The inverse problem associated to the Davenport constant for $C_2 \oplus C_2 \oplus C_{2n}$, and applications to the arithmetical characterization of class groups

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Abstract

The inverse problem associated to the Davenport constant for some finite abelian group is the problem of determining the structure of all minimal zero-sum sequences of maximal length over this group, and more generally of long minimal zero-sum sequences. Results on the maximal multiplicity of an element in a long minimal zero-sum sequence for groups with large exponent are obtained. For groups of the form $C_2^{r-1} \oplus C_{2n}$ the results are optimal up to an absolute constant. And, the inverse problem, for sequences of maximal length, is solved completely for groups of the form $C_2^2 \oplus C_{2n}$.

Some applications of this latter result are presented. In particular, a characterization, via the system of sets of lengths, of the class group of rings of algebraic integers is obtained for certain types of groups, including $C_2^2 \oplus C_{2n}$ and $C_3 \oplus C_{3n}$; and the Davenport constants of groups of the form $C_4^2 \oplus C_{4n}$ and $C_6^2 \oplus C_{6n}$ are determined.

Keywords: Davenport constant, zero-sum sequence, zero-sumfree sequence, inverse problem, non-unique factorization, Krull monoid, class group

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1 Introduction

Let G be an additive finite abelian group. The Davenport constant of G, denoted $\mathsf{D}(G)$, can be defined as the maximal length of a minimal zero-sum sequence over G, that is the largest ℓ such that there exists a sequence $g_1 \dots g_\ell$ with $g_i \in G$ such that $\sum_{i=1}^{\ell} g_i = 0$ and $\sum_{i\in I} g_i \neq 0$ for each $\emptyset \neq I \subsetneq \{1,\dots,\ell\}$. Another common way to define this constant is via zero-sum free sequences, i.e., one defines $\mathsf{d}(G)$ as the maximal length of a zero-sum free sequence; clearly $\mathsf{D}(G) = \mathsf{d}(G) + 1$.

The problem of determining this constant was popularized by P. C. Baayen, H. Davenport, and P. Erdős in the 1960s. Still its actual value is only known for a few types of groups. If $G \cong \oplus_{i=1}^{r} C_{n_i}$ with cyclic group C_{n_i} of order n_i and $n_i \mid n_{i+1}$, then let $\mathsf{D}^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1)$. It is well-known and not hard to see that $\mathsf{D}(G) \geq \mathsf{D}^*(G)$. Since the end of the 1960s it is known that in fact $\mathsf{D}(G) = \mathsf{D}^*(G)$ in case G is a p-group or G has rank at most two (see [44, 45, 53]). Yet, already at that time it was noticed that $\mathsf{D}(G) = \mathsf{D}^*(G)$ does not hold for all finite abelian groups. The first example asserting inequality is due to P.C. Baayen (cf. [53]) and, now, it is known that for each $r \geq 4$ infinitely many groups with rank r exist such that this equality does not hold (see [34], and also see [19] for further examples).

There are presently two main additional classes of groups for which the equality $\mathsf{D}(G) = \mathsf{D}^*(G)$ is conjectured to be true, namely groups of rank three and groups of the form C_n^r (see, e.g., [23, Conjecture 3.5] and [1]; the problems are also mentioned in [41, 4]). Both conjectures are only confirmed in special cases. The latter conjecture is confirmed only if $r = 3$ and $n = 2p^k$ for prime p, if $r = 3$ and $n = 32^k$ (see [53, 54] as a special case of results for groups of rank three), and if n is a prime power or $r \leq 2$ by the above mentioned results. Since to summarize all results asserting equality for groups of rank three in a brief and concise way seems impossible, we now only mention—additional information on results towards this conjecture is recalled in Section 4 and see [53, 54, 18, 11, 7, 6]—that it is well-known to hold true for groups of the form $C_2^2 \oplus C_{2n}$ (see [53]), was only recently determined for groups of the form $C_3^2 \oplus C_{3n}$ (see [7]), and is established in the present paper for $C_4^2 \oplus C_{4n}$ and $C_6^2 \oplus C_{6n}$ as an application of our inverse result for $C_2^2 \oplus C_{2n}$ (cf. below).

For groups of rank greater than three there is not even a conjecture regarding the precise value of $\mathsf{D}(G)$. The equality $\mathsf{D}(G) = \mathsf{D}^*(G)$ is known to hold for p-groups (as mentioned above), for groups of the form $C_2^3 \oplus C_{2n}$ (see [3]), and groups that are in a certain sense similar to groups of rank two, cf. (3.2). However, for $G = C_2^{r-1} \oplus C_{2n}$ with $r \geq 5$ and n odd it is known that $\mathsf{D}(G) > \mathsf{D}^*(G)$; we refer to [42] for lower bounds for the gap between these two constants. And, we mention that, via a computer-aided yet not purely computational argument (see [46]), it is known that $\mathsf{D}(G) = \mathsf{D}^*(G) + 1$ for $C_2^{r-1} \oplus C_6$ where $r \in \{5, 6, 7\}$, for $C_2^4 \oplus C_{10}$, and for $C_3^3 \oplus C_6$; and $\mathsf{D}(G) = \mathsf{D}^*(G) + 2$ for $C_2^7 \oplus C_6$.

In addition to the direct problem of determining the Davenport constant the associated inverse problem, i.e., the problem of determining the structure of minimal zero-sum sequences over G of length $\mathsf{D}(G)$ (and more generally long minimal zero-sum sequences)—essentially equivalently, the problem of determining the structure of maximal length (and long) zero-sum free sequences—received considerable attention as well (see, e.g., [23] for an overview). On the one hand, it is traditional to study inverse problem associated to the various direct problems of Combinatorial Number Theory. On the other hand, in certain applications knowledge on the inverse problem is crucial (cf. below).

An answer to this inverse problem is well-known, and not hard to obtain, in case G is cyclic; yet, the refined problem of determining the structure of minimal zero-sum sequences over cyclic groups that are long, yet do not have maximal length, recently received considerable attention see [48, 55, 43, 28]. Moreover, the structure of minimal zero-sum sequence over elementary 2-groups (of arbitrary length) is well-known and easy to establish.

Yet, already for elementary p-groups of rank two the inverse problem is not yet solved, tough there is at least a well-supported conjecture and various partial results towards this conjecture. And, assuming this conjecture holds true the inverse problem is solved for all groups of rank two (see Section 3.2 for details, and [21] and [13] for earlier unconditional results for $C_2 \oplus C_{2n}$ and $C_3 \oplus C_{3n}$, respectively).

For groups of rank three or greater, except of course elementary 2-groups, so far no results and not even conjectures are known. In this paper we solve this inverse problem for groups of the form $C_2^2 \oplus C_{2n}$, the first class of groups of rank three. Our actual result is quite lengthy, thus we defer the precise statement to Section 3.5. Moreover, our investigations of this problem are imbedded in more general investigations on the maximal multiplicity of an element in long minimal zero-sum sequences, i.e., the height of the sequence, over certain types of groups, expanding on investigations of this type carried out in [19] and [6] (for details see the Section 3).

The investigations on this and other inverse zero-sum problems are in part motivated by applications to Non-Unique Factorization Theory, which among others is concerned with the various phenomena of non-uniqueness arising when considering factorizations of algebraic integers, or more generally elements of Krull monoids, into irreducibles (see, e.g., the monograph [32], the lecture notes [31], and the proceedings [10], for detailed information on this subject; and see [25] for a recent application of the above mentioned results on cyclic groups to Non-Unique-Factorization Theory). For an overview of other applications of the Davenport constant and related problems see, e.g., [23, Section 1]. In Section 5 we present an application of the above mentioned result to a central problem in Non-Unique Factorization Theory, namely to the problem of characterizing the ideal class group of the ring of integers of an algebraic number field by its system of sets of lengths (see [32, Chapter 7]). We refer to Sections 2 and 5 for terminology and a more detailed discussion of this problem. For the moment, we only point out why the inverse problem associated to $C_2^2 \oplus C_{2n}$ is relevant to that problem. We need the solution of this inverse problem to distinguish the system of sets of lengths of the ring of integers of an algebraic number field with class group of the form $C_2^2 \oplus C_{6n}$ from that of one with class group of the form $C_3 \oplus C_{6n}$. The relevance of distinguishing precisely these two types of groups is due to the fact that a priori the likelihood that the system of sets of lengths in this case are not distinct was exceptionally high; a detailed justification for this assertion is given in Section 5.

In addition, in Section 4, we discuss some other applications of our inverse result, in particular (as already mentioned) we use it to determine the value of the Davenport constant for two new types of groups (of rank three), and discuss our results in the context of the problem of determining the order of elements in long minimal zero-sum sequences and the cross number, i.e., a weighted length, of these sequences (see [19, 21, 36, 37] for results on this problem).

2 Preliminaries

We recall some terminology and basic facts. We follow [32, 23, 31] to which we refer for further details.

We denote the non-negative and positive integers by $\mathbb{N}_0$ and $\mathbb{N}$, respectively. By $[a, b]$ we always mean the interval of integers, that is the set $\{z \in \mathbb{Z} \colon a \le z \le b\}$. We set $\max \emptyset = 0$.

By C_n we denote a cyclic group of order n; by C_n^r we denote the direct sum of r groups C_n. Let G be a finite abelian group; throughout we use additive notation for finite abelian groups. For $g \in G$, the order of g is denoted by $\operatorname{ord}(g)$. For a subset $G_0 \subset G$, the subgroup generated by G_0 is denote by $\langle G_0 \rangle$. A subset $E \subset G \setminus \{0\}$ is called independent if $\sum_{e \in E} a_e e = 0$, with $a_e \in \mathbb{Z}$, implies that $a_e e = 0$ for each $e \in E$. An independent generating subset of G is called a basis of G. We point out that if $G_0 \subset G \setminus \{0\}$ and $\prod_{g \in G_0} \operatorname{ord}(g) = |\langle G_0 \rangle|$, then G_0 is independent. There exist uniquely determined $1 < n_1 \mid \cdots \mid n_r$ and prime powers $q_i \neq 1$ such that $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r} \cong C_{q_1} \oplus \cdots \oplus C_{q_{r^*}}$. Then $\exp(G) = n_r$, $\mathsf{r}(G) = r$, and $\mathsf{r}^*(G) = r^*$ is called the exponent, rank, and total rank of G, respectively; moreover, for a prime p the number of q_is that are powers of this p is called the p-rank of G, denoted $\mathsf{r}_p(G)$. The group G is called a p-group if its exponent is a prime power, and it is called an elementary group if its exponent is squarefree. For subset $A, B \subset G$, we denote by $A \pm B = \{a \pm b \colon a \in A,\, b \in B\}$ the sum-set and the difference-set of A and B, respectively.

A sequence S over G is an element of the multiplicatively written free abelian monoid over G, which is denoted by $\mathcal{F}(G)$, that is $S = \prod_{g \in G} g^{v_g}$ with $v_g \in \mathbb{N}_0$. Moreover, for each sequence S there exist up to ordering uniquely determined $g_1, \ldots, g_\ell \in G$ such that $S = \prod_{i=1}^{\ell} g_i$. The neutral element of $\mathcal{F}(G)$ is called the empty sequences, and denoted by 1. Let $S = \prod_{g \in G} g^{v_g} \in \mathcal{F}(G)$. A divisor $T \mid S$ is called a subsequence of S; the subsequence T is called proper if $T \neq S$. If $T \mid S$, then $T^{-1}S$ denotes the co-divisor of T in S, i.e., the unique sequence fulfilling $T(T^{-1}S) = S$. Moreover, for sequences $S_1, S_2 \in \mathcal{F}(G)$, the notation $\gcd(S_1, S_2)$ is used to denote the greatest common divisor of S_1 and S_2 in $\mathcal{F}(G)$, which is well-defined, since $\mathcal{F}(G)$ is a free monoid. One calls $\mathsf{v}_g(S) = v_g$ the multiplicity of g in S, $|S| = \sum_{g \in G} \mathsf{v}_g(S)$ the length of S, $\mathsf{k}(S) = \sum_{g \in G} \mathsf{v}_g(S)/\operatorname{ord}(g)$ the cross number of S, $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \colon g \in G\}$ the height of S, and $\sigma(S) = \sum_{g \in G} \mathsf{v}_g(S)g$ the sum of S. The sequence $S \in \mathcal{F}(G)$ is called short if $1 \le |S| \le \exp(G)$ and it is called squarefree if $\mathsf{v}_g(S) \le 1$ for each $g \in G$. The set of subsums of S is $\Sigma(S) = \{\sigma(T) \colon 1 \neq T \mid S\}$, and the support of S is $\operatorname{supp}(S) = \{g \in G \colon \mathsf{v}_g(S) \ge 1\}$. The sequence S is called zero-sumfree if $0 \notin \Sigma(S)$. For $S = \prod_{i=1}^{\ell} g_i$, the notation $-S$ is used to denote the sequence $\prod_{i=1}^{\ell}(-g_i)$, and for $f \in G$, $f + S$ denotes the sequence $\prod_{i=1}^{\ell}(f + g_i)$. One says that S is a zero-sum sequence if $\sigma(S) = 0$, and one denotes the set of all zero-sum sequences over G by $\mathcal{B}(G)$; the set $\mathcal{B}(G)$ is a submonoid of $\mathcal{F}(G)$. A non-empty zero-sum sequences S is called a minimal zero-sum sequence if $\sigma(T) \neq 0$ for each non-empty and proper subsequence of S, and the set of all minimal zero-sum sequences is denoted by $\mathcal{A}(G)$. Clearly, each map $f : G \to G'$ between abelian groups G and G' can be extended in a unique way to a monoid homorphism of $\mathcal{F}(G) \to \mathcal{F}(G')$, which we also denote by f; if f is a group homomorphism, then $f(\mathcal{B}(G)) \subset \mathcal{B}(G')$.

We recall some definitions on factorizations over monoids. Let M be an atomic monoid, i.e., M is a commutative cancelative semigroup with neutral element (i.e., an abelian monoid) such that each non-invertible element $a \in M$ is the product of finitely many irreducible elements (atoms). If $a = u_1 \dots u_n$ with $u_i \in M$ irreducible, then n is called the length of this factorization of a. Moreover, the set of lengths of a, denoted $\mathsf{L}(a)$, is the set of all n such that a has a factorization into irreducibles of length n. For $e \in M$ an invertible element, one defines $\mathsf{L}(e) = \{0\}$. The set $\mathcal{L}(M) = \{\mathsf{L}(a) \colon a \in M\}$ is called the system of sets of lengths of M. Note that $\mathcal{B}(G)$ is an atomic monoid and its irreducible elements are the minimal zero-sum sequences, i.e., the elements of $\mathcal{A}(G)$. For convenience of notation, we write $\mathcal{L}(G)$ instead of $\mathcal{L}(\mathcal{B}(G))$ and refer to it as the system of sets of lengths of G. We exclusively use the term factorization to refer to a factorization into irreducible elements (of some atomic monoid that is mentioned explicitly or clear from context). In particular, if we say that for a zero-sum sequence $B \in \mathcal{B}(G)$ we consider a factorization $B = \prod_{i=1}^{\ell} A_i$ we always mean a factorization into irreducible elements in the monoid $\mathcal{B}(G)$, i.e., $A_i \in \mathcal{A}(G)$ for each i. Yet, if we consider, for some $S \in \mathcal{F}(G)$, a product decomposition $S = \prod_{i=1}^{\ell} S_i$ with sequences $S_i \in \mathcal{F}(G)$ this is not a factorization (except if $|S_i| = 1$ for each i) and we thus refer to it as a decomposition.

Next, we recall some definitions and results on the Davenport constant and related notions.

Let G be a finite abelian group. Let $\mathsf{D}(G) = \max\{|A| \colon A \in \mathcal{A}(G)\}$ denote the Davenport constant and let $\mathsf{K}(G) = \max\{\mathsf{k}(A) \colon A \in \mathcal{A}(G)\}$ denote the cross number of G. Moreover, for $k \in \mathbb{N}$, let $\mathsf{D}_k(G) = \max\{|B| \colon B \in \mathcal{B}(G), \max \mathsf{L}(B) \le k\}$ denote the generalized Davenport constants introduced in [39] in the context of Analytic Non-Unique Factorization Theory; for the relevance in the present context, originally noticed in [14], see (3.1). For an overview on results on this constant see [32] and for recent results [7] and [17]. Observe that $\mathsf{D}_1(G) = \mathsf{D}(G)$. Additionally, let $\eta(G)$ denote the smallest $\ell \in \mathbb{N}$ such that each $S \in \mathcal{F}(G)$ with $|S| \ge \ell$ has a short zero-sum subsequence. Essentially by definition, we have $\mathsf{D}(G) \le \eta(G)$. We recall that $\eta(G) \le |G|$, which is sharp for cyclic groups and elementary 2-groups; see [29] for this bound, also see [31, 32] for proofs of this and other results on $\eta(G)$; and, e.g., [16, 15] for lower bounds.

It is well known that, with n_i and q_i as above,

$$\mathsf{D}(G) \ge \mathsf{D}^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1) \text{ and } \mathsf{K}(G) \ge \frac{1}{\exp(G)} + \sum_{i=1}^{r^*} \frac{q_i - 1}{q_i}\,. \tag{2.1}$$

For G a p-group equality holds in both inequalities, and for $\mathsf{r}(G) \le 2$ equality holds for the Davenport constant. And, we recall the well-known upper bound $\mathsf{K}(G) \le 1/2 + \log |G|$ (see [35]).

Moreover, we recall that for finite abelian groups G_1 and G_2, we have $\mathsf{D}(G_1 \oplus G_2) \ge \mathsf{D}(G_1) + \mathsf{D}(G_2) - 1$, and if $G_1 \subsetneq G_2$ then $\mathsf{D}(G_1) < \mathsf{D}(G_2)$. In particular, the support of a minimal zero-sum sequence of lengths $\mathsf{D}(G)$ is a generating set of G. Additionally, we recall the lower bound $\mathsf{D}(G) \geq 4\,\mathsf{r}^*(G) - 3\,\mathsf{r}(G) + 1$, which is relevant in Section 5 (see [17]).

We recall some results on $\mathsf{D}_k(G)$. Setting

$$\mathsf{D}_0'(G) = \max\{\mathsf{D}(G) - \exp(G), \eta(G) - 2\exp(G)\}$$

and letting G_1 denote a group such that $G \cong G_1 \oplus C_{\exp(G)}$, we have

$$k\exp(G) + (\mathsf{D}(G_1) - 1) \leq \mathsf{D}_k(G) \leq k\exp(G) + \mathsf{D}_0'(G) \tag{2.2}$$

for each $k \in \mathbb{N}$. Moreover, there exists some $\mathsf{D}_0(G)$ such that for all sufficiently large k, depending on G, $\mathsf{D}_k(G) = k\exp(G) + \mathsf{D}_0(G)$. Clearly, we have $\mathsf{D}_0(G) \leq \mathsf{D}_0'(G)$. Also, note that by the bounds recalled above $\mathsf{D}_0'(G) \leq |G| - \exp(G)$. For groups of rank at most two and in closely related situations both inequalities in (2.2) are in fact equalities (see [39, 32]), yet in general neither one is an equality (see, e.g., [17] and cf. below). In particular, in general the precise value of $\mathsf{D}_k(G)$ and $\mathsf{D}_0(G)$ are not known, not even for p-groups; see [7] for recent precise results for C_3^3.

In case G is an elementary 2-group it is known for all k that $\mathsf{D}_k(G) \leq k\exp(G) + \mathsf{D}_0(G)$. Moreover, it is known that $\mathsf{D}_0(C_2^r) = 2^r/3 + O(2^{r/2})$, where explicit bounds for the implied constant are known and one thus can infer that $\mathsf{D}_0(C_2^r) < 2^{r-1}$ for each $r \in \mathbb{N}$, which is more convenient though less precise for our applications. Additionally, we recall that $\mathsf{D}_k(C_2^3) = 2k + 3$ for each $k \geq 2$ (see [14]); for similar results for $r \in \{4, 5\}$ and the upper bound see [17].

