Higher-order class groups and block monoids of Krull monoids with torsion class group

Wolfgang A. Schmid

Institute of Mathematics and Scientific Computing,
University of Graz, Heinrichstraße 36, 8010 Graz, Austria

wolfgang.schmid@uni-graz.at

Abstract

Extensions of the notion of a class group and a block monoid associated to a Krull monoid with torsion class group are introduced and investigated. Instead of assigning to a Krull monoid only one abelian group (the class group) and one monoid of zero-sum sequences (the block monoid), we assign to it a recursively defined family of abelian groups, the first being the class group, and do alike for the block monoid. These investigations are motivated by the aim of gaining a more detailed understanding of the arithmetic of Krull monoids, including Dedekind and Krull domains, both from a technical and conceptual point of view. To illustrate our method, some first arithmetical applications are presented.

Keywords: Dedekind domain, half-factorial, Krull monoid, non-unique factorization, simply presented group, zero-sum sequence

2000 MSC: 13C20, 13F05, 11R27, 20K10

Short title: Higher-order class groups and block monoids

1 Introduction

A commutative and cancelative semigroup with identity element, in this paper we refer to such a structure as a monoid, is called a Krull monoid if it is v-noetherian and completely integrally closed (see the monographs [22, 23, 20] for detailed information on Krull monoids, also cf. Section 2 for a brief overview on this and related notions used below). The multiplicative monoid of a Dedekind or Krull domain are classical examples of Krull monoids. Thus, all the constructions and results that are formulated for Krull monoids in this paper are valid for Dedekind domains and Krull domains as well; the class group, as defined for Krull monoids (cf. below), in this case coincides with the ideal or divisor class group, respectively. However, for the present investigations the purely multiplicative point of view is crucial. Moreover, Krull monoids that are not the multiplicative monoid of some domain arise in various ways. The term Krull monoid was introduced by L. Chouinard [8], investigating under which conditions semi-group rings are Krull domains. Moreover, the monoid of isomorphy classes of certain modules under direct sum decomposition are Krull monoids (see, e.g., [11, 12, 24]). Finally, to get a precise understanding of the arithmetic of the ring of algebraic integers of number fields, and more generally Dedekind and Krull domains, it is necessary to understand the arithmetic of certain submonoids, which are Krull monoids (see [20, Sections 4.3 and 9.4] and, e.g., [7] for a recent contribution).

It is well-known that a monoid H is a Krull monoid if and only if there exists a free (commutative) monoid F and a monoid homomorphism $\varphi : H \to F$ such that for all $a, b \in H$ one has that $a \mid b$ if and only if $\varphi(a) \mid \varphi(b)$; a homomorphism with this property is called a divisor homomorphism. In this paper, we work exclusively with this characterization of Krull monoids. Additionally, there exists an essentially unique "smallest" free monoid F with this property, namely the monoid of non-empty divisorial ideals of H. Let H be a Krull monoid and $\varphi : H \to F$ the divisor homomorphism into this "smallest" free monoid, which is called a divisor theory. Then, $\mathsf{q}(F)/\mathsf{q}(\varphi(H))$, where $\mathsf{q}(\cdot)$ denotes the quotient group, is called the class group of H, denoted $\mathcal{C}(H)$. Moreover, $\mathcal{D}(H) \subset \mathcal{C}(H)$ denotes the subset of classes containing prime divisors, i.e., prime elements of F.

A Krull monoid H is factorial if and only if $\mathcal{C}(H)$ is trivial. Moreover, the complexity of the arithmetic of a Krull monoid is to some extent governed by its class group. However, it is well-known that knowledge of the class group only does not yield much information on the actual arithmetic of the Krull monoid, e.g., there exists Krull monoids with arbitrary finite class group and even infinite class group that are half-factorial (for an overview regarding the property half-factorial see the survey articles by S.T. Chapman and J. Coykendall [4, 9], for the definition see Section 2). A main reason for this phenomenon, which is in sharp contrast with the fact that the monoid of principal ideals of the ring of integers of an algebraic number field is determined up to isomorphy by the class group, is the fact that for a general Krull monoid it is not guaranteed that each class contains a prime divisor, indeed the sole restriction on this set is that it generates the class group as a monoid (see, e.g., [6, Example 1.6] for a natural example of a Krull monoid whose class group is isomorphic to $\mathbb{Z}$ and only two classes contain prime divisors). Thus, to gain more detailed information on the complexity of the arithmetic of a Krull monoid it is common to consider the class group and in addition the subset of classes containing prime divisors. Knowledge of these two already determines the arithmetic of a Krull monoid to a large extent. Indeed, if H is a Krull monoid and B the block monoid associated to H, i.e., the monoid of zero-sum sequences over the set $\mathcal{D}(H)$, then there exists a natural homomorphism $\beta_H : H \to B$, called the block homomorphism. And, this homomorphism is a transfer homomorphism and thus preserves a lot of arithmetical properties, in particular it preserves all information regarding lengths of factorizations (cf. Section 2). The reason that not all information on factorizations is preserved by the block homomorphism is that the number of prime divisors contained in each class is not taken into account; we recall that up to units and isomorphy a Krull monoid is determined by its class group, the subset of classes containing prime divisors, and the number of prime divisors in each class (see, e.g., [20, Theorem 2.5.4]). The notion of a block monoid was introduced by W. Narkiewicz [26] to investigate the arithmetic of rings of algebraic integers. Meanwhile, it has been extended to arbitrary Krull monoids and has become a common tool in Non-Unique Factorization Theory; moreover, the idea of transferring problems on factorizations to auxiliary monoids has been applied for other types of domains and monoids as well (see the monograph [20] and the conference proceedings [1, 3] for an overview).

Thus, if one wishes to explore certain arithmetical properties of H, e.g., its system of sets of lengths, one can consider the analogous problem in B, which is typically much simpler to investigate than the original Krull monoid.

In this paper, we extend the notion of a class group and a block monoid for Krull monoids with torsion class group, by assigning to a Krull monoid not only a single abelian group (the class group) and the block monoid (a monoid of zero-sum sequences over a subset of the class group), but a recursively defined family of abelian groups and monoids of zero-sum sequences over subsets of these groups, which we call higher-order class groups and higher-order block monoids, respectively.

The basic idea underlying our construction is simple. The above mentioned block monoid B is itself a Krull monoid. Thus, one can consider its class group and form the block monoid B' associated to B; we refer to B' as the refinement of B. Clearly, one can iterated this construction, considering the refinement of B' and so on; thus, defining a sequence of groups and monoids of zero-sum sequences associated to a Krull monoid. However, this basic construction of a sequence of groups is not yet satisfactory, e.g., since for infinite class group these sequences are not necessarily eventually constant (cf. Example 4.17), and thus we explore a transfinite version of this construction and construct a family (indexed by ordinals) of groups and monoids of zero-sum sequences. The need for this transfinite construction is a main reason for the restriction to the case that the class group of the Krull monoid is a torsion group, since in this case we are able to obtain an explicit understanding of the above mentioned iteration, which presently seems to be needed for the transfinite construction. Besides, having applications in Non-Unique Factorization Theory in mind, which we illustrate by some simple examples in the present paper (see Subsection 6.2), an explicit construction is desirable.

The motivation for our considerations is two-fold. On the one hand, we seek technical simplifications for investigations in Non-Unique Factorization Theory. A main motivation for considering the block monoid associated to a Krull monoid, and more generally for the method of transferring to auxiliary monoids, is the fact that these are in general simpler to investigate than the original monoid. And, in our recursive construction the family of groups turns out to be a descending chain of subgroups of the class group and each of the monoids of zero-sum sequences in this family is potentially simpler, and definitely not more complicated, than the preceding ones while still containing essentially all the arithmetical information. Note that with B, B' as above one obtains a block homomorphism β' from B to its refinement B'. The composition of transfer homomorphisms being again a transfer homomorphism, we get a transfer homomorphism $\beta' \circ \beta$ from H to B'. And, an analogous statement will be obtained for the transfinite version. Moreover, these families are eventually constant and a monoid of zero-sum sequences in this constant part can be shown to have a property (cf. Lemma 4.5) that is of significant technical advantage in investigations of Non-Unique Factorization Theory. As an example, we present a strengthening of a technical result in Non-Unique Factorization Theory that is an immediate consequence of our investigations (see Corollary 4.13). Regarding applications in Non-Unique Factorization Theory, we point out that our construction tends to yield significant simplifications only in case the subset of classes containing prime divisors is rather sparse. Thus, at first it might seem almost useless when investigating, for instance, the ring of integers of an algebraic number field (where each class contains a prime divisor). Yet, as recalled above, various results in Non-Unique Factorization Theory on rings of algebraic integers depend on detailed knowledge of the arithmetic of certain submonoids. And, these submonoids can have the property that the subset of classes containing prime divisors is sparse. Indeed, it is rather common that the submonoids that are most interesting in this context, since they yield extremal values for certain parameters, have this property.

On the other hand, the notion of a higher-order class group might be of some conceptual interest as well. To illustrate this, we explore, expanding on the fact that a Krull monoid is factorial if and only if its class group is trivial, consequences of the triviality of higher-order class groups. And, we use the thus defined notions to reconsider certain constructions of A. Geroldinger and R. Göbel [17] made in the context of the investigation of the class groups of half-factorial monoids with infinite class group (see Section 7).

1.1 Organization of the paper

The organization of this paper is as follows. In Section 2, we recall some key notions and results. Then, in Section 3, we investigate the class groups, assuming they are torsion groups, associated to divisor homomorphisms of a Krull monoid H into a free monoid. These investigations build on results of S.T. Chapman, F. Halter-Koch, and U. Krause [5] on outside factorial monoids. In particular, we see that the class group of H is, up to a natural identification, a subgroup of each class group associated to a divisor homomorphism. More precisely, the class group associated to a divisor homomorphism is an extension of the class group by a group that is the direct sum, over all prime divisors, of cyclic groups whose orders are determined in terms of valuations of elements of H. Having this result at hand, we give an explicit description of the refinement of a block monoid (over a subset of a torsion group). Building on this description, we define and investigate higher-order block monoids in an abstract setting (see Section 4). And, we show how this construction yields (as mentioned above), in a natural way, an extension of a technical result in Non-Unique Factorization Theory. In Section 5, we investigate a finitary analog of higher-order block monoids that has certain technical advantages. Having the abstract versions of our constructions at hand, we introduce higher-order class groups and block monoids of Krull monoids, and of divisor homomorphisms of Krull monoids into free monoids (see Definition 6.1). It turns out that the higher-order class groups and block monoids of infinite order are in fact independent of the divisor homomorphism (as long as its class group is a torsion group). Moreover, we give an example that illustrates how these notions can be applied in Non-Unique Factorization Theory (see Subsection 6.2). Finally, in Section 7, expanding on the above mentioned characterization that a Krull monoid is factorial if and only if its class group is trivial, we introduce the notion of a σ-pseudo factorial and a finitary-pseudo factorial Krull monoid, defined by the property that the higher-order class group of order σ or the finitary analog, respectively, is trivial. Having the results of Section 6, it is easy to see that σ-pseudo factorial and finitary-pseudo factorial Krull monoids are half-factorial monoids. And, we investigate these monoids, focusing on the case that the class group is a p-group. In particular, in analogy with the well-known problem (see [21, Problem CG1]) of determining whether for each abelian group G there exists a half-factorial Krull monoid (or equivalently a half-factorial Dedekind domain) whose class group is isomorphic to G, we explore which p-groups are isomorphic to class groups of finitary-pseudo factorial Krull monoids. We build on the above mentioned work of A. Geroldinger and R. Göbel [17] and obtain a complete answer to our variant, namely that these groups are precisely the simply presented p-groups.

2 Preliminaries

We recall some key notions and terminology. Our notation mainly follows the monograph [20] to which we refer for a complete exposition of the ideas recalled briefly below.

We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the set of positive integers and non-negative integers, respectively. For $a, b \in \mathbb{Z}$, we denote by $[a, b] = \{z \in \mathbb{Z} \colon a \le z \le b\}$ the interval of integers. We denote the smallest infinite ordinal by ω. We identify finite ordinals and cardinals, in case it is convenient and no confusion is to be expected.

2.1 Groups

Throughout the paper, all groups we consider are abelian. Thus, by a group we always mean an abelian group. We use additive notation for groups. Let G be a group. For $G_0 \subset G$, we denote by $\langle G_0 \rangle$ the subgroup generated by G_0; we call G_0 a generating set if $G = \langle G_0 \rangle$. We use the convention that $\langle \emptyset \rangle = \{0\}$. We call a subset G_0 independent if $0 \notin G_0$ and $\sum_{g \in G_0} a_g g = 0$, with $a_g \in \mathbb{Z}$ (almost all 0), implies that $a_g g = 0$ for each $g \in G_0$. We denote the order of $g \in G$ by $\mathrm{ord}(g)$ and call G a torsion group if each $g \in G$ has finite order. Moreover, for p a prime, we say that G is a p-group if $\mathrm{ord}(g)$ is a power of p for each $g \in G$, and we say that G is an elementary p-group if each non-zero element has order p. For p a prime, we set $pG = \{pg \colon g \in G\}$, $p^0 G = G$, $p^{\sigma+1} G = p(p^\sigma G)$, and $p^\sigma G = \cap_{\rho < \sigma} p^\rho G$ for σ a limit ordinal. For $n \in \mathbb{N}$, we denote by C_n a cyclic group with n elements. For p a prime, we denote by $\mathbb{Z}(p^\infty) = \{a/p^n + \mathbb{Z} \colon a \in \mathbb{Z}, n \in \mathbb{N}\} \subset \mathbb{Q}/\mathbb{Z}$ the quasicyclic p-group. By the rank of a p-group we mean the maximal cardinality of an independent subset.

2.2 Monoids

As mentioned in Section 1, throughout this paper a monoid is a cancelative commutative semigroup with identity element, which we denote by 1. We use multiplicative notation for monoids. Let H be a monoid. We denote its invertible elements by $H^\times$. A monoid is called reduced if $H^\times = \{1\}$. For $a, b \in H$, we write $a \simeq b$, and call a and b associates, if $aH^\times = bH^\times$. For a subset $A \subset H$, we denote by $[A]$ the submonoid of H generated by A. By $\mathsf{q}(H)$ we denote a quotient group of H such that $H \subset \mathsf{q}(H)$. Moreover, if $H' \subset H$ is a submonoid, then we assume that $\mathsf{q}(H') \subset \mathsf{q}(H)$.

A monoid F is called free if there exists a subset $P \subset F$ such that for each $f \in F$ there exist uniquely determined $\mathsf{v}_p(f) \in \mathbb{N}_0$ (almost all 0) such that $f = \prod_{p \in P} p^{\mathsf{v}_p(f)}$; the set P is called the basis of F and $\mathsf{v}_p(f)$ the p-valuation of f. For some set P, we denote by $\mathcal{F}(P)$ the free monoid with basis P. For $f \in \mathcal{F}(P)$, let $\operatorname{supp}(f) = \{p \in P \colon \mathsf{v}_p(f) > 0\}$ the support of f.

For G a group and $G_0 \subset G$, let $\sigma : \mathcal{F}(G_0) \to G$ denote the monoid homomorphism that is identical on G_0, i.e., $\sigma(\prod_{g \in G_0} g^{v_g}) = \sum_{g \in G_0} v_g g$. Furthermore, let $\mathcal{B}(G_0) = \ker \sigma$. Then, $\mathcal{B}(G_0)$ is a submonoid of $\mathcal{F}(G_0)$ and it is called the block monoid over G_0. It is common to refer to the elements of $\mathcal{F}(G_0)$ as sequences over G_0 and to the elements of $\mathcal{B}(G_0)$ as zero-sum sequences. For $S = \prod_{g \in G_0} g^{v_g} \in \mathcal{F}(G_0)$, let $|S| = \sum_{g \in G_0} v_g$ denote the length of S; moreover, if G_0 consists of torsion elements, then let $\mathsf{k}(S) = \sum_{g \in G_0} v_g / \operatorname{ord}(g)$ denote the cross number of S.

2.3 Factorizations and transfer homomorphisms

An element $a \in H$ is called irreducible, or an atom, if $a = bc$ implies that b or c is invertible. The set of atoms in H is denoted by $\mathcal{A}(H)$. A monoid is called atomic [factorial] if each non-invertible element of H is the product of atoms [in an essentially unique way, i.e., up to ordering and associates]. Clearly, free monoids are factorial and conversely a monoid is factorial if and only if $H/H^\times$ is free.

