THE CATENARY DEGREE OF KRULL MONOIDS I

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Abstract. Let H be a Krull monoid with finite class group G such that every class contains a prime divisor (for example, a ring of integers in an algebraic number field or a holomorphy ring in an algebraic function field). The catenary degree $\mathsf c(H)$ of H is the smallest integer N with the following property: for each $a \in H$ and each two factorizations z, z' of a, there exist factorizations $z = z_0, \ldots, z_k = z'$ of a such that, for each $i \in [1, k]$, z_i arises from z_{i-1} by replacing at most N atoms from z_{i-1} by at most N new atoms. Under a very mild condition on the Davenport constant of G, we establish a new and simple characterization of the catenary degree. This characterization gives a new structural understanding of the catenary degree. In particular, it clarifies the relationship between $\mathsf c(H)$ and the set of distances of H and opens the way towards obtaining more detailed results on the catenary degree. As first applications, we give a new upper bound on $\mathsf c(H)$ and characterize when $\mathsf c(H) \le 4$.

1. Introduction

In this paper we study the arithmetic of Krull monoids, focusing on the case that the class group is finite, and in addition, we often suppose that every class contains a prime divisor. This setting includes, in particular, rings of integers in algebraic number fields and holomorphy rings in algebraic function fields (more examples are given in Section 2). Let H be a Krull monoid with finite class group. Then sets of lengths of H have a well-defined structure: they are AAMPs (almost arithmetical multiprogressions) with universal bounds on all parameters (see [19, Section 4.7] for an overview). Moreover, a recent realization theorem reveals that this description of the sets of lengths is best possible (see [34]).

Here we focus on the catenary degree of H. This invariant considers factorizations in a more direct way and not only their lengths, and thus has found strong attention in the recent development of factorization theory (see [8, 20, 6, 17, 3]). The catenary degree $\mathsf c(H)$ of H is defined as the smallest integer N with the following property: for each $a \in H$ and each two factorizations z and z' of a, there exist factorizations $z = z_0, \ldots, z_k = z'$ of a such that, for each $i \in [1, k]$, z_i arises from z_{i-1} by replacing at most N atoms from z_{i-1} by at most N new atoms. The definition reveals immediately that H is factorial if and only if its catenary degree equals zero. Furthermore, it is easy to verify that the finiteness of the class group implies the finiteness of the catenary degree, and that the catenary degree depends only on the class group (under the assumption that every class contains a prime divisor). However, apart from this straightforward information, there is up to now almost no insight into the structure of the concatenating chains of factorizations and no information on the relationship between the catenary degree and other invariants such as the set of distances. Almost needless to say, apart from very simple cases, the precise value of the catenary degree—in terms of the group invariants of the class group—is unknown.

The present paper brings some light into the nature of the catenary degree. To do so, we introduce a new arithmetical invariant, $\daleth(H)$, which is defined as follows (see Definition 3.1): for each two atoms $u, v \in H$, we look at a factorization having the smallest number of factors besides two, say $uv = w_1 \cdot \ldots \cdot w_s$, where $s \ge 3$, $w_1, \ldots, w_s$ are atoms of H and uv has no factorization of length k with $2 < k < s$. Then $\daleth(H)$ denotes the largest possible value of s over all atoms $u, v \in H$. By definition, we have $\daleth(H) \le \mathsf c(H)$,

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and Examples 3.3 offer a list of well-studied monoids where $\daleth(H)$ is indeed strictly smaller than $\mathsf c(H)$. But the behavior is different for Krull monoids H with finite class group and every class containing a prime divisor. Under a very mild condition on the Davenport constant of the class group, we show that the catenary degree is equal to $\daleth(H)$ (see Corollary 4.3 and Remark 4.4), which immediately implies that the catenary degree equals the maximum of the set of distances plus two.

Since $\daleth(H)$ is a much more accessible invariant than the original condition given in the definition of the catenary degree, the equality $\daleth(H) = \mathsf c(H)$ widely opens the door for further investigations of the catenary degree, both for explicit computations as well as for more abstract studies based on methods from Additive and Combinatorial Number Theory (the latter is done in [18], with a focus on groups with large exponent). Exemplifying this, in Section 5, we derive an upper bound on $\daleth(H)$, and thus on $\mathsf c(H)$ as well, and then characterize Krull monoids with small catenary degree (Corollary 5.6).