Finally, we point out that by the definition of $\mathsf{D}_k(G)$, we know, for each $k \in \mathbb{N}$, that if $|A| > \mathsf{D}_k(G)$, then $\max \mathsf{L}(A) > k$. In particular, we get that

$$\text{if} \quad \frac{|A| - \mathsf{D}_0'(G)}{\exp(G)} > k\,, \quad \text{then} \quad \max \mathsf{L}(A) > k\,. \tag{2.3}$$

In case we know that $\mathsf{D}_k(G) \leq k\exp(G) + \mathsf{D}_0(G)$, in particular for elementary 2-groups, we can replace $\mathsf{D}_0'(G)$ by $\mathsf{D}_0(G)$ in this inequality.

3 On the structure of long minimal zero-sum sequences

We start by giving an overview of the results to be established in this section. To put them into context and since it is relevant for the subsequent discussion, we recall some known results; including a brief, and thus rather ahistorical, discussion of the direct problem.

As mentioned in Section 1, the problem of determining the Davenport constant for p-groups was solved at the end of the 1960s. Yet, since that time the method used to prove this result was neither generalized to more general types of groups nor modified to yield an answer to the inverse problem. In fact, now for p-groups other proofs and refinements of that proof are known (see, e.g., [1, 32, 24]), but the same limitations seem to apply.

Thus, to obtain information on the Davenport constant for other types of groups one tries to leverage the information available for p-groups (and cyclic groups), via an 'inductive' argument, reducing the problem of determining $\mathsf{D}(G)$, or the associated inverse problem, to a problem over a subgroup H of G, a problem over the factor group G/H, and the problem of recombining the information, i.e., on tries to combine knowledge on groups G_1 and G_2 to gain information on a group G that is an extension of G_1 and G_2. This is one of the most frequently applied and classical techniques in the investigation of the Davenport constant and the associated inverse problems (see [47, 45, 53] for classical contributions, in particular, for groups of rank two, and [32] for an overview). In fact, essentially all results on the exact value of the Davenport constant for non-p-groups—cyclic groups and isolated examples obtained by purely computational means seem to be the only exceptions—and various bounds were obtained via some form of this method (see [23] and [32] for an overview).

To discuss the inductive method in more detail, we fix some notation. Let G be a finite abelian group, let $H \subset G$ be a subgroup, and let $\varphi : G \to G/H$ denote the canonical map. In applications frequently the factor group G/H is 'fixed' and only H 'varies.' Say, for some group K investigations are carried out for all the groups G_n that are extensions—to be precise, typically only extensions fulfilling some additional condition are considered, see the discussion below—of K by groups of the same type but with a varying parameter n, e.g., cyclic groups of order n or groups of the form C_n^2 (cf. the types of groups mentioned in in Sections 1, 3.4, and 4). In view of this, the present setup, which makes the 'fixed' group G/H depend on the two 'varying' groups G and H, is somewhat counter-intuitive. Yet, to use this setup, rather than the dual one, has several technical advantages that (it is hoped) outweigh this. Thus, we are mainly interested in the situation that $|H|$ is large relative to $|G/H|$; in fact, as detailed below, we are mainly concerned with the situation that even the exponent of H is large relative to $|G/H|$.

We recall the following key-formula (see [14]), which encodes several classical applications of inductive arguments (cf. below and see Step 1 of the Proof of Theorem 3.1 for a related reasoning),

$$\mathsf{D}(G) \le \mathsf{D}_{\mathsf{D}(H)}(G/H). \tag{3.1}$$

The relevance of this formula is at least twofold. On the one hand, for certain types of groups G and a suitably chosen proper subgroup H the inequality in (3.1) is in fact an equality. And, the subproblems of determining the Davenport constant of H and the generalized Davenport constants of G/H can be solved; e.g., by iteratively applying this formula to eventually attain a situation where all groups are p-groups or cyclic. To assert this equality, one combines the formula with the well-known lower bound for $\mathsf{D}(G)$ to obtain the chain of inequalities $\mathsf{D}^*(G) \le \mathsf{D}(G) \le \mathsf{D}_{\mathsf{D}(H)}(G/H)$. In this way, the problem of determining the Davenport constant of groups of rank at most two, can be reduced to a problem on elementary p-groups of rank at most three; groups of rank three are used, to determine the generalized Davenport constants via an imbedding argument. Indeed, this is the original—and still the only known—argument, slightly rephrased, to determine the Davenport constant for groups of rank two. A similar approach still works in related situations. In particular, it can be used to show that

$$\mathsf{D}(G' \oplus C_n) = \mathsf{D}^*(G' \oplus C_n) \tag{3.2}$$

where G' is a p-group with $\mathsf{D}(G') \le 2\exp(G') - 1$ and n is co-prime to $\exp(G')$ (see [53], and [11] for a generalization).

On the other hand, this formula is useful to decide which choice for the subgroup H is 'suitable' and to highlight limitations of this form—strictly limiting to the consideration of direct problems—of the inductive approach. We recall, cf. (2.2), that $\mathsf{D}_{\mathsf{D}(H)}(G/H) \ge \exp(G/H)(\mathsf{D}(H) - 1) + \mathsf{D}^*(G/H)$. So, at least $\exp(G/H)(\mathsf{D}^*(H) - 1) + \mathsf{D}^*(G/H) \le \mathsf{D}^*(G)$ should hold. Recalling that we are mainly interested in the case that (the exponent of) H is large relative to G/H, we see that in our context we effectively have to restrict to considering subgroups H such that $\exp(G) = \exp(H)\exp(G/H)$, since otherwise the upper bound in (3.1) can be much too large. Conversely, if $\exp(G) = \exp(H)\exp(G/H)$ and H is cyclic, then we see that $\exp(G/H)(\mathsf{D}^*(H) - 1) + \mathsf{D}^*(G/H) = \mathsf{D}^*(G)$ and thus any error in the estimate (3.1) is only due to the inaccuracy of the lower bound (2.2) and thus can be bounded in terms of G/H only, i.e., in our context is relatively small. However, as discussed, for groups of rank greater than two the lower bound in (2.2) is often not accurate. For example, for the group $G = C_2^2 \oplus C_{2p}$ for some odd prime p, we get by the result on $\mathsf{D}_k(C_2^3)$ recalled in Section 2 (also, note that all other choices of subgroups will result in much worse estimates)

$$2p + 2 = \mathsf{D}^*(G) \le \mathsf{D}(G) \le \mathsf{D}_{\mathsf{D}(C_p)}(C_2^3) = 2p + 3.$$

Thus, $\mathsf{D}(C_2^2 \oplus C_{2p})$ cannot be determined by (3.1) alone.

However, it is known that a refined inductive argument allows to prove that $\mathsf{D}(C_2^2 \oplus C_{2n}) = 2n+2$ for each $n \in \mathbb{N}$ (cf. Section 1). Yet, some information on the inverse problems associated to the subproblems in C_2^3 and C_n is required; for example, knowing $\nu(C_n)$ (so that Proposition 4.2, a result given in [53, 54], is applicable) and having some information on the inverse problem associated to the generalized Davenport constant for C_2^3 (to prove this proposition) allows to prove this.

More recently, results were obtained that solve the inverse problem associated to the Davenport constant via inductive arguments, or at least give conditional or partial answers to this problem. The first results of this form are due to W.D. Gao and A. Geroldinger (see [21, 22]), where this problem is solved for $C_2 \oplus C_{2n}$ and C_{2n}^2, in the latter case assuming n has Property **B**, i.e., a solution to the inverse problem for C_n^2 (see Section 3.2 for the definition). In Section 3.2 we recall very recent results obtained via the inductive method, fully reducing the inverse problem for groups of rank two to the case of elementary p-groups of rank two; for recent progress on this remaining problem see, e.g., [40, 27, 5].

The purpose of our investigations on the inverse problem is twofold. On the one hand, we obtain a full solution to the inverse problem for groups of the form $C_2^2 \oplus C_{2n}$ for each $n \in \mathbb{N}$. The motivation for and relevance of these investigations already has been discussed in Section 1; additionally we recall that, for this class of groups, in contrast to groups of rank at most two, it is necessary to operate below the upper bound that can be inferred from (3.1). On the other hand, we imbed these investigations into a more general investigation of one main aspect of the structure of long minimal zero-sum sequences, namely their height, over certain types of groups. In Section 4 we briefly discuss implications of our results for the two other main aspects, namely the cardinality of the support and the order of elements in the sequence (see [23]). We recall that to impose some condition on the relative size of the exponent is essentially inevitable when considering this question; for example, for G an elementary p-groups it is known that if the rank is large relative to the exponent (yet, not imposing any absolute upper bound on the exponent), then there exist minimal zero-sum sequence of maximal length that are squarefree, i.e., have height 1 (see [19] for this and more general results of this type).

Investigations of this type were started in [19]. And, in the recent decidability result for the Davenport constant of groups of the form $C_m^{r-1} \oplus C_{mn}$ with $\gcd(m,n) = 1$ (see [6]) this question was investigated as well, since it was relevant for that argument. First, we consider this problem in a very general setting, expanding on known results of this form. We highlight which parameters are relevant and discuss in which ways this result can be improved in specific situations. Second, we restrict to the case that G has a large exponent (in a relative sense), mainly focusing on the case that G has a cyclic subgroup H such that $|H|$ is large relative to $|G/H|$, implementing some of the improvements only sketched for the general case. Third, we turn to a more restricted class of groups, namely groups of the form $C_2^{r-1} \oplus C_{2n}$. In this case, we establish bounds for the height of long minimal zero-sum sequences that are optimal up to an absolute constant; inspecting our proof, yields 7 as the value for this constant (and this could be slightly improved). One reason for focusing on this particular class of groups is the fact that, for reasons explained above, we want a precise understanding of the inverse problem associated to $C_2^2 \oplus C_{2n}$. However, this is not the only reason. This type of groups is an interesting extremal case. We apply the inductive method with H cyclic and G/H an elementary 2-group. On the one hand, this combines, when considering the relative size of exponent versus rank, the two most extreme cases; and, from a theoretical point of view, the case that G/H is an elementary 2-group can thus be considered as a worst-case scenario. On the other hand, from a practical point of view, certain of the arising subproblems are easier to address or better understood for elementary 2-groups than, say, for arbitrary elementary p-groups. Finally, we apply the thus gained insight with some ad hoc arguments to obtain a complete solution of the inverse problem for $C_2^2 \oplus C_{2n}$ (for sequences of maximal length).

3.1 General groups

We start the investigations by considering the problem of establishing lower bounds for the height in the general situation. Our result, Theorem 3.1—to be precise, refinements of it—turns out to be fairly accurate in certain cases. Yet, as discussed above, due to the nature of the problem, the result has to be essentially empty if we do not impose restrictions on the group G, the subgroup H, and the length of the sequence A; the result depends on the length of A via the size of the elements of $\mathsf{L}(\varphi(A))$, cf. (2.3). Additionally, our arguments in the general case are not optimized (see below for a discussion of refinements).

To formulate our results we introduce some notions. Let G be a finite abelian group. For $\ell \in [1, \mathsf{D}(G)]$, let $\mathsf{h}(G, \ell) = \min\{\mathsf{h}(A) \colon A \in \mathcal{A}(G), \, |A| \ge \ell\}$ denote the minimal height of a minimal zero-sum sequences of lengths at least ℓ over G; though not explicitly named, this quantity has been investigated frequently (see below). For $k \in \mathbb{Z}$, let $\operatorname{supp}_k(S) = \{g \in G \colon \mathsf{v}_g(S) \ge k\}$ denote the support of level k; for $k = 1$, this yields the usual definition of the support of a sequence, and for $k \le 0$ we have $\operatorname{supp}_k(S) = G$. For $\ell \in [1, \mathsf{D}(G)]$ and $\delta \in \mathbb{N}_0$, let $\mathsf{ci}(G, \ell, \delta) = \max\{|\operatorname{supp}_{\mathsf{h}(A)-\delta}(A)| \colon A \in \mathcal{A}(G), \, |A| \ge \ell\}$ denote the maximal cardinality of the set of $-\delta$-important elements for minimal zero-sum sequences of length at least ℓ; this terminology is inspired by [6] where elements occurring with high multiplicity are called important, also cf. [26, Section 3] for the relevance of elements appearing with high multiplicity in this context. In Section 3.2, we point out information that is available on these quantities via known results, illustrating that this result is actually applicable (in suitable situations).

Theorem 3.1. *Let G be a finite abelian group and let $\{0\} \ne H \subsetneq G$ be a subgroup, and $\varphi : G \to G/H$ the canonical map. Let $A \in \mathcal{A}(G)$ and $k \in \mathsf{L}(\varphi(A))$. With $\delta_0 = 1$ if $2 \nmid |H|$ and $\delta_0 = 2$ if $2 \mid |H|$, we have*

$$\mathsf{h}(A) \ge \frac{\mathsf{h}(H, k) - \mathsf{D}(G/H)|G/H|}{(2\,\mathsf{ci}(H, k, \delta_0) - 1)|G/H|}.$$

Since similar general results are already known (see [19, 6]), we point out the main novelty of our result. We take the situation that there can be more than one important element in long minimal zero-sum sequences over H into account, via the parameter $\mathsf{ci}(H, k, \delta_0)$. This additional generality is useful, since it allows to apply the result for non-cyclic H and additionally makes it applicable in the situation that the subgroup H is cyclic yet the sequence A is not long enough to guarantee the existence of some $k \in \mathsf{L}(\varphi(A))$ for which $\mathsf{ci}(H, k, \delta_0) = 1$ (see Section 3.2 for details). In other aspects our result, as formulated, is weaker than the other general results, yet after its proof we discuss that these weaknesses can be overcome with some modifications (yet, of course, not achieving the precision of certain non-general results, such as [26, 49], where various facts specific to the situation at hand are taken into account); we do not take these modifications into account in the result, since we believe that to introduce even more parameters is not desirable. Yet, we take them into account in our more specialized investigations in the subsequent sections.

We write the proof of Theorem 3.1 in a structured way, since we frequently refer to this proof in the proofs of more specific result, to avoid redoing identical arguments.

Proof of Theorem 3.1.

Step 1, Generating minimal zero-sum sequences over H:
Since $k \in \mathsf{L}(\varphi(A))$, there exist $F_1, \dots, F_k \in \mathcal{F}(G)$ such that $A = F_1 \dots F_k$ and $\varphi(F_1) \dots \varphi(F_k)$ is a factorization of $\varphi(A)$; in particular, we have $\sigma(F_i) \in H$ for each $i \in [1, k]$. We note that $C = \prod_{i=1}^k \sigma(F_i) \in \mathcal{A}(H)$, since $\sum_{i \in J} \sigma(F_i) = 0$ for some $J \subset [1, k]$ is equivalent to $\sigma(\prod_{i \in J} F_i) = 0$.

Step 2, Choosing a minimal zero-sum sequence over H:
Let $\prod_{i=1}^k \sigma(F_i) = \prod_{i=1}^s h_i^{v_i}$ with pairwise distinct elements h_i such that $v_1 \ge \dots \ge v_s > 0$, and let $t \in [1, s]$ be maximal such that $v_i = v_1$ for each $i \in [1, t]$. We assume that the F_i are chosen in such a way that the sequence, in the traditional sense, $(v_1, \dots, v_s, 0, \dots)$ is minimal, in the lexicographic order, among all these sequences defined via decompositions $A = F_1' \dots F_k'$ such that $\varphi(F_1') \dots \varphi(F_k')$ is a factorization of $\varphi(A)$; in particular, $v_1 = \mathsf{h}(\prod_{i=1}^k \sigma(F_i))$ is minimal and moreover t is minimal among all sequences that yield this minimal v_1.

Step 3, Identifying a 'large fibre':
Since $C \in \mathcal{A}(H)$ and since $v_1 = \mathsf{h}(C)$, we have $v_1 \geq \mathsf{h}(H,k)$. Moreover, for $\delta \in \{1,2\}$ let $t_\delta \in [1,s]$ be maximal such that $v_i \geq v_1 - \delta$ for each $i \in [1,t_\delta]$; note that $t_\delta \in [1, \mathsf{ci}(H,k,\delta)]$.

Let $I \subset [1,k]$ such that $\prod_{i\in I} \sigma(F_i) = h_1^{v_1}$. Let $\overline{g} \in G/H$ such that $\mathsf{v}_{\overline{g}}(\varphi(\prod_{i\in I} F_i)) = \mathsf{h}(\varphi(\prod_{i\in I} F_i))$. Clearly, $\mathsf{h}(\varphi(\prod_{i\in I} F_i)) \geq |\prod_{i\in I} F_i|/|G/H|$.

Step 4, Investigating the 'large fibre':
Let $g_1 \mid \prod_{i\in I} F_i$, say $g_1 \mid F_{k_1}$, with $\varphi(g_1) = \overline{g}$.

Let $k_2 \in I \setminus \{k_1\}$ such that there exists some $g_2 \mid F_{k_2}$ with $\varphi(g_2) = \overline{g}$. We note that since $|F_{k_1}| \leq \mathsf{D}(G/H)$ and $\mathsf{v}_{\overline{g}}(\varphi(\prod_{i\in I} F_i)) \geq |\prod_{i\in I} F_i|/|G/H| \geq v_1/|G/H|$, our claim is trivially true if such a k_2 does not exist.

Let $F'_{k_i} = g_i^{-1} g_j F'_{k_i}$ for $\{i,j\} = \{1,2\}$ and let $F_i = F'_i$ for $i \in [1,k] \setminus \{k_1,k_2\}$. We note that $\sigma(F'_{k_1}) = h_1 - (g_1 - g_2)$ and that $\sigma(F'_{k_2}) = h_1 + (g_1 - g_2)$; since $g_1 - g_2 \in H$, both sums are elements of H.

We consider $D = \prod_{i=1}^k \sigma(F'_i) \in \mathcal{A}(H)$. We have $D = C h_1^{-2} \sigma(F'_{k_1}) \sigma(F'_{k_2})$. By our constraints on $\mathsf{h}(C)$ and t, it follows that at least one of the following two statements has to hold (for clarity, we disregard some slight improvements achievable by distinguishing more cases).

- $\sigma(F'_{k_i}) \in \{h_1, \dots, h_{t_1}\}$ for some $i \in \{1,2\}$.
- $\sigma(F'_{k_1}) = \sigma(F'_{k_2}) \in \{h_{t_1+1}, \dots, h_{t_2}\}$.

We note that the second statement can only hold if $g_1 - g_2$ has order 2, i.e., only if $2 \mid |H|$.

Let $H_0 = \{h_1, \dots, h_{t_{\delta_0}}\}$. We get that $\sigma(F'_{k_1}) = h_1 - (g_1 - g_2) \in H_0$ or $\sigma(F'_{k_2}) = h_1 + (g_1 - g_2) \in H_0$. Thus, $(g_2 - g_1) \in (-h_1 + H_0) \cup (h_1 - H_0) = H'_0$. We have $|H'_0| \leq 2|H_0| - 1 = 2t_{\delta_0} - 1$.

Thus, it follows that

$$\varphi^{-1}(\overline{g}) \cap \operatorname{supp}(\prod_{i\in I\setminus\{k_1\}} F_i) \subset g_1 + H'_0. \tag{3.3}$$

Thus, there exists some $g' \in G$ with $\varphi(g') = \overline{g}$ such that

$$\begin{aligned}\mathsf{v}_{g'}(\prod_{i\in I\setminus\{k_1\}} F_i) &\geq \frac{\mathsf{v}_{\overline{g}}(\varphi(\prod_{i\in I\setminus\{k_1\}} F_i))}{|H'_0|} \geq \frac{(|\prod_{i\in I} F_i|/|G/H|) - \mathsf{D}(G/H)}{2t_{\delta_0} - 1} \\ &\geq \frac{v_1 - \mathsf{D}(G/H)|G/H|}{|G/H|(2t_{\delta_0} - 1)}.\end{aligned}$$

Recalling that $v_1 \geq \mathsf{h}(H,k)$ and $t_{\delta_0} \leq \mathsf{ci}(H,k,\delta_0)$, the claim follows (obviously, we can ignore the scenario that the numerator is negative). □

Next, we discuss how this result can be expanded and improved (if more assumptions are imposed).