Let $a \in H \setminus H^\times$ and $a = u_1 \dots u_n$ be a factorization of a into irreducible elements $u_i \in \mathcal{A}(H)$. Then, n is called the length of this factorization. By $\mathsf{L}(a)$ we denote the set of all $n \in \mathbb{N}$ such that a has a factorization into irreducible elements of length n. For $a \in H^\times$ we set $\mathsf{L}(a) = \{0\}$. Moreover, $\mathcal{L}(H) = \{\mathsf{L}(a) \colon a \in H\}$ is called the system of sets of lengths of H. If $|\mathsf{L}(a)| = 1$ for each $a \in H$, then H is called half-factorial.

A monoid homomorphism $\theta : H \to D$ is called a transfer homomorphism if it satisfies the following two conditions.

(T1) $D = \theta(H) D^\times$ and $H^\times = \theta^{-1}(D^\times)$.

(T2) If $u \in H$ and $b, c \in D$ such that $\theta(a) = bc$, then there exist $v, w \in H$ such that $\theta(v) \simeq b$, $\theta(w) \simeq c$, and $u = vw$.

Transfer homomorphisms are important, since they preserve a lot of information on factorizations of elements. Let $\theta : H \to D$ be a transfer homomorphism of atomic monoids. Then, $\theta(\mathcal{A}(H)) D^\times = \mathcal{A}(D)$ and $\mathsf{L}(a) = \mathsf{L}(\theta(a))$ for each $a \in H$, in particular $\mathcal{L}(H) = \mathcal{L}(D)$. The composition of transfer homomorphisms is again a transfer homomorphism.

For detailed information on transfer homomorphisms see [20, Section 3.2].

2.4 Class groups and divisor homomorphisms

Let H and D be monoids and $\varphi : H \to D$ a monoid homomorphism.

The group $\mathcal{C}(\varphi) = \mathsf{q}(D) / \mathsf{q}(\varphi(H))$ is called the class group of φ. We use additive notation for this group and for each $a \in \mathsf{q}(D)$ the class containing a is denoted by $[a]_\varphi$.

The homomorphism φ is called a divisor homomorphism if $a \mid b$ if and only if $\varphi(a) \mid \varphi(b)$. And, φ is called cofinal if for each $d \in D$ there exists some

$a \in H$ such that $d \mid \varphi(a)$. If $\mathsf{q}(D)/\,\mathsf{q}(\varphi(H))$ is a torsion group, then φ is cofinal. Moreover, if $H \subset D$ is a submonoid and $H \hookrightarrow D$ is a divisor homomorphism, then H is called a saturated submonoid of D.

As mentioned in Section 1, a monoid H is called a Krull monoid if there exists some free monoid F such that there exists a divisor homomorphism $\varphi : H \to F$. For each Krull monoid there exists an essentially unique (it is unique up to unique isomorphism) "smallest" free monoid with the above property. More precisely the following is well-known. A divisor homomorphism $\varphi : H \to \mathcal{F}(P)$ is called a divisor theory if for each $p \in P$ there exists a finite set $\emptyset \neq X \subset H$ such that $p = \gcd \varphi(X)$. And, each Krull monoid has a unique (up to unique isomorphism) divisor theory. The class group of a divisor theory of a Krull monoid is called the class group of the Krull monoid and is denoted by $\mathcal{C}(H)$.

Krull monoids are atomic and a Krull monoid is factorial if and only if its class group is trivial (see, e.g., [20, Corollary 2.3.13]).

We recall and introduce some additional notation regarding divisor homomorphisms into free monoids. Let H be a Krull monoid and let $\varphi : H \to \mathcal{F}(P)$ be a divisor homomorphism. The elements of P are called prime divisors and the set $\mathcal{D}(\varphi) = \{[p]_\varphi \colon p \in P\} \subset \mathcal{C}(\varphi)$ is called the subset of classes containing prime divisors. One has $\langle \mathcal{D}(\varphi) \rangle = \mathcal{C}(\varphi)$. Furthermore, let $\tilde{\beta} : \mathcal{F}(P) \to \mathcal{F}(\mathcal{D}(\varphi))$ be the homomorphism defined by $p \mapsto [p]_\varphi$. Then, $\operatorname{im}(\tilde{\beta} \circ \varphi) = \mathcal{B}(\mathcal{D}(\varphi))$ and $\beta_\varphi = \tilde{\beta} \circ \varphi : H \to \mathcal{B}(\mathcal{D}(\varphi))$ is a transfer homomorphism. This map is called the block homomorphism associated to φ and $\mathcal{B}(\mathcal{D}(H))$ is called the block monoid associated to φ. Additionally, let $\mathcal{F}_\varphi = \{\gcd \varphi(X) \colon \emptyset \neq X \subset H\}$. In [20, Theorem 2.4.7] this set is denoted by F_0. There, it is proved that $\mathcal{F}_\varphi$ is a monoid and that there exists an epimorphism from $\{[a]_\varphi \colon a \in \mathcal{F}_\varphi\}$, which is a submonoid of $\mathcal{C}(\varphi)$, to $\mathcal{C}(H)$. If φ is cofinal, thus in particular if $\mathcal{C}(\varphi)$ is a torsion group, then, for $p \in P$, we set $m_p(\varphi) = \min(\{\mathsf{v}_p(\varphi(a)) \colon a \in H\} \setminus \{0\})$.

If we just refer to the block monoid or block homomorphism associated to a Krull monoid, without specifying a divisor homomorphism, we mean one associated to a divisor theory of H, which is again essentially unique. And, in this case we use the notation $\mathcal{D}(H)$ and β_H to denote the subset of classes containing prime divisors and the block homomorphism, respectively.

For detailed information see [20, Chapter 2].

3 Divisor homomorphisms with torsion class group

As mentioned in Section 2, it is well-known that every Krull monoid has an essentially unique divisor theory. However, besides divisor theories there exist numerous distinct divisor homomorphisms into free monoids. In this section, we investigate divisor homomorphisms into free monoids with torsion class group, i.e., outside factorial monoids (see [5]). Our investigation builds on known results (cf. [20, Theorem 2.4.7]), valid without the condition that the class group is a torsion group. Yet, in our more restricted situation a more explicit approach is possible, leading to a quite precise understanding of these divisor homomorphisms. We apply this information in our investigation of higher-order block monoids. In particular, we use it to show that the class groups and block monoids of order σ for infinite σ are independent of the divisor homomorphism (see Theorem 6.3). For a related result, investigating the class groups of certain submonoids of a Krull monoid, see [25].

Theorem 3.1. *Let $\varphi : H \to \mathcal{F}(P)$ be a divisor homomorphism such that $\mathcal{C}(\varphi)$ is a torsion group. Then, up to a natural identification, $\mathcal{C}(H)$ is a subgroup of $\mathcal{C}(\varphi)$. More precisely, there exists an exact sequence*

$$0 \to \mathcal{C}(H) \hookrightarrow \mathcal{C}(\varphi) \to \oplus_{p \in P} C_{m_p(\varphi)} \to 0.$$

And, for each $p \in P$, we have $k[p]_\varphi \in \mathcal{C}(H)$ if and only if $m_p(\varphi) \mid k$, and $\mathcal{D}(H) = \{m_p(\varphi)[p]_\varphi : p \in P\}$.

We start with a preparatory lemma.

Lemma 3.2. *Let $\varphi : H \to \mathcal{F}(P)$ be a divisor homomorphism such that $\mathcal{C}(\varphi)$ is a torsion group.*

1. *The monoid $\mathcal{F}_\varphi$ is free. More precisely, $\mathcal{F}_\varphi = [\{p^{m_p(\varphi)} : p \in P\}]$. In particular, $m_p(\varphi) = \gcd\{\mathsf{v}_p(\varphi(a)) : a \in H\}$ and $m_p(\varphi) \mid \operatorname{ord}([p]_\varphi)$ for each $p \in P$.*
2. *$\iota(\varphi) : \mathcal{F}_\varphi \hookrightarrow \mathcal{F}(P)$ is a divisor homomorphism.*
3. *For each $p \in P$, $\operatorname{ord}([p]_{\iota(\varphi)}) = m_p(\varphi)$ and $\mathcal{C}(\iota(\varphi)) = \bigoplus_{p \in P} \langle [p]_{\iota(\varphi)} \rangle$.*
4. *The map $\varphi_r : H \to \mathcal{F}_\varphi$, defined via $\varphi_r(a) = \varphi(a)$, is a divisor theory.*

Proof. 1. Let $p \in P$ and $M_p = \operatorname{ord}([p]_\varphi)$, and let $u_p \in H$ such that $\varphi(u_p) = p^{M_p}$. We first assert that $m_p(\varphi) \mid M_p$. Assume not. Let $a \in H$ with $\mathsf{v}_p(\varphi(a)) = m_p(\varphi)$. Since $p^{M_p} \mid \varphi(a)^{\lceil M_p / m_p(\varphi) \rceil}$ and φ is a divisor homomorphism, it follows that $u_p \mid a^{\lceil M_p / m_p(\varphi) \rceil}$. Yet, $0 < \mathsf{v}_p(\varphi(a u_p^{-1})) < m_p(\varphi)$, a contradiction.

We observe that $\gcd \varphi(\{u_p, a\}) = p^{m_p(\varphi)} \in \mathcal{F}_\varphi$. We assert that $m_p(\varphi) = \gcd\{\mathsf{v}_p(\varphi(a)) : a \in H\}$. Let $a_1, a_2 \in H$ and $d = \gcd\{\mathsf{v}_p(\varphi(a_1)), \mathsf{v}_p(\varphi(a_2))\}$. We have to show that there exists some $a \in H$ with $\mathsf{v}_p(\varphi(a)) = d$. We observe that for all sufficiently large k the element kd is contained in the submonoid of $\mathbb{N}_0$ generated by $\mathsf{v}_p(\varphi(a_1))$ and $\mathsf{v}_p(\varphi(a_2))$. Let $k = (1 + \ell M_p / d)$ for some $\ell \in d\mathbb{N}$ such that $kd = x_1 \mathsf{v}_p(\varphi(a_1)) + x_2 \mathsf{v}_p(\varphi(a_2))$ for $x_1, x_2 \in \mathbb{N}$. Then, as above, $u_p^{-\ell} a_1^{x_1} a_2^{x_2} \in H$ and $\mathsf{v}_p(\varphi(u_p^{-\ell} a_1^{x_1} a_2^{x_2})) = d$. The claim is now immediate.
2. By the first part, we know that for $f \in \mathcal{F}(P)$ we have $f \in \mathcal{F}_\varphi$ if and only if $m_p(\varphi) \mid \mathsf{v}_p(f)$ for each $p \in P$. Thus, if $f_1, f_2 \in \mathcal{F}_\varphi$ and $f_1 \mid f_2$ in $\mathcal{F}(P)$, then $f_1^{-1} f_2 \in \mathcal{F}_\varphi$.
3. By the above reasoning, we have for $f_1, f_2 \in \mathcal{F}(P)$ that $f_1 \in f_2 \, \mathsf{q}(\mathcal{F}_\varphi)$ if and only if $\mathsf{v}_p(f_1) \equiv \mathsf{v}_p(f_2) \pmod{m_p(\varphi)}$ for each $p \in P$. Thus, the claim follows.
4. By the first part, $\mathcal{F}_\varphi$ is free. The map φ_r is a divisor homomorphism, since φ is a divisor homomorphism. The condition that each element of $\mathcal{F}_\varphi$ is the greatest common divisor of the image of finitely many elements of H is true by definition, since in a free monoid the greatest common divisor of an infinite set is already attained by a finite subset. $\square$

Proof of Theorem 3.1. By Lemma 3.2, we know that $\varphi_r : H \to \mathcal{F}_\varphi$ is a divisor theory. Thus, $\mathcal{C}(H)$ is (up to unique isomorphism) $\mathsf{q}(\mathcal{F}_\varphi) / \mathsf{q}(\varphi_r(H))$. We have that $\mathsf{q}(\mathcal{F}_\varphi) \subset \mathsf{q}(\mathcal{F}(P))$ is a subgroup and that $\mathsf{q}(\varphi_r(H)) = \mathsf{q}(\varphi(H))$. Thus, $\mathsf{q}(\mathcal{F}_\varphi) / \mathsf{q}(\varphi_r(H)) \subset \mathsf{q}(\mathcal{F}(P)) / \mathsf{q}(\varphi(H))$ is a subgroup. The former is $\mathcal{C}(H)$ and the latter $\mathcal{C}(\varphi)$, establishing the first claim.

Since $(\mathsf{q}(\mathcal{F}(P)) / \mathsf{q}(\varphi(H))) / (\mathsf{q}(\mathcal{F}_\varphi) / \mathsf{q}(\varphi_r(H))) \cong \mathsf{q}(\mathcal{F}(P)) / \mathsf{q}(\mathcal{F}_\varphi)$, we have $\mathcal{C}(\varphi) / \mathcal{C}(H) \cong \mathcal{C}(\iota(\varphi))$. By Lemma 3.2, $\mathcal{C}(\iota(\varphi)) \cong \oplus_{p \in P} C_{m_p(\varphi)}$ and the existence of the exact sequence is established.

The remaining result on $\mathcal{D}(H)$ follows once the following claim is established. *Claim:* Let $f_1, f_2 \in \mathcal{F}(P)$, $f_{i,0} = \operatorname{lcm}\{f \in \mathcal{F}_\varphi : f \mid f_i\}$ and $f_i' = f_{i,0}^{-1} f_i$ for $i \in [1,2]$. Then, $[f_1]_\varphi = [f_2]_\varphi$ if and only if $f_1' = f_2'$ and $[f_{1,0}]_\varphi = [f_{2,0}]_\varphi$.
Proof of Claim: Suppose $[f_1]_\varphi = [f_2]_\varphi$. That is, there exists some $h \in \mathsf{q}(\varphi(H))$ such that $f_1 = hf_2$. Since $\varphi(H) \subset \mathcal{F}_\varphi$, we know that $\mathsf{v}_p(h) \in m_p(\varphi)\mathbb{Z}$ for each $p \in P$. We observe that $f_i' \in [0, m_p(\varphi) - 1]$ for each $p \in P$ and conclude that $f_1' = f_2'$. Thus, we have $f_{1,0} = hf_{2,0}$, that is $[f_{1,0}]_{\varphi_0} = [f_{2,0}]_{\varphi_0}$. The converse direction is obvious. $\square$

4 Abstract version of higher-order constructions

In this section, we develop the notion of a higher-order block monoid and a higher-order class group in an abstract fashion. That is, we only consider block monoids and frequently even just investigate subsets of torsion groups without making any direct reference to monoids.

4.1 The refinement of a block monoid

Using Theorem 3.1, we give an explicit description of the block monoid associated to $\mathcal{B}(G_0)$ for G_0 a subset of a torsion group, which we call the refinement of $\mathcal{B}(G_0)$. To avoid any ambiguity, we reiterate that the refinement of $\mathcal{B}(G_0)$ is defined via a divisor theory of $\mathcal{B}(G_0)$. Yet, the obvious imbedding $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is in general only a divisor homomorphism. The block monoid associated to the imbedding $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is, up to standard identification, $\mathcal{B}(G_0)$ (see [20, Proposition 2.5.6]).

Proposition 4.1. *Let G be a torsion group and let $G_0 \subset G$. The refinement of $\mathcal{B}(G_0)$ is, up to identification, $\mathcal{B}(H_0)$ where $H_0 = \{n(g)g : g \in G_0\}$ and $n(g) \in \mathbb{N}$ is minimal with $n(g)g \in \langle G_0 \setminus \{g\}\rangle$. In particular, the class group of $\mathcal{B}(G_0)$ is $\langle H_0 \rangle$, a subgroup of $\langle G_0 \rangle$.*

Proof. As recalled above, the imbedding $\varphi : \mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is a divisor homomorphism, $\mathcal{C}(\varphi) = \langle G_0 \rangle$ and $\mathcal{D}_\varphi(\mathcal{B}(G_0)) = G_0$. We observe that, for $n \in \mathbb{N}$ and $g \in G_0$, the following statements are equivalent:

- There exists some $B \in \mathcal{B}(G_0)$ with $\mathsf{v}_g(B) = n$.
- $ng \in \langle G_0 \setminus \{g\}\rangle$.

Thus, indeed $n(g) = m_g(\varphi)$ and the result follows by Theorem 3.1. $\square$

In view of this proposition, we make the following definitions.

Definition 4.2. Let G be a torsion group and let $G_0 \subset G$. Let $g \in G_0$.

1. $n_{G_0}(g) = \min\{n \in \mathbb{N} : ng \in \langle G_0 \setminus \{g\}\rangle\}$.
2. $\gamma_{G_0} : G_0 \to G$ is defined by $\gamma_{G_0}(g) = n_{G_0}(g)g$.
3. $\Gamma(G_0) = \gamma_{G_0}(G_0)$.

