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QUASI-HALF-FACTORIAL SUBSETS OF ABELIAN TORSION GROUPS

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Abstract. If G is an abelian torsion group with generating subset G_0, then by a classical result in the theory of non-unique factorizations, the block monoid $\mathcal{B}(G_0)$ is a half-factorial monoid if each of its atoms has cross number 1. In this case, G_0 is called a *half-factorial set*. In this note, we introduce the notion of a *k-quasi-half-factorial set* and show for many abelian torsion groups that G_0 k-quasi-half-factorial implies that G_0 is half-factorial. We moreover show in general that G_0 k-quasi-half-factorial implies that G_0 is weakly half factorial, a condition which has been of interest in the recent literature.

1. Introduction

A monoid (or a domain) is called half-factorial if every (non-zero and) non-invertible element is a product of irreducible elements and for each element the number of factors in such a factorization is unique; see the following section for a more formal definition and explanation of other terminology. Initiated by a paper of L. Carlitz [2], the study of half-factorial structures is a main subject of non-unique factorization theory (cf., e.g., the survey articles [3, 7]).

It is well-known that whether a Krull monoid, thus in particular a Dedekind domain, is half-factorial just depends on its class group and the subset of classes containing (non-empty, divisorial) prime ideals; more precisely, a Krull monoid with class group G and subset of classes containing prime ideals G_0 is half-factorial if and only if G_0 is a half-factorial set, which means that the block monoid $\mathcal{B}(G_0)$ is half-factorial. We refer to the monographs [16], in particular Chapter 9, and [10], in particular to Section 3.4, for a detailed treatment and a discussion of the

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historical development, originating in the works of L. Carlitz, H. Davenport, and W. Narkiewicz.

Thus, one is interested in investigating half-factorial subsets of abelian groups. It should be noted that indeed for each abelian group G and each strongly generating subset $G_0 \subset G$ (i.e., the semigroup generated by G_0 is equal to G) some Krull monoid (even Dedekind domain) exists such that G is (isomorphic to) the class group of this Krull monoid and G_0 is the subset of classes containing prime ideals (see, e.g., [10, Theorem 2.5.4], and [12] for a more detailed analysis for Dedekind domains). Consequently it is reasonable to study half-factorial subsets, and related notions, of (abstract) abelian groups without any further reference to some particular Krull monoid. Frequently, as is done in this paper, one even disregards the restriction that the set has to be strongly generating; on the one hand, for instance in investigations of quantitative aspects of non-unique factorizations, non-generating half-factorial subsets are of interest as-well (see, e.g., [10, Theorem 9.4.6]), and on the other hand for most investigations it is no actual restriction.

For torsion groups the following characterization, due to L. Skula [20] and A. Zaks [22], of half-factorial sets is well-known and decisive in their investigation.

Theorem 1.1. *Let G be an abelian torsion group, and let $G_0 \subset G$. The set G_0 is half-factorial if and only if for each minimal zero-sum sequence $A = g_1 \dots g_n$ over G_0 the cross number $\mathsf{k}(A) = \sum_{i=1}^{n} 1/\operatorname{ord} g_i$ is equal to 1.*

However, despite this characterization an explicit characterization of half-factorial subsets is only known for special types of groups (cf. [10, Section 6.7]). In investigations on half-factorial sets, and more generally of the arithmetic of Krull monoids, it has turned out to be useful to generalize the notion 'half-factorial set' in various ways (see, e.g., [4, 5] and the discussion on whf sets below).

In this paper we introduce a new type of generalization, namely k-quasi-half-factorial sets, where $k \in \mathbb{N}$, which is motivated by the characterization recalled above; in particular, it is only feasible for subsets of torsion groups. Since some notation is needed to state this definition in a convenient way, we defer it to Section 3. For now, we just mention that, by Theorem 1.1, half-factorial sets are characterized by the fact that a certain semi-length function, namely the cross number, is constant on the set of atoms (cf. [1]), and in this paper we investigate subsets $G_0 \subset G$ for which this semi-length function is constant on certain other subsets of $\mathcal{B}(G_0)$. Moreover, 1-quasi-half-factorial sets are those sets that fulfil the condition of Theorem 1.1; thus, by this theorem '1-quasi-half-factorial' is equivalent to 'half-factorial'. Conversely, it will be easy to see that half-factorial sets are k-quasi-half-factorial for every $k \in \mathbb{N}$ (cf. Lemma 3.2).