Remark 3.2. In a more restricted context one can assert that the lengths of most of the sequences F_i are equal to $\exp(G/H)$ (see Lemma 3.7). Thus, the estimate $|\prod_{i\in I} F_i| \geq v_1$ can be improved, almost by a factor of $\exp(G/H)$.

In the important special case $\mathsf{ci}(H,k,\delta_0) = 1$ the following improvement is possible.

Remark 3.3. If $|H_0| = 1$, i.e., $H'_0 = \{0\}$, then we can repeat the argument of Step 4 with k_2 (instead of k_1) as 'distinguished' index, to get that also $\varphi^{-1}(\overline{g}) \cap \operatorname{supp}(F_{k_1}) = \{g_1\}$; note that in this case we know already $g_2 = g_1$. Thus, in this case we get $\mathsf{h}(H,k)$ instead of $\mathsf{h}(H,k) - \mathsf{D}(G/H)|G/H|$ in the numerator of our lower bound for $\mathsf{h}(A)$. Yet, note that then we have to impose some (in our context) mild additional assumption to guarantee the existence of two distinct $k_1, k_2 \in I$ with $\overline{g} \in \operatorname{supp}(F_{k_i})$, e.g., assuming that $\mathsf{h}(H,k) > \mathsf{D}(G/H)|G/H|$ guarantees this.

In Theorem 3.13 we see, on the one hand, that some condition such as $\overline{g} \in \mathrm{supp}(F_{k_i})$ for distinct k_1, k_2 is essential to guarantee that elements with the same image under φ are actually equal or closely related; and on the other hand, that the actual condition can be weakened in that context.

Moreover, not only information on the height of the sequence can be obtained in this way.

Remark 3.4. Inspecting the proof of Theorem 3.1 the following assertions are clear.

1. The assertion made in (3.3) holds for each element $\overline{g} \in G/H$. And, in the situation of Remark 3.3, for each $\overline{g} \in G/H$ with $\mathsf{v}_{\overline{g}}(\varphi(\prod_{i\in I} F_i)) > \mathsf{D}(G/H)$. Thus, we could gain information on all elements of the 'large fibre' with at most $\mathsf{D}(G/H)|G/H|$ exceptions, i.e., a number that just depends on G/H and thus in our context is small.

2. If there is more than one 'large fibre,' i.e., $t > 1$, then we can apply the argument to each of these fibres (yet, note that H_0' depends on the fibre).

Thus, via this method more detailed insight, beyond the height, into the structure of the sequences could be obtained. Indeed, one can expand on the second assertion by noting that the argument can even be expanded to the product of all 'large fibres'; yet, instead of the set H_0' we need to consider the set $H_0 - H_0$, again ignoring slight improvements. Thus, using $|H_0 - H_0| \le |H_0|(|H_0| - 1) + 1$, we see that depending on the relative size of t and t_{δ_0}, this can yield a better or a worse result. And, in case one has detailed knowledge on the structure of long minimal zero-sum sequences over H, it is possible to extend these considerations to fibres corresponding to elements with high yet not maximal multiplicity in C (cf. the proof of Theorem 3.6). Finally, we add that apparently the structure of the set H_0 is relevant too, e.g., since with such knowledge better bounds for $|H_0 - H_0|$ might be obtained, or additional restrictions inferred. However, examples show that without imposing additional restrictions, the structure of H_0 can be drastically different; namely, all elements of H_0 can be independent but they can also form an 'interval' (see Section 3.2), which are both rather extreme examples regarding $|H_0 - H_0|$, yet at opposite ends of the spectrum. Thus, we do not pursue these ideas any further in this general setting; yet, this is considered in our investigations for cyclic H.

Remark 3.5. Somewhat oversimplifying, for certain types of groups G/H the size of $\max \mathsf{L}(\varphi(A))$ (relative to $|A|$) is 'large' if $\mathrm{supp}(\varphi(A))$ is 'large' and conversely. In situations where this is the case one can get improved results via taking this correlation into account, since then one can argue that $\max \mathsf{L}(\varphi(A))$ is not as small as possible (among all sequences $B \in \mathcal{B}(G/H)$ of length $|A|$) or $\mathrm{supp}(\varphi(A))$ is not as large as possible (among all sequences $B' \in \mathcal{B}(G/H)$ of length $|A|$), and each of these has a positive effect on the estimates for the height.

We refer to [22, Theorem 7.1] for a result of this form for C_m^2 and to [49] for an application of it in this context, and to [26, Section 4]. Yet, elementary 2-groups do not have this property and only a minimal improvement could be achieved in this way. Thus, in this case we give a different type of argument that in combination with the above reasoning still allows to assert that for sufficiently long A the support of $\varphi(A)$ is not too large (see Section 3.4).

3.2 On $\mathsf{h}(H, k)$ and $\mathsf{ci}(H, k, \delta)$

Let H be a finite abelian group, $k \in [1, \mathsf{D}(H)]$, and $\delta \in \mathbb{N}_0$. Apparently, the two parameters $\mathsf{h}(H, k)$ and $\mathsf{ci}(H, k, \delta)$ are crucial for the quality of the estimate in Theorem 3.1. We summarize some results on these invariants.

It is clear that $\mathsf{h}(H, k) \le \exp(H)$ and if equality holds then $k = \exp(H)$. Thus, equality holds if and only H is cyclic and $k = |H|$, $\exp(H) = 2$ and $k = 2$, or $\exp(H) = 1$ and $k = 1$. Moreover, for $\delta < \mathsf{h}(H, k)$, we have $\mathsf{ci}(H, k, \delta) \le (\mathsf{D}(H) - \delta)/(\mathsf{h}(H, k) - \delta)$.

Over cyclic groups the structure of long minimal zero-sum sequences is well-understood. A zero-sum sequence B over C_n is said to have index 1 if there exists some generating element $e \in C_n$ and $b_1, \dots b_{|B|} \in [1, n]$

$$\text{with} \quad \sum_{i=1}^{|B|} b_i = n \quad \text{such that} \quad B = \prod_{i=1}^{|B|} (b_i e). \tag{3.4}$$

Each zero-sum sequence of index 1 is a minimal zero-sum sequences, yet the converse is in general not true. However, all long minimal zero-sum sequences have index 1 and recently in [48] and [55] (improving on various earlier results, originating in a result of [8], and see [31] for an overview; and cf. Section 1 for references to further results) the precise threshold-value was determined. Namely, it is known that if A is a minimal zero-sum sequence over C_n and $|A| \ge \lfloor n/2 \rfloor + 2$, then A has index 1, and this bound on the length is best possible (except for $n \in [1,7] \setminus \{6\}$, since in these cases all minimal zero-sum sequences have index 1). From this result one can infer (see the above mentioned papers for details) that for $k \ge (n+3)/2$ we have $\mathsf{h}(C_n, k) \ge (3k - n)/3$ and $\mathsf{ci}(C_n, k, 2) \le 2$, and for $k \ge (2n+3)/3$ we have $\mathsf{h}(C_n, k) = 2k - n$ and $\mathsf{ci}(C_n, k, 2) = 1$. Moreover, for each $A \in \mathcal{A}(C_n)$ with $|A| \ge (n+3)/2$ we have that $\operatorname{supp}_{\mathsf{h}(A)-2} \subset \{e, 2e\}$ for some generating element $e \in C_n$, with the single exception $n = 6$ and $A = e^3(3e)$.

Over non-cyclic groups much less is known on the structure of minimal zero-sum sequences and thus on $\mathsf{h}(H, k)$ and $\mathsf{ci}(H, k, \delta)$; yet, partial results document that these invariants remain relevant beyond the case of cyclic groups. We discuss the present state of knowledge for groups of rank two. We recall that $n \in \mathbb{N}$ is said to have Property **B** if $\mathsf{h}(C_n^2, \mathsf{D}(C_n^2)) = n - 1$. If n has Property **B**, then a short argument yields a full characterization of all minimal zero-sum sequences of maximal length over C_n^2, and it is conjectured that each $n \in \mathbb{N}$ has Property **B** (see, e.g., [23, 22]). By [26] it is known that if each prime divisor of n has Property **B**, then so does n. Thus, in combination with results of [5] it is known that Property **B** holds for each $n \in \mathbb{N}$ that is not divisible by a prime greater than 23. And, by [49] it follows, for $m, n \in \mathbb{N} \setminus \{1\}$, that if m has Property **B**, then $\mathsf{h}(C_m \oplus C_{mn}, \mathsf{D}(C_m \oplus C_{mn})) = \max\{m - 1, n + 1\}$. Also, note that if $n \ge 5$ has Property **B**, then $\mathsf{ci}(C_n^2, \mathsf{D}(C_n^2), 2) = 2$; that 2 is an upper bound follows by the general inequality given above and recall that for independent e_1, e_2 of order n the sequence $e_1^{n-1} e_2^{n-1}(e_1 + e_2)$ is a minimal zero-sum sequence.

Moreover, it is known by [5] that there exists some positive constant $\overline{\delta}$ such that for each (sufficiently large) prime p we have $\mathsf{h}(C_p^2, \mathsf{D}(C_p^2)) \ge \overline{\delta} p$; indeed, it is even known that for each $\varepsilon > 0$ there exists some $\delta_\varepsilon > 0$ such that $\mathsf{h}(C_p^2, k) \ge \delta_\varepsilon p$ for $k \ge (1 + \varepsilon)p$ for all sufficiently large primes p. We point out that for our applications knowledge on $\mathsf{h}(H, k)$ for k (slightly) below $\mathsf{D}(H)$, such as provided by that result is of particular relevance. The class of groups for which, using the notation of Theorem 3.1, there exists some $k \in \mathsf{L}(\varphi(A))$ such that k is close to $\mathsf{D}(H)$ (in a relative sense) is much larger than the class of groups for which such a k with $k = \mathsf{D}(H)$ exists (cf. the discussion at the beginning of this section). Extrapolating from the cyclic case, one can hope that $\mathsf{h}(C_n^2, \mathsf{D}(C_n^2) - \ell) = n - 1 - 2\ell$ for each $\ell \le cn$ for some positive constant c; at least, it seems quite likely that $\mathsf{h}(C_n^2, \mathsf{D}(C_n^2) - \ell)$ is still close to $n - 1$ for sufficiently small $\ell \in \mathbb{N}$.

Additional information on $\mathsf{h}(H, k)$ for k close to $\mathsf{D}(H)$ for groups with large exponent is available via results in [19].

Finally, note that the structure of minimal zero-sum sequences over elementary 2-groups is completely understood, namely A is a minimal zero-sum sequence if and only if $A = (e_1 + \dots + e_s) \prod_{i=1}^{s} e_i$ for independent elements e_i. So, we have $\mathsf{h}(C_2^r, \mathsf{D}(C_2^r)) = 1$ for $r \ge 2$. Hence, we typically cannot (in a meaningful way) apply Theorem 3.1 (or related results) with H an elementary 2-group. Moreover, note that replacing $\mathsf{h}(\cdot)$ and $\mathsf{ci}(\cdot)$ by different parameters describing the structure of minimal zero-sum sequence will not change this. The actual problem is the fact that long minimal zero-sum sequences over elementary 2-groups (and more generally groups with large rank) can be much less rigid than long minimal zero-sum sequences over groups with large exponent. For example, consider a zero-sum free sequence S of length $\mathsf{D}(H) - 2$; if H is cyclic, then S can be extended to a minimal zero-sum sequence in at most two ways, whereas if H is an elementary 2-group of rank $r \geq 2$, then this can be done in $1+2^{r-2}$ ways. Our parameters are merely a way to quantify this phenomenon.

3.3 Groups with large exponent

In this section we obtain refined results on the height of long minimal zero-sum sequences over groups with 'large exponent'. We mainly focus on the case that G has a cyclic subgroup H such that $|H|$ is large relative to $|G/H|$, since in this case precise information on the structure of minimal zero-sum sequences over H is available. Additionally, we consider the case that G has a large subgroup of the form C_p^2 for prime p.

Theorem 3.6. *Let G be a finite abelian group, $\{0\} \neq H \subsetneq G$ be a cyclic subgroup such that* $\exp(G) = \exp(H)\exp(G/H)$.

1. *For each $\ell \in [1, \mathsf{D}(G)]$ with*

$$\ell > \frac{\exp(G/H)}{\exp(G/H)+1}\exp(G) + \mathsf{D}_0'(G/H) + \frac{(|G/H|+1)\,\mathsf{D}(G/H)}{\exp(G/H)+1},$$

   we have

$$\mathsf{h}(G,\ell) > \frac{\exp(G)}{|G/H|} - \frac{(\exp(G/H)+1)}{|G/H|}(\exp(G)-\ell) - (\exp(G/H)+1).$$

2. *Suppose that $|H| \geq 12$. For each $\ell \in [1, \mathsf{D}(G)]$ with*

$$\ell > \frac{\exp(G)}{2} + \mathsf{D}_0'(G/H) + \exp(G/H)\,\mathsf{D}(G/H)|G/H|,$$

   we have

$$\mathsf{h}(G,\ell) \geq \frac{2\exp(G)}{3\exp(G/H)|G/H|} - \frac{\exp(G)-\ell}{\exp(G/H)|G/H|} - \frac{2}{\exp(G/H)}.$$

Note that the trivial bound $\mathsf{D}(G) \geq \exp(G)$ and the fact that $\mathsf{D}_0'(G/H) < \eta(G/H) \leq |G/H|$ (see Section 2) readily implies that ℓ fulfilling the condition actually exist if $\exp(G)$ is 'large' relative to $|G|$ (and H is chosen in a suitable way), yet this is not the case without such a condition. The condition $|H| \geq 12$ is a purely technical condition to avoid corner-cases in the argument; in view of the above assertion, imposing it is almost no loss.

The two statements of the result address orthogonal issues. The aim of the first statement is to establish a good lower bound (see Example 3.8 for some details on the quality of this bound) on the height of fairly long minimal zero-sum sequences over G; however, note that even this statement is valid for sequences of length slightly less than the exponent of G, as usual assuming that the exponent is large. Whereas the aim of the second statement is to establish some bound for considerably shorter sequences. To establish the former statement, we use Lemma 3.7, implementing Remark 3.2 (note that in the lemma we do not require that H is cyclic); to establish the latter one, we basically use Theorem 3.1 in combination with the results on cyclic groups recalled in Section 3.2, and in particular use knowledge on the structure of the set H_0 to improve the result, cf. the discussion after Remark 3.4.

Lemma 3.7. *Let G be a finite abelian group an $H \subset G$ a subgroup. Let $A \in \mathcal{A}(G)$ and $A = F_1 \dots F_k$ such that $\varphi(F_1)\dots\varphi(F_k)$ is a factorization of $\varphi(A)$. Let $I_>$, $I_<$, and $I_=$ denote the subsets of $[1,k]$ such that for i in the respective subset we have $|F_i|$ is greater than, less than, and equal to, resp., the exponent of G/H.*

1. *Then* $\max \mathsf{L}(\prod_{i \in I_> \cup I_=} \varphi(F_i)) + |I_<| \leq \mathsf{D}(H)$. *In particular,* $|I_<| \leq (\mathsf{D}(H)\exp(G/H) + \mathsf{D}_0'(G/H)) - |A|$.

2. If $k = \max \mathsf{L}(\varphi(A))$, then $|\prod_{i \in I_>} \varphi(F_i)| \le \mathsf{D}_{|I_>|}(G/H)$; in particular, $|I_>| \le \mathsf{D}_0'(G/H)$.

In this lemma, we can replace $\mathsf{D}_0'(G/H)$ by $\mathsf{D}_0(G/H)$ for the same groups for which we can do so in (2.3).

Proof. We recall that $\prod_{i=1}^k \sigma(F_i) \in \mathcal{A}(H)$.

1. Let $\ell \in [0,k]$ such that, say, $I_< = [\ell+1,k]$. Let $B = \prod_{i=1}^{\ell} F_i$ and let $B = F_1' \dots F_{\ell'}'$ such that $\varphi(F_1') \dots \varphi(F_{\ell'}')$ is a factorization of $\varphi(B)$ and $\ell' = \max \mathsf{L}(\varphi(B))$. We note that $\prod_{i=1}^{\ell'} \sigma(F_i') \prod_{j=\ell+1}^{k} \sigma(F_i)$ is a minimal zero-sum sequence over H. Thus, $\ell' + (k-\ell) \le \mathsf{D}(H)$, establishing the claim. It remains to assert the additional statement. Since $\max \mathsf{L}(\varphi(B)) \le \mathsf{D}(H) - |I_<|$, it follows by (2.3) that

$$\frac{|\varphi(B)| - \mathsf{D}_0'(G/H)}{\exp(G/H)} \le \mathsf{D}(H) - |I_<|.$$

Noting that $|\varphi(B)| \ge |A| - (\exp(G/H)-1)|I_<|$ and combining the inequalities, the claim follows.

2. If $k = \max \mathsf{L}(\varphi(A))$, then $\max \mathsf{L}(\prod_{i \in I_>} \varphi(F_i)) = |I_>|$, and the claim follows by definition of $\mathsf{D}_{|I_>|}(G/H)$. The additional claim follows by using the upper bound (2.2) for $\mathsf{D}_{|I_>|}(G/H)$ and noting that $|\prod_{i \in I_>} \varphi(F_i)| \ge (\exp(G/H)+1)|I_>|$. □

Of course, this lemma is only relevant if $(\mathsf{D}(H)\exp(G/H) + \mathsf{D}_0'(G/H)) - |A|$ is small. Yet, this is the case, in particular, if H is a large cyclic subgroup with $\exp(G) = \exp(H)\exp(G/H)$ and $|A|$ is not too much smaller than $\mathsf{D}(G)$ (cf. (3.1) and the subsequent discussion).

Proof of Theorem 3.6. Let $\varphi : G \to G/H$ denote the canonical map. Let $\ell \in [1, \mathsf{D}(G)]$ fulfilling the respective condition on its size and let $A \in \mathcal{A}(G)$ with $|A| \ge \ell$. Let $k = \max \mathsf{L}(\varphi(A))$. We note that $k \ge (|A| - \mathsf{D}_0'(G/H))/\exp(G/H)$ (see (2.2)).

1. We note that by our assumption on $|A|$ we have $k \ge (2|H|+3)/3$ and thus $\mathsf{h}(H,k) = 2k - |H|$ and $\mathsf{ci}(H,k,2) = 1$ (see Section 3.2). First, we use the exact same argument as in Steps 1–3 in the proof of Theorem 3.1; we continue using the notation of that proof below. Yet, in Step 4 we estimate $|\prod_{i \in I} F_i|$ in another way. Namely, we note that by Lemma 3.7 at most $(\mathsf{D}(H)\exp(G/H) + \mathsf{D}_0'(G/H)) - |A| = (\exp(G) + \mathsf{D}_0'(G/H)) - |A|$ of the sequences F_i do not have length at least $\exp(G/H)$. Thus, $|\prod_{i \in I} F_i| \ge \exp(G/H)|I| - (\exp(G/H)-1)(\exp(G) + \mathsf{D}_0'(G/H) - |A|)$. Using the fact that $|I| \ge \mathsf{h}(H,k)$ and the assertions made above, we get $|\prod_{i \in I} F_i| \ge (\exp(G/H)+1)(|A| - \mathsf{D}_0'(G/H)) - \exp(G/H)\exp(G)$.

By the assumption on $|A|$, we get $|\prod_{i \in I} F_i|/|G/H| > \mathsf{D}(G/H)$. Thus, as in Step 4 of the proof of Theorem 3.1 and taking Remark 3.3 into account we get

$$\begin{aligned}
\mathsf{h}(A) &\ge \frac{|\prod_{i \in I} F_i|}{|G/H|} \\
&\ge \frac{(\exp(G/H)+1)(|A| - \mathsf{D}_0'(G/H)) - \exp(G/H)\exp(G)}{|G/H|} \\
&= \frac{\exp(G)}{|G/H|} + \frac{(\exp(G/H)+1)(|A| - \exp(G) - \mathsf{D}_0'(G/H))}{|G/H|}.
\end{aligned}$$

Recalling that $\mathsf{D}_0'(G/H) < |G/H|$, the claim follows.