Since we need it frequently, we explicitly point out a property of $n_{G_0}(g)$ that is clear by Lemma 3.2 and the proof of Proposition 4.1 (also cf. [20, Lemma 6.7.10]).

Remark 4.3. Let G be a torsion group, let $G_0 \subset G$, and let $g \in G_0$. Then, $n_{G_0}(g) = \gcd\{\mathsf{v}_g(B) \colon B \in \mathcal{B}(G_0)\}$. In particular, $n_{G_0}(g) \mid \mathrm{ord}(g)$.

Using the terminology introduced above, we reformulate and refine Proposition 4.1.

Corollary 4.4. *Let G be a torsion group and let $G_0 \subset G$. We have $\mathcal{D}(\mathcal{B}(G_0)) = \Gamma(G_0)$. The block homomorphism is given by*

$$\beta_{\mathcal{B}(G_0)} : \begin{cases} \mathcal{B}(G_0) & \to \mathcal{B}(\Gamma(G_0)) \\ B & \mapsto \prod_{h \in \Gamma(G_0)} h^{\mathrm{ord}(h)\,\mathsf{k}(F_h)}, \ \text{where } F_h = \prod_{g \in \gamma_{G_0}^{-1}(h)} g^{\mathsf{v}_g(B)} \end{cases}.$$

Proof. The first statement is clear by Proposition 4.1 and Definition 4.2. To get the result on the block homomorphism, we note that, for $B \in \mathcal{B}(G_0)$ and $g \in G_0$, we have $\mathrm{ord}(\gamma_{G_0}(g)) = \mathrm{ord}(n_{G_0}(g)g) = \mathrm{ord}(g)/n_{G_0}(g)$. Thus, $\mathrm{ord}(h)\,\mathsf{k}(F_h) = \sum_{g \in \gamma_{G_0}^{-1}(h)} \mathsf{v}_g(B)(\mathrm{ord}(g)/n_{G_0}(g))$. Now, the claim is clear by the proof of Proposition 4.1. □

Since it is relevant for our following work, we state the following simple lemma.

Lemma 4.5. *Let G be a torsion group and let $G_0 \subset G$. The following statements are equivalent.*

1. $n_{G_0}(g) = 1$ *for each* $g \in G_0$, *i.e.,* $\gamma_{G_0} = \mathrm{id}_{G_0}$.
2. $\Gamma(G_0) = G_0$.
3. $\langle \Gamma(G_0) \rangle = \langle G_0 \rangle$.

We point out that the groups $\langle \Gamma(G_0) \rangle$ and $\langle G_0 \rangle$ actually have to be equal; it is not sufficient that they are isomorphic (cf. Example 4.17).

Proof. Since the implications from 1. to 2. to 3. are trivial, we only have to show that 3. implies 1.

Suppose $\langle \Gamma(G_0) \rangle = \langle G_0 \rangle$. Let $g \in G_0$. We show that $n_{G_0}(g) = 1$. Assume not. We have $g \in \langle \Gamma(G_0) \rangle$, i.e., $g = \sum_{h \in \Gamma(G_0)} a_h h$ with $a_h \in \mathbb{Z}$ (almost all 0). This implies that $g = \sum_{f \in G_0} a_f n_{G_0}(f) f$ with $a_f \in \mathbb{Z}$ (almost all 0). It follows that $(1 - a_g n_{G_0}(g))g \in \langle G_0 \setminus \{g\} \rangle$. Since by assumption $n_{G_0}(g) > 1$ and by Remark 4.3 $n_{G_0}(g) \mid \mathrm{ord}(g)$, we have $\gcd\{1 - a_g n_{G_0}(g), \mathrm{ord}(g)\} = 1$. Consequently, $g \in \langle G_0 \setminus \{g\} \rangle$, contradicting the assumption $n_{G_0}(g) > 1$. □

4.2 Main definitions and basic results

We recursively define quantities that are fundamental in the definition of higher-order class groups and block monoids.

Definition 4.6. Let G be a torsion group and let $G_0 \subset G$. Let σ be some ordinal.

- If $\sigma = 0$, then $n_{G_0}^{\sigma}(g) = 1$, $\gamma_{G_0}^{\sigma} = \mathrm{id}_{G_0}$, and $\Gamma^{\sigma}(G_0) = G_0$.
- $\Gamma^{\sigma+1}(G_0) = \Gamma(\Gamma^{\sigma}(G_0))$, $n_{G_0}^{\sigma+1}(g) = n_{\Gamma^{\sigma}(G_0)}(\gamma_{G_0}^{\sigma}(g)) n_{G_0}^{\sigma}(G)$, and $\gamma_{G_0}^{\sigma+1} = \gamma_{\gamma^{\sigma}(G_0)} \circ \gamma_{G_0}^{\sigma}$.

- If σ is a limit ordinal, then $n^\sigma_{G_0}(g) = \sup_{\rho<\sigma} n^\rho_{G_0}(g)$. Moreover, $\gamma^\sigma_{G_0}(g) = n^\sigma_{G_0}(g)g$ and $\Gamma^\sigma(G_0) = \gamma^\sigma_{G_0}(G_0)$.

For completeness, we add that we use the convention that $n^\sigma_{G_0}(g)g = 0$ in case $n^\sigma_{G_0}(g)$ is infinite; yet, as is shown in the lemma below, $n^\sigma_{G_0}(g)$ is always finite.

It is apparent, by the definition and Proposition 4.1, that for finite σ the block monoid $\mathcal{B}(\Gamma^\sigma(G_0))$ is just the monoid obtained by constructing the refinement, starting from $\mathcal{B}(G_0)$, σ-times. Thus, there exists a natural transfer homomorphism from $\mathcal{B}(G_0)$ to $\mathcal{B}(\Gamma^\sigma(G_0))$, namely the composition of the respective block homomorphisms.

Our next aim is to construct, for each ordinal σ, a transfer homomorphism from $\mathcal{B}(G_0)$ to $\mathcal{B}(\Gamma^\sigma(G_0))$. For finite σ this transfer homomorphism coincides with the composition of the block homomorphisms.

We start with a technical lemma.

Lemma 4.7. *Let G be a torsion group and let $G_0 \subset G$. Let $\sigma \le \tau$ be ordinals.*

1. $n^\sigma_{G_0}(g) \mid n^\tau_{G_0}(g)$.
2. $n^\sigma_{G_0}(g) \mid \operatorname{ord}(g)$.

Proof. The first statement is clear by definition. To get the second one we induct on σ. For $\sigma = 0$ this is trivial, and for $\sigma = 1$ the fact that $\operatorname{ord}(g)g \in \langle G_0 \setminus \{g\}\rangle$, implies the claim (cf. Remark 4.3). We consider $\sigma+1$. We have $n^{\sigma+1}_{G_0}(g) = n_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g))n^\sigma_{G_0}(g)$. By the induction hypothesis, $\operatorname{ord}(\gamma^\sigma_{G_0}(g)) = \operatorname{ord}(g)/n^\sigma_{G_0}(g)$. And, by the case $\sigma = 1$, $n_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g)) \mid \operatorname{ord}(\gamma^\sigma_{G_0}(g))$. For σ a limit ordinal the claim is immediate by the definition of $n^\sigma_{G_0}(g)$ and the induction hypothesis. $\square$

To construct our transfer homomorphisms, we need to consider the following specific decomposition of elements of $\mathcal{B}(G_0)$, which generalizes the one used in Corollary 4.4.

Notation 4.8. Let G be a torsion group and let $G_0 \subset G$. Let σ be an ordinal. For $h \in \Gamma^\sigma(G_0)$ and $B \in \mathcal{B}(G_0)$, let $F^\sigma_h(B) = \prod_{g\in(\gamma^\sigma_{G_0})^{-1}(h)} g^{\mathsf{v}_g(B)}$.

We establish some basic facts about $F^\sigma_h(B)$.

Lemma 4.9. *Let G be a torsion group and let $G_0 \subset G$. Let σ be an ordinal. Let $B \in \mathcal{B}(G_0)$. For each $h \in \Gamma^\sigma(G_0)$ we have $\operatorname{ord}(h)\,\mathsf{k}(F^\sigma_h(B)) \in \mathbb{N}_0$. Moreover, $\prod_{h\in\Gamma^\sigma(G_0)} h^{\operatorname{ord}(h)\,\mathsf{k}(F^\sigma_h(B))} \in \mathcal{B}(\Gamma^\sigma(G_0))$.*

Proof. Let $h \in \Gamma^\sigma(G_0)$. For simplicity, we write F^σ_h instead of $F^\sigma_h(B)$. Let $g \in (\gamma^\sigma_{G_0})^{-1}(h)$. This means that $n^\sigma_{G_0}(g)g = h$. Since, by Lemma 4.7, $n^\sigma_{G_0}(g) \mid \operatorname{ord}(g)$ this implies that $\operatorname{ord}(g) = n^\sigma_{G_0}(g)\operatorname{ord}(h)$.

We induct on σ. For $\sigma = 0$ the claims are trivial. We consider $\sigma+1$. By definition, $(\gamma^{\sigma+1}_{G_0})^{-1}(h) = (\gamma^\sigma_{G_0})^{-1}(\gamma^{-1}_{\Gamma^\sigma(G_0)}(h))$. Thus, $F^{\sigma+1}_h = \prod_{h'\in\gamma^{-1}_{\Gamma^\sigma(G_0)}(h)} F^\sigma_{h'}$. By the induction hypothesis, we have $\operatorname{ord}(h')\,\mathsf{k}(F^\sigma_{h'}) \in \mathbb{N}_0$. We observe that $\operatorname{ord}(h') = n_{\Gamma^\sigma(G_0)}(h')\operatorname{ord}(h)$ and, by Remark 4.3, $n_{\Gamma^\sigma(G_0)}(h') \mid \mathsf{v}_{h'}(\beta^\sigma_{G_0}(B)) = \operatorname{ord}(h')\,\mathsf{k}(F^\sigma_{h'})$. Thus,

$$\operatorname{ord}(h)\,\mathsf{k}(F^\sigma_{h'}) = \frac{\operatorname{ord}(h')}{n_{\Gamma^\sigma(G_0)}(h')}\,\mathsf{k}(F^\sigma_{h'}) \in \mathbb{N}_0,$$

implying that $\mathrm{ord}(h)\,\mathsf{k}(F_h^{\sigma+1}) = \sum_{h' \in \gamma_{\Gamma^\sigma(G_0)}^{-1}(h)} \mathrm{ord}(h)\,\mathsf{k}(F_{h'}^\sigma) \in \mathbb{N}_0$. Moreover, we observe that $\sigma(F_h^{\sigma+1}) = \sum_{h' \in \gamma_{\Gamma^\sigma(G_0)}^{-1}(h)} \sigma(F_{h'}^\sigma)$, implying that $\sigma(\beta_{G_0}^{\sigma+1}(B)) = \sigma(\beta_{G_0}^\sigma(B)) = 0$.

Let σ be a limit ordinal. Let $\rho < \sigma$ such that $n_{G_0}^\rho(g) = n_{G_0}^\sigma(g)$ for each $g \in \mathrm{supp}(B)$; since $\mathrm{supp}(B)$ is finite, the existence of such a ρ is guaranteed. We note that $F_h^\rho = F_h^\sigma$ and $\prod_{h \in \Gamma^\rho(G_0)} h^{\mathrm{ord}(h)\,\mathsf{k}(F_h^\rho)} = \prod_{h \in \Gamma^\sigma(G_0)} h^{\mathrm{ord}(h)\,\mathsf{k}(F_h^\sigma)}$. The claim follows by the induction hypothesis. $\square$

Having these preparatory results at hand, we construct the transfer homomorphisms.

Theorem 4.10. *Let G be a torsion group and let $G_0 \subset G$. Let σ be an ordinal. The map*

$$\beta_{G_0}^\sigma : \begin{cases} \mathcal{B}(G_0) & \to \mathcal{B}(\Gamma^\sigma(G_0)) \\ B & \mapsto \prod_{h \in \Gamma^\sigma(G_0)} h^{\mathrm{ord}(h)\,\mathsf{k}(F_h^\sigma(B))} \end{cases}$$

is a transfer homomorphism. For finite $\sigma \geq 1$ it is a composition of the block homomorphisms.

Proof. We induct on σ. For $\sigma = 0$ the map is the identity and the claim is trivial. For $\sigma = 1$ we observe that $\beta_{G_0}^1$ is the block homomorphism (cf. Corollary 4.4). We consider $\sigma + 1$. By the proof of Lemma 4.9, we see that $\beta_{G_0}^{\sigma+1} = \beta_{\Gamma^\sigma(G_0)}^1 \circ \beta_{G_0}^\sigma$. It is thus a transfer homomorphism, as composition of two transfer homomorphisms.

Now, suppose that σ is a limit ordinal. By the definition, it is easy to see that $\beta_{G_0}^\sigma$ is a homomorphism. We show that $\beta_{G_0}^\sigma$ is surjective. Let $C \in \mathcal{B}(\Gamma^\sigma(G_0))$. For $h \in \mathrm{supp}(C)$, let $g_h \in (\gamma_{G_0}^\sigma)^{-1}(h)$. We set $B = \prod_{h \in \mathrm{supp}(C)} g_h^{n_{G_0}^\sigma(g_h)\,\mathsf{v}_C(h)}$. Then, $F_h^\sigma(B) = g_h^{n_{G_0}^\sigma(g_h)\,\mathsf{v}_C(h)}$ and

$$\mathrm{ord}(h)\,\mathsf{k}(F_h^\sigma(B)) = \frac{\mathrm{ord}(h)\ n_{G_0}^\sigma(g_h)\,\mathsf{v}_h(B)}{\mathrm{ord}(g_h)} = \mathsf{v}_h(B).$$

Thus, $\beta_{G_0}^\sigma(B) = C$. Since $\mathcal{B}(G_0)$ and $\mathcal{B}(\Gamma^\sigma(G_0))$ are reduced, this shows that (T1) holds.

Next, we note that for each B there exists some $\rho < \sigma$ such that $\beta^\rho(B) = \beta^\sigma(B)$ and the same holds true for each each $B' \mid B$. Using this observation and noting that, by the induction hypothesis, (T2) holds for $\beta_{G_0}^\rho$, we get that (T2) holds for $\beta_{G_0}^\sigma$. $\square$

We want to show that the family $(\Gamma^\sigma(G_0))_\sigma$ is eventually constant. To this end, we consider the groups $\langle \Gamma^\sigma(G_0) \rangle$. Since we need it in Section 7, we prove a result that is stronger than required for the present purpose.

Lemma 4.11. *Let G be a torsion group and let $G_0 \subset G$. Let σ be a limit ordinal. Then, $\langle \Gamma^\sigma(G_0) \rangle = \bigcap_{\rho < \sigma} \langle \Gamma^\rho(G_0) \rangle$.*

Proof. Since, by Lemma 4.7, $n_{G_0}^\rho(g) \mid n_{G_0}^\sigma(g)$ for each $\rho < \sigma$, it is clear that $\langle \Gamma^\sigma(G_0) \rangle \subset \langle \Gamma^\rho(G_0) \rangle$ for each $\rho < \sigma$.

Now, let $g \in \bigcap_{\rho < \sigma} \langle \Gamma^\rho(G_0) \rangle$. We have to show that $g \in \langle \Gamma^\sigma(G_0) \rangle$. Considering $\rho = 0$, we know that $g \in \langle G_0 \rangle$. Let $G_1 \subset G_0$ be a finite set such that $g \in \langle G_1 \rangle$.

If $g \in \langle \gamma_{G_0}^{\rho}(G_1) \rangle$ for each $\rho < \sigma$, then, since the finiteness of G_1 implies that $\gamma_{G_0}^{\tau}|_{G_1} = \gamma_{G_0}^{\sigma}|_{G_1}$ for some $\tau < \sigma$, we have $g \in \langle \gamma_{G_0}^{\sigma}(G_1) \rangle \subset \langle \Gamma^{\sigma}(G_0) \rangle$, implying the claim.

Thus, we may assume that there exists some $\tau' < \sigma$ such that $g \notin \langle \gamma_{G_0}^{\tau'}(G_1) \rangle$; and we assume that τ' is minimal with this property. By the same argument as above, we note that τ' is a successor ordinal, say $\tau' = \tau + 1$. We have $g \in \langle \gamma_{G_0}^{\tau}(G_1) \rangle$, i.e., $g = \sum_{h \in G_1} a_h n_{G_0}^{\tau}(h) h$ with $a_h \in \mathbb{Z}$ (almost all 0), but $g \notin \langle \gamma_{G_0}^{\tau+1}(G_1) \rangle = \langle \{ n_{G_0}^{\tau+1}(h) h \colon h \in G_1 \} \rangle$. Without restriction we may assume that $\gamma_{G_0}^{\tau}|_{G_1}$ is injective.