2. Preliminaries

Our notation and terminology are consistent with [19]. We briefly gather some key notions. We denote by $\mathbb N$ the set of positive integers, and we put $\mathbb N_0 = \mathbb N \cup \{0\}$. For real numbers $a, b \in \mathbb R$, we set $[a, b] = \{x \in \mathbb Z \mid a \le x \le b\}$, and we define $\sup \emptyset = \max \emptyset = \min \emptyset = 0$. By a *monoid*, we always mean a commutative semigroup with identity which satisfies the cancellation law (that is, if a, b, c are elements of the monoid with $ab = ac$, then $b = c$ follows). The multiplicative semigroup of non-zero elements of an integral domain is a monoid.

Let G be an additive abelian group and $G_0 \subset G$ a subset. Then $[G_0] \subset G$ denotes the submonoid generated by G_0 and $\langle G_0 \rangle \subset G$ denotes the subgroup generated by G_0. We set $G_0^\bullet = G_0 \setminus \{0\}$. A family $(e_i)_{i \in I}$ of *nonzero* elements of G is said to be *independent* if

$$\sum_{i \in I} m_i e_i = 0 \quad \text{implies} \quad m_i e_i = 0 \quad \text{for all } i \in I, \qquad \text{where } m_i \in \mathbb Z\,.$$

If $I = [1, r]$ and $(e_1, \ldots, e_r)$ is independent, then we simply say that $e_1, \ldots, e_r$ are independent elements of G. The tuple $(e_i)_{i \in I}$ is called a *basis* if $(e_i)_{i \in I}$ is independent and $\langle \{e_i \mid i \in I\} \rangle = G$.

Let $A, B \subset G$ be subsets. Then $A + B = \{a + b \mid a \in A, b \in B\}$ is their *sumset*. If $A \subset \mathbb Z$, then the *set of distances* of A, denoted $\Delta(A)$, is the set of all differences between consecutive elements of A, formally, all $d \in \mathbb N$ for which there exist $l \in A$ such that $A \cap [l, l + d] = \{l, l + d\}$. In particular, we have $\Delta(\emptyset) = \emptyset$.

For $n \in \mathbb N$, let C_n denote a cyclic group with n elements. If G is finite with $|G| > 1$, then we have

$$G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}\,, \quad \text{and we set} \quad \mathsf d^*(G) = \sum_{i=1}^{r} (n_i - 1)\,,$$

where $r = \mathsf r(G) \in \mathbb N$ is the *rank* of G, $n_1, \ldots, n_r \in \mathbb N$ are integers with $1 < n_1 \mid \ldots \mid n_r$ and $n_r = \exp(G)$ is the exponent of G. If $|G| = 1$, then $\mathsf r(G) = 0$, $\exp(G) = 1$, and $\mathsf d^*(G) = 0$.

Monoids and factorizations. Let H be a monoid. We denote by $H^\times$ the set of invertible elements of H, and we say that H is *reduced* if $H^\times = \{1\}$. Let $H_{\text{red}} = H/H^\times = \{aH^\times \mid a \in H\}$ be the associated reduced monoid and $\mathsf q(H)$ a quotient group of H. For a subset $H_0 \subset H$, we denote by $[H_0] \subset H$ the submonoid generated by H_0. Let $a, b \in H$. We say that *a divides b* (and we write $a \,|\, b$) if there is an element $c \in H$ such that $b = ac$, and we say that a and b are *associated* ($a \simeq b$) if $a \,|\, b$ and $b \,|\, a$.