2. Again, we proceed as in the proof of Theorem 3.1 and use the same notation. We note that by our assumption on $|A|$ we have $k \ge (|H|+3)/2$ and thus $\mathsf{h}(H,k) \ge (3k - |H|)/3$ and $\mathsf{ci}(H,k,2) \le 2$ (see Section 3.2). We get $|\prod_{i \in I} F_i| \ge |I| \ge (3k - |H|)/3 > |G/H|\,\mathsf{D}(G/H)$, the last inequality by our assumption on $|A|$. We distinguish two case.

Suppose $t_\delta = 1$. Then it follows that

$$\begin{aligned}\mathsf{h}(A) &\geq |\prod_{i\in I} F_i|/|G/H| \geq (3k - |H|)/(3|G/H|)\\ &\geq \frac{2\exp(G)}{3\exp(G/H)|G/H|} + \frac{|A| - \exp(G) - \mathsf{D}_0'(G/H)}{\exp(G/H)|G/H|}.\end{aligned}$$

Suppose $t_\delta = 2$. As discussed in Section 3.2 we know that $\{h_1, h_2\} = \{e, 2e\}$ for some generating element $e \in H$. Let $j \in \{1, 2\}$ such that $h_j = e$ and $J \subset [1, k]$ such that $\prod_{i\in J} \sigma(F_i) = h_j^{v_j}$. We know that $v_j \geq \mathsf{h}(H, k) - \delta$. By our assumption on $|A|$ and arguing as above we get that $|J| > |G/H|\,\mathsf{D}(G/H)$.

We argue analogously to the beginning of Step 4 in the proof of Theorem 3.1 where $h_j^{v_j}$ has the role of the 'large fiber'. Yet, note that possibly h_j is not the element with maximal multiplicity in $\prod_{i\in I} \sigma(F_i)$ However, since by the results mentioned in Section 3.2 we know that the multiplicity of the element with the third highest multiplicity in this sequence is less than $v_j - 2$, we can still apply this argument (cf. the discussion after Remark 3.4).

We define F'_{k_1} and F'_{k_2} analogously as in that proof. Yet, here we can infer that $\sigma(F'_{k_1}) = \sigma(F'_{k_2}) = e$ has to hold, since otherwise, by the minimality assumption on the v_i and in view of the above remark on the third highest multiplicity, we get that, say, $\sigma(F'_{k_1}) = 2e$ and thus $\sigma(F'_{k_2}) = 0$, which is absurd as A is a minimal zero-sum sequences. Thus, we get

$$\begin{aligned}\mathsf{h}(A) &\geq \frac{\prod_{i\in J} F_i}{|G/H|} \geq \frac{|J|}{|G/H|} \geq \frac{3k - |H| - 3\delta}{3|G/H|}\\ &\geq \frac{2\exp(G)}{3\exp(G/H)|G/H|} + \frac{|A| - \exp(G) - \mathsf{D}_0'(G/H) - 2\exp(G/H)}{\exp(G/H)|G/H|}.\end{aligned}$$

Noting in each case that $\mathsf{D}_0'(G/H) + \exp(G/H) \leq |G/H|$, the claim follows. □

To discuss the quality of our result, we point out the following examples.

Example 3.8. Let $G = G' \oplus \langle f\rangle$ with $\text{ord}(f) = \exp(G)$, and let $\ell \in [\exp(G), \mathsf{D}^*(G)]$. We observe that there exist sequences $S_1, S_2 \in \mathcal{F}(G')$ with $|S_i| = \exp(G)$, $\mathsf{h}(S_i) \leq 1 + \max\{\lfloor \frac{\exp(G)}{|G'|} \rfloor, 1\}$, and $\text{ord}(\sigma(S_1)) = \exp(G')$ and $\sigma(S_2) = 0$. In case $\ell > \exp(G)$, let $T \in \mathcal{F}(G')$ be a zero-sum free sequence with $|T| = \ell - \exp(G)$ and $\sigma(T) = \sigma(S)$, which exists due to the condition on the order of $\sigma(S)$. Then, $T(f + S_1)$ and $(f + S_2)$ are minimal zero-sum sequence over G with length ℓ and $\exp(G)$, respectively, and height at most $\lfloor \frac{\exp(G)}{|G'|} \rfloor + 1$.

Thus, we see that the bound established in Theorem 3.6, for sequence of length in $[\exp(G), \mathsf{D}^*(G)]$, is off by approximately a factor of $\exp(G/H)$ (assuming that $\exp(G)$ is large). In Section 3.4, we improve this bound for groups of the form $C_2^{r-1} \oplus C_{2n}$.

Now, we consider a different type of group. Here, it is crucial that we can deal with the situation that minimal zero-sum sequences over the subgroup H can contain more than one important element.

Theorem 3.9. *Let $n_1, n_2 \in \mathbb{N}$ with $n_1 \mid n_2$ and let p be a prime. Let $G = G' \oplus C_{n_1 p} \oplus C_{n_2 p}$ with $\exp(G') \mid n_1$ and let $K = G' \oplus C_{n_1} \oplus C_{n_2}$. For each positive ε there exist positive δ', δ'' (depending only on ε) such that if p is sufficiently large (depending on ε and K), then for each $\ell \in [1, \mathsf{D}(G)]$ with $\ell \geq (1 + \varepsilon)\exp(G) + \mathsf{D}_0'(K)$ we have*

$$\mathsf{h}(G, \ell) \geq \frac{\delta' \exp(G)}{\exp(K)|K|} - \delta''\,\mathsf{D}(K).$$

Note that since $\mathsf{D}(G) \geq (n_1 + n_2)p - 1$ elements ℓ fulfilling our conditions actually exist for $\varepsilon < n_1/n_2$ (and sufficiently large p).

Proof. Let H be a subgroup of G isomorphic to C_p^2 such that $G/H \cong K$ and let $\varphi : G \to G/H$ denote the canonical map. Let $\varepsilon > 0$ and let $\ell \in [1, \mathsf{D}(G)]$ fulfilling the assumption on it size. Let $A \in \mathcal{A}(G)$ with $|A| \ge \ell$ and let $k = \max \mathsf{L}(\varphi(A))$.

By (2.3), we know that $k \ge (|A| - \mathsf{D}_0'(K))/\exp(K) \ge (1+\varepsilon)p$. We apply Theorem 3.1, to get that (we assume $p > 2$)

$$\mathsf{h}(A) \ge \frac{\mathsf{h}(C_p^2, k) - \mathsf{D}(K)|K|}{(2\,\mathsf{ci}(C_p^2, k, 1) - 1)|K|}.$$

As recalled in Section 3.2, by [5], there exists some δ (depending on ε only) such that if p is sufficiently large, then $\mathsf{h}(H, k) \ge \delta p$. Moreover, we get that $\mathsf{ci}(C_p^2, k, 1) \le (2p-1)/(\delta p - 1) \le c/\delta$ for any $c > 2$ and sufficiently large p. So, we have (assuming p is sufficiently large that the numerator is positive)

$$\mathsf{h}(A) \ge \frac{\delta p - \mathsf{D}(K)|K|}{(2c/\delta - 1)|K|} = \frac{(\delta p - \mathsf{D}(K)|K|)\delta/(2c)}{|K|} = \frac{\delta^2 p/(2c)}{|K|} - \delta\,\mathsf{D}(K)/(2c).$$

Setting $\delta' = \delta^2/(2c)$ and $\delta'' = \delta/(2c)$, the claim follows. □

From the proof it readily follows that we can choose for δ' any value that is less than $\delta^2/4$ where δ has to fulfil $\mathsf{h}(C_p^2, k) \ge \delta p$ for $k \ge (1+\varepsilon)p$, and likewise for δ'' any value less than $\delta/4$. Presently, $\mathsf{h}(C_p^2, k) \ge \delta p$ is only known to hold for very small δ even for $k = \mathsf{D}(C_p^2)$, and thus our result is presently only interesting from a qualitative point of view; thus, we directly applied Theorem 3.1 and, e.g., disregarded Lemma 3.7. Yet, as discussed in Section 3.2 it is fairly likely that for k close to $\mathsf{D}(C_p^2)$ the value of $\mathsf{h}(C_p^2, k)$ is actually close to $p-1$, i.e., δ is close to 1. Recall that for $n_1 = n_2$ and, say, $|A| = \mathsf{D}^*(G)$, the difference $\mathsf{D}(C_p^2) - \max \mathsf{L}(\varphi(A))$ is bounded above by a value independent of p.

3.4 Groups of the form $C_2^{r-1} \oplus C_{2n}$

We improve the estimate for $\mathsf{h}(G, k)$ obtained in Theorem 3.13 for G of the form $C_2^{r-1} \oplus C_{2n}$ with $r, n \in \mathbb{N}$. We see in Corollary 3.12 that for $k \in [\exp(G), \mathsf{D}^*(G)]$ our result is optimal up to an absolute constant.

Theorem 3.10. *Let $r, n \in \mathbb{N}$ with $n \ge 8$ and $G = C_2^{r-1} \oplus C_{2n}$. For each $\ell \in [1, \mathsf{D}(G)]$ with $\ell \ge 2\exp(G)/3 + 2 + \mathsf{D}_0(C_2^r)$, we have*

$$\mathsf{h}(G, \ell) > \frac{\exp(G)}{2^{r-1}} - \frac{\exp(G) - |A|}{2^{r-3}} - 6.$$

Again, the result is only relevant if n is large relative to r, and it is thus essentially no loss, yet helpful in the proof, to impose the condition $n \ge 8$. The key to this improvement is to apply the following observation. Additionally, we can perform certain estimates in a more precise way, since in this case more is known on $\mathsf{D}_k(G/H)$ than in the general case.

Lemma 3.11. *Let $r, n \in \mathbb{N}$, $G = C_2^{r-1} \oplus C_{2n}$, and let $H \subset G$ be a cyclic subgroup of order n such that $G/H \cong C_2^r$. Let $T \in \mathcal{F}(G)$ such that there exists some $e \in H$ with $2g = e$ for each $g \mid T$. If $F \mid T$ such that $\sigma(F) \in H$, then,*

1. *in case n is even, $|F|$ is even and $\sigma(F) \in \{\frac{|F|}{2}e, \frac{|F|+n}{2}e\}$.*
2. *in case n is odd, $\sigma(F) = \frac{|F|}{2}e$ if $|F|$ is even, and $\sigma(F) = \frac{|F|+n}{2}e$ if $|F|$ is odd.*

Proof. Let $F \mid T$ such that $\sigma(F) \in H$. We consider $\sigma(F^2)$. We note, since $2g = e$ for each $g \mid T$, that $\sigma(F^2) = |F|e$. Thus, $2\sigma(F) = |F|e$, and the claim follows. □

Clearly, analogues of this lemma hold for more general classes of groups. Yet, their application to our problem would be less direct, and we thus restrict to considering this special case.

Proof of Theorem 3.10. Let $H \subset G$ be a cyclic subgroup of order n such that $G/H \cong C_2^r$, and let $\varphi : G \to G/H$ denote the canonical map. Let $\ell \in [1, \mathsf{D}(G)]$ fulfilling the condition on the size, and let $A \in \mathcal{A}(G)$ with $|A| \ge \ell$. Let $k = \max \mathsf{L}(\varphi(A))$. We note that $k \ge (|A| - \mathsf{D}_0(C_2^r))/2$ (as discussed in Section 2, we can use here and below $\mathsf{D}_0(\cdot)$ instead of $\mathsf{D}_0'(\cdot)$, since G/H is an elementary 2-group). In particular, $k \ge (2n+3)/3$. Thus, $v_1 \ge 2k - n$ and $\mathsf{ci}(H, k, 2) = 1$. Again, we proceed as in the proof of Theorem 3.1 and use the same notation. We note that by Lemma 3.7, with $I_<$, $I_>$, and $I_=$ as defined there, we get that $|I_<| \le 2n + \mathsf{D}_0(C_2^r) - |A|$ and $|I_>| \le \mathsf{D}_0(C_2^r)$. Thus, all except at most $2n + 2\,\mathsf{D}_0(C_2^r) - |A|$ of the sequences F_i have length 2, i.e., $\varphi(F_i) = f^2$ for some $f \in G/H \setminus \{0\}$. Let $I' = I \cap I_=$, i.e., the maximal subset of I such that $|F_i| = 2$ for each $i \in I'$. We note that $|I'| \ge 2|A| - 3n - 3\,\mathsf{D}_0(C_2^r)$. We assert that $\varphi(\operatorname{supp}(\prod_{i \in I'} F_i))$ is sumfree, i.e., the equation $x + y = z$ has no solution in that set. Assume to the contrary, there exist f_1, f_2, f_3 such that $f_1 + f_2 = f_3$. Since $0 \notin \varphi(\operatorname{supp}(\prod_{i \in I'} F_i))$, it follows that f_1, f_2, f_3 are pairwise distinct. Let $j_1, j_2, j_3 \in I'$ such that $\varphi(F_{j_i}) = f_i^2$ for $i \in [1,3]$. We apply Lemma 3.11 with $f_1 f_2 f_3 \mid \prod_{i \in I'} F_i$. It follows that n is odd and $\sigma(f_1 f_2 f_3) = \frac{n+3}{2} h_1$. Yet, this is impossible since $(\frac{n+3}{2} h_1)^2 (\prod_{i \in [1,k] \setminus \{j_1, j_2, j_3\}} \sigma(F_i))$ has length at least $(n+3)/2$, recall $n \ge 9$, but does not have index 1 (cf. Section 3.2); this is obvious with respect to the generating element h_1, yet is also true with respect to each other generating element.

Thus $\varphi(\operatorname{supp}(\prod_{i \in I'} F_i))$ is sumfree. Since the maximal cardinality of a sumfree subset of C_2^r is $|C_2^r|/2$, we get that there exists some $\overline{g} \in G/H$ such that $\mathsf{v}_{\overline{g}}(\varphi(\prod_{i \in I'} F_i)) \ge |\prod_{i \in I'} F_i| / (|G/H|/2)$. Hence, as in Step 4 of the proof Theorem 3.1, and cf. Remark 3.3 we get (now, at first, we consider again the full 'large fibre'),

$$\begin{aligned} \mathsf{h}(A) \ge \mathsf{v}_{\overline{g}}(\varphi(\prod_{i \in I} F_i)) &\ge \frac{|\prod_{i \in I'} F_i|}{|G/H|/2} = \frac{2|I'|}{|G/H|/2} \\ &\ge \frac{4(2|A| - 3n - 3\,\mathsf{D}_0(C_2^r))}{|G/H|} \\ &= \frac{\exp(G)}{2^{r-1}} + \frac{|A| - \exp(G)}{2^{r-3}} - \frac{12\,\mathsf{D}_0(C_2^r)}{2^r}. \end{aligned}$$

Recalling that $\mathsf{D}_0(C_2^r) < 2^{r-1}$ (see Section 2), the claim follows. □

We now assert that Theorem 3.10 is quite precise.

Corollary 3.12. *We have*

$$\mathsf{h}(C_2^{r-1} \oplus C_{2n}, k) = \frac{n}{2^{r-2}} + O(1)$$

for $n, r \in \mathbb{N}$ and $k \in [2n, 2n + r - 1]$.

Proof. We may assume $n \ge 8$. On the one hand, by Example 3.8 we know that $\mathsf{h}(C_2^{r-1} \oplus C_{2n}, k) \le \max\{\lfloor \frac{n}{2^{r-2}} \rfloor + 1, 2\}$ for $k \in [2n, 2n + r - 1]$. On the other hand, by Theorem 3.10 we know that if $2n \ge \frac{2}{3} 2n + 2 + \mathsf{D}_0(C_2^r)$, then $\mathsf{h}(C_2^{r-1} \oplus C_{2n}, k) > \frac{2n}{2^{r-1}} - 6$ for $k \in [2n, 2n + r - 1]$. Yet, if $2n < \frac{2}{3} 2n + 2 + \mathsf{D}_0(C_2^r)$, then $\frac{2n}{3} < \mathsf{D}_0(C_2^r) < 2^{r-1}$, implying that $\max\{\lfloor \frac{n}{2^{r-2}} \rfloor + 1, 2\} \le 3$, which in combination with the trivial lower bound $\mathsf{h}(C_2^{r-1} \oplus C_{2n}, k) \ge 1$ implies the claim. □

Indeed, inspecting the proof and using the trivial lower bound of 1 for the height for $n \le 7$, we see that $0 \le \max\{\lfloor \frac{n}{2^{r-2}} \rfloor + 1, 2\} - \mathsf{h}(C_2^{r-1} \oplus C_{2n}, k) \le 7$. Recalling for $n \le 7$ the results of Section 3.2 for $r \le 2$, this bound can be improved to 6 and using that $\frac{12\,\mathsf{D}_0(C_2^r)}{2^r} = 4 + o(1)$ (instead of using the estimate 6), a further slight improvement for large r would be possible; the latter is the case for Theorem 3.10 as well.

We end by pointing out two related facts. By (3.2) we know that for each r there exist infinitely many n such that $\mathsf{D}(C_2^{r-1} \oplus C_{2n}) = \mathsf{D}^*(C_2^{r-1} \oplus C_{2n})$, namely all n divisible by a sufficiently high power of 2. For these n, our result provides a quite satisfactory answer, since it addresses the structure of all sufficiently long minimal zero-sum sequences. Yet, for example, if $r \geq 5$ and n is odd, then $\mathsf{D}(C_2^{r-1} \oplus C_{2n}) > \mathsf{D}^*(C_2^{r-1} \oplus C_{2n})$ (see Section 1) and thus though Theorem 3.10 also yields a lower bound on the height of sequences of length greater than $\mathsf{D}^*(C_2^{r-1} \oplus C_{2n})$ we cannot apply Example 3.8 to get an upper bound for the height of these sequences. Indeed, it might well be the case that the structure of these exceptionally long sequences is more restricted and thus they have a larger height. The author considers the question whether this is the case or not to be an interesting one, which however will not be pursued here. Yet, he hopes (and believes) that some insight on it can be obtained, based on the thus presented methods and the very recent results of [17] that are in part motivated by this problem.

3.5 Groups of the form $C_2^2 \oplus C_{2n}$

Using the methods and results outlined in the preceding sections and some ad hoc arguments, we derive an explicit description of the structure of minimal zero-sum sequences of maximal length over $C_2^2 \oplus C_{2n}$. As mentioned in Section 1 $\mathsf{D}(C_2^2 \oplus C_{2n}) = 2n+2$ is well-known; yet, since it causes essentially no additional effort, we formulate our proof in such a way that it does not make use of this fact, and thus contains a proof of this result as well.

Theorem 3.13. *Let $n \in \mathbb{N}$ and $G = C_2^2 \oplus C_{2n}$. Then $A \in \mathcal{F}(G)$ is a minimal zero-sum sequence of length $\mathsf{D}(G)$ if and only if there exists a basis $\{f_1, f_2, f_3\}$ of G, where $\operatorname{ord}(f_1) = \operatorname{ord}(f_2) = 2$ and $\operatorname{ord}(f_3) = 2n$, such that A is equal to one of the following sequences:*

1. $f_3^{v_3}(f_3 + f_2)^{v_2}(f_3 + f_1)^{v_1}(-f_3 + f_2 + f_1)$ *with $v_i \in \mathbb{N}$ odd $v_3 \geq v_2 \geq v_1$ and $v_3 + v_2 + v_1 = 2n + 1$.*
2. $f_3^{v_3}(f_3 + f_2)^{v_2}(af_3 + f_1)(-af_3 + f_2 + f_1)$ *with $v_2, v_3 \in \mathbb{N}$ odd $v_3 \geq v_2$ and $v_2 + v_3 = 2n$ and $a \in [2, n-1]$.*
3. $f_3^{2n-1}(af_3 + f_2)(bf_3 + f_1)(cf_3 + f_2 + f_1)$ *with $a + b + c = 2n + 1$ where $a \leq b \leq c$, and $a, b \in [2, n-1]$, $c \in [2, 2n-3] \setminus \{n, n+1\}$.*
4. $f_3^{2n-1-2v}(f_3 + f_2)^{2v}f_2(af_3 + f_1)((1-a)f_3 + f_2 + f_1)$ *with $v \in [0, n-1]$ and $a \in [2, n-1]$.*
5. $f_3^{2n-2}(af_3 + f_2)((1-a)f_3 + f_2)(bf_3 + f_1)((1-b)f_3 + f_1)$ *with $a, b \in [2, n-1]$ and $a \geq b$.*
6. $\prod_{i=1}^{2n}(f_3 + d_i)f_2f_1$ *where $S = \prod_{i=1}^{2n} d_i \in \mathcal{F}(\langle f_1, f_2 \rangle)$ with $\sigma(S) = f_1 + f_2$.*

Introducing more redundancy in the classification of the sequences, we could relax the conditions on the parameters a, b and v, v_i in the above description; however, the parity of the v_i is crucial. Yet, besides avoiding redundancy, to have these restrictive conditions is convenient when applying this result (see Section 4). We point out that there is still some redundancy in this classification, e.g., since we do not restrict the sequences S in 6., which however could be avoided easily at the expense of an even longer classification. Moreover, the case $n = 1$ is included for the sake of completeness only; it is of course well-known.