Since $g \notin \langle \{ n_{G_0}^{\tau+1}(h) h \colon h \in G_1 \} \rangle$, we know that there exists some $f \in G_1$ such that $n_{G_0}^{\tau+1}(f) \nmid a_f n_{G_0}^{\tau}(f)$.

Since $g \in \langle \Gamma^{\tau+1}(G_0) \rangle$, we know that there exist $b_h \in \mathbb{Z}$ (almost all 0) such that $g = \sum_{h \in G_0} b_h n_{G_0}^{\tau+1}(h) h$. We may assume that if $n_{G_0}^{\tau+1}(h) h = n_{G_0}^{\tau+1}(f) f$ for some $h \neq f$, then $b_h = 0$. We have $\sum_{h \in G_1} a_h n_{G_0}^{\tau}(h) h = \sum_{h \in G_0} b_h n_{G_0}^{\tau+1}(h) h$. Implying that, with $a_h = 0$ for $h \notin G_1$,

$$(a_f n_{G_0}^{\tau}(f) - b_f n_{G_0}^{\tau+1}(f)) f = \sum_{h \in G_0 \setminus \{f\}} (b_h n_{G_0}^{\tau+1}(h) - a_h n_{G_0}^{\tau}(h)) h.$$

We recall that $n_{G_0}^{\tau+1}(h) = n_{\Gamma^{\tau}(G_0)}(\gamma_{G_0}^{\tau}(h)) n_{G_0}^{\tau}(h)$ for each $h \in G_1$. Thus, we have

$$(a_f - b_f n_{\Gamma^{\tau}(G_0)}(\gamma_{G_0}^{\tau}(f))) \gamma_{G_0}^{\tau}(f) = \sum_{h \in G_0 \setminus \{f\}} (b_h n_{\Gamma^{\tau}(G_0)}(\gamma_{G_0}^{\tau}(f)) - a_h) \gamma_{G_0}^{\tau}(h).$$

By our assumption on G_1 and b_h, we know that $b_h n_{\Gamma^{\tau}(G_0)}(\gamma_{G_0}^{\tau}(f)) - a_h = 0$ for each $h \in (\gamma_{G_0}^{\tau})^{-1}(\gamma_{G_0}^{\tau}(f)) \setminus \{f\}$. Consequently, we have

$$(a_f - b_f n_{\Gamma^{\tau}(G_0)}(\gamma_{G_0}^{\tau}(f))) \gamma_{G_0}^{\tau}(f) \in \langle \Gamma^{\tau}(G_0) \setminus \{\gamma_{G_0}^{\tau}(f)\} \rangle.$$

Yet, the condition $n_{G_0}^{\tau+1}(f) \nmid a_f n_{G_0}^{\tau}(f)$ is equivalent to $n_{\Gamma^{\tau}(G_0)}(\gamma_{G_0}^{\tau}(f)) \nmid a_f$, and thus $n_{\Gamma^{\tau}(G_0)}(\gamma_{G_0}^{\tau}(f)) \nmid (a_f - b_f n_{\Gamma^{\tau}(G_0)}(\gamma_{G_0}^{\tau}(f)))$, a contradiction (cf. Remark 4.3). This contradiction completes the argument. □

We point out that for the finitary analog of this construction such a result does not hold (see Example 5.5).

Proposition 4.12. *Let G be a torsion group and let $G_0 \subset G$. There exists a smallest ordinal σ such that $\Gamma^{\sigma}(G_0) = \Gamma^{\tau}(G_0)$ for each $\tau \geq \sigma$. If G_0 is finite, then σ is finite.*

Proof. Let σ be an ordinal. If, for each $\rho < \sigma$, $\Gamma^{\rho}(G_0) \neq \Gamma^{\rho+1}(G_0)$, then by Lemma 4.5 and Lemma 4.11, $(\langle \Gamma^{\rho}(G_0) \rangle)_{\rho \leq \sigma}$ is a properly descending chain of subgroups of $\langle G_0 \rangle$. Since the lengths of such a properly descending chain is bounded by the cardinality of $\langle G_0 \rangle$, the claim follows. □

We point out that this result directly yields a generalization of a result that proved to be useful in Non-Unique Factorization Theory (cf. [20, Theorem 6.7.11] where this result is proved for finite sets, [15, 27] for earlier versions, and, e.g., [28] for an application).

Corollary 4.13. *Let G be a torsion group and let $G_0 \subset G$. There exists some set $G_0^* \subset G$ with $g \in \langle G_0^* \setminus \{g\}\rangle$ for each $g \in G_0^*$ and a transfer homomorphism*

$$\theta : \mathcal{B}(G_0) \to \mathcal{B}(G_0^*).$$

Proof. By Proposition 4.12, there exists some σ such that $\Gamma^{\sigma+1}(G_0) = \Gamma^{\sigma}(G_0)$. We set $G_0^* = \Gamma^{\sigma}(G_0)$. By Theorem 4.10, there exists a transfer homomorphism from $\mathcal{B}(G_0)$ to $\mathcal{B}(G_0^*)$. We have $\Gamma(G_0^*) = \Gamma^{\sigma+1}(G_0) = \Gamma^{\sigma}(G_0) = G_0^*$, which, by Lemma 4.5, implies $g \in \langle G_0^* \setminus \{g\}\rangle$ for each $g \in G_0^*$. $\square$

The finitary construction, to be given in the following section, in particular Proposition 5.7, yields another way to get such a set G_0^*.

4.3 Examples

We give some simple examples of the effect of $\Gamma^{\sigma}(\cdot)$. On the one hand, we do so for illustration, yet on the other hand we refer to these examples elsewhere in the paper.

The first example characterizes sets with $\Gamma^1(G_0) \subset \{0\}$, that is $\mathcal{B}(G_0)$ is factorial; clearly, $\Gamma^{\sigma}(G_0) = \emptyset$ only in the trivial case that $G_0 = \emptyset$. We recover the characterization given in [19, Propostion 3], also cf. Lemma 7.2.

Example 4.14. Let G be a torsion group and let $G_0 \subset G$. Then, $\Gamma^1(G_0) \subset \{0\}$ if and only if $G_0 \setminus \{0\}$ is independent. To see this, it suffices to note that $\Gamma^1(G_0) \subset \{0\}$ is equivalent to $n_{G_0}(g) = \mathrm{ord}(g)$ for each $g \in G$, that is $kg \notin \langle G_0 \setminus \{g\}\rangle$ for each $k \notin \mathrm{ord}(g)\mathbb{Z}$.

In the following examples, we study subsets of p-groups. Later, in Section 7, we undertake a systematic investigation of subsets of p-groups with $\Gamma^{\sigma}(G_0) = \{0\}$.

Example 4.15. Let p be a prime and $k \in \mathbb{N}$. Let $C_{p^k} = \langle e\rangle$ and let $\emptyset \neq G_0 \subset \{p^j e \colon j \in [0,k]\}$. Let $n = |G_0 \setminus \{0\}|$, then $\Gamma^n(G_0) = \{0\}$ and $\Gamma^m(G_0) \neq \{0\}$ for each $0 \leq m < n$. To see this, we first consider the element $g \in G_0$ of maximal order. If $g = 0$, the claim is clear, and we assume $g \neq 0$. We have $g \notin \langle G_0 \setminus \{g\}\rangle$, yet $n_{G_0}(g)g \in G_0 \setminus \{g\}$. For each $h \in G_0 \setminus \{g\}$, we have $n_{G_0}(h) = 1$, since $h = -\mathrm{ord}(g)/\mathrm{ord}(h)g \in \langle g\rangle \subset \langle G_0 \setminus \{g\}\rangle$. And, the claim follows by induction on $|G_0 \setminus \{0\}|$.

Example 4.16. Let p be a prime. Let $G_0 = \{p^{-n} + \mathbb{Z} \colon n \in \mathbb{N}_0\} \subset \mathbb{Z}(p^{\infty})$. Then, $\Gamma^{\sigma}(G_0) = G_0$. To see this, note that for each $g \in G_0$ there exists some $h \in G_0 \setminus \{g\}$ with $ph = g$, thus $g = \langle h\rangle \subset \langle G_0 \setminus \{g\}\rangle$ and $n_{G_0}(g) = 1$.

Example 4.17. Let $G = \oplus_{k\in\mathbb{N}}\langle e_k\rangle$ with $\mathrm{ord}(e_k) = p^k$ and let $G_0 = \{p^j e_k \colon j \in \mathbb{N}_0,\, k \in \mathbb{N}\}$. Then, $\Gamma^{\omega}(G_0) = \{0\}$, yet $\Gamma^{\rho}(G_0) \neq \{0\}$ for each $\rho < \omega$. Indeed, $\langle \Gamma^{\rho}(G_0)\rangle \cong G$ for each $\rho < \omega$ (cf. the remark after Lemma 4.5). To see this, it suffices to note that each "coordinate" can be considered separately and to apply Example 4.15.

The following example shows that $\Gamma^{\sigma}(\cdot)$ does not preserve inclusions.

Example 4.18. Let p be an odd prime. Let $C_{p^2} = \langle e\rangle$ and let $G_0 = \{e, -e, pe\}$ and $G_0' = \{-e, pe\}$. Then, $\Gamma^{\sigma}(G_0) = G_0$ and $\Gamma^{\sigma}(G_0') = \{pe, -pe\}$ for each $\sigma \geq 1$. To see this, note that $\pm e \in \langle \mp e\rangle$ and $pe \in \langle e\rangle$. Yet, $-e \notin \langle pe\rangle$, but $p(-e) \in \langle pe\rangle$.

We end with an example that is relevant in Lemma 7.8.

Example 4.19. Let p, q distinct primes, $C_{pq} = \langle e \rangle$, and $G_0 = \{e, pe, qe\}$. Then, $\Gamma^\sigma(G_0) = G_0$. To see this, note that $e \in \langle \{pe, qe\} \rangle$.

4.4 Some technical results

In this subsection, we prove some technical results that are needed in the following sections. We make frequent use of the following notation.

Notation 4.20. Let G be a torsion group. For $g, h \in G$, let $n_{g,h}(g)$ denote the divisor of $\mathrm{ord}(g)$ with the property that $\langle n_{g,h}(g) g \rangle = \langle g \rangle \cap \langle h \rangle$.

The relevance of the following lemma is due to the fact that it yields a condition regarding the non-equality of $\gamma^\sigma_{G_0}(g)$ and $\gamma^\sigma_{G_0}(h)$ that is independent of G_0.

Lemma 4.21. *Let G be a torsion group and let $G_0 \subset G$. Let σ be an ordinal. Let $h, g \in G_0$. We have*

- $n^\sigma_{G_0}(g) \mid n_{g,h}(g)$ *and* $n^\sigma_{G_0}(h) \mid n_{g,h}(h)$, *or*
- $\gamma^\sigma_{G_0}(g) = \gamma^\sigma_{G_0}(h)$.

Moreover, if $n_{g,h}(g)g \neq n_{g,h}(h)h$, then $n^\sigma_{G_0}(g) \mid n_{g,h}(g)$, $n^\sigma_{G_0}(h) \mid n_{g,h}(h)$, and $\gamma^\sigma_{G_0}(g) \neq \gamma^\sigma_{G_0}(h)$.

Proof. We induct on σ. The case $\sigma = 0$ is trivial. We consider $\sigma+1$. If $\gamma^\sigma_{G_0}(g) = \gamma^\sigma_{G_0}(h)$, clearly $\gamma^{\sigma+1}_{G_0}(g) = \gamma^{\sigma+1}_{G_0}(h)$. Thus, suppose $\gamma^\sigma_{G_0}(g) \neq \gamma^\sigma_{G_0}(h)$. By the induction hypothesis, we know that $n^\sigma_{G_0}(g) \mid n_{g,h}(g)$ and $n^\sigma_{G_0}(h) \mid n_{g,h}(h)$. Since $\gamma^\sigma_{G_0}(h) \in \Gamma^\sigma(G_0) \setminus \{\gamma^\sigma_{G_0}(g)\}$ and since $n^\sigma_{G_0}(h) \mid n_{g,h}(h)$, it follows that

$$\frac{n_{g,h}(g)}{n^\sigma_{G_0}(g)} \gamma^\sigma_{G_0}(g) = n_{g,h}(g) g \in \langle \Gamma^\sigma(G_0) \setminus \{\gamma^\sigma_{G_0}(g)\} \rangle.$$

Thus,

$$n_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g)) \mid \frac{n_{g,h}(g)}{n^\sigma_{G_0}(g)}$$

and $n^{\sigma+1}_{G_0}(g) = n_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g)) n^\sigma_{G_0}(g) \mid n_{g,h}(g)$. Suppose σ is a limit ordinal. If $n^\rho_{G_0}(g) \mid n_{g,h}(g)$ and $n^\rho_{G_0}(h) \mid n_{g,h}(h)$ for each $\rho < \sigma$, then clearly $n^\sigma_{G_0}(g) \mid n_{g,h}(g)$ and $n^\sigma_{G_0}(h) \mid n_{g,h}(h)$. Yet, if $\gamma^\rho_{G_0}(g) = \gamma^\rho_{G_0}(h)$ for some $\rho < \sigma$, then $\gamma^{\rho'}_{G_0}(g) = \gamma^{\rho'}_{G_0}(h)$ for each $\rho' \geq \rho$.

It remains to prove the "moreover"-statement. Suppose that $n_{g,h}(g)g \neq n_{g,h}(h)h$. It suffices to show that $\gamma^\sigma_{G_0}(g) \neq \gamma^\sigma_{G_0}(h)$. Assume to the contrary that $\gamma^\tau_{G_0}(g) = \gamma^\tau_{G_0}(h)$ for some τ, and suppose τ is minimal with this property. We note that τ is a successor ordinal, say $\tau = \sigma+1$. Thus, we have $\gamma^\sigma_{G_0}(g) \neq \gamma^\sigma_{G_0}(h)$ and consequently $n^\sigma_{G_0}(g) \mid n_{g,h}(g)$ and $n^\sigma_{G_0}(h) \mid n_{g,h}(h)$. By the above argument, we have $n^\tau_{G_0}(g) \mid n_{g,h}(g)$ and $n^\tau_{G_0}(h) \mid n_{g,h}(h)$. If at least one of the divisibility relations is proper, then $\gamma^\tau_{G_0}(g) \neq \gamma^\tau_{G_0}(h)$ by the definition of $n_{g,h}(g)$ and $n_{g,h}(h)$. Yet, if $n^\tau_{G_0}(g) = n_{g,h}(g)$ and $n^\tau_{G_0}(h) = n_{g,h}(h)$, then $\gamma^\tau_{G_0}(g) \neq \gamma^\tau_{G_0}(h)$ by assumption, a contradiction. $\square$

Using this lemma, we establish a relation among $n^\sigma_{G_1}(g)$ and $n^\sigma_{G_0}(g)$ for $G_1 \subset G_0$.

Proposition 4.22. *Let G be a torsion group and let $G_1 \subset G_0 \subset G$. Let σ be an ordinal. Then, $n^\sigma_{G_0}(g) \mid n^\sigma_{G_1}(g)$ for each $g \in G_1$.*

Proof. We induct on σ. For $\sigma = 0$ the claim is obvious. We consider $\sigma + 1$. We start by showing the following auxiliary claim: $G_1 \setminus (\gamma^\sigma_{G_1})^{-1}(\gamma^\sigma_{G_1}(g)) \subset G_0 \setminus (\gamma^\sigma_{G_0})^{-1}(\gamma^\sigma_{G_0}(g))$. Let $h \in G_1 \setminus (\gamma^\sigma_{G_1})^{-1}(\gamma^\sigma_{G_1}(g))$, i.e., $\gamma^\sigma_{G_1}(h) \neq \gamma^\sigma_{G_1}(g)$. By Lemma 4.21, we thus know that $n^\sigma_{G_1}(g) \mid n_{g,h}(g)$ and $n^\sigma_{G_1}(h) \mid n_{g,h}(h)$. By the induction hypothesis, we know that $n^\sigma_{G_0}(g) \mid n^\sigma_{G_1}(g) \mid n_{g,h}(g)$ and $n^\sigma_{G_0}(h) \mid n^\sigma_{G_1}(h) \mid n_{g,h}(h)$. Since equality in the chain of divisibilty relations cannot hold, $n^\sigma_{G_0}(g)g \neq n^\sigma_{G_0}(h)h$ and $h \in G_0 \setminus (\gamma^\sigma_{G_0})^{-1}(\gamma^\sigma_{G_0}(g))$.