The aim of this paper is to explore to what extent these observations can be extended. More precisely, we seek to answer whether, for each $k \in \mathbb{N}$, every k-quasi-half-factorial set is already half-factorial. Though, this question, in full generality, remains unanswered, we settle various special cases. Among others, one of our results (Theorem 5.1) provides a positive answer to this question in case the set under consideration is the subset of classes containing prime ideals of a Krull monoid containing a prime element, which of course includes (the multiplicative monoids of) rings of integers of algebraic number fields, the archetypal objects in non-unique factorization theory.

From a technical point of view a key-result and of some interest in its own right is Proposition 4.3, where it is proved that if G_0 is k-quasi-half-factorial for some $k \in \mathbb{N}$, then G_0 is weakly half-factorial (see Section 3 for the definition). Weakly half-factorial sets were introduced by J. Śliwa [21], using different terminology, in his investigations on half-factorial sets; also see [10, Definition 6.7.2] or [18].

2. Preliminaries

We recall some notation and terminology, which is common in non-unique factorization theory; essentially we follow the usage in the monograph [10]. A monoid H is an abelian cancellative semigroup with identity element 1; we use multiplicative notation for monoids. The set of invertible elements of H is denoted by $H^\times$. An element $a \in H \setminus H^\times$ is called an atom if $a = bc$ with $b, c \in H$ implies that b or c is invertible; and it is called prime if $a \mid bc$ implies $a \mid b$ or $a \mid c$. We denote the set of atoms and primes of H by $\mathcal{A}(H)$ and $\mathcal{P}(H)$, respectively. Let $h \in H \setminus H^\times$. If $h = a_1 \dots a_n$ with $a_i \in \mathcal{A}(H)$, then h is said to have a factorization (into atoms) of length n. The set $\mathsf{L}(h) = \{n \colon h \text{ has a factorization of length } n\}$ is called the set of lengths of h; for $h \in H^\times$, let $\mathsf{L}(h) = \{0\}$. The monoid is called atomic if $\mathsf{L}(h) \neq \emptyset$ for every $h \in H$, and it is called half-factorial if $|\mathsf{L}(h)| = 1$ for every $h \in H$.

Let G be an additively written abelian torsion group. We denote its exponent by $\exp(G) \in \mathbb{N} \cup \{\infty\}$. We denote by C_n a cyclic group of order $n \in \mathbb{N}$, and for a prime p, by C_{p^∞} a quasi-cyclic group of type p^∞.

For a subset $G_0 \subset G$, let $\mathcal{F}(G_0)$ denote the, multiplicatively written, free abelian monoid over G_0. An element $S \in \mathcal{F}(G_0)$ is called a sequence over G_0; by definition $S = \prod_{g \in G_0} g^{v_g}$ with $v_g \in \mathbb{N}_0$ where almost all v_gs equal 0, and $S = \prod_{i=1}^{l} g_i$ for some $l \in \mathbb{N}_0$ and $g_i \in G_0$, which are uniquely determined up to permutation. One calls $\sigma(S) = \sum_{i=1}^{n} g_i \in G$ the sum of S, $|S| = n$ its length, and $\mathsf{k}(S) = \sum_{i=1}^{n} 1/\operatorname{ord} g_i$ its cross number. Moreover, $\mathsf{v}_g(S) = v_g$ is called the multiplicity of g in S. The monoid $\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0)\colon \sigma(S) = 0\} \subset \mathcal{F}(G_0)$ is called the block monoid over G_0; its elements are called zero-sum sequences. For convenience we write $\mathcal{A}(G_0)$ instead of $\mathcal{A}(\mathcal{B}(G_0))$ and do analogously for similar notation. The elements of $\mathcal{A}(G_0)$ are the minimal zero-sum sequences over G_0.