Proof of Theorem 3.13. For $n = 1$ the claim is well-known and simple (cf. the discussion at the end of Section 3.2). We assume $n \geq 2$. It is clear that all the listed sequences have length $2n + 2$ and have sum 0. First, we show that they are indeed minimal zero-sum sequences. We only address the case that the sequence is of the form given in 1. and 2. as example, the other cases are fairly analogous; and for 6. also see Example 3.8. For $i \in [1, 3]$, let $\pi_i : G \to \langle f_i \rangle$ denote the projection with respect to the basis $\{f_1, f_2, f_3\}$. Let A be of the form given in 1., and let $1 \neq U \mid A$ a zero-sum sequence. If $(-f_3 + f_2 + f_1) \nmid U$, then $2 \mid \mathsf{v}_{f_3+f_i}(U)$ for $i \in \{1, 2\}$, since otherwise $\sigma(\pi_i(U)) \neq 0$. Yet, this implies $\mathsf{v}_{f_3+f_1}(U) + \mathsf{v}_{f_3+f_2}(U) + \mathsf{v}_{f_3}(U) < 2n$, and thus $\sigma(\pi_3(U)) \neq 0$, a contradiction. Thus, suppose $(-f_3 + f_2 + f_1) \mid U$. Then, then

$2 \nmid \mathsf{v}_{f_3+f_i}(U)$ for $i \in \{1,2\}$. Thus, $\sigma(\pi_3(U)) = 0$, implies $\mathsf{v}_{f_3+f_1}(U) + \mathsf{v}_{f_3+f_2}(U) + \mathsf{v}_{f_3}(U) = 2n+1$, i.e., $U = A$.

Let A be of the form given in 2., and let $1 \neq U \mid A$ a zero-sum sequence. First, suppose $(af_3 + f_1)(-af_3+f_2+f_1) \mid U$. Then $(f_3+f_2) \mid U$, since otherwise $\sigma(\pi_2(U)) \neq 0$. Thus $\mathsf{v}_{f_3+f_2}(U)+\mathsf{v}_{f_3}(U) = 2n$, i.e., $U = A$. Second, suppose $(af_3 + f_1)(-af_3 + f_2 + f_1) \nmid U$. If $(af_3 + f_1) \mid U$ or $(-af_3 + f_2 + f_1) \mid U$, then $(af_3 + f_1)(-af_3 + f_2 + f_1) \mid U$, since otherwise $\sigma(\pi_1(U)) \neq 0$. So, we have $U = f_3^{w_3}(f_3 + f_2)^{w_2}$. We note that $2 \mid w_2$. Yet, this implies $\mathsf{v}_{f_3+f_2}(U) + \mathsf{v}_{f_3}(U) < 2n$, a contradiction.

Thus, to complete the proof our result it remains to show that each minimal zero-sum sequences of maximal lengths over G is indeed of the form given in 1. to 6., in particular we have to show that its length is $2n + 2$.

Let H be a subgroup of G isomorphic to C_n such that $G/H \cong C_2^3$ and let $\varphi : G \to G/H$ denote canonical map. Let $A \in \mathcal{A}(G)$ with $|A| = \mathsf{D}(G)$. By (2.1), or the above argument, we have $|A| \geq 2n + 2$. Conversely, by (3.1) and the result on $\mathsf{D}_k(C_2^3)$ recalled in Section 2, we have $|A| \leq 2n + 3$.

We start by investigating the structure of $B = \varphi(A)$. By (2.3) and $\mathsf{D}_0(C_2^3) = 3$ we get that $\max \mathsf{L}(B) = n$. Let $B = S_1 \dots S_k T_1 \dots T_\ell$ be a factorization, where the S_i denote the short minimal zero-sum sequence and the, possibly empty, zero-sum sequence $T = T_1 \dots T_\ell$ is not divisible by a short zero-sum sequence. We have that T is squarefree and $0 \nmid T$. Note that since $|T| \leq 7$, we get $k + \ell = n$. Moreover, let $A = F_1 \dots F_k R_1 \dots R_\ell$ such that $\varphi(F_i) = S_i$ and $\varphi(R_j) = T_j$; furthermore set $F = F_1 \dots F_k$ and $R = R_1 \dots R_\ell$.

Since $n \geq k \geq (|B| - |T|)/2$, we have $|T| \neq 0$, and thus in fact $n-1 \geq k \geq (|B| - |T|)/2$. This implies that $|T| \geq 4$ and so $|T| \in \{4, 7\}$, since there are no squarefree zero-sum sequences of length 5 or 6 over C_2^3 that do not contain 0. Additionally, note that if $|A| = 2n + 3$, then $|T| = 7$.

We assert that $0 \nmid B$, i.e., $|S_i| = 2$ for each i, and that $|A| = 2n + 2$, i.e., $\mathsf{D}(G) = 2n + 2$. Suppose that $0 \mid B$. By Lemma 3.7 we get that $|A| = 2n + 2$ and $\mathsf{v}_0(B) = 1$. Moreover, we have $n - 2 \geq k - 1 \geq (|B| - 1 - |T|)/2$ and thus $|T| = 7$.

Thus, if $0 \mid B$ or $|A| = 2n + 3$, then $|T| = 7$. We assume that $|T| = 7$, i.e., $\operatorname{supp}(T) = G/H \setminus \{0\}$.

We observe that $\sigma(F_1) \dots \sigma(F_{n-2})\, \sigma(R_1)\, \sigma(R_2) = g^n$ for some $g \in H$ with $H = \langle g \rangle$ (see Section 3.2). We use the following notation. Let $R = \prod_{\emptyset \neq I \subset \{1,2,3\}} g_I$ where $\varphi(g_I) = \sum_{i \in I} e_i$ and $\{e_1, e_2, e_3\}$ is a basis of G/H; yet, we write g_i instead of $g_{\{i\}}$ for $i \in \{1, 2, 3\}$. In the same way we see that if $R = R_1' R_2'$ with non-empty R_i' such that $\sigma(R_i') \in H$, i.e., $\sigma(\varphi(R_i')) = 0$, then $\sigma(R_i') = g$. Consequently, $g_{\{1,2,3\}} + \sum_{i=1}^3 g_i = g_{\{i,j\}} + g_k + g_{\{1,2,3\}}$ for $\{i, j, k\} = \{1, 2, 3\}$. Thus, $g_{\{i,j\}} = g_i + g_j$. Moreover, $g_i + g_j + g_{\{i,j\}} = g$ and thus $2g_{\{i,j\}} = g$. Yet, $g_{\{1,2\}} + g_{\{1,3\}} + g_{\{2,3\}} = g$ as well. This implies that $3g = 2g$, a contradiction.

Consequently, we have $|A| = 2n + 2$ and $0 \nmid B$. Moreover, $|T| = 4$ and T is a minimal zero-sum sequence; in particular, $k = n - 1$ and $\ell = 1$. Note that for each $T' \mid T$ of length 3 the set $\operatorname{supp}(T')$ is a basis of G/H.

Again, we have $\sigma(F_1) \dots \sigma(F_{n-1})\, \sigma(R) = g^n$ for some generating element g of H. For convenience of notation we set $F_n = R$.

Next, we show that if $\varphi(h) = \varphi(h')$ for $hh' \mid A$ then $h = h'$. First, suppose h and h' occur in distinct subsequences, i.e., $h \mid F_i$ and $h' \mid F_j$ for $i \neq j$. In this case the assertion follows as in Step 4 of the proof of Theorem 3.1.

Now, suppose $hh' \mid F_i$ for some i. We note that $i \neq n$, say $i = n - 1$. There exists some $U \mid F_n$ such that $\sigma(\varphi(U)) = -\varphi(h)$. Let $U' = U^{-1} F_n$. Then $\sigma(\varphi(U')) = \sigma(\varphi(U))$. Thus, we consider $F_{n-1}' = hU$ and $F_n' = h'U'$ as well as $F_{n-1}'' = h'U$ and $F_n'' = hU'$. As above, we get $\sigma(F_{n-1}') = \sigma(F_n') = g$ and $\sigma(F_{n-1}'') = \sigma(F_n'') = g$. Thus, $\sigma(F_n') = \sigma(F_n'')$ and the claim follows.

We point out two consequences of the above reasoning.

C1 The elements in $\operatorname{supp}(R)$ occur with odd multiplicity in A and the multiplicities of all other elements are even. Thus, the decomposition $A = FR$ is unique. Moreover, the decomposition $F = F_1 \dots F_{n-1}$ is unique (up to ordering) as well.

C2 For each $h \in \text{supp}(F)$ we have $\text{ord}(2h) = n$ and, since $\varphi(h) \neq 0$, the order of h is even. Thus $\text{ord}(h) = 2n$. Moreover, there exists some generating element $g \in H$ such that we have, for each i, $\sigma(F_i) = g$ and $\sigma(R) = g$.

In a similar way we establish the following additional facts, which we use frequently in the remainder of the proof.

F1 If $\varphi(h_0) = \varphi(h_1) + \varphi(h_2)$ with $h_0 \mid F$ and $h_1 h_2 \mid R$, then $h_0 = h_1 + h_2$.

F2 $\text{supp}(\varphi(F))$ is sumfree, i.e., the equation $x + y = z$ has no solution in $\text{supp}(\varphi(F))$.

F3 For each $h \in \text{supp}(F) \cap \text{supp}(R)$ we have $h = \sigma(h^{-1}R)$ and moreover for each $R' \mid R$ with $|R'| = 3$ and $h \mid R'$ we have $G = \langle \text{supp}(R') \rangle$.

Ad F1. Suppose $\varphi(h_0) = \varphi(h_1) + \varphi(h_2)$ with $h_0 \mid F$ and $h_1 h_2 \mid R$, say $h_0 \mid F_{n-1}$, i.e., $h_0^2 = F_{n-1}$. Let $h_3 h_4 = (h_1 h_2)^{-1} R$. We note that $\varphi(h_1) + \varphi(h_2) = \varphi(h_3) + \varphi(h_4)$. We set $F'_{n-1} = h_0 h_1 h_2$ and $F'_n = h_0 h_3 h_4$. Then $\sigma(F_1) \dots \sigma(F_{n-2})\, \sigma(F'_{n-1})\, \sigma(F'_n) = g^n$. In particular, $\sigma(F'_n) = \sigma(R)$ and thus $h_0 = h_1 + h_2$.

Ad F2. Compare Lemma 3.11.

Ad F3. Suppose $h \in \text{supp}(F)$. Then $h^2 \mid F$ and we thus have $2h = g = \sigma(R)$, implying the first part of the claim. Now, let $h \mid R' \mid R$ where $|R'| = 3$, and let $h' \mid R$ such that $R = R'h'$. We have $h' = \sigma(R) - \sigma(R') = 2h - \sigma(R') \in \langle \text{supp}(R') \rangle$. Thus, $\text{supp}(R) \subset \langle \text{supp}(R') \rangle$. Moreover, each non-zero element of G/H is the sum of two distinct elements of $\text{supp}(\varphi(R))$, implying by F1, that $\text{supp}(F) \subset \text{supp}(R) + \text{supp}(R) \subset \langle \text{supp}(R') \rangle$. Recalling that $\text{supp}(A)$ is a generating set of G (see Section 2), the claim follows.

Having established these facts we start the detailed investigation of the sequence A. We distinguish several case according to the number of elements in $\text{supp}(F) \cap \text{supp}(R)$. Let $N = |\text{supp}(F) \cap \text{supp}(R)|$. Note that in case $n = 2$ we have $|\text{supp}(F)| = 1$ and thus $N \leq 1$.

Suppose $N = 4$. By this assumption we have $R^2 \mid F$. By C2, on the one hand $\sigma(R^2) = \sigma(F_{i_1}) + \sigma(F_{i_2}) + \sigma(F_{i_3}) + \sigma(F_{i_4}) = |R|g = 4g$, yet on the other hand $\sigma(R^2) = 2\,\sigma(R) = 2g$, a contradiction. (Also, compare Lemma 3.11.)

Suppose $N = 3$. Let $g_1 g_2 g_3 = \gcd(F, R)$ such that $\mathsf{v}_{g_3}(A) \geq \mathsf{v}_{g_2}(A) \geq \mathsf{v}_{g_1}(A)$ and $g_{\{1,2,3\}} = \gcd(F, R)^{-1} R$. Moreover, by F2 (and F1) and since by assumption $g_{\{1,2,3\}} \nmid F$, we know that $\text{supp}(F) = \{g_1, g_2, g_3\}$. We set $f_3 = g_3$ and $f_2 = g_2 - g_3$, $f_1 = g_1 - g_3$. Since $2g_i = g$ for each $i \in \{1, 2, 3\}$, we have $\text{ord}(f_1) = \text{ord}(f_2) = 2$. Moreover, by F2 $\text{ord}(f_3) = 2n$ and by F3 it follows that $\{f_1, f_2, f_3\}$ is a generating set of G and, due to the orders of the elements (see the remark in Section 2), a basis. Recalling that by F3 we have $g_{\{1,2,3\}} = g_3 - g_2 - g_1$, we get

$$A = f_3^{v_3} (f_3 + f_2)^{v_2} (f_3 + f_1)^{v_1} (-f_3 + f_2 + f_1),$$

where $v_3 \geq v_2 \geq v_1$ by assumption and each v_i is odd by C1. Thus, A is of the form given in 1.

Suppose $N = 2$. Let $g_2 g_3 = \gcd(F, R)$ and $g_1 g_{\{1,2,3\}} = \gcd(F, R)^{-1} R$. If there exists some $g' \in \text{supp}(F) \setminus \{g_2, g_3\}$, then, by F2, $\varphi(g') \neq \varphi(g_2) + \varphi(g_3)$. Thus, $\varphi(g') = \varphi(g_i) + \varphi(g_J)$ with $i \in \{2, 3\}$ and $J \in \{1, \{1, 2, 3\}\}$. Without restriction we assume that, in case $\text{supp}(F) \setminus \{g_2, g_3\} \neq \emptyset$, this set contains an element $g_{\{1,3\}}$ with $\varphi(g_{\{1,3\}}) = \varphi(g_1) + \varphi(g_3)$. By F1 we have $g_{\{1,3\}} = g_1 + g_3$.

Similarly as above, we set $f_3 = g_3$ and $f_2 = g_2 - g_3$. Since $2g_3 = 2g_2$, we have $\text{ord}(f_2) = 2$, and again $g_{\{1,2,3\}} = g_3 - g_2 - g_1$. There exists some $a \in [0, n-1]$ such that the order of $g_1 - af_3 = f_1$ is two (note that it cannot be one). Again, the set $\{f_1, f_2, f_3\}$ is a generating set for G and thus a basis.

If $|\text{supp}(F)| = 2$, then

$$A = f_3^{v_3} (f_3 + f_2)^{v_2} (af_3 + f_1)(-af_3 + f_2 + f_1)$$

where again $v_i \geq 3$ is odd. Possibly changing the basis, we obtain $v_3 \geq v_2$. We note that in case $a = 0$ or $a = 1$ the sequence is of the form given in 6. and 1., resp., and otherwise it is of the form given in 2.

Now, suppose $|\operatorname{supp}(F)| = 3$. By assumption, the third element in $\operatorname{supp}(F)$ is $g_{\{1,3\}} = g_1 + g_3$. Moreover since $2g_{\{1,3\}} = 2g_3$, it follows that $2g_1 = 0$ and thus $a = 0$. Therefore,

$$A = f_3^{v_3}(f_3 + f_2)^{v_2}(f_3 + f_1)^{v_1} f_1(f_2 + f_1)$$

where $v_2, v_3 \geq 3$ odd, and $v_1 \geq 2$ even. Thus, the sequence is, after change of basis, of the form given in 6.

Finally, if $|\operatorname{supp}(F)| = 4$, then again by assumption $g_{\{1,3\}} \in \operatorname{supp}(F)$ and as above we get that the fourth element in $\operatorname{supp}(F)$ is equal to $g_1 + g_2$, that is

$$A = f_3^{v_3}(f_3 + f_2)^{v_2}(f_3 + f_1)^{v_1}(f_3 + f_2 + f_1)^{v_4} f_1(f_2 + f_1)$$

$v_2, v_3 \geq 3$ odd, and $v_1, v_4 \geq 2$ even. Thus again the sequence is, after change of basis, of the form given in 6.

Suppose $N = 1$. Let $g_3 = \gcd(F, R)$. We know that each element of $\operatorname{supp}(F) \setminus \{g_3\}$ is the sum of two distinct elements of $\operatorname{supp}(R)$, in fact it is the sum of g_3 and some other element. If $|\operatorname{supp}(F)| \geq 2$, then let $g_2 \mid g_3^{-1}R$ such that $g_{\{2,3\}} = g_2 + g_3 \in \operatorname{supp}(F)$ and if $|\operatorname{supp}(F)| = 3$, then let additionally $g_1 \mid (g_2g_3)^{-1}R$ such that $g_{\{1,3\}} = g_1 + g_3 \in \operatorname{supp}(F)$. Note that by F2 we have $|\operatorname{supp}(F)| \leq 3$. We denote the remaining element(s) in $\operatorname{supp}(R)$ by $g_1, g_2, g_{\{1,2,3\}}$; $g_1, g_{\{1,2,3\}}$; or $g_{\{1,2,3\}}$, respectively.

Let $f_3 = g_3$. As above there exist $a, b \in [0, n-1]$ such that the order of $g_2 - af_3 = f_2$ and of $g_1 - bf_3 = f_1$ are two. The set $\{f_1, f_2, f_3\}$ is a basis of G. Again, by F3 we have $g_3 = g_1 + g_2 + g_{\{1,2,3\}}$. Thus, if $|\operatorname{supp}(F)| = 1$, then

$$A = f_3^{2n-1}(af_3 + f_2)(bf_3 + f_1)(cf_3 + f_2 + f_1)$$

where $c \in [0, 2n-1]$ and $(a+b+c)f_3 = f_3$. Possibly changing the basis, we obtain $a \leq b \leq c$. To show that the sequence is of the form 3., it remains to discuss some special cases. If $a = b = 0$, then the sequence is of the form given in 6. If $a = 0$ and $b \geq 2$ (note that $a = 0$ and $b = 1$ is impossible), it is of the form 4. If $a = b = 1$, then it is of the form 1. If $a = 1$ and $b \geq 2$, then it is if the form 2. It remains to consider the case $a \geq 2$; note that this implies $a + b + c = 2n + 1$. If $c = n$ or $c = n + 1$, then we get that the sequence is of the form given in 4. and 2., resp., with respect to the basis $\{f_1' = f_2, f_2' = nf_3 + f_2 + f_1, f_2, f_3' = f_3\}$.

Suppose that $|\operatorname{supp}(F)| \geq 2$. Since $2g_{\{3,2\}} = 2g_3$, we have $\operatorname{ord}(g_2) = 2$, that is $a = 0$. If $|\operatorname{supp}(F)| = 2$, we thus have

$$A = f_3^{2n-1-2v}(f_3 + f_2)^{2v} f_2(bf_3 + f_1)(cf_3 + f_2 + f_1)$$

with $(b + c)f_3 = f_3$. If $b \in \{0, 1\}$, the sequence is if the form 6., and otherwise it is of the form 4.