We know that $n^{\sigma+1}_{G_1}(g)$ is the minimal $n \in \mathbb{N}$ fulfilling the two properties $ng \in \langle \Gamma^\sigma(G_1) \setminus \{\gamma^\sigma_{G_1}(g)\}\rangle$ and $n^\sigma_{G_1}(g) \mid n$. We denote the set of all $n \in \mathbb{N}$ fulfilling the former and the latter property by I_1 and J_1, respectively. In the same way we have $n^{\sigma+1}_{G_0}(g)$ is minimal with the properties $ng \in \langle \Gamma^\sigma(G_0) \setminus \gamma^\sigma_{G_0}(g)\rangle$ and $n^\sigma_{G_0}(g) \mid n$, and we denote the respective sets by I_0 and J_0. By the induction hypothesis, we know that $I_1 \subset I_0$. We show that $J_1 \subset J_0$ as well, which implies the claim.

Let $n \in J_1$. Thus, for suitable $a_f, a_h, a_{f'} \in \mathbb{Z}$ (almost all 0) and $d_h \in \mathbb{N}$, we have

$$\begin{aligned} ng &= \sum_{f \in \Gamma^\sigma(G_1) \setminus \{\gamma^\sigma_{G_1}(g)\}} a_f f = \sum_{h \in G_1 \setminus (\gamma^\sigma_{G_1})^{-1}(\gamma^\sigma_{G_1}(g))} a_h n^\sigma_{G_1}(h) h \\ &= \sum_{h \in G_1 \setminus (\gamma^\sigma_{G_1})^{-1}(\gamma^\sigma_{G_1}(g))} a_h d_h n^\sigma_{G_0}(h) h = \sum_{f' \in \gamma^\sigma_{G_0}(G_1 \setminus (\gamma^\sigma_{G_1})^{-1}(\gamma^\sigma_{G_1}(g)))} a_{f'} f', \end{aligned}$$

where the penultimate equality holds by the induction hypothesis. Since, by the auxiliary claim,

$$\gamma^\sigma_{G_0}(G_1 \setminus (\gamma^\sigma_{G_1})^{-1}(\gamma^\sigma_{G_1}(g))) \subset \gamma^\sigma_{G_0}(G_0 \setminus (\gamma^\sigma_{G_0})^{-1}(\gamma^\sigma_{G_0}(g))) = \Gamma^\sigma(G_0) \setminus \{\gamma^\sigma_{G_0}(g)\},$$

we have $ng \in \langle \Gamma^\sigma(G_0) \setminus \{\gamma^\sigma_{G_0}(g)\}\rangle$, i.e., $n \in J_0$.

It remains to consider the case that σ is a limit ordinal. This case is obvious, since there exists some $\rho < \sigma$ with $n^\rho_{G_0}(g) = n^\sigma_{G_0}(g)$ and $n^\rho_{G_1}(g) = n^\sigma_{G_1}(g)$. $\square$

The following lemma is used in Theorem 6.3, where we show that the class groups of infinite order are independent of the divisor homomorphism.

Lemma 4.23. *Let G be a torsion group and let $G_0 \subset G$. For each $g \in G$, let $d_g \in \mathbb{N}$ with $d_g \mid n_{G_0}(g)$, and let $H_0 = \{d_g g \colon g \in G_0\}$. Then, for each $g \in G_0$, $n^\sigma_{G_0}(g) \mid d_g n^\sigma_{H_0}(d_g g) \mid n^{\sigma+1}_{G_0}(g)$ for $\sigma < \omega$ and $n^\sigma_{G_0}(g) = d_g n^\sigma_{H_0}(d_g g)$ for $\sigma \geq \omega$. In particular, $\gamma^\sigma_{G_0}(g) = \gamma^\sigma_{H_0}(d_g g)$ for $\sigma \geq \omega$.*

Proof. Let $g \in G_0$. For $\sigma = 0$ the claim is merely the condition on d_g.

We consider $\sigma + 1$ for $\sigma < \omega$. We set $d^\sigma_g = d_g n^\sigma_{H_0}(d_g g)/n^\sigma_{G_0}(g)$. By the induction hypothesis, we have $d^\sigma_g \in \mathbb{N}$ and $d^\sigma_g \mid n_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g))$. Moreover, we have $\Gamma^\sigma(H_0) = \{d^\sigma_g \gamma^\sigma_{G_0}(g) \colon g \in G_0\}$.

We assert that if $\gamma^\sigma_{G_0}(g_1) = \gamma^\sigma_{G_0}(g_2)$ for $g_1, g_2 \in G_0$, then $d^\sigma_{g_1} = d^\sigma_{g_2}$. For $\sigma = 0$ this is trivial, and we assume $\sigma \geq 1$. Thus, we know that $d_g \mid n^\sigma_{G_0}(g)$.

Suppose $\gamma^\sigma_{G_0}(g_1) = \gamma^\sigma_{G_0}(g_2)$ for $g_1, g_2 \in G_0$. We have

$$\frac{n^\sigma_{G_0}(g_1)}{d_{g_1}}(d_{g_1}g_1) = \frac{n^\sigma_{G_0}(g_2)}{d_{g_2}}(d_{g_2}g_2).$$

Consequently, for $i \in [1, 2]$,

$$n_{d_{g_1}g_1, d_{g_2}g_2}(d_{g_i}g_i) \mid \frac{n^\sigma_{G_0}(g_1)}{d_{g_1}} \mid n^\sigma_{H_0}(d_{g_i}g_i),$$

where the latter holds by the induction hypothesis.

Thus, by Lemma 4.21, we have $\gamma^\sigma_{H_0}(d_{g_1}g_1) = \gamma^\sigma_{H_0}(d_{g_2}g_2)$. Since $d_{g_i} \mid \mathrm{ord}(g_i)$, we have $n^\sigma_{H_0}(d_{g_i}g_i)d_{g_i} \mid \mathrm{ord}(g_i)$. Thus, $n^\sigma_{H_0}(d_{g_1}g_1)d_{g_1}g_1 = n^\sigma_{H_0}(d_{g_2}g_2)d_{g_2}g_2$ together with $n^\sigma_{G_0}(g_1)g_1 = n^\sigma_{G_0}(g_2)g_2$, implies that $d^\sigma_{g_1} = d^\sigma_{g_2}$.

By the just asserted fact, we have

$$\Gamma^\sigma(H_0) \setminus \{d^\sigma_g \gamma^\sigma_{G_0}(g)\} \subset \{d^\sigma_h \gamma^\sigma_{G_0}(h) \colon h \in G_0 \setminus (\gamma^\sigma_{G_0})^{-1}(\gamma^\sigma_{G_0}(g))\},$$

and thus $\langle \Gamma^\sigma(H_0) \setminus \{d^\sigma_g \gamma^\sigma_{G_0}(g)\}\rangle \subset \langle \Gamma^\sigma(G_0) \setminus \{\gamma^\sigma_{G_0}(g)\}\rangle$. Therefore, we have $n_{\Gamma^\sigma(H_0)}(d^\sigma_g \gamma^\sigma_{G_0}(g))d^\sigma_g \gamma^\sigma_{G_0}(g) \in \langle \Gamma^\sigma(G_0) \setminus \{g_\sigma\}\rangle$, implying that

$$n_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g)) \mid n_{\Gamma^\sigma(H_0)}(d^\sigma_g \gamma^\sigma_{G_0}(g))d^\sigma_g.$$

Multiplying the above relation by $n^\sigma_{G_0}(g)$, we get

$$n_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g))n^\sigma_{G_0}(g) \mid n_{\Gamma^\sigma(H_0)}(d^\sigma_g \gamma^\sigma_{G_0}(g))d^\sigma_g n^\sigma_{G_0}(g).$$

The first term is merely $n^{\sigma+1}_{G_0}(g)$ and, using the definition of d^σ_g, we get that the second one equals $n_{\Gamma^\sigma(H_0)}(n^\sigma_{H_0}(d_g g)d_g g)d_g n^\sigma_{H_0}(d_g g) = d_g n^{\sigma+1}_{H_0}(d_g g)$. This completes the argument for the first relation.

To show the other relation, we proceed similarly. We observe that if, for $g_1, g_2 \in G_0$, we have $d^\sigma_{g_1}\gamma^\sigma_{G_0}(g_1) = d^\sigma_{g_2}\gamma^\sigma_{G_0}(g_2)$, then, by Lemma 4.21, since $n_{\gamma^\sigma_{G_0}(g_1), \gamma^\sigma_{G_0}(g_2)}(\gamma^\sigma_{G_0}(g_i)) \mid d^\sigma_{g_i} \mid n_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g_i))$, we have

$$\gamma^{\sigma+1}_{G_0}(g_1) = \gamma_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g_1)) = \gamma_{\Gamma^\sigma(G_0)}(\gamma^\sigma_{G_0}(g_2)) = \gamma^{\sigma+1}_{G_0}(g_2).$$

Thus,

$$\langle \Gamma^{\sigma+1}(G_0) \setminus \{\gamma^{\sigma+1}_{G_0}(g)\}\rangle \subset \langle \Gamma^\sigma(H_0) \setminus \{d^\sigma_g \gamma^\sigma_{G_0}(g)\}\rangle.$$

This inclusion implies $n_{\Gamma^{\sigma+1}(G_0)}(\gamma^{\sigma+1}(g))\gamma^{\sigma+1}_{G_0}(g) \in \langle \Gamma^\sigma(H_0) \setminus \{d^\sigma_g \gamma^\sigma_{G_0}(g)\}\rangle$. Noting that

$$n_{\Gamma^{\sigma+1}(G_0)}(\gamma^{\sigma+1}(g))\gamma^{\sigma+1}_{G_0}(g) = n_{\Gamma^{\sigma+1}(G_0)}(\gamma^{\sigma+1}(g))\frac{n_{\Gamma^\sigma(G_0)}(\gamma^\sigma(g))}{d^\sigma_g}(d^\sigma_g \gamma^\sigma_{G_0}(g)),$$

it follows that

$$n_{\Gamma^\sigma(H_0)}(d^\sigma_g \gamma^\sigma_{G_0}(g)) \mid n_{\Gamma^{\sigma+1}(G_0)}(\gamma^{\sigma+1}(g))\frac{n_{\Gamma^\sigma(G_0)}(\gamma^\sigma(g))}{d^\sigma_g}.$$

Multiplying by $d_g n^\sigma_{H_0}(d_g g)$, we get

$$d_g n^\sigma_{H_0}(d_g g)n_{\Gamma^\sigma(H_0)}(d^\sigma_g \gamma^\sigma_{G_0}(g)) \mid n_{\Gamma^{\sigma+1}(G_0)}(\gamma^{\sigma+1}(g))n_{\Gamma^\sigma(G_0)}(\gamma^\sigma(g))n^\sigma_{G_0}(g),$$

that is $d_g n^{\sigma+1}_{H_0}(d_g g) \mid n^{\sigma+2}_{G_0}(g)$.

Now, let $\sigma \geq \omega$. It suffices to consider $\sigma = \omega$. The two divisibilty relations and the definition readily yield that $n^\omega_{G_0}(g) = d_g n^\omega_{H_0}(d_g g)$.

The "in particular"-statement is clear. $\square$

5 Abstract version of the finitary analog

Having applications in Non-Unique Factorization Theory in mind, it is natural to consider finitary analogs of the constructions done in the previous section, in the hope that they yield still simpler sets than can be obtained by the constructions of the preceding section. To be more specific, properties of factorizations of an element B of a block monoid just depend on $\mathcal{B}(\mathrm{supp}(B))$ and $\mathrm{supp}(B)$ is a finite set. Thus, an immediate idea would be to consider $\cup_{G_1 \subset G_0,\, |G_1|<\infty}\Gamma^{\sigma}(G_1)$ instead of $\Gamma^{\sigma}(G_0)$, or in other words to carry out our construction for each divisor-closed finitely generated submonoid of $\mathcal{B}(G_0)$ and then to put the results together in a naive way. Unfortunately, this approach has the drawback that $\Gamma^{\sigma}(\cdot)$ does not preserve inclusions (cf. Example 4.18) and thus the set $\cup_{G_1 \subset G_0,\, |G_1|<\infty}\Gamma^{\sigma}(G_1)$ is in general too large. The following modification avoids this problem.

Definition 5.1. Let G be a torsion group and let $G_0 \subset G$. Let σ be an ordinal. For $g \in G_0$, let $n_{G_0}^{\sigma,\mathrm{fin}}(g) = \min\{n_{G_1}^{\sigma}(g) \colon g \in G_1 \subset G_0,\, |G_1| < \infty\}$, $\gamma_{G_0}^{\sigma,\mathrm{fin}}(g) = n_{G_0}^{\sigma,\mathrm{fin}}(g)g$, and $\Gamma^{\sigma,\mathrm{fin}}(G_0) = \gamma_{G_0}^{\sigma,\mathrm{fin}}(G_0)$.

First, we collect some simple facts about the just introduced notions and relate them to the notions introduced in the preceding section. Then, we show that they are indeed meaningful, in the sense that there exists a transfer homomorphism from $\mathcal{B}(G_0)$ to $\mathcal{B}(\Gamma^{\sigma,\mathrm{fin}}(G_0))$.

Lemma 5.2. *Let G be a torsion group, let $G_0 \subset G$, and let $g \in G_0$. Let σ be an ordinal.*

1. $n_{G_0}^{\sigma,\mathrm{fin}}(g) = \gcd\{n_{G_1}^{\sigma}(g) \colon g \in G_1 \subset G_0,\, |G_1| < \infty\}$.
2. $n_{G_0}^{\sigma}(g) \mid n_{G_0}^{\sigma,\mathrm{fin}}(g)$.
3. $n_{G_0}^{\sigma,\mathrm{fin}}(g) \mid n_{G_1}^{\sigma,\mathrm{fin}}(g)$ *for* $G_1 \subset G_0$.

Proof. 1. Let $G_2, G_3 \subset G_0$ be finite sets containing g. By Proposition 4.22, $n_{G_2 \cup G_3}^{\sigma}(g) \mid n_{G_i}^{\sigma}(g)$ for $i \in \{2,3\}$. Thus, the claim follows.
2. Let $G_2 \subset G_0$ be a finite set containing with g such that $n_{G_2}^{\sigma}(g) = n_{G_0}^{\sigma,\mathrm{fin}}(g)$. By Proposition 4.22, we know that $n_{G_0}^{\sigma}(g) \mid n_{G_2}^{\sigma}(g)$. Thus, the claim follows.
3. We note that $n_{G_0}^{\sigma,\mathrm{fin}}(g) = n_{G_2}^{\sigma}(g)$ and $n_{G_1}^{\sigma,\mathrm{fin}}(g) = n_{G_3}^{\sigma}(g)$ for finite sets $G_2 \subset G_0$ and $G_3 \subset G_1$ containing g. Moreover, by Proposition 4.22 and the definition of $n_{G_0}^{\sigma,\mathrm{fin}}(g)$, we know that $n_{G_2 \cup G_3}^{\sigma}(g) = n_{G_0}^{\sigma,\mathrm{fin}}(g)$ as well. Again by Proposition 4.22, $n_{G_2 \cup G_3}^{\sigma}(g) \mid n_{G_3}^{\sigma}(g)$, implying the claim. □

The following proposition shows that, as is to be expected, for finite G_0 the finitary construction coincides with the usual one. Moreover, it shows that the finitary construction is actually only interesting for the ordinal ω.

Proposition 5.3. *Let G be a torsion group, let $G_0 \subset G$, and let $g \in G_0$. Let σ be an ordinal.*

1. *If G_0 is finite, then* $n_{G_0}^{\sigma,\mathrm{fin}}(g) = n_{G_0}^{\sigma}(g)$.
2. *If $\sigma < \omega$, then* $n_{G_0}^{\sigma,\mathrm{fin}}(g) = n_{G_0}^{\sigma}(g)$.

3. If $\sigma \geq \omega$, then $n_{G_0}^{\sigma,\mathrm{fin}}(g) = n_{G_0}^{\omega,\mathrm{fin}}(g)$.

Proof. 1. By Proposition 4.22, we have $n_{G_0}^{\sigma}(g) \mid n_{G_1}^{\sigma}(g)$ for each $G_1 \subset G_0$ with $g \in G_1$. By the definition of $n_{G_0}^{\sigma,\mathrm{fin}}(g)$, the claim follows.
2. We induct on σ. For $\sigma = 0$ the claim is trivial.