A monoid F is called *free* (*abelian, with basis* $P \subset F$) if every $a \in F$ has a unique representation of the form

$$a = \prod_{p \in P} p^{\mathsf v_p(a)} \quad \text{with} \quad \mathsf v_p(a) \in \mathbb N_0 \quad \text{and} \quad \mathsf v_p(a) = 0 \;\; \text{for almost all} \;\; p \in P\,.$$

We set $F = \mathcal{F}(P)$ and call

$$|a|_F = |a| = \sum_{p \in P} \mathsf{v}_p(a) \quad \text{the } \textit{length} \text{ of } a\,.$$

We denote by $\mathcal{A}(H)$ the *set of atoms* of H, and we call $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\rm red}))$ the *factorization monoid* of H. Further, $\pi \colon \mathsf{Z}(H) \to H_{\rm red}$ denotes the natural homomorphism given by mapping a factorization to the element it factorizes. For $a \in H$, the set

$$\mathsf{Z}(a) = \mathsf{Z}_H(a) = \pi^{-1}(aH^\times) \subset \mathsf{Z}(H) \quad \text{is called the } \textit{set of factorizations} \text{ of } a,$$

$$\mathsf{L}(a) = \mathsf{L}_H(a) = \big\{ |z| \;\big|\; z \in \mathsf{Z}(a) \big\} \subset \mathbb{N}_0 \quad \text{is called the } \textit{set of lengths} \text{ of } a, \text{ and}$$

$$\Delta(H) = \bigcup_{a \in H} \Delta\big(\mathsf{L}(a)\big) \ \subset \mathbb{N} \qquad \text{denotes the } \textit{set of distances} \text{ of } H\,.$$

The monoid H is called

- *atomic* if $\mathsf{Z}(a) \ne \emptyset$ for all $a \in H$ (equivalently, every non-unit of H may be written as a finite product of atoms of H).
- *factorial* if $|\mathsf{Z}(a)| = 1$ for all $a \in H$ (equivalently, every non-unit of H may be written as a finite product of primes of H).

Two factorizations $z,\, z' \in \mathsf{Z}(H)$ can be written in the form

$$z = u_1 \cdot \ldots \cdot u_l v_1 \cdot \ldots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_l w_1 \cdot \ldots \cdot w_n$$

with

$$\{v_1, \ldots, v_m\} \cap \{w_1, \ldots, w_n\} = \emptyset,$$

where $l,\, m,\, n \in \mathbb{N}_0$ and $u_1, \ldots, u_l,\, v_1, \ldots, v_m,\, w_1, \ldots, w_n \in \mathcal{A}(H_{\rm red})$. Then $\gcd(z, z') = u_1 \cdot \ldots \cdot u_l$, and we call $\mathsf{d}(z, z') = \max\{m,\, n\} = \max\{|z \gcd(z,z')^{-1}|, |z' \gcd(z,z')^{-1}|\} \in \mathbb{N}_0$ the *distance* between z and z'.

Krull monoids. The theory of Krull monoids is presented in the monographs [25, 24, 19]. We briefly summarize what is needed in the sequel. Let H and D be monoids. A monoid homomorphism $\varphi \colon H \to D$ is called

- a *divisor homomorphism* if $\varphi(a) \mid \varphi(b)$ implies $a \,|\, b$, for all $a, b \in H$.
- *cofinal* if, for every $a \in D$, there exists some $u \in H$ such that $a \,|\, \varphi(u)$.
- a *divisor theory* (for H) if $D = \mathcal{F}(P)$ for some set P, φ is a divisor homomorphism, and for every $p \in P$ (equivalently, for every $a \in \mathcal{F}(P)$), there exists a finite subset $\emptyset \ne X \subset H$ satisfying $\gcd\big(\varphi(X)\big) = p$.

Note that, by definition, every divisor theory is cofinal. We call $\mathcal{C}(\varphi) = \mathsf{q}(D)/\mathsf{q}(\varphi(H))$ the class group of φ and use additive notation for this group. For $a \in \mathsf{q}(D)$, we denote by $[a] = [a]_\varphi = a\,\mathsf{q}(\varphi(H)) \in \mathsf{q}(D)/\mathsf{q}(\varphi(H))$ the class containing a. If $\varphi \colon H \to \mathcal{F}(P)$ is a cofinal divisor homomorphism, then

$$G_P = \{[p] = p\mathsf{q}(\varphi(H)) \mid p \in P\} \subset \mathcal{C}(\varphi)$$

is called the *set of classes containing prime divisors*, and we have $[G_P] = \mathcal{C}(\varphi)$. If $H \subset D$ is a submonoid, then H is called *cofinal* (*saturated*, resp.) in D if the imbedding $H \hookrightarrow D$ is cofinal (a divisor homomorphism, resp.).