Now, suppose $|\operatorname{supp}(F)| = 3$. Then, additionally, by the same argument $\operatorname{ord}(g_1) = 2$, that is $b = 0$. Thus,

$$A = f_3^{2n-1-2v-2w}(f_3 + f_2)^{2v}(f_3 + f_1)^{2w} f_2 f_1(f_3 + f_2 + f_1)$$

and the sequence is of the form given in 6.

Suppose $N = 0$. Let $g_3 \mid R$. By assumption and F1, we know that each element of $\operatorname{supp}(F)$ is the sum of g_3 and some other element in $\operatorname{supp}(R)$. Moreover, we know that $|\operatorname{supp}(F)| \leq 2$. Thus, let $g_2 \mid g_3^{-1}R$ such that $g_{\{2,3\}} = g_2 + g_3 \in \operatorname{supp}(F)$ and, in case $|\operatorname{supp}(F)| = 2$ let $g_1 \in \operatorname{supp}(R) \setminus \{g_2, g_3\}$ such that $g_1 + g_3 \in \operatorname{supp}(F)$. We denote the remaining element(s) of $\operatorname{supp}(R)$ by g_1, $g_{\{1,2,3\}}$, or $g_{\{1,2,3\}}$, respectively.

Let $f_3 = g_{\{2,3\}}$ and $f_1, f_2 \in G$ such that $\{f_1, f_2, f_3\}$ is a basis of G. For $I \in \{1, 2, 3, \{1, 2, 3\}\}$, let $g_I = a_I f_3 + b_I f_2 + c_I f_1$ with $a_I \in [0, 2n-1]$ and $b_I, c_I \in \{0, 1\}$. Since by F2 $g_{\{2,3\}} = g_2 + g_3 = g_1 + g_{\{1,2,3\}}$, it follows that $a_2 + a_3 \equiv 1 \pmod{2n}$, $b_2 = b_3$, and $c_2 = c_3$; as well as $a_1 + a_{\{1,2,3\}} \equiv 1 \pmod{2n}$, $b_1 = b_{\{1,2,3\}}$, and $c_1 = c_{\{1,2,3\}}$. Moreover, $\{g_1, g_2, g_3\}$ is a generating set of G.

Since neither g_2 nor g_3 is an element of H, it follows that $(b_3, c_3) \neq (0,0)$. By change of basis, we may assume $b_3 = 1$ and $c_3 = 0$. Since $\{g_1, g_2, g_3\}$ is a generating set of G, it follows that $c_1 \neq 0$, and by change of basis, we may assume that $b_1 = 0$.

If $\operatorname{supp}(F) = \{g_{\{2,3\}}\}$, then

$$A = f_3^{2n-2}(a_3 f_3 + f_2)((1-a_3)f_3 + f_2)(a_1 f_3 + f_1)((1-a_1)f_3 + f_1).$$

Possibly changing the basis, we obtain $a_1, a_3 \in [0, n-1]$ and $a_3 \geq a_1$. If $a_3 \in \{0,1\}$ the sequence is of the form 6., if $a_3 \geq 2$ and $a_1 \in \{0,1\}$ it is of the form 4., and otherwise it is of the form 5.

Now, suppose $|\operatorname{supp}(F)| = 2$. By assumption this means $g_{\{1,3\}} = g_1 + g_3 \in \operatorname{supp}(F)$. Let $g_{\{1,3\}} = a_{\{1,3\}} f_3 + b_{\{1,3\}} f_2 + c_{\{1,3\}} f_1$ with $a_{\{1,3\}} \in [0, 2n-1]$ and $b_{\{1,3\}}, c_{\{1,3\}} \in \{0,1\}$. We have $2g_{\{1,3\}} = 2g_{\{2,3\}}$ and $g_{\{1,3\}} = g_1 + g_3 = g_2 + g_{\{1,2,3\}}$. Thus $a_{\{1,3\}} \in \{1, 1+n\}$ and $b_{\{1,3\}} = c_{\{1,3\}} = 1$.

We observe that $\sigma(g_{\{1,3\}} g_1 g_2) \in \langle f_3 \rangle$. Let $k \in \mathbb{N}$ such that $2k = \mathsf{v}_{f_3}(A)$. We observe that $\Sigma(g_{\{1,3\}}^{-2} F) \cap \langle f_3 \rangle = \{i f_3 + j(2f_3) \colon (i,j) \in [0,2k] \times [0, n-2-k] \setminus \{(0,0)\}\} = \{j f_3 \colon j \in [1, 2n-4]\}$. Since $-\sigma(g_{\{1,3\}} g_1 g_2) \notin \Sigma(g_{\{1,3\}}^{-2} F) \cup \{0\}$, it follows that $\sigma(g_{\{1,3\}} g_1 g_2) \in \{f_3, 2f_3, 3f_3\}$. Using $g_{\{1,3\}} = g_1 + g_3$ and $a_2 f_3 = (1-a_3) f_3$, it follows that $\sigma(g_{\{1,3\}} g_1 g_2) = (1+2a_1) f_3$. Consequently, $a_1 \in \{0, 1, n, 1+n\}$. Moreover, if $a_1 \in \{\delta, \delta+n\}$ for $\delta \in \{0,1\}$, then, since $a_{\{1,3\}} \in \{1, 1+n\}$, we have $a_3 \in \{1-\delta, 1-\delta+n\}$. Let $a_1 = \delta + \varepsilon n$ and $a_3 = 1 - \delta + \varepsilon' n$ with $\varepsilon, \varepsilon' \in \{0,1\}$. Changing the basis to $\{f_1' = f_1 + \varepsilon n f_3, f_2' = f_2 + \varepsilon' n f_3, f_3\}$ and recalling that $g_{\{1,3\}} = g_1 + g_3$, we have

$$A = f_3^{2v}(f_3 + f_2' + f_1')^{2n-2-2v}(f_3 + f_2') f_2' (f_3 + f_1') f_1'$$

and the sequence is of the form 6. □

The examples of minimal zero-sum sequences over $C_2^2 \oplus C_{2n}$ can readily be 'extrapolated' to $C_2^{r-1} \oplus C_{2n}$ for each $r \geq 4$ to yield numerous examples of minimal zero-sum sequences of length $\mathsf{D}^*(C_2^{r-1} \oplus C_{2n})$, which is known to equal $\mathsf{D}(C_2^{r-1} \oplus C_{2n})$ for suitable n. In Section 4, we give an example how potentially interesting examples can be constructed in this way. This extrapolation also yields an informal 'lower bound' on the length a characterization at the level of detail of Theorem 3.13, even for fairly small $r > 3$, has to have. And, this 'lower bound' is definitely not sharp, since for each $r \geq 5$ the characterization of minimal zero-sum sequences of maximal length cannot be so uniform in n anymore, as it is known that their lengths, as a function of n, also depends on the 2-valuation of n and not just the size of n. Thus, in the author's opinion, results giving only a somewhat rougher classification than Theorem 3.13 seem the more feasible and relevant way to expand on this result. Indeed, a main reason for giving a description at this level of detail for $C_2^2 \oplus C_{2n}$ at all is an immediate application where these details are helpful; simplifications of Theorem 3.13 would almost directly cause complications in the proof of Lemma 4.7.

4 Applications of Theorem 3.13

In this section we discuss applications of Theorem 3.13. First, we show that this result in combination with classical results essentially directly yields the exact value of $\mathsf{D}(C_4^2 \oplus C_{4n})$ and $\mathsf{D}(C_6^2 \oplus C_{6n})$ for each $n \in \mathbb{N}$. Second, we discuss implications of this result to questions, other than the height, on the structure of long minimal zero-sum sequences over groups of the form $C_2^{r-1} \oplus C_{2n}$. Finally, we prove a result on the system of set of lengths of $C_2^2 \oplus C_{2n}$; this result is very technical, yet crucial in Section 5, indeed to get this result was a main motivation for proving Theorem 3.13.

4.1 The Davenport constant for some groups of rank three

As mentioned in Section 1 it is conjectured that $\mathsf{D}(G) = \mathsf{D}^*(G)$ for groups of rank three. However, this conjecture is wide open and so far was only confirmed for several special types of groups (see below for an overview); we contribute two new special types of groups (to be precise, the first assertion is new for odd $n \geq 5$ and the second one is new except for n a multiple of 64, a multiple of 81, a power of 2, or a power of 3; cf. below).

Theorem 4.1. *Let $n \in \mathbb{N}$.*

1. $\mathsf{D}(C_4^2 \oplus C_{4n}) = \mathsf{D}^*(C_4^2 \oplus C_{4n})$.
2. $\mathsf{D}(C_6^2 \oplus C_{6n}) = \mathsf{D}^*(C_6^2 \oplus C_{6n})$.

Our proof combines a classical method with Theorem 3.13. We recall this method and related notions.

Let G be a finite abelian group. Let $\nu(G)$ denote the smallest $\ell \in \mathbb{N}$ such that for each zero-sum free $S \in \mathcal{F}(G)$ with $|S| \geq \ell$ we have $G \setminus (\Sigma(S) \cup \{0\}) \subset a + N$ for some subgroup $N \subsetneq G$ and some $a \in G \setminus N$.

The group G is said to have Property **Q** if $\nu(G) = \mathsf{D}^*(G) - 2$ and for each zero-sum free $S \in \mathcal{F}(G)$ with $|S| \geq \nu(G)$ we have $G \setminus (\Sigma(S) \cup \{0\}) \subset a + N$ for some subgroup $N \subsetneq G$ of index two and some $a \in G \setminus N$.

It is known that

$$\mathsf{D}(G) - 2 \leq \nu(G) \leq \mathsf{D}(G) - 1 \tag{4.1}$$

and conjectured that equality always holds at the lower bound, except for the trivial group (see [18]). This conjecture is known to hold true for p-groups and cyclic groups (see [53]). Moreover, it is known to hold for certain groups of rank two, and if Property **B** is true for all $n \in \mathbb{N}$—in fact, Property **C** that is implied by Property **B** would suffice—then it holds for all groups of rank two (see [53] and [18]). Clearly, Property **Q** can only hold for groups of even order. It is known that it holds for 2-groups, cyclic groups of even order, and for certain groups of rank two whose 2-rank is two, and again assuming Property **B** for all n, it is know to hold for all groups of rank two whose 2-rank is two (cf. [54]).

Yet, the only non-p-group of rank greater than two for which this conjecture was confirmed is the group $C_2^2 \oplus C_6$ (see [54]). As a direct consequence of Theorem 3.13, we can confirm this conjecture for $C_2^2 \oplus C_{2n}$ for each $n \in \mathbb{N}$ and assert that they have Property **Q** (see Lemma 4.3).

The relevance of these notions is due to the following result established by P. C. Baayen, J. H. van Lint, and P. van Emde Boas, D. Kruyswijk, respectively (see [53, 54]).

Proposition 4.2. *Let $G = \oplus_{i=1}^3 C_{n_i}$ with $n_1 \mid n_2 \mid n_3$.*

1. *If $\nu(G) = \mathsf{D}^*(G) - 2$, then $\mathsf{D}(\oplus_{i=1}^3 C_{2n_i}) = \mathsf{D}^*(\oplus_{i=1}^3 C_{2n_i})$.*
2. *If G has Property **Q**, then $\mathsf{D}(\oplus_{i=1}^3 C_{3n_i}) = \mathsf{D}^*(\oplus_{i=1}^3 C_{3n_i})$.*

Note that the condition $\nu(G) = \mathsf{D}^*(G) - 2$, and thus also Property **Q**, implies that $\mathsf{D}(G) = \mathsf{D}^*(G)$.

A considerable part of all known results on the equality $\mathsf{D}(G) = \mathsf{D}^*(G)$ for groups of rank three is obtained via combining this result with the results on $\nu(\cdot)$ and Property **Q** recalled above. In addition to the groups for which the equality $\mathsf{D}(G) = \mathsf{D}^*(G)$ can be established in this way, the equality is known for the following groups:

- p-groups (by the general result on p-groups).
- groups of rank three of the form $G' \oplus C_n$ with G' a p-group with $\mathsf{D}(G') \leq 2\exp(G') - 1$ and n co-prime to $\exp(G)$ (see (3.2) and the discussion there) and if $G \cong C_{n_1} \oplus C_{n_2} \oplus C_{n_3 m}$ with $n_1 \mid n_2 \mid n_3$ and $m \in \mathbb{N}$ and it is known that $\mathsf{D}(\oplus_{i=1}^3 C_{n_i}) = \mathsf{D}^*(\oplus_{i=1}^3 C_{n_i})$ and $(n_1 n_2^2 - 2n_2 - n_1 - 2) \leq n_3$ (see [11]).
- $C_3^2 \oplus C_{3n}$ and $C_3 \oplus C_{3n}^2$, the latter assuming n has Property **B** and n is co-prime to 6 (see [7, 6]).

For specific n these results allow to determine $C_4^2 \oplus C_{4n}$ and $C_6^2 \oplus C_{6n}$ (cf. the n we mentioned above), yet not for general n. Thus, we prove the following result for $C_2^2 \oplus C_{2n}$.

Lemma 4.3. *Let $n \in \mathbb{N}$. Then $\nu(C_2^2 \oplus C_{2n}) = \mathsf{D}^*(C_2^2 \oplus C_{2n}) - 2$ and more precisely $C_2^2 \oplus C_{2n}$ has Property **Q**.*

Proof. By (4.1), it suffices to show the following. If $S \in \mathcal{F}(C_2^2 \oplus C_{2n})$ with $|S| \ge \mathsf{D}^*(C_2^2 \oplus C_{2n}) - 2$, then there exists a subgroup $N \subset C_2^2 \oplus C_{2n}$ of index 2 and some $y \notin N$ such that $C_2^2 \oplus C_{2n} \setminus (\Sigma(S) \cup \{0\}) \subset y + N$. We assume that $\Sigma(S) \ne C_2^2 \oplus C_{2n} \setminus \{0\}$, since otherwise the claim is trivial. Thus, there exists some $g \in C_2^2 \oplus C_{2n}$ such that gS is zero-sum free, and hence $(-\sigma(gS))gS$ is a minimal zero-sum sequence. Since $\mathsf{D}(C_2^2 \oplus C_{2n}) \ge |(-\sigma(gS))gS| = 2 + |S| \ge \mathsf{D}^*(C_2^2 \oplus C_{2n}) = \mathsf{D}(C_2^2 \oplus C_{2n})$. We get that S is a subsequence of length $\mathsf{D}(C_2^2 \oplus C_{2n}) - 2$ of a minimal zero-sum sequence of length $\mathsf{D}(C_2^2 \oplus C_{2n})$. By Theorem 3.13 we know the structure of all these minimal zero-sum sequences explicitly. Thus, we merely have to check, via determining their set of subsums, that all these sequences actually fulfil these conditions.

We distinguish cases according to the type of minimal zero-sum sequences and then subcases according to the type of the two missing elements. Additionally, we note that, say, for $w_3, w_2 \in \mathbb{N}$ we have $\Sigma(f_3^{w_3}(f_3 + f_2)^{2w_2}) \supset \{f_3, 2f_3, 3f_3, \dots, (2w_2 + w_3)f_3\}$. Thus, the set of subsums of the subsequence of elements occurring with high multiplicity depends only in a mild way on the actual multiplicities of the elements and this set of subsums contains almost the entire subgroup $\langle f_3 \rangle$ (or some other cyclic subgroup of order $2n$). The remaining details of the argument are a completely routine but long computation. Thus, we omit them. □

Now, Theorem 4.1 follows directly.

Proof of Theorem 4.1. Clear, by Proposition 4.2 and Lemma 4.3. □

4.2 Some further implications of Theorem 3.13

We discuss implications of Theorem 3.13 regarding typical questions on the structure of minimal zero-sum sequences (see [23]). We exclude the case $n = 1$ from our considerations, since this case is well-known and to include it would require to treat it separately.

We start with a result on the support and the maximal multiplicity of an element in minimal zero-sum sequences of maximal lengths.

Corollary 4.4. *Let $n \ge 2$. Let $A \in \mathcal{A}(C_2^2 \oplus C_{2n})$ with $|A| = \mathsf{D}(C_2^2 \oplus C_{2n})$.*

1. *$|\operatorname{supp}(A)| \in [4, 6]$. This is optimal for $n \ge 3$, yet for $n = 2$ we have $|\operatorname{supp}(A)| \le 5$.*
2. *There exists some $g \in \operatorname{supp}(A)$ such that $\mathsf{v}_g(A) > 2n/4$, and this bound is best possible.*

Proof. We use the notation introduced in Theorem 3.13.
1. If A is of the form as given by 1.–5. of Theorem 3.13 it is clear that $4 \le |\operatorname{supp}(A)| \le 5$. Suppose A is of the form 6. Since $\sigma(S) \ne 0$ it follows that $|\operatorname{supp}(S)| > 1$, and clearly $|\operatorname{supp}(S)| \le 4$, thus $4 \le |\operatorname{supp}(A)| \le 6$. Moreover, note that in case $n = 2$ we have $|\operatorname{supp}(S)| \le 3$. To see that the result is optimal, it suffices to consider the sequences $0^{2n-1}(f_1 + f_2)$, $0^{2n-2}f_1f_2$, and $0^{2n-4}f_1f_2(f_1 + f_2)^2$, which, for $n \ge 3$, shows that the support of the sequences of the form 6. indeed can be any of 4, 5, or 6; where as for $n = 2$ we get 4 and 5.
2. Let $g \in \operatorname{supp}(A)$ such that $w = \mathsf{v}_g(A)$ is maximal. Inspecting the classification in Theorem 3.13, we see that w is at least as large as claimed in the case 1.–5. and for 6. we directly get $w \ge n/2$. Yet, we note that in case n is even, the only sequence $S \in \mathcal{F}(\langle f_1, f_2 \rangle)$ compatible with $w = n/2$ is $(0f_1f_2(f_1 + f_2))^{n/2}$, which has sum 0. Thus, in 6. actually $w > n/2$ holds. The optimality is clear by Example 3.8. □

W. Gao and A. Geroldinger [19, 21] started to investigate the order of elements in minimal zero-sum sequences of maximal lengths; recently these investigations have been expanded by B. Girard [36, 37]. We explore how our result relates to results and conjectures obtained in the context of these investigations. For $A \in \mathcal{A}(G)$ let $S_A = \prod_{g \in \operatorname{supp}(A),\, \operatorname{ord}(g) = \exp(G)} g^{\mathsf{v}_g(A)}$ the subsequences of elements of order equal to the exponent.

Corollary 4.5. *Let $n \geq 2$, and let $A \in \mathcal{A}(C_2^2 \oplus C_{2n})$ with $|A| = \mathsf{D}(C_2^2 \oplus C_{2n})$.*

1. *$|S_A| \geq 2n-2$, in particular there exists some $g \in \text{supp}(A)$ with $\text{ord}(g) = \exp(G)$.*
2. *$\mathsf{k}(A) \leq 2$.*

Proof. We use the notation introduced in Theorem 3.13.
1. Clear, by Theorem 3.13.
2. For A of the form 1., 2., and 6. in Theorem 3.13 this is clear. In case A is of the form 3. we note that the order of each of the elements af_3+f_2, bf_3+f_1, and $cf_3+f_2+f_1$ is a multiple of 2, and by the conditions on a, b, c none of these orders is equal to 2, thus $\mathsf{k}((af_3+f_2)(bf_3+f_1)(cf_3+f_2+f_1)) \leq 3/4$, and the claim follows. In case A is of the form 4. or 5., it suffices to show that $\mathsf{k}((af_3+f_2)((1-a)f_3+f_2)) \leq 1/2$ for each $a \in [2, n-1]$. Again, we have that the order of af_3+f_2 and of $(1-a)f_3+f_2$ is a multiple of 2 but not equal to 2, and the claim follows. □

The first statement of this corollary, for this type of groups, confirms [19, Conjecture 6.1], stating that each minimal zero-sum sequence of maximal length contains some element of order $\exp(G)$; additionally, we note that our lower bound $2n-2$, for certain n, cannot be improved (also cf. Corollary 4.6). The second statement confirms, for this types of groups, [36, Conjecture 1.2], stating that if $S \in \mathcal{F}(\oplus_{i=1}^r C_{n_i})$, where $n_i \mid n_{i+1}$, and S is zero-sumfree with $|S| \geq \sum_{i=1}^r (n_i - 1)$, then $\mathsf{k}(S) \leq \sum_{i=1}^r (n_i-1)/n_i$; note that $2 = (2n-1)/(2n) + 1/2 + 1/2 + 1/(2n)$, and that we consider a minimal zero-sum sequence of length $1 + \sum_{i=1}^r (n_i - 1)$, which explains the additional $1/(2n)$.

