frac{\operatorname{ord} g - 1}{\operatorname{ord} g} + \mathsf{k}(G) + \left(\frac{1}{\operatorname{ord} g} + \mathsf{k}(G)\right) = 2\mathsf{k}(G) + 1. \end{aligned}$$

This yields the claimed bound, since $\min \Delta(G_0) \leq \max \mathsf{L}(W^k) - \min \mathsf{L}(W^k)$. □

Proof of Theorem 5.4. By definition we have to show that $\min \Delta(G_0) \leq 2\mathsf{k}(G) - 1$ for every whf set $G_0 \subset G$ that is not half-factorial. In case such a subset does not exist, the result holds trivially. By definition, non-half-factorial whf sets fulfill the condition of Lemma 5.5 and the result follows.

The 'in particular'-statement follows, since $\mathsf{k}(G) \leq \log |G|$ (see [21, Theorem 2]). □

For special classes of groups, for instance for p-groups, (more) precise results on $\mathsf{k}(G)$ are known. If one applies these results, instead of the general upper bound for $\mathsf{k}(G)$, one obtains better explicit bounds for $\max \Delta_0^*(G)$ for these groups; for instance, $\max \Delta_0^*(G) \leq 2r\frac{\exp(G)-1}{\exp(G)} - 1$ for p-groups of rank r. (Note that for p-groups with $p \geq 2r - 1$ we obtain a further improvement in Theorem 5.7.)

Moreover, Lemma 5.5 gives a new, more elegant proof of the estimate $\max \Delta^*(G) \leq \max\{\exp(G) - 2, 2\mathsf{k}(G) - 1\}$ (cf. [30, Theorem 3.1]), since it is known (see [10, Lemma 5.4]) that the existence of an atom $A \in \mathcal{A}(G_0)$ with $\mathsf{k}(A) < 1$ yields $\exp(G) - 2$ as an upper bound for $\min \Delta(G_0)$.

For special types of groups more precise results on $\Delta^*(G)$ are known. We obtain similar results for $\Delta_0^*(G)$. We recall (see [30, Theorem 4.1]) that for

$p \in \mathbb{P}$ and $r \in \mathbb{N}$,

$$\max \Delta^*(C_p^r) = \max\{p-2, r-1\}. \tag{4}$$

Theorem 5.6. *Let G be an elementary p-group of rank r.*

1. *If p is odd, then $[1, \lfloor \frac{r-1}{2} \rfloor] \cup p \cdot [1, \lfloor \frac{r-1}{p} \rfloor] \subset \Delta_0^*(G) \subset [1, r-1]$.*

2. *If $p = 2$, then $\Delta_0^*(G) = [1, \lfloor \frac{r-1}{3} \rfloor] \cup 2 \cdot [1, \lfloor \frac{r-1}{2} \rfloor]$.*

In particular, if $r \neq 1$ and $r \equiv 1 \pmod p$, then $\max \Delta^*(G) = \max \Delta_0^*(G)$.

Proof of Theorem 5.6. First, we note that the 'in particular'-statement is a direct consequence of (4) and the other assertions on $\Delta_0^*(G)$ of this theorem. For $r \le 2$, we have $\Delta_0^*(G) = \emptyset$ and the result holds. Thus, assume $r \ge 3$.

1. The left inclusion is just Proposition 5.1. Let $G_0 \subset G$ be a whf and non-half-factorial set. We need to show that $\min \Delta(G_0) \le r-1$. If $r \ge p$ this is immediate by (4); thus assume $p > r$. We note that, again by (4), $\min \Delta(G_0) < p$. It is known (see [10, Lemma 5.4]) that

$$\min \Delta(G_0) \mid \gcd\{\exp(G)(\mathsf{k}(A) - 1) : A \in \mathcal{A}(G_0)\}.$$

Since $p \nmid \min \Delta(G_0)$ and since $\mathsf{k}(A) - 1$ is an integer, we have $\min \Delta(G_0) \mid \gcd\{\mathsf{k}(A) - 1 : A \in \mathcal{A}(G_0)\}$ and consequently $\min \Delta(G_0) \le \mathsf{k}(A) - 1$ for each A with $\mathsf{k}(A) > 1$. Since $\mathsf{k}(A) \le \frac{1}{p} + r\frac{p-1}{p} < r$ (cf. the remark after (3)), the result follows.