The monoid H is called a *Krull monoid* if it satisfies one of the following equivalent conditions ([19, Theorem 2.4.8]) :

- H is v-noetherian and completely integrally closed.
- H has a divisor theory.
- $H_{\rm red}$ is a saturated submonoid of a free monoid.

In particular, H is a Krull monoid if and only if H_{red} is a Krull monoid. Let H be a Krull monoid. Then a divisor theory $\varphi\colon H \to \mathcal{F}(P)$ is unique up to unique isomorphism. In particular, the class group $\mathcal{C}(\varphi)$ defined via a divisor theory of H and the subset of classes containing prime divisors depend only on H. Thus it is called the *class group* of H and is denoted by $\mathcal{C}(H)$.

An integral domain R is a Krull domain if and only if its multiplicative monoid $R\setminus\{0\}$ is a Krull monoid, and a noetherian domain is Krull if and only if it is integrally closed. Rings of integers, holomorphy rings in algebraic function fields, and regular congruence monoids in these domains are Krull monoids with finite class group such that every class contains a prime divisor ([19, Section 2.11]). Monoid domains and power series domains that are Krull are discussed in [23, 28, 29].

Zero-sum sequences. Let $G_0 \subset G$ be a subset and $\mathcal{F}(G_0)$ the free monoid with basis G_0. According to the tradition of combinatorial number theory, the elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0. For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G_0} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G_0)\,,$$

we call $\mathsf{v}_g(S)$ the *multiplicity* of g in S,

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ the } \textit{length} \text{ of } S\,, \quad \mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G \text{ the } \textit{support} \text{ of } S\,,$$

$$\sigma(S) = \sum_{i=1}^{l} g_i \text{ the } \textit{sum} \text{ of } S \quad \text{and} \quad \Sigma(S) = \Big\{ \sum_{i \in I} g_i \mid \emptyset \ne I \subset [1,l] \Big\} \text{ the } \textit{set of subsums} \text{ of } S\,.$$

The sequence S is called

- *zero-sum free* if $0 \notin \Sigma(S)$,
- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *minimal zero-sum sequence* if it is a nontrivial zero-sum sequence and every proper subsequence is zero-sum free.

The monoid

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\}$$

is called the *monoid of zero-sum sequences* over G_0, and we have $\mathcal{B}(G_0) = \mathcal{B}(G) \cap \mathcal{F}(G_0)$. Since $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is saturated, $\mathcal{B}(G_0)$ is a Krull monoid (the atoms are precisely the minimal zero-sum sequences). Its significance for the investigation of general Krull monoids is demonstrated by Lemma 3.6.

For every arithmetical invariant $*(H)$ defined for a monoid H, we write $*(G_0)$ instead of $*(\mathcal{B}(G_0))$. In particular, we set $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ and $\Delta(G_0) = \Delta(\mathcal{B}(G_0))$. We define the *Davenport constant* of G_0 by

$$\mathsf{D}(G_0) = \sup\big\{|U| \;\big|\; U \in \mathcal{A}(G_0)\big\} \in \mathbb{N}_0 \cup \{\infty\}\,,$$

and the following properties will be used throughout the manuscript without further mention. If G_0 is finite, then $\mathsf{D}(G_0)$ is finite ([19, Theorem 3.4.2]). Suppose that $G_0 = G$ is finite. Then

$$1 + \mathsf{d}^*(G) \le \mathsf{D}(G)\,, \tag{2.1}$$

and equality holds if G is a p-group or $\mathsf{r}(G) \le 2$ (see [19, Chapter 5] and [17, Section 4.2]).

3. The catenary degree and its refinements

We recall the definition of the catenary degree $\mathsf{c}(H)$ of an atomic monoid H and introduce, for all $k \in \mathbb{N}$, the refinements $\mathsf{c}_k(H)$.

Definition 3.1. Let H be an atomic monoid and $a \in H$.