3. Basic definitions and first results

Let H be an atomic monoid. To state the definition of k-quasi-half-factorial sets, we first introduce k-quasi atoms. For $k \in \mathbb{N}$, we call $h \in H$ a k-quasi-atom if $\max \mathsf{L}(h) = k$; we denote the set of k-quasi-atoms of H by $\mathcal{A}_k(H)$. This modifies a definition of F. Halter-Koch [13] (also see [10, Definition 9.0]), namely $\mathcal{M}_k(H)$ the set of $h \in H$ with $\max \mathsf{L}(h) \le k$. Obviously $\mathcal{A}_k(H) = \mathcal{M}_k(H) \setminus \mathcal{M}_{k-1}(H)$. Moreover, $a \in \mathcal{A}(H)$ if and only if $\max \mathsf{L}(a) = 1$ (equivalently $\mathsf{L}(a) = \{1\}$) and $\mathcal{A}_k(H) \subset \mathcal{A}(H)^k$, but in general this inclusion may be proper, and more precisely $\mathcal{A}_k(H) = \mathcal{A}(H)^k \setminus \bigcup_{l \ge k+1} \mathcal{A}(H)^l$. Furthermore, we note that if $S \subset H$ is a divisor-closed submonoid of H, i.e., S is a submonoid such that for each $s \in S$ all divisors of s in H are in fact elements of S, then $\mathcal{A}_k(S) = \mathcal{A}_k(H) \cap S$; in particular, for $G_0 \subset G$ we have $\mathcal{A}_k(G_0) = \mathcal{A}_k(G) \cap \mathcal{B}(G_0)$. Finally, recall that $\mathcal{A}_2(G)$ is the set of almost minimal zero-sum sequences, which were studied in [6].

As already mentioned (see Theorem 1.1) half-factorial subsets of torsion groups are characterized by the cross number of the atoms of the block monoid. We extend this definition by considering quasi-atoms instead of atoms.

Definition 3.1. Let G be an abelian torsion group and let $G_0 \subset G$. For $k \in \mathbb{N}$, we call G_0 a k-quasi-half-factorial (k-qhf) set if $\mathsf{k}(A) = k$ for each $A \in \mathcal{A}_k(G_0)$.

Moreover, we recall the definition of a whf set. A subset $G_0 \subset G$ of an abelian torsion group is called weakly half-factorial (whf) if $\mathsf{k}(A) \in \mathbb{N}$ for each $A \in \mathcal{A}(G_0)$.

The following basic lemma relates the qhf-properties for different values of k.

Lemma 3.2. *Let G be an abelian torsion group and let $G_0 \subset G$. Further let $k, \ell \in \mathbb{N}$. If G_0 is k-qhf and ℓ-qhf, then G_0 is $(k+\ell)$-qhf. In particular, if G_0 is half-factorial, then G_0 is k-qhf for every $k \in \mathbb{N}$.*

Proof. Let $B \in \mathcal{A}_{k+\ell}(G_0)$. By definition we have $\max \mathsf{L}(B) = k + \ell$; let $B = A_1 \dots A_{k+\ell}$ with $A_i \in \mathcal{A}(G_0)$. By the maximality of $k + \ell$, it follows that $B_1 = A_1 \dots A_k \in \mathcal{A}_k(G_0)$ and $B_2 = A_{k+1} \dots A_{k+\ell} \in \mathcal{A}_\ell(G_0)$. Since G_0 is k-qhf and ℓ-qhf, we have $\mathsf{k}(B_1) = k$ and $\mathsf{k}(B_2) = \ell$ and consequently $\mathsf{k}(B) = k + \ell$. Thus G_0 is $(k + \ell)$-qhf.

Since, by Theorem 1.1, G_0 is half-factorial if and only if G_0 is 1-qhf, the second statement is a direct consequence of the first one. $\square$

4. Arbitrary subsets of special groups

In this section we consider arbitrary subsets of particular types of abelian groups. For groups of small total rank we obtain the following result.

Theorem 4.1. *Let G be (isomorphic to) a subgroup of $C_{p^\infty} \oplus C_{q^\infty}$ for (possibly equal) primes p and q or of $C_2^2 \oplus C_{p^\infty}$ for some odd prime p. If $G_0 \subset G$ is a k-qhf set for some $k \in \mathbb{N}$, then G_0 is half-factorial.*

We also consider groups of small exponent.

Theorem 4.2. *Let G be an abelian torsion group and let $G_0 \subset G$.*

1. *If $\exp(G) = 2$ and G_0 is k-qhf for some $k \in \mathbb{N}$, then G_0 is half-factorial.*
2. *If $\exp(G) = 3$ and G_0 is k-qhf for some $k \in \{2, 3\}$, then G_0 is half-factorial.*
3. *If $\exp(G) = 4$ and G_0 is 2-qhf, then G_0 is half-factorial.*

To prove these results we first prove some more technical results. The following proposition is a key-tool in the proof of Theorem 4.1.