We end with a result, obtained via extrapolating an example of a minimal zero-sum sequences found in Theorem 3.13, that gives an example of a group for which minimal zero-sum sequence of maximal length can contain relatively few elements of order equal to the exponent.

Corollary 4.6. *For each $N \in \mathbb{N}$ there exists a finite abelian group G with $\exp(G) \geq N$ that has the following property. There exists some $A \in \mathcal{A}(G)$ with $|A| = \mathsf{D}(G)$ such that $|S_A| \leq 2\exp(G)/3 + 1$.*

Proof. Let $N \in \mathbb{N}$ and suppose $N \geq 3$. Let $n = 2^{\ell-1}3$ with $\ell \geq \log_2 N$. Let $r = 2^\ell$ and $G = C_2^{r-1} \oplus C_{2n} = \oplus_{i=1}^r \langle f_i \rangle$ with $\text{ord}(f_r) = 2n$ and $\text{ord}(f_i) = 2$ for $i \in [1, r-1]$. By (3.2), note that $G \cong G' \oplus C_3$ with $G' = C_2^{2^\ell - 1} \oplus C_{2^\ell}$, we know that $\mathsf{D}(G) = \mathsf{D}^*(G)$. Let, cf. the sequence of type 5. in Theorem 3.13,

$$A = f_r^{2n-(r-1)} \prod_{i=1}^{r-1} (4f_r + f_i)(-3f_r + f_i).$$

Then $A \in \mathcal{A}(G)$ with $|A| = \mathsf{D}(G)$ and $S_A = f_r^{2n-(r-1)}$. Thus $|S_A| = 2^\ell 3 - r + 1 = 2^{\ell+1} + 1 = 2\exp(G)/3 + 1$. □

This result is in sharp contrast with a recent result of B. Girard [37], asserting that for G a p-group and $A \in \mathcal{A}(G)$ with $|A| = \mathsf{D}(G)$ one has $|S_A| \geq \exp(G)$. A construction of exceptionally long minimal zero-sum sequences (of lengths greater than $\mathsf{D}^*(G)$) over $C_2^{r-1} \oplus C_{12}$ containing few elements of maximal order was recently given by S. Griffiths [38].

Moreover, note that if we impose the condition that $|A| = \mathsf{D}^*(G)$ instead of $|A| = \mathsf{D}(G)$, then S_A can be empty, since in the above construction no condition on r needs to be imposed—we do not need to apply (3.2)—and we thus can choose r to equal $2n+1$. These observations reinforce the believe that to confirm [19, Conjecture 6.1], mentioned above, in general is difficult, as such an argument, at least implicitly, has to contain fairly detailed information on the phenomenon $\mathsf{D}(G) > \mathsf{D}^*(G)$; and this not only for groups of the form $C_2^{r-1} \oplus C_{2n}$ as the above construction can be generalized, e.g., consider the sequence $f_r^{pn-(r-1)} \prod_{i=1}^{r-1} (p^2 f_r + f_i)^{p-1}((1-(p-1)p^2)f_r + f_i)$ over $C_p^{r-1} \oplus C_{pn} = \oplus_{i=1}^r \langle f_i \rangle$ where $p(1-(p-1)p^2) \mid n$.

4.3 A result on $\mathcal{L}(C_2^2 \oplus C_{2n})$

Intensely using Theorem 3.13, we prove the following result, which is crucial in Section 5.

Lemma 4.7. *Let $n \in \mathbb{N}$. Then $\{2, 3, 2n, 2n+1, 2n+2\} \notin \mathcal{L}(C_2^2 \oplus C_{2n})$.*

We recall two technical results used in its proof (for the first see [30] or [52, Lemma 9.4], for the second see [32, Lemma 6.4.5]).

Lemma 4.8. *Let $B \in \mathcal{B}(G)$. If $\{2, \mathsf{D}(G)\} \subset \mathsf{L}(B)$ then $B = (-A)A$ with $A \in \mathcal{A}(G)$ and $|A| = \mathsf{D}(G)$. Moreover, if additionally $\mathsf{D}(G) - 1 \in \mathsf{L}(B)$, then there exists (possibly equal) $g, h \in G$ with $gh(g+h) \mid A$.*

Lemma 4.9. *Let $A \in \mathcal{A}(G)$ with $|A| \ge 2$. Let $W \in \mathcal{A}(G)$ such that $W \mid (-A)A$. Then $|A| - |W| + 2 \in \mathsf{L}((-A)A)$.*

Proof of Lemma 4.7. Assume to the contrary that there exists some $B \in \mathcal{B}(C_2^2 \oplus C_{2n})$ with $\mathsf{L}(B) = \{2, 3, 2n, 2n+1, 2n+2\}$. Since $\mathsf{D}(C_2^2 \oplus C_{2n}) = 2n+2$ and $\{2, 2n+2\} \subset \mathsf{L}(B)$ it follows by Lemma 4.8 that $B = (-A)A$ with $A \in \mathcal{A}(C_2^2 \oplus C_{2n})$ and $|A| = 2n+2$. By Theorem 3.13 we have precise information on the structure of A; we use the notation introduced there. Additionally, we observe that, since $2n+1 \in \mathsf{L}(B)$ and by Lemma 4.8, there exist $g, h \in C_2^2 \oplus C_{2n}$ such that $gh(g+h) \mid A$. Thus we may assume that $n \ge 2$, since for $n = 1$ we have $\sigma(gh(g+h)) = 0$, a contradiction.

First, we assert that A is not of the form given 1., 2., and 3. of Theorem 3.13, by showing that A does not have a subsequence of the form $gh(g+h)$. For 1. this is clear. For 2. we note that $(f_3 + f_2) + (af_3 + f_1) = -af_3 + f_2 + f_1$ is equivalent to $(2a-1)f_3 = 0$, which is impossible; the other cases are analogous. The reasoning for 3. is analogous to the one for 2.

Now, suppose A is of the form given in 4. Let $W = f_2((1-a)f_3 + f_2 + f_1)(-(af_3 + f_1))f_3^v(f_3 + f_2)^w$ where $v + w = 2a - 1$ and $2 \mid w$. Then $W \in \mathcal{A}(C_2^2 \oplus C_{2n})$ and $W \mid (-A)A$. We have $|W| = 2a+2$ and thus by Lemma 4.9 $|A| - |W| + 2 = 2n+2-(2a+2)+2 \in \mathsf{L}(B)$. So, we have $2(n-a+1) \in \{2, 3, 2n, 2n+1, 2n+2\}$ and consequently $n - a + 1 \in \{1, n, n+1\}$. Yet, this means that $a \in \{n, 1, 0\}$, a contradiction, since $a \in [2, n-1]$.

Next, suppose A is of the form given in 5. We proceed similarly as above. Let $W = ((1-a)f_3 + f_2)(-(af_3 + f_2))f_3^{2a-1}$. We have $W \in \mathcal{A}(C_2^2 \oplus C_{2n})$, $W \mid (-A)A$, and $|W| = 2a+1$. Consequently, $2n + 3 - 2a \in \mathsf{L}(B)$, implying that $a \in \{1, n\}$, a contradiction.

Finally, suppose A is of the form given in 6. We show that $3 \notin \mathsf{L}(B)$. Assume to the contrary $B = A_1A_2A_3$ with $A_i \in \mathcal{A}(C_2^2 \oplus C_{2n})$. Let $\pi : C_2^2 \oplus C_{2n} \to \langle f_3 \rangle$ denote the canonical projection with respect to the basis $\{f_1, f_2, f_3\}$.

First, we assert that $A_i \notin \{f_1^2, f_2^2\}$ for each i. Assume to the contrary that, say, $A_1 = f_1^2$. We note that $A_i \ne f_2^2$ for $i \in \{2, 3\}$, since otherwise the remaining minimal zero-sum sequence would have length $4n > 2n+2$, a contradiction. Thus, $f_2 \mid A_2$ and $f_2 \mid A_3$. We note that $\mathsf{v}_{f_3}(\pi(A_i)) \ne n$ for $i \in \{2, 3\}$. Thus $\mathsf{v}_{f_3}(\pi(A_i)) = \mathsf{v}_{-f_3}(\pi(A_i))$ and the length of A_i is odd for $i \in \{2, 3\}$. Consequently $|A_2| = |A_3| = 2n+1$. Let $g_1g_2h_1h_2 \mid A_3$ such that $\pi(g_i) = f_3$ and $\pi(h_i) = -f_3$. Then $(g_i + h_i) \in \langle \{f_1, f_2\} \rangle$, and the sequence $(g_1 + h_1)(g_2 + h_2)f_2$ has a zero-sum subsequence, which yields a zero-sum subsequence of A_3. Since A_3 is a minimal zero-sum subsequence, it follows that $A_3 = g_1g_2h_1h_2f_2$, implying $n = 2$. Clearly $\operatorname{supp}(A_2)$ cannot contain two elements that are inverse to each other. Thus, it follows that $(g_1 - f_3)(g_2 - f_3)(h_1 + f_3)(h_2 + f_3) = S$, with S as defined in Theorem 3.13, and thus $\sigma(S) = f_2$, a contradiction to $\sigma(S) = f_1 + f_2$.

So, we may assume that $f_1 \mid A_1$, $f_2 \mid A_2$, and $f_1f_2 \mid A_3$. We note that, for each $i \in \{1, 2, 3\}$, $A_i \notin \{-A, A\}$, thus $\mathsf{v}_{f_3}(\pi(A_i)) = \mathsf{v}_{-f_3}(\pi(A_i)) > 0$. Similarly as above, let $gh \mid A_3$ such that $\pi(g) = f_3$ and $\pi(h) = -f_3$. It follows that $(g+h)f_1f_2$ has a zero-sum subsequence, which by the minimality of A_3 implies that $A_1 = ghf_1f_2$. Since $|A_1| + |A_2| = 4n$ and the lengths of A_1 and A_2 is odd, we may assume that $|A_2| \ge 2n+1$. Again, let $g_1g_2h_1h_2 \mid A_2$ such that $\pi(g_i) = f_3$ and $\pi(h_i) = -f_3$. As above, it follows that $A_2 = g_1g_2h_1h_2f_2$, yielding a contradiction. ☐

5 Characterization of class groups

As mentioned in Section 1 we apply Theorem 3.13 to the problem of characterizing the class group of a Krull monoid with finite class group where each class contains a prime divisor via the system of sets of lengths, for specific types of groups. We imbed these investigation into a more general analysis of this problem for groups of large exponent. In this more general case, we do not obtain a full answer, yet we can show that at least the exponent of the group is determined by the system of sets of length.

We refer to, e.g., [32, 31] for detailed information on Krull monoids (as well as other notions briefly discussed below) and we recall that the multiplicative monoid of the ring of algebraic integers of a number field (and its ideal class group) are the classical example of a Krull monoid with the above properties; for further examples see, e.g., [32], in particular Example 2.3.2, Sections 2.10 and 2.11, also cf. Example 7.4.2.

It is well-known that for M a Krull monoid with class group G where each class contains a prime divisor, one has $\mathcal{L}(M) = \mathcal{L}(\mathcal{B}(G))$. A first version of this result is due to W. Narkiewicz and the latter developments mainly due to A. Geroldinger and F. Halter-Koch (see, e.g., [32, Chapter 3] for this and related results). Thus, the problem of characterizing the class group of M via the system of sets of lengths is reduced to the problem of characterizing G via $\mathcal{L}(\mathcal{B}(G))$. Recall that we write $\mathcal{L}(G)$ instead of $\mathcal{L}(\mathcal{B}(G))$ and refer to it as the system of sets of lengths of G.

The problem for which types of finite abelian groups the system of sets determines the group, i.e., for which G the fact that $\mathcal{L}(G) = \mathcal{L}(G')$ implies that $G \cong G'$, was originally considered by A. Geroldinger [30]. Various of the investigations on sets of lengths undertaken since that time are motivate by the aim of making progress on this problem (see, e.g., [32, Chapters 6 and 7]). For detailed information on the general problem of giving arithmetical definitions of class groups (not necessarily restricted to sets of lengths only), a problem raised by W. Narkiewicz, we refer to [32, Chapter 7]. For recent related investigations see, e.g., [2, 12].

In [30] a characterization via the system of sets of lengths was obtained in case the group is a cyclic group, an elementary 2-group, or of the form $C_2 \oplus C_{2n}$, and its Davenport constant is at least 4; and additionally in case the Davenport constant of the class group is at most 7. For further results on this problem see [52, 51], where this problem is solved for groups of the form C_n^2 and in case the Davenport constant is at most 10. For the four groups whose Davenport constant is less than 4, the situation is slightly different. Namely, it is only possible to determine from the system of sets of lengths whether the group is isomorphic to one of the groups C_1 and C_2, and whether it is isomorphic to one of the groups C_2^2 and C_3; the first is essentially due to L. Carlitz [9] the latter due to A. Geroldinger [30]. Presently, these two pairs of groups are the only known examples for the phenomenon that non-isomorphic groups yield the same system of sets of lengths, and it is thus an open problem whether all other types of groups are characterized by the system of sets of length or whether there are more 'exceptions'.

Here, we obtain such a characterization via the system of sets of lengths for several other types of groups (see Theorems 5.3 and 5.6). As indicated in Section 1 the author believes that the most relevant aspect of these results is the fact that $C_2^2 \oplus C_{6n}$ and $C_3 \oplus C_{6n}$ can be distinguished by the system of sets of length. In this case the 'large' group C_{6n} is only slightly 'perturbed' in two distinct ways in such a way that both 'perturbations' have the same effect on those invariants that were used in essentially all characterization results established so far, namely the Davenport constant and the large elements of the set $\Delta_1(G)$; additionally, note that in this case the 'perturbations' C_2^2 and C_3 even have the same system of sets of lengths. No other result of this form was known so far; note that for $C_{2n} \cong C_1 \oplus C_{2n}$ and $C_2 \oplus C_{2n}$ the Davenport constants are different and C_n^2 can be treated as one 'large' group that remains 'unperturbed'. Thus, to address this problem for $C_2^2 \oplus C_{6n}$ and $C_3 \oplus C_{6n}$ for $n \in \mathbb{N}$ seems of particular relevance in this context.

The proofs of such characterization results are often informally split into two steps. First, via general considerations, it is asserted that only a few groups can have the same system of sets of lengths as the (type of) group under consideration. Second, via more explicit arguments, one distinguishes these few remaining groups.

For our investigations we need some additional notation and results. We recall them very briefly, for details see, e.g., [32, Section 7.3] and [52].

Let G be a finite abelian group. It is well-know that the Davenport constant of G, for $|G| \geq 2$, is determined by $\mathcal{L}(G)$; recall that $\mathsf{D}(G) = \max\{\max L \colon L \in \mathcal{L}(G), 2 \in L\}$ (see [30, Lemma 7]). The set $\Delta_1(G)$, introduced in [30], is the set of all $d \in \mathbb{N}$ such that $\mathcal{L}(G)$ contains arbitrarily long almost arithmetical progressions with difference d; almost arithmetical progression informally means an arithmetical progression where a globally bounded number of elements at the beginning and the end of the arithmetical progressions may be missing. Thus, obviously $\Delta_1(G)$ is determined by $\mathcal{L}(G)$.

In particular, we have that if G and G' are finite abelian groups with at least two elements such that $\mathcal{L}(G) = \mathcal{L}(G')$, then

$$\mathsf{D}(G) = \mathsf{D}(G') \text{ and } \Delta_1(G) = \Delta_1(G'). \tag{5.1}$$

The relevance of the set $\Delta_1(G)$ in this context is due to the fact that via the Structure Theorem for Sets of Lengths (see, e.g., [32, Chapter 4]) this set $\Delta_1(G)$ is known to be closely linked to a set $\Delta^*(G)$—we omit the definition—which can be investigated more directly than $\Delta_1(G)$ itself. Namely, $\Delta^*(G) \subset \Delta_1(G)$ and $\Delta_1(G)$ consist of divisors of elements of $\Delta^*(G)$; thus, all elements of $\Delta_1(G)$ that are greater than half of the maximum of this set are directly determined by $\Delta^*(G)$, yet not all its elements (see [20]).

We recall some results on $\Delta_1(G)$ that we need in our investigations (all these results are actually result on $\Delta^*(G)$, suitably transcribed).

It is well-known that $\Delta_1(G) \neq \emptyset$ if and only if $|G| \geq 3$; specifically, if $|G| \geq 3$, then $1 \in \Delta_1(G)$ (see [30]). Moreover, it is known that $[1, \mathsf{r}(G) - 1] \subset \Delta_1(G)$, $\exp(G) - 2 \in \Delta_1(G)$ if $\exp(G) \geq 3$, and $\max \Delta_1(G) \leq \mathsf{D}(G) - 2$ except for the trivial group (see, e.g., [32]). Thus, if G is cyclic with $|G| \geq 3$, then $\max \Delta_1(G) = \exp(G) - 2$, and more generally it is known (see [20]) that if $|G| \leq \exp(G)^2$ and $\exp(G) \geq 3$, then $\max \Delta_1(G) = \exp(G) - 2$ (for more precise results on $\Delta_1(C_n)$ cf. [33] and for recent results related to the latter assertion cf. [51]).

Furthermore, we have

$$\max \Delta_1(G) = \max\{\exp(G) - 2, \mathsf{m}(G)\} \tag{5.2}$$

where $\mathsf{m}(G)$ is a certain constant that fulfills $\mathsf{m}(G) \leq \max\{\mathsf{r}^*(G) - 1, \mathsf{K}(G) - 1\}$. And, if G is a p-group, then $\mathsf{m}(G) = \mathsf{r}(G) - 1$, in particular $\max \Delta_1(G) = \max\{\exp(G) - 2, \mathsf{r}(G) - 1\}$; additionally, if $\mathsf{r}(G) \geq \exp(G) - 1$, then $\Delta_1(G) = [1, \mathsf{r}(G) - 1]$ (see [51]).

Additionally, we recall that if G does not have a subgroup isomorphic to $C_{\exp(G)}^2$ and $\exp(G) \geq 5$, then (cf. [51, Theorem 3.2])

$$\max(\Delta_1(G) \setminus \{\exp(G) - 2\}) = \max\{\lfloor \frac{\exp(G)}{2} \rfloor - 1, \mathsf{m}(G)\}. \tag{5.3}$$

Using the fact that $\mathcal{L}(G) = \mathcal{L}(G')$ implies that $\mathsf{D}(G) = \mathsf{D}(G')$, excluding the trivial group, and the fact that only finitely many (up to isomorphy) groups can have the same Davenport constant, some restriction on the groups that possibly can have the same system of sets of lengths can readily and generally be inferred. However, except when limiting to the consideration of groups with a very small Davenport constant this restriction alone is in general too weak, to allow to address all groups fulfilling it via more explicit considerations. And, an analogous statement holds true replacing $\mathsf{D}(G) = \mathsf{D}(G')$ by $\max \Delta_1(G) = \max \Delta_1(G')$.

Yet, combining information on $\mathsf{D}(G)$ and $\max \Delta_1(G)$, in certain cases, considerably more restrictive conditions can be inferred.