We consider $\sigma+1$. Let $g \in G_0$. We know that $n_{\Gamma^\sigma G_0}(\gamma_{G_0}^{\sigma}(g))\gamma_{G_0}^{\sigma}(g) \in \langle \Gamma^\sigma(G_0) \setminus \{\gamma_{G_0}^{\sigma}(g)\}\rangle$. There exists a finite subset $H_0^\sigma \subset \Gamma^\sigma(G_0) \setminus \{\gamma_{G_0}^{\sigma}(g)\}$ such that $n_{\Gamma^\sigma(G_0)}(\gamma_{G_0}^{\sigma}(g))\gamma_{G_0}^{\sigma}(g) \in \langle H_0^\sigma \rangle$. For each $h \in H_0^\sigma$ there exists, by the induction hypothesis, some finite set G_h and some $g_h \in G_h$ such that $\gamma_{G_h}^{\sigma}(g_h) = \gamma_{G_0}^{\sigma}(g_h) = h$. Moreover, there exists some set G_g such that $\gamma_{G_g}^{\sigma}(g) = \gamma_{G_0}^{\sigma}(g)$. We set $G_1 = G_g \cup \bigcup_{h \in H_0^\sigma} G_h$ and $G_2 = \{g\} \cup \{g_h \colon h \in H_0^\sigma\}$. Then, by Proposition 4.22, $\gamma_{G_1}^{\sigma}|_{G_2} = \gamma_{G_0}^{\sigma}|_{G_2}$. Thus, $n_{\Gamma^\sigma(G_1)}(\gamma_{G_1}^{\sigma}(g))\gamma_{G_1}^{\sigma}(g) \in \langle \Gamma^\sigma(G_1) \setminus \{\gamma_{G_1}^{\sigma}(g)\}\rangle$, implying that $n_{\Gamma^\sigma(G_1)}(\gamma_{G_1}^{\sigma}(g)) \mid n_{\Gamma^\sigma(G_0)}(\gamma_{G_0}^{\sigma}(g))$. In combination with the induction hypothesis and Proposition 4.22, this implies the claim.
3. By Proposition 4.12, we know that for each finite set $G_1 \subset G_0$ with $g \in G_1$, $n_{G_1}^{\sigma}(g) = n_{G_1}^{\omega}(g)$. Thus, the claim follows. □

Now, we show the existence of a transfer homomorphism.

Theorem 5.4. *Let G be a torsion group and let $G_0 \subset G$. There exists a transfer homomorphism $\beta_{G_0}^{\omega,\mathrm{fin}} : \mathcal{B}(G_0) \to \mathcal{B}(\Gamma^{\omega,\mathrm{fin}}(G_0))$ given by $\beta_{G_0}^{\omega,\mathrm{fin}}(B) = \prod_{h \in \Gamma^{\omega,\mathrm{fin}}(G_0)} h^{\mathrm{ord}(h)\mathsf{k}(F_h^{\omega,\mathrm{fin}})}$, where $F_h^{\omega,\mathrm{fin}} = \prod_{g \in (\gamma_{G_0}^{\omega,\mathrm{fin}})^{-1}(h)} g^{\mathsf{v}_g(B)}$.*

Proof. As in Lemma 4.9, we see that $\beta_{G_0}^{\omega,\mathrm{fin}}$ is a surjective homomorphism.

Let $B \in \mathcal{B}(G_0)$. For each $g \in \mathrm{supp}(B)$, there exists a finite set $G_g \subset G_0$ such that $n_{G_0}^{\omega,\mathrm{fin}}(g) = n_{G_g}^{\omega}(g)$. We set $G_1 = \cup_{g \in \mathrm{supp}(B)} G_g$. By Proposition 4.22, $n_{G_g}^{\omega}(g) = n_{G_1}^{\omega}(g)$ for each $g \in G_1$. By Proposition 4.12, there exists some $\rho < \omega$ such that $n_{G_1}^{\omega}(g) = n_{G_1}^{\rho}(g)$ for each $g \in \mathrm{supp}(B)$. Thus, $\beta_{G_0}^{\omega,\mathrm{fin}}(B) = \beta_{G_1}^{\rho}(B)$ for each B and the same is true for each $B' \mid B$. The claim thus follows by Theorem 4.10. □

We give a simple example that shows a strength of the finitary construction (cf. Example 4.16).

Example 5.5. Let p be a prime. Let $G_0 = \{p^{-n} + \mathbb{Z} \colon n \in \mathbb{N}_0\} \subset \mathbb{Z}(p^\infty)$. Then, $\Gamma^{\omega,\mathrm{fin}}(G_0) = \{0\}$. To see this, note that, by Example 4.15, $\Gamma^\omega(G_1) = \{0\}$ for each finite subset $G_1 \subset G_0$.

In Section 7 we investigate subsets of p-groups with $\Gamma^{\omega,\mathrm{fin}}(G_0) = \{0\}$ in detail. We point out that the above example shows that an analog of Lemma 4.11 does not hold for the finitary constructions, since $\Gamma^{\rho,\mathrm{fin}}(G_0) = G_0$ for each $\rho < \omega$ and consequently $\cap_{\rho<\omega} \langle \Gamma^{\rho,\mathrm{fin}}(G_0)\rangle = \mathbb{Z}(p^\infty) \neq \{0\} = \langle \Gamma^{\omega,\mathrm{fin}}(G_0)\rangle$.

We want to show a result analogous to Proposition 4.12. To this end, we need the following lemma.

Lemma 5.6. *Let G be a torsion group and let $G_0 \subset G$. Further, let $G_1 \subset G_0$ be finite. For each ordinal σ and for each $g \in G_1$, we have*

$$n^{\sigma}_{\gamma_{G_0}^{\omega,\mathrm{fin}}(G_1)}(\gamma_{G_0}^{\omega,\mathrm{fin}}(g)) n_{G_0}^{\omega,\mathrm{fin}}(g) \mid n_{G_1}^{\omega}(g).$$

Proof. Let $H_1 = \gamma^{\omega,\mathrm{fin}}_{G_0}(G_1)$ and let $H_2 = \Gamma^{\omega}(G_1)$. By Lemma 5.2, we know that $n^{\omega,\mathrm{fin}}_{G_0}(g) \mid n^{\omega}_{G_1}(g)$. We claim that if $g, g' \in G_1$ and $\gamma^{\omega,\mathrm{fin}}_{G_0}(g) = \gamma^{\omega,\mathrm{fin}}_{G_0}(g')$, then $\gamma^{\omega}_{G_1}(g) = \gamma^{\omega}_{G_1}(g')$. To see this, it suffices to note that if $\gamma^{\omega,\mathrm{fin}}_{G_0}(g) = \gamma^{\omega,\mathrm{fin}}_{G_0}(g')$, then $n_{g,g'}(g) \mid n^{\omega,\mathrm{fin}}_{G_0}(g)$ and $n_{g,g'}(g') \mid n^{\omega,\mathrm{fin}}_{G_0}(g')$, and thus $n_{g,g'}(g) \mid n^{\omega}_{G_1}(g)$ and $n_{g,g'}(g') \mid n^{\omega}_{G_1}(g')$, implying the claim by Lemma 4.21.

Since $n^{\omega}_{G_1}(g), n^{\omega,\mathrm{fin}}_{G_0}(g)$ and $n^{\omega}_{G_1}(g'), n^{\omega,\mathrm{fin}}_{G_0}(g')$ divide the order of g and g', respectively, it follows, by the above claim, that if $\gamma^{\omega,\mathrm{fin}}_{G_0}(g) = \gamma^{\omega,\mathrm{fin}}_{G_0}(g')$, then $n^{\omega}_{G_1}(g)/n^{\omega,\mathrm{fin}}_{G_0}(g) = n^{\omega}_{G_1}(g')/n^{\omega,\mathrm{fin}}_{G_0}(g')$. For $h \in H_1$ we thus define $d(h) = n^{\omega}_{G_1}(g)/n^{\omega,\mathrm{fin}}_{G_0}(g)$ for some $g \in (\gamma^{\omega,\mathrm{fin}}_{G_0})^{-1}(h) \cap G_1$. We have to show that $n^{\sigma}_{H_1}(h) \mid d(h)$ for each $h \in H_1$. We note that $H_2 = \{d(h)h \colon h \in H_1\}$ and we recall that, by Proposition 4.12 and Lemma 4.5, $f \in \langle H_2 \setminus \{f\}\rangle$ for each $f \in H_2$.

Now, we induct on σ. For $\sigma = 0$ we have to show that $n^{\omega,\mathrm{fin}}_{G_0}(g) \mid n^{\omega}_{G_1}(g)$. This is stated above.

We consider $\sigma + 1$. Let $h \in H_1$. We suppose that $n^{\sigma}_{H_1}(h) \mid d(h)$. Thus, we have

$$\frac{d(h)}{n^{\sigma}_{H_1}(h)}(n^{\sigma}_{H_1}(h)h) \in \langle H_2 \setminus \{d(h)h\}\rangle.$$

It thus remains to show that $\langle H_2 \setminus \{d(h)h\}\rangle \subset \langle \Gamma^{\sigma}(H_1) \setminus \{\gamma^{\sigma}_{H_1}(h)\}\rangle$. To see this, it suffices to show that if $\gamma^{\sigma}_{H_1}(h) = \gamma^{\sigma}_{H_1}(h')$, then $d(h)h = d(h')h'$. Let $h, h' \in H_1$ such that $\gamma^{\sigma}_{H_1}(h) = \gamma^{\sigma}_{H_1}(h')$. By the induction hypothesis, we know that $n^{\sigma}_{H_1}(h)n^{\omega,\mathrm{fin}}_{G_0}(g) \mid n^{\omega}_{G_1}(g)$ for some $g \in (\gamma^{\omega,\mathrm{fin}}_{G_0})^{-1}(h) \cap G_1$ and the analogous statement holds for h' for some $g' \in (\gamma^{\omega,\mathrm{fin}}_{G_0})^{-1}(h') \cap G_1$. For these elements g, g' we thus have $n^{\sigma}_{H_1}(h)n^{\omega,\mathrm{fin}}_{G_0}(g)g = n^{\sigma}_{H_1}(h')n^{\omega,\mathrm{fin}}_{G_0}(g')g'$. Thus, $n_{g,g'}(g)$ and $n_{g,g'}(g')$ divide $n^{\sigma}_{H_1}(h)n^{\omega,\mathrm{fin}}_{G_0}(g)$ and $n^{\sigma}_{H_1}(h')n^{\omega,\mathrm{fin}}_{G_0}(g')$, respectively. Thus, $n_{g,g'}(g) \mid n^{\omega}_{G_1}(g)$ and $n_{g,g'}(g') \mid n^{\omega}_{G_1}(g')$. By Lemma 4.21, this implies that $n^{\omega}_{G_1}(g)g = n^{\omega}_{G_1}(g')g'$, that is $d(h)h = d(h')h'$ as claimed.

For σ a limit ordinal the argument is immediate. □

Now, the above mentioned result follows easily.

Proposition 5.7. *Let G be a torsion group and let $G_0 \subset G$. Let σ be an ordinal.*

1. $\Gamma^{\omega,\mathrm{fin}}(\Gamma^{\omega,\mathrm{fin}}(G_0)) = \Gamma^{\omega,\mathrm{fin}}(G_0)$.
2. $\Gamma^{\sigma}(\Gamma^{\omega,\mathrm{fin}}(G_0)) = \Gamma^{\omega,\mathrm{fin}}(G_0)$ *for each ordinal σ.*

Proof. 1. Let $H_0 = \Gamma^{\omega,\mathrm{fin}}(G_0)$ and let $h \in H_0$. Let $g \in (\gamma^{\omega,\mathrm{fin}}_{G_0})^{-1}(h)$. There exists some finite subset $G_1 \subset G_0$ such that $n^{\omega}_{G_1}(g) = n^{\omega,\mathrm{fin}}_{G_0}(g)$. Let $H_1 = \gamma^{\omega,\mathrm{fin}}_{G_0}(G_1)$. Obviously, H_1 is a finite subset of H_0. Since $n^{\omega}_{G_1}(g) = n^{\omega,\mathrm{fin}}_{G_0}(g)$, we get, by Lemma 5.6, $n^{\sigma}_{H_1}(h) = 1$ for each σ. Thus, $n^{\sigma,\mathrm{fin}}_{H_0}(h) = 1$, implying the claim.

2. By Lemma 5.2, $n^{\sigma}_{H_0}(h) \mid n^{\sigma,\mathrm{fin}}_{H_0}(h)$, implying the claim by 1. □

We establish an analog of Lemma 4.23 that we need in the following section.

Lemma 5.8. *Let G be a torsion group and let $G_0 \subset G$. For each $g \in G_0$, let $d_g \in \mathbb{N}$ with $d_g \mid n_{G_0}(g)$, and let $H_0 = \{d_g g \colon g \in G_0\}$. Then, $\gamma^{\omega,\mathrm{fin}}_{G_0}(g) = \gamma^{\omega,\mathrm{fin}}_{H_0}(d_g g)$.*

Proof. We show that $n^{\omega,\mathrm{fin}}_{G_0}(g) = n^{\omega,\mathrm{fin}}_{H_0}(d_g g) d_g$.

Let $G_1 \subset G_0$ finite. We set $H_1 = \{d_g g \colon g \in G_1\}$. By Proposition 4.22, we have $d_g \mid n_{G_1}(g)$ for each $g \in G_1$. Thus, by Lemma 4.23, we get that $n^{\omega}_{G_1}(g) = n^{\omega}_{H_1}(d_g g) d_g$. Consequently, $n^{\omega,\mathrm{fin}}_{G_0}(g) \ge n^{\omega,\mathrm{fin}}_{H_0}(d_g g) d_g$.

Conversely, let $H_2 \subset H_0$ finite. Let $G_2 \subset G_0$ finite such that $H_2 = \{d_g g \colon g \in G_2\}$. Again, we have $d_g \mid n_{G_2}(g)$ for each $g \in G_2$ and we get $n^{\omega}_{G_2}(g) = n^{\omega}_{H_2}(d_g g) d_g$, implying that $n^{\omega,\mathrm{fin}}_{G_0}(g) \le n^{\omega,\mathrm{fin}}_{H_0}(d_g g) d_g$. □

6 Results for Krull monoids

Having the abstract versions of our constructions at hand, the application to Krull monoids is fairly straightforward.

6.1 General results

We make the following definitions.

Definition 6.1. Let H be a Krull monoid with torsion class group and let $\varphi : H \to \mathcal{F}(P)$ be a divisor homomorphism such that $\mathcal{C}(\varphi)$ is a torsion group.

1. Let σ be an ordinal. Let $\mathcal{C}^{\sigma}(H) = \langle \Gamma^{\sigma}(\mathcal{D}(H)) \rangle$ and $\mathcal{C}^{\sigma}(\varphi) = \langle \Gamma^{\sigma}(\mathcal{D}(\varphi)) \rangle$. We call these groups the class group of order σ of H and φ, respectively. Moreover, we call $\mathcal{B}(\Gamma^{\sigma}(\mathcal{D}(H)))$ and $\mathcal{B}(\Gamma^{\sigma}(\mathcal{D}(\varphi)))$ the block monoid of order σ associated to H and φ, respectively.
2. Let $\mathcal{C}^{\omega,\mathrm{fin}}(H) = \langle \Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(H)) \rangle$ and $\mathcal{C}^{\omega,\mathrm{fin}}(\varphi) = \langle \Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(\varphi)) \rangle$. We call these groups the finitary higher-order class group of H and φ, respectively. Moreover, we call $\mathcal{B}(\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(H)))$ and $\mathcal{B}(\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(\varphi)))$ the finitary higher-order block monoid associated to H and φ, respectively.

By the already recalled and established results, it follows quite directly that the notion of a higher-order block monoid is meaningful, in the sense that there exists a transfer homomorphism from the original Krull monoid to each of its higher-order block monoids, and thus much information on the arithmetic of the original Krull monoid is still encoded in the higher-order block monoids. We summarize this in the following result.

Theorem 6.2. *Let $\varphi : H \to \mathcal{F}(P)$ be a divisor homomorphism such that $\mathcal{C}(\varphi)$ is a torsion group. Let σ be an ordinal. There exists a transfer homomorphism*

$$\beta^{\sigma}_{\varphi} : H \to \mathcal{B}(\Gamma^{\sigma}(\mathcal{D}(\varphi))).$$

Moreover, there exists a transfer homomorphism

$$\beta^{\omega,\mathrm{fin}}_{\varphi} : H \to \mathcal{B}(\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(\varphi))).$$

Proof. As noted in Section 2, there exists a transfer homomorphism $\beta_{\varphi} : H \to \mathcal{B}(\mathcal{D}(\varphi))$. By Theorem 4.10, there exists a transfer homomorphism $\beta^{\sigma}_{\mathcal{D}(\varphi)} : \mathcal{B}(\mathcal{D}(\varphi)) \to \mathcal{B}(\Gamma^{\sigma}(\mathcal{D}(\varphi)))$, and by Theorem 5.4, there exists a transfer homomorphism $\beta^{\omega,\mathrm{fin}}_{\mathcal{D}(\varphi)} : \mathcal{B}(\mathcal{D}(\varphi)) \to \mathcal{B}(\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(\varphi)))$.