Proposition 4.3. *Let G be an abelian torsion group and let $G_0 \subset G$. If G_0 is a k-qhf set, for some $k \in \mathbb{N}$, then G_0 is a whf set.*

Proof. We suppose that G_0 is k-qhf, for $k \in \mathbb{N}$, and we assume to the contrary that G_0 is not whf. Thus, there exists some $A' \in \mathcal{A}(G_0)$ such that $\mathsf{k}(A') \notin \mathbb{N}$. We set $G_0' = \operatorname{supp} A'$, which is a finite set. Let $A \in \mathcal{A}(G_0')$ such that $\mathsf{k}(A)$ is minimal with $\mathsf{k}(A) \notin \mathbb{N}$ among all elements of $\mathcal{A}(G_0')$.

First, we assume that $\mathsf{k}(A) < 1$. We assert that $A^k \in \mathcal{A}_k(G)$, contradicting the assumption that G_0 is k-qhf. Obviously $k \in \mathsf{L}(A^k)$ and, since $A^k \in \mathcal{B}(G_0')$ and A has the minimal cross number among all elements of $\mathcal{A}(G_0')$, it follows that indeed $\max \mathsf{L}(A^k) = k$.

So, we assume that $\mathsf{k}(A) > 1$. Thus, the cross number of each element of $\mathcal{A}(G_0')$ is at least 1. Let $T \in \mathcal{A}(G_0')$ with $\mathsf{k}(T) = 1$, e.g., $T = g^{\operatorname{ord} g}$ for some $g \in G_0'$. Let $B = AT^{k-1} \in \mathcal{B}(G_0')$. We assert that $B \in \mathcal{A}_k(G)$, again a contradiction, since $\mathsf{k}(B) = \mathsf{k}(A) + k - 1 > k$. It is obvious that $k \in \mathsf{L}(B)$. Let $B = A_1 \dots A_j$ be a factorization into atoms. It remains to show that $j \le k$. Since $\mathsf{k}(B) \notin \mathbb{N}$, there exists some $1 \le m \le j$ with $\mathsf{k}(A_m) \notin \mathbb{N}$. By assumption, $\mathsf{k}(A_m) \ge \mathsf{k}(A)$ and thus $\mathsf{k}(A_m^{-1}B) \le k - 1$. Since $\mathsf{k}(A_i) \ge 1$ for each $1 \le i \le j$, we have $j - 1 \le k - 1$, completing the argument. $\square$

Though, by Theorem 5.1 and the well-known fact that there exist whf sets that are not half-factorial or by Theorem 4.2 and the fact that such sets exist for elementary 2-groups, it is clear that whf sets exist that are not k-qhf for each $k \in \mathbb{N}$, we give the following simple example for illustration.

Example 4.4. Let $G = C_2^3 = \langle e, e_1, e_2 \rangle$. It is well-known (see, e.g., [10, Theorem 6.7.5]), and easy to see, that the set $G_0 = e + \langle e_1, e_2 \rangle$ is whf. But, for each $k \in \mathbb{N}$ we have $e^{2k-1}(e+e_1)(e+e_2)(e+e_1+e_2) \in \mathcal{A}_k(G_0)$ and thus G_0 is not k-qhf.

The following result, which extends [18, Proposition 4.4] (also see [10, Theorem 5.1.14, Corollary 5.7.18]), allows us to conclude that for certain groups whf sets are even half-factorial, which by Proposition 4.3 implies that k-qhf sets are half-factorial. First, we recall the definition of the cross number of an abelian torsion group G: we call $\mathsf{K}(G) = \sup\{\mathsf{k}(A) \colon A \in \mathcal{A}(G)\}$ the cross number of G.