Proposition 5.1. *Let $G \neq \{0\}$ be a finite abelian group. Let $d = \max \Delta_1(G)$ and $R = \mathsf{D}(G) - d$. At least one of the following statements holds.*

1. *There exists a finite abelian group G_1 with $\mathsf{D}(G_1) \leq R - 1$ and $\exp(G_1) \mid (d + 2)$ such that $G \cong G_1 \oplus C_{d+2}$.*

2. $\mathsf{r}(G) \ge \min\{8d - 6R + 13, d - R/3 + 5/3\}$ *and* $\exp(G) < d + 2$.

Proof. By (5.2) we have $d = \max\{\exp(G) - 2, \mathsf{m}(G)\}$.

First, suppose that $d = \exp(G) - 2$. Then, there exists a finite abelian group G_1 with $\exp(G_1) \mid (d+2)$ such that $G \cong G_1 \oplus C_{d+2}$. Since $d + R = \mathsf{D}(G) \ge (d + 2) + \mathsf{D}(G_1) - 1$, the first statement holds true.

Second, suppose that $d \ne \exp(G) - 2$. Then $d = \mathsf{m}(G)$ and $\mathsf{m}(G) > \exp(G) - 2$, in particular $\exp(G) < d + 2$. Then, by the remark after (5.2), we have $d \le \mathsf{K}(G) - 1$ or $d \le \mathsf{r}^*(G) - 1$.

Suppose the former holds true. We get $R - 1 \ge \mathsf{D}(G) - \mathsf{K}(G)$. Let $A = \prod_{i=1}^{\ell} g_i \in \mathcal{A}(G)$ with $\mathsf{k}(A) = \mathsf{k}(G)$ and assume that $0 \nmid A$. It follows that

$$R - 1 \ge |A| - \mathsf{k}(A) = \sum_{i=1}^{\ell} \frac{\operatorname{ord}(g_i) - 1}{\operatorname{ord}(g_i)}.$$

Suppose that ℓ_2 of the g_is have order 2. Then, we get $R - 1 \ge \ell_2/2 + 2(\ell - \ell_2)/3$, and using the fact that $\ell \ge 2\,\mathsf{k}(A)$ and the assumption $d \le \mathsf{K}(G) - 1$, this yields that $\ell_2 \ge 8d - 6R + 14$. We observe that $\mathsf{r}(G) \ge \mathsf{r}_2(G) \ge \ell_2 - 1$. Now, suppose $d \le \mathsf{r}^*(G) - 1$. Recalling that $\mathsf{D}(G) \ge 4\,\mathsf{r}^*(G) - 3\,\mathsf{r}(G) + 1$ (see Section 2) and since $d = \mathsf{D}(G) - R$, it follows that $\mathsf{r}(G) \ge d - R/3 + 5/3$. Thus, $\mathsf{r}(G) \ge \min\{8d - 6R + 13, d - R/3 + 5/3\}$. □

Note that, except for the trivial group, since $\max \Delta_1(G) \le \mathsf{D}(G) - 2$ we have $R \ge 2$. In Proposition 5.5 we expand on this result for $R = 4$; the case $R = 2$ and $R = 3$ are not considered in detail, since they correspond to the problem of characterizing cyclic groups, elementary 2-groups, and groups of the form $C_2 \oplus C_{2n}$, which is well-known. However, note that the characterization of C_n^2 does not directly correspond to the case $R = n + 1$, though closely related arguments are used.

Next, we apply Proposition 5.1 to groups with a relatively large exponent.

Proposition 5.2. *Let G be a finite abelian group with $\exp(G) = n \ge 4$, say $G \cong G_1 \oplus C_n$. Let G' be a finite abelian group with $\mathcal{L}(G') = \mathcal{L}(G)$. Suppose at least one of the following conditions holds.*

1. *The rank of G is at most two.*
2. $\mathsf{D}(G) \le 7n/4 - (3 \log_2 n + 18)/4$.

Then, $\exp(G) = \exp(G')$ *and more precisely* $G' \cong G_1' \oplus C_n$ *with* $\exp(G_1') \mid n$ *and* $\mathsf{D}(G_1') \le \mathsf{D}(G) - (n - 1)$.

The first condition is almost, yet not entirely, a special case of the second one.

Proof. 1. Suppose the first condition holds. Let $m \in \mathbb{N}$ such that $G \cong C_m \oplus C_n$.

If $m = n$ or $n = 4$, the claim follows by [51] and [30], respectively; indeed, in this case we have $G \cong G'$ (cf. the discussion at the beginning of this section). Thus, we assume that $m \ne n$ and that $n \ge 5$.

Since G is a group of rank at most two, we know that $\mathsf{D}(G) = \mathsf{D}^*(G) = n + m - 1$, $\max \Delta_1(G) = n - 2$, and $\lfloor n/2 \rfloor \notin \Delta_1(G)$ (see (5.3)). Since $\mathcal{L}(G) = \mathcal{L}(G')$ and by (5.1), we know that $\Delta_1(G) = \Delta_1(G')$ and that $\mathsf{D}(G) = \mathsf{D}(G')$.

We apply Proposition 5.1 to G'. We need to assert that the first statement holds. Assume to the contrary that $\mathsf{r}(G') \ge \min\{8(n - 2) - 6R + 13, (n - 2) - R/3 + 5/3\}$ where $R = m + 1$. Noting that $m \le n/2$, it follows that $\mathsf{r}(G') \ge \lfloor n/2 \rfloor + 1$. By (5.1) we get that $\lfloor n/2 \rfloor \in \Delta_1(G')$, a contradiction.

2. Suppose the second condition holds. The argument is similar. We start by bounding $\mathsf{m}(G)$. As recalled above, we know that $\mathsf{m}(G) \le \max\{\mathsf{K}(G) - 1, \mathsf{r}^*(G) - 1\}$. We recall the upper bounds $\mathsf{K}(G) \le 1/2 + \log |G|$ (see Section 2) and $\mathsf{r}^*(G) \le \log_2 |G|$. For $G_1 \cong \oplus_{i=1}^{s} C_{m_i}$ with $m_i \mid m_{i+1}$ we have $\mathsf{D}(G) - n \ge \mathsf{D}(G_1) - 1 \ge \sum_{i=1}^{s}(m_i - 1)$. Since $m - 1 \ge \log_2 m \ge \log m$ for each $m \in \mathbb{N}$, we get that $\mathsf{m}(G) \le \max\{\log n + (\mathsf{D}(G) - n) - 1/2, \log_2 n + (\mathsf{D}(G) - n) - 1\} = \log_2 n + \mathsf{D}(G) - n + 1$. In particular, by our assumption on $\mathsf{D}(G)$, we have $\mathsf{m}(G) \le n - 4$. Again, by the condition on $\mathsf{D}(G)$, we know that G has no subgroup isomorphic to C_n^2, and thus by (5.2) and (5.2) we have that $\max \Delta_1(G) = n - 2$ and $\max(\Delta_1(G) \setminus \{n - 2\}) = \max\{\mathsf{m}(G), \lfloor n/2 \rfloor - 1\}$. We set $R = \mathsf{D}(G) - (n - 2)$.

Again, we have $\Delta_1(G) = \Delta_1(G')$ and that $\mathsf{D}(G) = \mathsf{D}(G')$, and we apply Proposition 5.1 to G'. It suffices to show that the second statement cannot hold. Assume to the contrary that the second statement holds, i.e., $\mathsf{r}(G') \ge \min\{8(n-2) - 6R + 13, (n-2) - R/3 + 5/3\} = (n-2) - R/3 + 5/3$. Again, we show that in this case $\Delta_1(G')$ contains some element not contained in $\Delta_1(G)$, yielding a contradiction and thus establishing our result. Since $[1, \mathsf{r}(G') - 1] \subset \Delta_1(G')$, it suffices to show that $\mathsf{r}(G') - 1 \ge \max\{\mathsf{m}(G) + 1, \lfloor n/2 \rfloor\}$; recall that $\max\{\mathsf{m}(G) + 1, \lfloor n/2 \rfloor\} < n - 2$ and the result on $\Delta_1(G)$ established above. By the above upper bound on $\mathsf{m}(G)$, the lower bound on $\mathsf{r}(G')$, and the condition on R implied by the assumption on $\mathsf{D}(G') = \mathsf{D}(G)$, this inequality holds. □

Now, we formulate the first of our characterization results.

Theorem 5.3. *Let G and G' be be finite abelian groups such that $\mathcal{L}(G) = \mathcal{L}(G')$.*

1. *If $G \cong C_2^2 \oplus C_{2n}$ for some $n \in \mathbb{N}$, then $G \cong G'$.*
2. *If $G \cong C_3 \oplus C_{3m}$ for some $m \in \mathbb{N}$, then $G \cong G'$.*
3. *If $G \cong C_2^{r-1} \oplus C_4$ for some $r \in \mathbb{N}$, then $G \cong G'$.*

Applying Proposition 5.1, we see in Proposition 5.5 that the groups appearing in this result are precisely the groups with $\mathsf{D}(G) = \max \Delta_1(G) + 4$. This can be seen as the 'first step', in the informal sense mentioned above. Yet, additional arguments are required to distinguish these three types of groups. To distinguish $C_2^{r-1} \oplus C_4$ from the two other groups Proposition 5.2 is almost sufficient, yet it is not applicable for $C_2^2 \oplus C_{2n}$ for some small n. Thus, we use a result on the Δ_1-sets instead (see below), which we need anyway for proving Proposition 5.5. Yet, to distinguish $C_2^2 \oplus C_{2n}$ and $C_3 \oplus C_{3m}$ requires more detailed knowledge on the system of sets of lengths (available via Lemma 4.7).

Lemma 5.4. *Let $m, n, r \in \mathbb{N}$.*

1. *If $n \ge 3$, then $\max \Delta_1(C_2^2 \oplus C_{2n}) = 2n - 2$ and $2n - 3 \notin \Delta_1(C_2^2 \oplus C_{2n})$.*
2. *$\max \Delta_1(C_3 \oplus C_{3m}) = 3m - 2$ and $3m - 3 \notin \Delta_1(C_3 \oplus C_{3m})$.*
3. *$\max \Delta_1(C_2^{r-1} \oplus C_4) = \max\{2, r-1\}$ and more precisely $\Delta_1(C_2^{r-1} \oplus C_4) = [1, \max\{2, r-1\}]$.*

Proof. 1. For $n = 3$ this is proved in [52, Propostion 9.1]. For $n \ge 4$, using the inequalities $\mathsf{r}^*(C_2^2 \oplus C_{2n}) \le 3 + \log n$ and $\mathsf{K}(C_2^2 \oplus C_{2n}) \le 1/2 + \log(8n)$ (see Section 2), we get that $\mathsf{m}(C_2^2 \oplus C_{2n}) < 2n - 3$. Thus, the result on $\max \Delta_1(C_2^2 \oplus C_{2n})$ follows by (5.2). The other claim follows by (5.3).
2. For $m = 1$, this follows by the results on Δ_1 for p-groups recalled above; note that $0 \notin \Delta_1(C_3^2)$ by definition. For $m \ge 2$, see [50, Corollary 6.4]; or, for $m \ge 3$, this can be obtained similarly to 1.
3. This follows by the results on Δ_1 for p-groups and the fact $1 \in \Delta_1(C_2^{r-1} \oplus C_4)$ recalled above; also see [30]. □

Proposition 5.5. *Let G be a finite abelian group. Let $d \in \mathbb{N}$. The following statements are equivalent.*

1. *$\max \Delta_1(G) = d$ and $\mathsf{D}(G) = d + 4$.*
2. *G is isomorphic to $C_2^2 \oplus C_{d+2}$ with $2 \mid d$, $C_3 \oplus C_{d+2}$ with $3 \mid (d+2)$, or $C_2^d \oplus C_4$ with $d \ge 2$.*

Proof. First, assume that G is of the form given in 2. We observe that $\mathsf{D}(G) = d + 4$ (cf. the remark after (2.1) and, e.g., Theorem 3.13). By Lemma 5.4 it follows that $\max \Delta_1(G) = d$. Thus, 2. implies 1.

Second, assume $\max \Delta_1(G) = d$ and $\mathsf{D}(G) = d + 4$. First, we observe that if $d = 1$, then by the results recalled at the beginning of this section we have $\exp(G) \le 3$ and $\mathsf{r}(G) \le 2$, and by assumption we have $\mathsf{D}(G) = 5$, implying that $G \cong C_3^2$. Thus, we assume $d \ge 2$. Since for $n, r \in \mathbb{N}$ we have

$\max \Delta_1(C_n) = \max\{0, n-2\}$ and $\max \Delta_1(C_2^r) = r-1$ whereas $\mathsf D(C_n) = n$ and $\mathsf D(C_2^r) = r+1$, we get that G is neither cyclic nor an elementary 2-group.

We apply Proposition 5.1. First, suppose that $\exp(G) = d+2$. Then $G \cong G_1 \oplus C_{d+2}$ with $\exp(G_1) \mid (d+2)$ and $\mathsf D(G_1) \le 3$. Thus, G_1 is isomorphic to C_1, C_2, C_2^2, or C_3. Yet, if G_1 is isomorphic to C_1 or C_2, then, by the remark after (2.1), $\mathsf D(G)$ is equal to $\exp(G)$ or $\exp(G)+1$, resp., a contradiction. So, G is isomorphic to $C_2^2 \oplus C_{d+2}$ with $2 \mid d$ or $C_3 \oplus C_{d+2}$ with $3 \mid (d+2)$.

Second, suppose that $\exp(G) < d+2$. Then, we get $\mathsf r(G) \ge d + 1/3$, and since $\mathsf r(G) - 1 \le d$ we have $\mathsf r(G) = d+1$. Since as recalled in Section 2, we have $\mathsf D(G) \ge 4\,\mathsf r^*(G) - 3\,\mathsf r(G) + 1$, it follows that $\mathsf r^*(G) = \mathsf r(G)$, i.e., G is a p-group. Since $\mathsf D(G) = \mathsf r(G) + 3$, we get by (2.1) that $G \cong C_2^d \oplus C_4$; recall that $\max \Delta_1(C_3^2) = 1$ and that G is neither cyclic nor an elementary 2-group. □

Now, we proof Theorem 5.3. In view of the preparatory results, it remains to distinguish the systems of sets of lengths of the three types of groups we want to characterize, i.e., the 'second step' of our argument. We recall that, since up to Davenport constant 10 the problem of characterization via the system of sets of lengths is solved (cf. above), we can assume that the Davenport constant of each involved group is at least 8; to make the stronger assumption that it is 11 is not helpful. In this proof we see the crucial role of Lemma 4.7.

Proof of Theorem 5.3. As noted above we may assume $\mathsf D(G) \ge 8$, i.e., $n \ge 3$, $m \ge 2$, and $r \ge 5$, respectively. By Proposition 5.5 we have, in each of the three cases, $\mathsf D(G) = \max \Delta_1(G) + 4$. We set $d = \max \Delta_1(G)$ and note that $d \ge 4$. Thus, by (5.1) $\mathsf D(G') = \max \Delta_1(G') + 4$. Again, by Proposition 5.5 G' is isomorphic to $C_2^2 \oplus C_{d+2}$ with $2 \mid d$, $C_3 \oplus C_{d+2}$ with $3 \mid (d+2)$, or $C_2^d \oplus C_4$. By Lemma 5.4 we get that $\Delta_1(C_2^d \oplus C_4)$ is an interval, whereas the Δ_1-sets for the two other groups are not intervals. Thus, $\mathcal L(C_2^d \oplus C_4)$ is distinct from the sets of lengths of the two other types of groups and it remains to show that $\mathcal L(C_3 \oplus C_{d+2}) \ne \mathcal L(C_2^2 \oplus C_{d+2})$ for $6 \mid (d+2)$. We recall that by [32, Lemma 6.6.4] we have $\{2, 3, 3\ell, 3\ell+1, 3\ell+2\} \in \mathcal L(C_3 \oplus C_{3\ell})$ for each $\ell \in \mathbb N$; one considers $\mathsf L((-A)A)$ where $A = e_2^{3\ell-1} e_1^2 (e_1 + e_2)$ and $C_3 \oplus C_{3\ell} = \langle e_1 \rangle \oplus \langle e_2 \rangle$ and the orders of e_1 and e_2 are 3 and 3ℓ, respectively. Thus $\{2, 3, 6k, 6k+1, 6k+2\} \in \mathcal L(C_3 \oplus C_{6k})$ for each $k \in \mathbb N$. Yet, by Lemma 4.7 with $n = 3k$ we get that $\{2, 3, 6k, 6k+1, 6k+2\} \notin \mathcal L(C_2^2 \oplus C_{6k})$ for each $k \in \mathbb N$, and the claim follows. □

We end with an additional characterization result and some discussion. Proposition 5.2 shows that for a group G with a relatively large exponent (in the sense of that proposition) if G' is a finite abelian group with $\mathcal L(G) = \mathcal L(G')$, then $\exp(G) = \exp(G')$. Moreover, we know that $\mathsf D(G) = \mathsf D(G')$. In the result below, we illustrate that in certain cases this information is sufficient to fully characterize the group via its system of sets of lengths.

Theorem 5.6. *Let p be a prime and $r \in \mathbb N$.*

1. *Let $n \in \mathbb N \setminus \{1\}$ such that $d \nmid n$ for each $d \in [2, p]$. If $\mathcal L(C_p \oplus C_{pn}) = \mathcal L(G)$ for some finite abelian group G, then $G \cong C_p \oplus C_{pn}$.*
2. *There exists some $N_{p,r} \in \mathbb N$ such that for each $n \in \mathbb N \setminus \{1\}$ with $d \nmid n$ for each $d \in [2, N_{p,r}]$ we have the following. If $\mathcal L(C_p^{r-1} \oplus C_{pn}) = \mathcal L(G)$ for some finite abelian group G, then $G \cong C_p^{r-1} \oplus C_{pn}$.*

Proof. 1. Suppose that $\mathcal L(C_p \oplus C_{pn}) = \mathcal L(G)$. By Proposition 5.2 we get that $G \cong G_1 \oplus C_{pn}$ with $\exp(G_1) \mid pn$ and $\mathsf D(G_1) \le p$. Since $\exp(G_1) = 1$ is easily seen to be impossible, e.g., compare the Davenport constants, and clearly $\exp(G_1) \le \mathsf D(G_1)$, we get by our assumption on n that $\exp(G_1) = p$, and thus $G_1 \cong C_p$ establishing the claim.
2. We set $N_{p,r} = 2p^r$, which is not optimized. We know that $\max \Delta_1(G) = np - 2$, e.g., observe that $|G| \le (np)^2$. Suppose that $\mathcal L(C_p^{r-1} \oplus C_{pn}) = \mathcal L(G)$. We observe that by (3.1) and (2.2) $\mathsf D(C_p^{r-1} \oplus C_{pn}) \le \mathsf D_{\mathsf D(C_n)}(C_p^r) \le np + \mathsf D_0'(C_p^r)$; and we recall $\mathsf D_0'(C_p^r) \le \eta(C_p^r) \le p^r$ (see Section 2). By Proposition 5.2, note that by our assumption on n and the above bound for the Davenport constant it is applicable, we get

that $G \cong G_1 \oplus C_{pn}$ with $\exp(G_1) \mid pn$ and $\mathsf{D}(G_1) \le \mathsf{D}_0'(C_p^r)$. As in 1. we see that $\exp(G_1) = p$. Thus $G \cong C_p^{s-1} \oplus C_{pn}$ for some $s \in \mathbb{N} \setminus \{1\}$. Yet, as recalled in Section 2 $\mathsf{D}(C_p^{s-1} \oplus C_{pn}) = \mathsf{D}(C_p^{r-1} \oplus C_{pn})$ only if $r = s$, the claim follows. □

We note that the only problem impeding a full generalization of the characterization result to all groups with large exponent (including groups of rank two), is the problem of distinguishing (or asserting the equality) of the system of sets of lengths of groups $G_1 \oplus C_n$ and $G_2 \oplus C_n$ where $\exp(G_i) \mid n$ and the Davenport constants are equal. Yet, to overcome this problem in general might be difficult; in particular, in the case that G_1 and G_2 are too large (in an absolute sense) to allow an explicit approach, as carried out in the present paper for C_2^2 and C_3, yet are too small relative to n to have a significant effect on the system of sets of lengths.

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