Thus, setting $\beta^{\sigma}_{\varphi} = \beta^{\sigma}_{\mathcal{D}(\varphi)} \circ \beta_{\varphi}$ and $\beta^{\omega,\mathrm{fin}}_{\varphi} = \beta^{\omega,\mathrm{fin}}_{\mathcal{D}(\varphi)} \circ \beta_{\varphi}$, the claim follows, since the composition of transfer homomorphisms is again a transfer homomorphism. □

We refer to β_φ^σ as block homomorphism of order σ associated to φ, and to $\beta_\varphi^{\omega,\mathrm{fin}}$ as finitary higher-order block homomorphism. By the proof of the above theorem and by Theorem 4.10, for finite σ, β_φ^σ is just the composition of $\sigma+1$ block homomorphisms. In particular, $\beta_\varphi^0 = \beta_\varphi$. Moreover, combining the explicit descriptions for the maps involved in the construction of the higher-order block homomorphism, an explicit definition of the higher-order block homomorphism could be given, yet we refrain from writing it down. Again, we write β_H^σ and $\beta_H^{\omega,\mathrm{fin}}$ to denote the higher-order block homomorphisms associated to a divisor theory of H.

By definition, the higher-order class groups and block monoids associated to a divisor homomorphism from H into a free monoid (with torsion class group) depend on the divisor homomorphism. However, below we see that this dependence is not too severe. It turns out that the higher-order constructions of infinite order, as well as the finitary one, are actually independent of the particular choice of the divisor homomorphism and just depend on the Krull monoid H.

Theorem 6.3. *Let $\varphi : H \to \mathcal{F}(P)$ be a divisor homomorphism such that $\mathcal{C}(\varphi)$ is a torsion group. Then, for $\sigma \geq \omega$ we have $\Gamma^\sigma(\mathcal{D}(\varphi)) = \Gamma^\sigma(\mathcal{D}(H))$. Moreover, $\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(\varphi)) = \Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(H))$.*

Proof. Clearly, it suffices to show the claim for $\sigma = \omega$.

We start by investigating $n_{\mathcal{D}(\varphi)}(g)$ for $g \in \mathcal{D}(\varphi)$. Let $d_g = m_p(\varphi)$ for some p with $[p]_\varphi = g$; we note that p is uniquely determined if $m_p(\varphi) > 1$, and thus $m_p(\varphi)$ is independent of the choice of p (see Theorem 3.1).

We know that $kg \in \langle \Gamma(\varphi) \setminus \{g\} \rangle$ implies that $d_g \mid k$. Thus, $d_g \mid n_{\Gamma(\varphi)}(g)$. We recall that, by Theorem 3.1, $\mathcal{D}(H) = \{d_g g \colon g \in \mathcal{D}(\varphi)\}$. Thus, we can apply Lemma 4.23 to get that $\Gamma^\omega(\mathcal{D}(\varphi)) = \Gamma^\omega(\mathcal{D}(H))$.

To get the result for the finitary version, we just apply Lemma 4.23 instead of Lemma 5.8. $\square$

For finite σ, $\Gamma^\sigma(\mathcal{D}(\varphi))$ and $\Gamma^\sigma(\mathcal{D}(H))$ can actually be distinct. For example, for G_0 a subset of a torsion group, $\Gamma^\sigma(\mathcal{D}(\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0))) = \Gamma^\sigma(G_0)$ whereas $\Gamma^\sigma(\mathcal{D}(\mathcal{B}(G_0)) = \Gamma^\sigma(\Gamma(G_0)) = \Gamma^{\sigma+1}(G_0)$ (cf. Propostion 4.1). The phenomenon that the relation is just given by a shift of σ, as in this example, is not a general one. Yet, it is "almost" true (cf. the statement of Lemma 4.23 for finite σ).

6.2 An application

We show an immediate way to apply Theorem 6.2 in Non-Unique Factorization Theory. A main aim of Non-Unique Factorization Theory is to quantify how much a certain type of monoid deviates from being factorial. A common way to do this is to study the system of sets of lengths and quantities derived from it. Two classical examples are the elasticity and the set of distances (see, e.g., [2] and [20]). Let H be an atomic monoid. For $a \in H \setminus H^\times$, let $\rho(a) = \sup \mathsf{L}(a) / \min \mathsf{L}(a)$ and set $\rho(a) = 1$ for $a \in H^\times$. And, if $\mathsf{L}(a) = \{\ell_1 < \ell_2 < \dots\}$, then let $\Delta(a) = \{\ell_2 - \ell_1, \ell_3 - \ell_2, \dots\}$ represent the set of successive distance. Moreover, let $\rho(H) = \sup\{\rho(a) \colon a \in H \setminus H^\times\}$ represent the elasticity of H and $\Delta(H) = \cup_{a \in H} \Delta(H)$ the set of distances of H. If H is a Krull monoid, then it is well-known that $\mathsf{L}(a)$ is a finite set, thus $\rho(a)$ and $\Delta(a)$ are finite. Yet, $\rho(H)$ and $\Delta(H)$ can be infinite.

Another common approach, which aims at a more precise understanding of the system of sets of lengths, is to investigate whether the Structure Theorem for Sets of Lengths holds. One says that the Structure Theorem for Sets of Lengths holds if for each $a \in H$ the set $\mathsf{L}(a)$ is an almost arithmetical multiprogression and its difference and its initial and end part are bounded by constants just depending on H (see [20, Chapter 4] for a precise definition). The first result of this type has been proved by A. Geroldinger [16] for Krull monoids with $\mathcal{D}(H)$ finite. Meanwhile, it is known to hold for various other classes of monoids (see [20, Section 4.7] for an overview).

Using higher-order block monoids and the well-known technique of "transferring" problems of the above type, we can easily establish the finiteness of $\rho(H)$ and $\Delta(H)$, and the validity of the Structure Theorem of Sets of Lengths for a new class of Krull monoids.

Corollary 6.4. *Let H be a Krull monoid with torsion class group. If $\Gamma^{\sigma}(\mathcal{D}(H))$ is finite for some ordinal σ, then $\rho(H) < \infty$, $|\Delta(H)| < \infty$, and the Structure Theorem for Sets of Lengths holds for H. Moreover, the same holds true if $\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(H))$ is finite.*

Proof. By Theorem 6.2, we know that there exits a transfer homomorphism from H to $\mathcal{B}(\Gamma^{\sigma}(\mathcal{D}(H)))$. Since a transfer homomorphism preserves sets of lengths (cf. the Section 2 or [20, Proposition 3.2.3]), and thus all quantities derived solely from sets of lengths, we have $\rho(H) = \rho(\mathcal{B}(\Gamma^{\sigma}(\mathcal{D}(H))))$, $\Delta(H) = \Delta(\mathcal{B}(\Gamma^{\sigma}(\mathcal{D}(H))))$, and the Structure Theorem for Sets of Lengths holds for H if and only if it holds for $\mathcal{B}(\Gamma^{\sigma}(\mathcal{D}(H)))$. If $\Gamma^{\sigma}(\mathcal{D}(H))$ is finite, then for $\mathcal{B}(\Gamma^{\sigma}(\mathcal{D}(H)))$ all three claims are well-known (cf. [20, Theorems 3.4.11 and 4.4.11]). The argument in case $\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(H))$ is finite is identical. □

We remark that examples of Krull monoids fulfilling the above condition, yet not covered by already known results of this type, actually exist. As recalled above, the conclusion of our result is well-known under the stronger condition that $\mathcal{D}(H)$ is finite. Moreover, under the weaker condition that only $\mathsf{D}(\mathcal{D}(H))$, the Davenport constant of $\mathcal{D}(H)$, is finite (i.e., $\sup\{|A| \colon A \in \mathcal{A}(\mathcal{B}(\mathcal{D}(H)))\} < \infty$), it is well-known that $\rho(H)$ and $\Delta(H)$ are finite (cf. [20, Theorem 3.4.11]). And, by a very recent result of A. Geroldinger and D. Grynkiewicz [18], it is also known that the Structure Theorem of Sets of Lengths holds. The condition $\mathsf{D}(\mathcal{D}(H)) < \infty$ neither implies nor is implied by our condition. Opposed to our result, these results do not require that the class group is a torsion group.

For illustration we write down a simple explicit example.

Example 6.5. Let p be a prime. Let $H_0 = \{p^{-n} + \mathbb{Z} \colon n \in \mathbb{N}_0\} \subset \mathbb{Z}(p^{\infty})$ and let $H_0' \subset \mathbb{Z}(p^{\infty})$ finite. We set $G_0 = H_0 \cup H_0'$. Then, $\mathcal{D}(\mathcal{B}(G_0)) = G_0$ is infinite and the Davenport constant of G_0 is infinite as well. Yet, $\Gamma^{\omega,\mathrm{fin}}(G_0)$ is contained in $\langle H_0' \rangle$ (cf. Example 5.5) and thus finite.

Additionally, we remark that the result of [18] can be used to strengthen Corollary 6.4.

Remark 6.6. Let H be a Krull monoid with $\mathcal{C}(H)$ torsion. If $\mathsf{D}(\Gamma^{\sigma}(\mathcal{D}(H))) < \infty$, for some ordinal σ, or $\mathsf{D}(\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(H))) < \infty$, then $\rho(H) < \infty$, $|\Delta(H)| < \infty$ and the Structure Theorem for Sets of Lengths holds.

7 Pseudo factorial monoids and a characterization of simply presented p-groups

As stated in Section 2, it is well-known that a Krull monoid is factorial if and only if its class group is trivial. Moreover, A. Geroldinger and F. Halter-Koch [19] investigated under which condition all factorizations of an element of a Krull monoid are block-equivalent, i.e., the image of the element under the block homomorphism has a unique factorization. In particular, each element of a Krull monoid H has this property if and only if $\mathcal{B}(\mathcal{D}(H))$ is a factorial monoid. We refer to such a monoid as block-unique factorization monoid. Using the notion of a higher-order class group, these considerations can be extended in a natural way.

Definition 7.1. Let H be a Krull monoid with torsion class group. For σ an ordinal, we say that H is σ-pseudo factorial if $\mathcal{C}^{\sigma}(H)$ is trivial. Moreoever, we say that H is finitary-pseudo factorial if $\mathcal{C}^{\omega,\mathrm{fin}}(H)$ is trivial.

In the following lemma, we collect some first facts about the just defined properties.

Lemma 7.2. *Let H be a Krull monoid with torsion class group.*

1. *If H is σ-pseudo factorial, then H is τ-pseudo factorial for each $\tau \geq \sigma$.*
2. *If H is σ-pseudo factorial, then H is finitary-pseudo factorial.*
3. *If H is finitary-pseudo factorial, then H is half-factorial.*
4. *H is 0-pseudo factorial if and only if H is factorial.*
5. *H is 1-pseudo factorial if and only if H is a block unique factorization monoid.*

Proof. 1. and 2. are clear by Lemma 4.7 and Lemma 5.2, respectively.
3. Theorem 6.2 yields a transfer homomorphism from H to $\mathcal{B}(\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(H))$. If H is finitary-pseudo factorial, then $\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(H)) \subset \{0\}$ and the latter monoid is obviously half-factorial, implying that H is half-factorial.
4. The class group of H is trivial if and only if H is factorial (see Section 2). Thus, the statement is clear.
5. By definition, H is a block-unique factorization monoid if and only if $\mathcal{B}(\mathcal{D}(H))$ is factorial. The claim follows by 4. $\square$

The notion of a σ-pseudo factorial and a finitary-pseudo factorial monoid thus give rise to a hierarchy of half-factorial monoids. By [19, Proposition 3], a Krull monoid with torsion class group is 1-pseudo factorial if and only if $\mathcal{D}(H) \setminus \{0\}$ is independent (also cf. Example 4.14). Below, we investigate $\mathcal{C}(H)$ and $\mathcal{D}(H)$ for σ-pseudo factorial and finitary-pseudo factorial H under the condition that $\mathcal{C}(H)$ is a p-group. It turns out that in this case the sets $\mathcal{D}(H)$ are closely connected to T-basis of $\mathcal{C}(H)$ (cf. below for a definition), and we can thus characterize the class groups of finitary-pseudo factorial H under the condition that it is a p-group. As mentioned in Section 1, these investigations are motivated and guided by [17].

Building on the results on $\mathcal{D}(H)$ for σ-pseudo factorial and finitary-pseudo factorial Krull monoids, it could be an interesting problem to investigate the arithmetical consequences of these properties in more detail. For example, one could study the number of essentially distinct factorizations of elements of σ-pseudo factorial and finitary-pseudo factorial monoids. Yet, in this paper we do not undertake such investigations.

We start by stating the above mentioned characterization.

Theorem 7.3. *Let G be a p-group. The following statements are equivalent.*

- *G is simply presented.*
- *There exists a finitary-pseudo factorial Krull monoid H with $\mathcal{C}(H) \cong G$.*

Remark 7.4. Calling a Dedekind domain finitary-pseudo factorial if its multiplicative monoid is finitary-pseudo factorial, the above result holds for 'Dedekind domain' instead of 'Krull monoid' as well.

We recall the definition of and some results on a T-basis of a p-group. Our terminology follows [29].

Let G be a p-group and $Y \subset G$. Then, Y is called a T-basis of G if the following conditions hold.

1. $G = \langle Y \rangle$.
2. $0 \notin Y$.
3. If $y \in Y$ and $py \neq 0$, then $py \in Y$.
4. For $y \in Y$ let $Z = \{z \in Y \colon p^n z \neq y \text{ for all } n \in \mathbb{N}_0\}$. Then, $y \notin \langle Z \rangle$.

We recall that if Y is a T-basis, then for each $g \in G$ there exist uniquely determined $a_y \in [0, p-1]$ (almost all 0) such that $g = \sum_{y \in Y} a_y y$. We recall that if $H \subset G$ is a subgroup and Y a T-basis of G, then $Y \cap H$ is a T-basis of H. A p-group is simply presented if and only if it has a T-basis (cf. [29, Lemma 2.1]). The notion of a simply presented p-group, i.e., a group that can be presented by generators and relations of the form $px = y$ and $px = 0$ only, has been introduced by P. Crawley and A.W. Hales [10] and is equivalent to the notion of a totally projective p-group. We refer to, e.g., [14] for a detailed account.

Next, we investigate $\Gamma^\sigma(G_0)$ for T-basis.

Proposition 7.5. *Let G be a simply presented p-group and let $G_0 \subset G$ such that $0 \in G_0$ and $G_0 \setminus \{0\}$ is a T-basis. For each ordinal σ, we have $\Gamma^\sigma(G_0) = G_0 \cap p^\sigma G$ and $n_{G_0}^\sigma(g) = 1$ for each $g \in G_0 \cap p^\sigma G$. In particular, $\langle \Gamma^\sigma(G_0) \rangle = p^\sigma G$ and $\Gamma^\sigma(G_0) \setminus \{0\}$ is a T-basis of $p^\sigma G$.*

Proof. We point out that the "in particular"-statement follows directly by the above mentioned property of a T-basis regarding subgroups.

We induct on σ. For $\sigma = 0$ the claim is trivial.

We consider $\sigma + 1$. Let $g \in \Gamma^\sigma(G_0)$, which equals $G_0 \cap p^\sigma G$ by the induction hypothesis.

First, assume $g \notin p^{\sigma+1} G$. Thus, for each $h \in \Gamma^\sigma(G_0) \subset p^\sigma G$, we have $p^n h \neq g$ for all $n \in \mathbb{N}$. Consequently, since by the induction hypothesis, $\Gamma^\sigma(G_0)$

is a T-basis, $g \notin \langle \Gamma^{\sigma}(G_0) \setminus \{g\} \rangle$. Yet, $pg \in \langle \Gamma^{\sigma}(G_0) \setminus \{g\} \rangle$. Thus, $n_{\Gamma^{\sigma}(G_0)}(g) = p$ and $n_{\Gamma^{\sigma}(G_0)}(g)g \in p^{\sigma+1}G$.

Second, assume $g \in p^{\sigma+1}G$. We may assume $g \neq 0$. There exists some $h \in p^{\sigma}G$ such that $ph = g$ and since, by the induction hypothesis, $\Gamma^{\sigma}(G_0) \setminus \{0\}$ is a T-basis of $p^{\sigma}G$, we may assume that $h \in \Gamma^{\sigma}(G_0)$. It thus follows that $g \in \langle \Gamma^{\sigma}(G_0) \setminus \{g\} \rangle$. Therefore, $n_{\Gamma^{\sigma}(G_0)}(g) = 1$. By the induction hypothesis, we know that $n^{\sigma}_{G_0}(g) = 1$ and consequently $n^{\sigma+1}_{G_0}(g) = 1$.