Proposition 4.5. *Let G be an abelian torsion group. If G is isomorphic to a subgroup of $C_{p^\infty} \oplus C_{q^\infty}$ for (possibly equal) primes p and q or of $C_2^2 \oplus C_{p^\infty}$ for some odd prime p, then $\mathsf{k}(A) < 2$ for every $A \in \mathcal{A}(G)$.*

Proof. Let $A \in \mathcal{A}(G)$. Clearly $\mathsf{k}(A) \le \mathsf{K}(\langle \operatorname{supp} A \rangle)$. Since the finite abelian group $\langle \operatorname{supp} A \rangle$ has total rank at most two or is isomorphic to $C_2^2 \oplus C_{p^n}$ for some odd prime p and $n \in \mathbb{N}$ the result follows by well-known results on the cross number due to A. Geroldinger and R. Schneider (cf. [10], in particular Theorem 5.5.9 and Theorem 5.7.17 for a unified treatment, and [9, 11] for the original results). $\square$

Indeed, it is not difficult to see that the converse of the above proposition holds as well. In view of [18, Proposition 4.4], we note that $\mathsf{K}(C_{p^\infty} \oplus C_{q^\infty}) = \mathsf{K}(C_2^2 \oplus C_{p^\infty}) = 2$. And, we point out that there exist some further groups with $\mathsf{K}(G) = 2$, namely C_{30}, $C_2 \oplus C_{12}$, $C_3 \oplus C_6$, and subgroups of rank 3 of $C_2^2 \oplus C_{2^\infty}$; this is a consequence of a lower bound on $\mathsf{K}(G)$ due to U. Krause and C. Zahlten [15] and again results of A. Geroldinger and R. Schneider [9, 11] (again, see [10, Chapter 5]). However, for these groups not every whf set is half-factorial (cf. [18, Theorem 4.5]).

Proof of Theorem 4.1. Since G_0 is k-qhf we know, by Proposition 4.3, that G_0 is whf. Since for each $A \in \mathcal{A}(G_0)$ on the one hand $\mathsf{k}(A) \in \mathbb{N}$ and on the other hand by Proposition 4.5 $\mathsf{k}(A) < 2$, it follows by Theorem 1.1 that G_0 is half-factorial. $\square$

Furthermore, it is known for some other types of groups that every whf set is already half-factorial. In particular, this holds true for $C_p^2 \oplus C_q$ and C_{2pq} where p

and q are primes and $q \equiv 1 \pmod p$, see [18, Proposition 4.8] and [17, Proposition 7.4] respectively. Again by Proposition 4.3, for these groups k-qhf sets are even half-factorial as-well.

The following result is used to prove Theorem 4.2.

Proposition 4.6. *Let G be an abelian torsion group and let $G_0 \subset G$ a k-qhf set for some $k \in \mathbb{N}$. If $A \in \mathcal{A}(G_0)$ with $\mathsf{k}(A) \neq 1$, then $\mathsf{v}_g(A) \le \operatorname{ord} g - 1 - (\operatorname{ord} g + 1)/k$ for every $g \in G_0$.*

PROOF. Suppose $A \in \mathcal{A}(G_0)$ with $\mathsf{k}(A) \neq 1$. By Proposition 4.3 we have $\mathsf{k}(A) > 1$. Let $g \in G_0$. We consider $B = g^{(k-1)\operatorname{ord} g} A$. We have $k \in \mathsf{L}(B)$ and, since $\mathsf{k}(B) \neq k$ and G_0 is k-qhf, we get $\max \mathsf{L}(B) \ge k+1$. Let $B = \prod_{i=1}^{\ell} A_i$ with $\ell > k$ and $A_i \in \mathcal{A}(G_0)$. We note that, for each $1 \le i \le \ell$, $A_i \neq A$ and thus $A_i \nmid A$; consequently $\mathsf{v}_g(A_i) \ge \mathsf{v}_g(A) + 1$. We have

$$(k-1)\operatorname{ord} g + \mathsf{v}_g(A) = \sum_{i=1}^{\ell} \mathsf{v}_g(A_i) \ge (k+1)(\mathsf{v}_g(A) + 1)$$

and the claim follows. □

A natural extension of this result is to consider $A' \in \mathcal{A}_m(G_0)$ with $\mathsf{k}(A') \neq m$ for some $m \le k$ instead of $A \in \mathcal{A}(G_0)$. However, this extension does not seem to yield further interesting results.