1. Let $z,\ z' \in \mathsf Z(a)$ be factorizations of a and $N \in \mathbb N_{\ge 0} \cup \{\infty\}$. A finite sequence $z_0,\ z_1, \ldots, z_k$ in $\mathsf Z(a)$ is called an *N-chain of factorizations* from z to z' if $z = z_0,\ \ z' = z_k$ and $\mathsf d(z_{i-1}, z_i) \le N$ for every $i \in [1, k]$.

   If there exists an N-chain of factorizations from z to z', we say that z and z' can be *concatenated* by an N-chain.
2. Let $\mathsf c_H(a) = \mathsf c(a) \in \mathbb N_0 \cup \{\infty\}$ denote the smallest $N \in \mathbb N_0 \cup \{\infty\}$ such that any two factorizations $z,\ z' \in \mathsf Z(a)$ can be concatenated by an N-chain.
3. For $k \in \mathbb N$, we set

$$\mathsf c_k(H) = \sup\{\mathsf c(a) \mid a \in H \text{ with } \min \mathsf L(a) \le k\} \in \mathbb N_0 \cup \{\infty\}\,,$$

   and we call

$$\mathsf c(H) = \sup\{\mathsf c(a) \mid a \in H\} \in \mathbb N_0 \cup \{\infty\}$$

   the *catenary degree* of H.
4. We set

$$\daleth(H) = \sup\big\{ \min\big(\mathsf L(uv) \setminus \{2\}\big) \mid u,\, v \in \mathcal A(H)\big\}\,,$$

   with the convention that $\min \emptyset = \sup \emptyset = 0$.

Let all notations be as above. Then $\daleth(H) = 0$ if and only if $\mathsf L(uv) = \{2\}$ for all $u, v \in \mathcal A(H)$. By definition, we have $\mathsf c(a) \le \sup \mathsf L(a)$. Let $z, z' \in \mathsf Z(a)$. Then, by definition of the distance, we have $z = z'$ if and only if $\mathsf d(z, z') = 0$. Thus, $\mathsf c(a) = 0$ if and only if a has unique factorization (that is, $|\mathsf Z(a)| = 1$), and hence H is factorial if and only if $\mathsf c(H) = 0$. Suppose that H is not factorial. Then there is a $b \in H$ having two distinct factorizations $y, y' \in \mathsf Z(b)$. A simple calculation (see [19, Lemma 1.6.2] for details) shows that

$$2 + \big||y| - |y'|\big| \le \mathsf d(y, y')\,, \quad \text{and hence} \quad 2 + \sup \Delta(\mathsf L(b)) \le \mathsf c(b)\,. \tag{3.1}$$

The following lemma gathers some simple properties of the invariants introduced in Definition 3.1.

Lemma 3.2. *Let H be an atomic monoid.*

1. *We have $0 = \mathsf c_1(H) \le \mathsf c_2(H) \le \ldots$ and*

$$\mathsf c(H) = \sup\{\mathsf c_k(H) \mid k \in \mathbb N\}\,.$$

2. *We have $\mathsf c(H) = \mathsf c_k(H)$ for all $k \in \mathbb N$ with $k \ge \mathsf c(H)$.*
3. *If $\mathsf c_k(H) > \mathsf c_{k-1}(H)$ for some $k \in \mathbb N_{\ge 2}$, then $\mathsf c_k(H) \ge k$.*
4. $\sup \Delta(H) \le \sup\{\mathsf c_k(H) - k \mid k \in \mathbb N$ *with* $2 \le k < \mathsf c(H)\}$. *Moreover, if $\mathsf c(H) \in \mathbb N$, then there is some minimal $m \in \mathbb N$ with $\mathsf c(H) = \mathsf c_m(H)$, and then*

$$\sup\{\mathsf c_k(H) - k \mid k \in \mathbb N_{\ge 2}\} = \max\{\mathsf c_k(H) - k \mid k \in [2, m]\}\,.$$

5. *For every $k \in \mathbb N$, we have*

$$\begin{aligned}\mathsf c_k(H) &\ge \sup\{\mathsf c(a) \mid a \in H \text{ with } k \in \mathsf L(a)\}\\ &\ge \sup\{\mathsf c(a) \mid a \in H \text{ with } k = \min \mathsf L(a)\}\,,\end{aligned}$$

   and equality holds if H contains a prime element.
6. *If H is not factorial, then*

$$\daleth(H) \le \min\big\{2 + \sup \Delta(H)\,, \mathsf c_2(H)\big\} \le \max\big\{2 + \sup \Delta(H)\,, \mathsf c_2(H)\big\} \le \mathsf c(H)\,. \tag{3.2}$$

Proof. 1. Obvious.