Now, let σ be a limit ordinal. Let $g \in G_0$. By Lemma 4.7, we know that $n^{\rho}_{G_0}(g) \mid n^{\sigma}_{G_0}(g)$ for each $\rho < \sigma$. By the induction hypothesis, we know that $n^{\rho}_{G_0}(g)g \in p^{\rho}G$ for each $\rho < \sigma$. It thus follows that $n^{\sigma}_{G_0}(g)g \in p^{\rho}G$ for each $\rho < \sigma$, that is $n^{\sigma}_{G_0}(g)g \in p^{\sigma}G$. Since $n^{\sigma}_{G_0}(g)$ is a power of p, being a divisor of $\operatorname{ord}(g)$, it follows that $n^{\sigma}_{G_0}(g)g \in G_0$. Thus, we know $\Gamma^{\sigma}(G_0) \subset G_0 \cap p^{\sigma}G$.

Suppose $g \in p^{\sigma}G$. We have to show that $n^{\sigma}_{G_0}(g) = 1$. Since $g \in p^{\rho}G$ for each $\rho < \sigma$, this follows by the induction hypothesis and the definition of $n^{\sigma}_{G_0}(g)$. $\square$

This result yields the following corollary, which in particular shows that it can be useful to consider higher-order class groups beyond the first infinite ordinal.

Corollary 7.6. *Let G be a simply presented p-group and let $G_0 \subset G$ such that $0 \in G_0$ and $G_0 \setminus \{0\}$ is a T-basis.*

1. *If G is reduced and of length σ, then $\Gamma^{\sigma}(G_0) = \{0\}$, but yet $\Gamma^{\rho}(G_0) \neq \{0\}$ for each $\rho < \sigma$.*
2. *$\Gamma^{\omega,\mathrm{fin}}(G_0) = \{0\}$.*

Proof. 1. This is clear by Proposition 7.5.
2. Let $G_1 \subset G_0$ be finite. We set $G_1' = \langle G_1 \rangle \cap G_0$. Then, $G_1' \setminus \{0\}$ is a T-basis of $\langle G_1 \rangle$. Since G_1 is finite, it is clearly reduced and has finite length. Thus, by the first statement, $\Gamma^{\omega}(G_1') = \{0\}$, that is $n^{\omega}_{G_1'}(g) = \operatorname{ord}(g)$ for each $g \in G_1'$. By Proposition 4.22, this implies $n^{\omega}_{G_1}(g) = \operatorname{ord}(g)$ for each $g \in G_1$. Thus, $n^{\omega,\mathrm{fin}}_{G_0}(g) = \operatorname{ord}(g)$ for each $g \in G_0$, implying the claim. $\square$

Next, we characterize generating subsets G_0 of p-groups with $\Gamma^{\omega,\mathrm{fin}}(G_0) = \{0\}$.

Proposition 7.7. *Let G be a p-group. Let $\emptyset \neq G_0 \subset G$ be a generating set and let $\overline{G_0} = \{p^n g \colon g \in G_0,\, n \in \mathbb{N}_0\}$. We have $\Gamma^{\omega,\mathrm{fin}}(G_0) = \{0\}$ if and only if $\overline{G_0} \setminus \{0\}$ is a T-basis of G.*

Proof. If $\overline{G_0} \setminus \{0\}$ is a T-basis of G, then, by Proposition 7.5, $\Gamma^{\omega,\mathrm{fin}}(\overline{G_0}) = \{0\}$. As in Corollary 7.6, it follows that $\Gamma^{\omega,\mathrm{fin}}(G_0) = \{0\}$ as well.

Suppose $\overline{G_0} \setminus \{0\}$ is not a T-basis of G. We have to show that $\Gamma^{\omega,\mathrm{fin}}(G_0) \neq \{0\}$. It is clear that $\overline{G_0} \setminus \{0\}$ fulfills the first three conditions in the definition of a T-basis. Thus, we know that the last one fails. This means that there exists some $h_0, h_1, \ldots, h_r \in \overline{G_0} \setminus \{0\}$ such that $h_0 = \sum_{i=1}^{r} a_i h_i$ with $a_i \in \mathbb{Z}$, and $p^n h_i \neq h_0$ for each $n \in \mathbb{N}_0$ and $i \in [1, r]$. We may assume that $p^n h_i \neq h_j$ for each $n \in \mathbb{N}_0$ and distinct $i, j \in [1, r]$. Moreover, we may assume that $\gcd\{a_i, p\} = 1$ for each $i \in [1, r]$.

Now, for $i \in [0, r]$, let $g_i \in G_0$ and $n_i \in \mathbb{N}_0$ such that $p^{n_i} g_i = h_i$; by our assumptions, $p^{\ell} g_i \neq p^k g_j$ for distinct $i, j \in [0, r]$, and $\ell \in [0, n_i]$ and $k \in [0, n_j]$. We set $G_1 = \{g_0, g_1, \ldots, g_r\}$.

We assert that $n^{\omega}_{G_1}(g_i) \mid p^{n_i}$ for each $i \in [0, r]$. We note that this holds if and only if $n^{\sigma}_{G_1}(g_i) \mid p^{n_i}$ for each finite σ. We induct on σ. If $\sigma = 0$, this is trivial. We consider $\sigma + 1$. By the induction hypothesis and our assumption, $\gamma^{\sigma}_{G_1}(g_i) = n^{\sigma}_{G_1}(g_i)g_i \neq n^{\sigma}_{G_1}(g_j)g_j = \gamma^{\sigma}_{G_1}(g_j)$ for distinct $i, j \in [0, r]$. We note that

$$\frac{p^{n_0}}{n^{\sigma}_{G_1}(g_0)}\gamma^{\sigma}_{G_1}(g_0) = \sum_{i=1}^{r} a_i \frac{p^{n_i}}{n^{\sigma}_{G_1}(g_i)}\gamma^{\sigma}_{G_1}(g_i)$$

and

$$\frac{p^{n_j}}{n^{\sigma}_{G_1}(g_j)}\gamma^{\sigma}_{G_1}(g_j) = b_j\big(\frac{p^{n_0}}{n^{\sigma}_{G_1}(g_0)}\gamma^{\sigma}_{G_1}(g_0) - \sum_{i=1, i\neq j}^{r} a_i \frac{p^{n_i}}{n^{\sigma}_{G_1}(g_i)}\gamma^{\sigma}_{G_1}(g_i)\big),$$

where $b_j \in \mathbb{Z}$ is an inverse of a_j modulo $\operatorname{ord}(p^{n_j}g_j)$. This implies $n_{\Gamma^{\sigma}(G_1)}(g_i) \mid \frac{p^{n_i}}{n^{\sigma}_{G_1}(g_i)}$ for each $i \in [0, r]$ and the assertion follows.

Since $n^{\omega,\mathrm{fin}}_{G_0}(g_0) \mid n^{\omega}_{G_1}(g_0)$, we have $n^{\omega,\mathrm{fin}}_{G_0}(g_0)g_0 \neq 0$ and $\Gamma^{\omega,\mathrm{fin}}(G_0) \neq \{0\}$. $\square$

Proof of Theorem 7.3. It is well-known (cf. [20, Theorem 3.7.8]) that for each group G and each subset $G_0 \subset G$ that generates G as a monoid, thus for each torsion group and each generating subset, there exists a Krull monoid H with $\mathcal{C}(H) \cong G$ and $\mathcal{D}(H) \subset \mathcal{C}(H)$ corresponding to G_0. Thus, in particular if G is a simply presented p-group and $G_0 \subset G$ a T-basis, then there exists a Krull monoid H such that $\mathcal{C}(H) \cong G$ and $\mathcal{D}(H) \subset \mathcal{C}(H)$ corresponds to G_0. By Proposition 7.5, and noting that $\Gamma^{\omega,\mathrm{fin}}(G_0) \subset \{0\}$ if and only if $\Gamma^{\omega,\mathrm{fin}}(G_0 \cup \{0\}) = \{0\}$, we get that $\Gamma^{\omega,\mathrm{fin}}(\mathcal{D}(H)) \subset \{0\}$, i.e., H is finitary-pseudo factorial.

Conversely, for each Krull monoid H we know that $\mathcal{D}(H)$ generates $\mathcal{C}(H)$. Thus, if H is finitary-pseudo factorial, then, by Proposition 7.7, $\overline{\mathcal{D}(H)} \setminus \{0\}$ is T-basis of $\mathcal{C}(H)$. Thus, $\mathcal{C}(H)$ is simply presented. $\square$

Using a classical result of L. Claborn (cf. [20, Theorem 3.7.8] or [13, Theorem 15.18]) on the existence of Dedekind domains with class groups containing particular distributions of prime ideals, one can restate Remark 7.4 in the obvious manner.

We point out that these results, specifically Theorem 7.3, do not yield any direct new insight into the problem whether each group is isomorphic to the class group of a half-factorial Krull monoid. For simply presented p-groups it is already known that they are isomorphic to the class group of some half-factorial Krull monoid (see [17]), and for most groups not each half-factorial subset G_0, i.e., a set G_0 such that $\mathcal{B}(G_0)$ is half-factorial, fulfills $\Gamma^{\omega,\mathrm{fin}}(G_0) \subset \{0\}$. Indeed, those few types of groups for which this is true can be characterized fairly easily, and all these groups are simply presented.

Lemma 7.8. *Let G be a torsion group. The following statements are equivalent.*

- *For each half-factorial subset G_0 of G, $\Gamma^{\omega,\mathrm{fin}}(G_0) \subset \{0\}$.*
- *G is an elementary 2-group or a p-group of rank one.*

Proof. Assume G is neither an elementary 2-group nor a p-group of rank one. Then, at least one of the following three statements is true:

- There exists some $e \in G$ with $\operatorname{ord}(e) = pq$ for distinct primes p and q. We set $G_1 = \{e, pe, qe\}$.

- There exist independent elements $e_1, e_2 \in G$ with $\mathrm{ord}(e_i) = p$ and p an odd prime. We set $G_2 = \{e_1 + je_2 \colon j \in [0, p-1]\}$.
- There exist independent elements $f_1, f_2 \in G$ with $\mathrm{ord}(f_1) = 2$, $\mathrm{ord}(f_2) = 4$. We set $G_3 = \{f_2, f_1 + f_2, f_1 + 2f_2\}$.

Each of the sets G_1, G_2, and G_3 is half-factorial (cf. [20, Corollaries 6.7.7 and 6.7.9]). Yet, Example 4.19 or an easy argument shows that none of them fulfills $\Gamma^{\omega,\mathrm{fin}}(G_i) = \{0\}$.

We show the converse direction. If G is an elementary 2-group, then $G_0 \subset G$ is half-factorial if and only if $G_0 \setminus \{0\}$ is independent (cf. [26, Problem II]). Thus, indeed $\Gamma^1(G_0) = \{0\}$ (see Example 4.14). If G is a p-group of rank one and $G_0 \subset G$ a half-factorial set, then, by [15, Corollary 5.4], each finite subset $G_1 \subset G_0$ is contained in a set $H_0 = \{p^n e \colon n \in [0, \mathrm{ord}(e)]\}$ for some $e \in G$. Thus, $\Gamma^\omega(H_0) \subset \{0\}$ (cf. Example 4.15) and the claim follows. □

Acknowledgement

The author would like to thank the referee for valuable suggestions, and the referee, F. Kainrath, and A. Philipp for numerous corrections. The author is supported by the FWF (Project number P18779-N13).

References

[1] D. D. Anderson, editor. *Factorization in integral domains*, volume 189 of *Lecture Notes in Pure and Applied Mathematics*, New York, 1997. Marcel Dekker Inc.

[2] D. F. Anderson. Elasticity of factorizations in integral domains: a survey. In *Factorization in integral domains (Iowa City, IA, 1996)*, volume 189 of *Lecture Notes in Pure and Appl. Math.*, pages 1–29. Dekker, New York, 1997.

[3] S. T. Chapman, editor. *Arithmetical properties of commutative rings and monoids*, volume 241 of *Lecture Notes in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[4] S. T. Chapman and J. Coykendall. Half-factorial domains, a survey. In *Non-Noetherian commutative ring theory*, volume 520 of *Math. Appl.*, pages 97–115. Kluwer Acad. Publ., Dordrecht, 2000.

[5] S. T. Chapman, F. Halter-Koch, and U. Krause. Inside factorial monoids and integral domains. *J. Algebra*, 252(2):350–375, 2002.

[6] S. T. Chapman, U. Krause, and E. Oeljeklaus. Monoids determined by a homogeneous linear Diophantine equation and the half-factorial property. *J. Pure Appl. Algebra*, 151(2):107–133, 2000.

[7] S. T. Chapman, W. A. Schmid, and W. W. Smith. On minimal distances in Krull monoids with infinite class group. *Bull. London Math. Soc.*, 40:613–618, 2008.

[8] L. G. Chouinard, II. Krull semigroups and divisor class groups. *Canad. J. Math.*, 33(6):1459–1468, 1981.

[9] J. Coykendall. Extensions of half-factorial domains: A survey. In *Arithmetical Properties of Commutative Rings and Monoids*, volume 241 of *Lecture Notes in Pure and Appl. Math.*, pages 46–70. CRC Press (Taylor & Francis Group), Boca Raton, 2005.

[10] P. Crawley and A. W. Hales. The structure of abelian p-groups given by certain presentations. *J. Algebra*, 12:10–23, 1969.

[11] A. Facchini and F. Halter-Koch. Projective modules and divisor homomorphisms. *J. Algebra Appl.*, 2(4):435–449, 2003.

[12] A. Facchini, W. Hassler, L. Klingler, and R. Wiegand. Direct-sum decompositions over one-dimensional Cohen-Macaulay local rings. In *Multiplicative ideal theory in commutative algebra*, pages 153–168. Springer, New York, 2006.

[13] R. M. Fossum. *The divisor class group of a Krull domain.* Springer-Verlag, New York, 1973.

[14] L. Fuchs. *Infinite abelian groups. Vol. II.* Academic Press, New York, 1973. Pure and Applied Mathematics. Vol. 36-II.

[15] W. Gao and A. Geroldinger. Half-factorial domains and half-factorial subsets of abelian groups. *Houston J. Math.*, 24(4):593–611, 1998.

[16] A. Geroldinger. Über nicht-eindeutige Zerlegungen in irreduzible Elemente. *Math. Z.*, 197(4):505–529, 1988.

[17] A. Geroldinger and R. Göbel. Half-factorial subsets in infinite abelian groups. *Houston J. Math.*, 29(4):841–858 (electronic), 2003.

[18] A. Geroldinger and D. J. Grynkiewicz. On the arithmetic of Krull monoids with finite Davenport constant. *J. Algebra*, 321(4):1256–1284, 2009.

[19] A. Geroldinger and F. Halter-Koch. Nonunique factorizations in block semigroups and arithmetical applications. *Math. Slovaca*, 42(5):641–661, 1992.

[20] A. Geroldinger and F. Halter-Koch. *Non-unique factorizations. Algebraic, Combinatorial and Analytic Theory.* Chapman & Hall/CRC, 2006.

[21] R. Gilmer. Some questions for further research. In *Multiplicative ideal theory in commutative algebra*, pages 405–415. Springer, New York, 2006.

[22] P. A. Grillet. *Commutative semigroups*, volume 2 of *Advances in Mathematics (Dordrecht).* Kluwer Academic Publishers, Dordrecht, 2001.

[23] F. Halter-Koch. *Ideal systems*, volume 211 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker Inc., New York, 1998.

[24] W. Hassler, R. Karr, L. Klingler, and R. Wiegand. Big indecomposable modules and direct-sum relations. *Illinois J. Math.*, 51(1):99–122 (electronic), 2007.

[25] K. Kuzara. A note on subsemigroups with divisor theory. *Monatsh. Math.*, 119(3):217–221, 1995.

[26] W. Narkiewicz. Finite abelian groups and factorization problems. *Colloq. Math.*, 42:319–330, 1979.

[27] W. A. Schmid. Arithmetic of block monoids. *Math. Slovaca*, 54(5):503–526, 2004.

[28] W. A. Schmid. Differences in sets of lengths of Krull monoids with finite class group. *J. Théor. Nombres Bordeaux*, 17(1):323–345, 2005.

[29] R. B. Warfield, Jr. Classification theory of abelian groups. II. Local theory. In *Abelian group theory (Oberwolfach, 1981)*, volume 874 of *Lecture Notes in Math.*, pages 322–349. Springer, Berlin, 1981.