PROOF OF THEOREM 4.2. We assume that G_0 is not half-factorial. Thus, by Theorem 1.1 there exists some $A \in \mathcal{A}(G_0)$ with $\mathsf{k}(A) \neq 1$. By Proposition 4.6 it follows that if G_0 is k-qhf, then $\mathsf{v}_g(A) \le \operatorname{ord} g - 1 - (\operatorname{ord} g + 1)/k$ for each $g \in G_0$. However, $\mathsf{v}_g(A) \ge 1$ for each $g \in \operatorname{supp} A \subset G_0$. In each case this yields a contradiction. □

We conclude this section by considering subsets of the group C_5^3, which is not covered by the results obtained so far.

Proposition 4.7. *Let $G_0 \subset C_5^3$ be a 2-qhf set. Then G_0 is half-factorial.*

Since it will be needed in the proof of this proposition, we recall a classical result of A. Kemnitz [14, Theorem 3], see [8] for recent developments on this type of problem.

Lemma 4.8. *Let $S \in \mathcal{F}(C_5^2)$ with $\mathsf{v}_g(S) \le 1$ for each $g \in C_5^2$. If $|S| \ge 9$, then there exists some $T \mid S$ such that $\sigma(T) = 0$ and $|T| = 5$.*

Proof of Proposition 4.7. We assume to the contrary that G_0 is not half-factorial. Thus, there exists some $A \in \mathcal{A}(G)$ with $\mathsf{k}(A) \neq 1$. By Proposition 4.3 we know that G_0 is whf. Thus $\mathsf{k}(A) \in \mathbb{N}$ and moreover $G_0 \subset \{0\} \cup (e+G')$ for some $G' \cong C_5^2$ and $\langle e, G' \rangle = C_5^3$ (cf., e.g., [10, Theorem 6.7.5] or [18, Theorem 3.2]). Since $0 \nmid A$ it follows that $|A| \geq 10$. Moreover, by Proposition 4.6, we known that $\mathsf{v}_g(A) \leq 1$ for each $g \in G_0$. Thus we have $A = \prod_{i=1}^{l}(e+g_i)$ with pairwise distinct $g_i \in G'$ and $l \geq 10$. We apply Lemma 4.8 to the sequence $\prod_{i=1}^{l} g_i$ and obtain that there exists a subset $I \subset \{1, \dots, l\}$ with $|I| = 5$ such that $\sum_{i \in I} g_i = 0$. Clearly $\sum_{i \in I}(e+g_i) = 0$ as well, a contradiction to $A \in \mathcal{A}(G_0)$. □

5. Special subsets of arbitrary groups

In this section we consider the problem for arbitrary abelian torsion groups, but impose certain conditions on the subsets we consider. In the following proposition we require that the set of prime elements in $\mathcal{B}(G_0)$ (denoted by $\mathcal{P}(G_0)$) is not empty. We recall (cf., e.g., [19, Proposition 3.3]) that $\mathcal{P}(G_0) \subset \{g^{\operatorname{ord} g} \colon g \in G_0\}$, in particular $\mathsf{k}(P) = 1$ for each $P \in \mathcal{P}(G_0)$. Moreover, if $0 \in G_0$, then $0 \in \mathcal{P}(G_0)$.

Theorem 5.1. *Let G be an abelian torsion group and let $G_0 \subset G$.*

(1) *Suppose G_0 is a k-qhf set for some $k \in \mathbb{N}$. If $\mathcal{P}(G_0) \neq \emptyset$, thus in particular if $0 \in G_0$, then G_0 is half-factorial.*
(2) *Suppose G_0 is a 2-qhf set. If there exists some $g \in G_0$ with $\operatorname{ord} g \leq 2$, then G_0 is half-factorial.*

Interpreting, as usual, $G_0 \subset G$ as the subset of classes containing prime ideals of the class group of some Krull monoid H, we have that $0 \in G_0$ is equivalent to $\mathcal{P}(H) \neq \emptyset$. However in general, $\mathcal{P}(G_0) \neq \emptyset$ does not imply $\mathcal{P}(H) \neq \emptyset$.