2. By Proposition 5.1 we have $2 \cdot [1, \lfloor \frac{r-1}{2} \rfloor] \subset \Delta_0^*(G)$. For $d \in [1, \lfloor \frac{r-1}{3} \rfloor]$ it follows by Lemma 5.3, with $t = 3d+1$, $I = [1, 2d+1]$, and $J = [1, d+1] \cup [2d+2, 3d+1]$ that $d \in \Delta_0^*(G)$. (Since both $|I|$ and $|J|$ are odd, the set is indeed whf.) Thus, it remains to show that there are no further elements in $\Delta_0^*(G)$. Let $G_0 \subset G$ a non-half-factorial whf set and $\{e_1, \dots, e_t\} \subset G_0$ a maximal independent set; without restriction we assume $t = r$. We denote $d = \min \Delta(G_0)$. For each $g \in G_0 \setminus (\{0\} \cup \{e_1, \dots, e_r\}) = G_1$ let $I_g \subset [1, r]$ such that $g = \sum_{i \in I_g} e_i$. Clearly $|I_g| \ge 2$ for each $g \in G_1$ and, since G_0 is whf, $|I_g|$ is odd. If $|G_1| = 1$, we have $\min \Delta(G_0) = |I_g| - 1$ by Lemma 5.3 and thus d is even and $d \le r-1$.

Thus assume there exist distinct $g, h \in G_1$. Now, we have $|I_g|-1, |I_h|-1 \in \Delta(G_0)$ and thus $d \mid \gcd(|I_g| - 1, |I_h| - 1) \le r-1$. If the sets I_g and I_h are disjoint, we have $\gcd(|I_g| - 1, |I_h| - 1) \le \frac{r}{2} - 1$ and thus d is even or $d \le \frac{r-2}{4}$.

Thus assume $I_g \cap I_h \neq \emptyset$. By Lemma 5.3 it follows that $d \mid \gcd(|I_g| - 1, |I_h| - 1, |I_g \cap I_h| - 1)$. If $|I_g| \neq |I_h|$, then $\gcd(|I_g| - 1, |I_h| - 1) \le \frac{r-1}{2}$. Thus in this case, d is even or $d \le \frac{r-1}{4}$.

Finally, assume $|I_g| = |I_h|$; observe that in this case $|I_g \cap I_h| < |I_g|$. If $|I_g \cap I_h| = 1$, then $|I_g| - 1 \leq \frac{r-1}{2}$ and again d is even or $d \leq \frac{r-1}{4}$. However, if $|I_g \cap I_h| > 1$, then d divides the positive integers $|I_g| - |I_g \cap I_h| = |I_h| - |I_g \cap I_h|$ and $|I_g \cap I_h| - 1$ and it follows that

$$3d \leq (|I_g| - |I_g \cap I_h|) + (|I_h| - |I_g \cap I_h|) + (|I_g \cap I_h| - 1) = |I_g \cup I_h| - 1 \leq r - 1,$$

yielding the result. □

In the following result we obtain an upper bound for $\max \Delta_0^*(G)$ in case G is a p-group where the prime p is "large" relative to the rank.

Theorem 5.7. *Let G be a p-group of rank r. If $p \geq 2r - 1$, then*

$$\max \Delta_0^*(G) \leq \mathsf{K}(G) - 1.$$

In particular, $\max \Delta_0^*(G) \leq r - 2$.

Proof. We can assume $r \geq 3$. Let $G_0 \subset G$ be a whf and non-half-factorial set and denote $d = \min \Delta(G_0)$. By [10, Lemma 5.4] we know

$$d \mid \gcd\{\exp(G)(\mathsf{k}(A) - 1) : A \in \mathcal{A}(G_0)\}$$

and by Theorem 5.4 we have $d \leq 2\mathsf{k}(G) - 1$. Known results on $\mathsf{k}(G)$ (cf. the remark after (3)) and our assumption yield $d < 2r - 1 \leq p$. It follows that $p \nmid d$ and therefore $d \leq \mathsf{k}(A) - 1$ for each $A \in \mathcal{A}(G_0)$, which implies the result. The 'in particular'-statement follows, since $\mathsf{K}(G) < r$. □

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