Proof of Theorem 5.1. 1. Let $P \in \mathcal{P}(G_0)$. Let $A \in \mathcal{A}(G_0)$. We consider $B = AP^{k-1}$. Since P is prime, each factorization of B is of the form $P^{k-1}R$ and obviously $A = R$. Thus, B has a unique factorization and $\mathsf{L}(B) = \{k\}$. Thus, since G_0 is k-qhf, it follows that $\mathsf{k}(B) = k$, and since $\mathsf{k}(P) = 1$, we have $\mathsf{k}(A) = 1$ and the result follows by Theorem 1.1.
2. Seeking a contradiction, we assume that $g \in G_0$ with $\operatorname{ord} g \leq 2$ but G_0 is not half-factorial. Thus, there exists some $A \in \mathcal{A}(G_0)$ with $\mathsf{k}(A) \neq 1$. By Proposition 4.6 we have $\mathsf{v}_g(A) \leq \operatorname{ord} g - 1 - (\operatorname{ord} g + 1)/2 < 0$. Yet, clearly we have $\mathsf{v}_g(A) \geq 0$, which yields the desired contradiction. □

References

[1] D. D. Anderson and D. F. Anderson. Elasticity of factorizations in integral domains. II. *Houston J. Math.*, 20(1):1–15, 1994.

[2] L. Carlitz. A characterization of algebraic number fields with class number two. *Proc. Amer. Math. Soc.*, 11:391–392, 1960.

[3] S. T. Chapman and J. Coykendall. Half-factorial domains, a survey. In *Non-Noetherian commutative ring theory*, volume 520 of *Math. Appl.*, pages 97–115. Kluwer Acad. Publ., Dordrecht, 2000.

[4] S. T. Chapman and W. W. Smith. On a characterization of algebraic number fields with class number less than three. *J. Algebra*, 135(2):381–387, 1990.

[5] S. T. Chapman and W. W. Smith. Finite cyclic groups and the k-HFD property. *Colloq. Math.*, 70(2):219–226, 1996.

[6] S. T. Chapman and W. W. Smith. A characterization of minimal zero-sequences of index one in finite cyclic groups. *Integers*, 5(1):A27, 5 pp. (electronic), 2005.

[7] J. Coykendall. Extensions of half-factorial domains: A survey. In *Arithmetical Properties of Commutative Rings and Monoids*, volume 241 of *Lecture Notes in Pure and Appl. Math.*, pages 46–70. CRC Press (Taylor & Francis Group), Boca Raton, 2005.

[8] W. D. Gao and R. Thangadurai. A variant of Kemnitz conjecture. *J. Combin. Theory Ser. A*, 107(1):69–86, 2004.

[9] A. Geroldinger. The cross number of finite abelian groups. *J. Number Theory*, 48(2):219–223, 1994.

[10] A. Geroldinger and F. Halter-Koch. *Non-unique factorizations. Algebraic, Combinatorial and Analytic Theory*. Chapman & Hall/CRC, 2006.

[11] A. Geroldinger and R. Schneider. The cross number of finite abelian groups. II. *European J. Combin.*, 15(4):399–405, 1994.

[12] R. Gilmer, W. Heinzer, and W. W. Smith. On the distribution of prime ideals within the ideal class group. *Houston J. Math.*, 22(1):51–59, 1996.

[13] F. Halter-Koch. A generalization of Davenport's constant and its arithmetical applications. *Colloq. Math.*, 63(2):203–210, 1992.

[14] A. Kemnitz. On a lattice point problem. *Ars Combin.*, 16(B):151–160, 1983.

[15] U. Krause and C. Zahlten. Arithmetic in Krull monoids and the cross number of divisor class groups. *Mitt. Math. Ges. Hamburg*, 12(3):681–696, 1991.

[16] W. Narkiewicz. *Elementary and analytic theory of algebraic numbers*. Springer-Verlag, Berlin, third edition, 2004.

[17] A. Plagne and W. A. Schmid. On the maximal cardinality of half-factorial sets in cyclic groups. *Math. Ann.*, 333(4):759–785, 2005.

[18] M. Radziejewski and W. A. Schmid. Weakly half-factorial sets in finite abelian groups. *Forum Math.*, to appear.

[19] W. A. Schmid. Arithmetic of block monoids. *Math. Slovaca*, 54(5):503–526, 2004.

[20] L. Skula. On c-semigroups. *Acta Arith.*, 31(3):247–257, 1976.

[21] J. Śliwa. Remarks on factorizations in algebraic number fields. *Colloq. Math.*, 46(1):123–130, 1982.

[22] A. Zaks. Half factorial domains. *Bull. Amer. Math. Soc.*, 82(5):721–723, 1976.

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