2. If $\mathsf{c}(H)$ is either zero or infinite, then the assertion is clear. Suppose that $\mathsf{c}(H) = m \in \mathbb{N}$. Then there is an $a \in H$ with factorizations $z = u_1 \cdot \ldots \cdot u_l \in \mathsf{Z}(a)$ and $z' = v_1 \cdot \ldots \cdot v_m \in \mathsf{Z}(a)$, where $l \in [1, m]$ and $u_1, \ldots, u_l, v_1, \ldots, v_m \in \mathcal{A}(H_{\text{red}})$, such that $\mathsf{d}(z, z') = \max\{l, m\} = m$ and z and z' cannot be concatenated by a d-chain of factorizations for any $d < m$. Since $\min \mathsf{L}(a) \le m$, we get, for all $k \ge m$, that

$$m \le \mathsf{c}(a) \le \mathsf{c}_m(H) \le \mathsf{c}_k(H) \le \mathsf{c}(H) = m\,,$$

and the assertion follows.

3. Suppose $k \in \mathbb{N}_{\ge 2}$ and $\mathsf{c}_k(H) > \mathsf{c}_{k-1}(H)$. Let $a \in H$ with $\min \mathsf{L}(a) \le k$ such that $\mathsf{c}(a) = \mathsf{c}_k(H)$. We note that actually $\min \mathsf{L}(a) = k$, as otherwise $\mathsf{c}_{k-1}(H) \ge \mathsf{c}(a)$, a contradiction. Let $z, z' \in \mathsf{Z}(a)$ such that $\mathsf{d}(z, z') = \mathsf{c}(a) = \mathsf{c}_k(H)$ and such that z and z' cannot be concatenated by an N-chain for $N < \mathsf{c}(a)$. Let $x = \gcd(z, z')$. We note that $\min\{|x^{-1}z|, |x^{-1}z'|\} \ge k$, as otherwise $x^{-1}z$ and $x^{-1}z'$ can be concatenated by a $\mathsf{c}_{k-1}(H)$-chain, implying that z and z' can be concatenated by such a chain. Thus, $\mathsf{d}(z, z') \ge k$, establishing the claim.

4. It suffices to show that, for every $d \in \Delta(H)$, there is a $k \in \mathbb{N}$ with $2 \le k < \mathsf{c}(H)$ and $d \le \mathsf{c}_k(H) - k$. Let $d \in \Delta(H)$. Then there is an element $a \in H$ and factorizations $z, z' \in \mathsf{Z}(a)$ such that $|z'| - |z| = d$ and $\mathsf{L}(a) \cap [|z|, |z'|] = \{|z|, |z'|\}$. For $N = \min\{|z'|, \mathsf{c}(H)\}$, there is an N-chain $z = z_0, \ldots, z_l = z'$ of factorizations from z to z'. We may suppose that this chain cannot be refined. This means that, for any $i \in [1, l]$, there is no d_i-chain concatenating z_{i-1} and z_i with $d_i < \mathsf{d}(z_{i-1}, z_i)$. There exists some $i \in [1, l]$ such that $|z_{i-1}| \le |z| < |z'| \le |z_i|$, say $z_{i-1} = xv_1 \cdot \ldots \cdot v_s$ and $z_i = xw_1 \cdot \ldots \cdot w_t$, where $x = \gcd(z_{i-1}, z_i)$, $s, t \in \mathbb{N}$ and $v_1, \ldots, v_s, w_1, \ldots, w_t \in \mathcal{A}(H_{\text{red}})$. We set $b = \pi(v_1 \cdot \ldots \cdot v_s)$, $k = \min \mathsf{L}(b)$ and get that

$$2 \le k \le s < t = \max\{s, t\} = \mathsf{d}(z_{i-1}, z_i) = \mathsf{d}(v_1 \cdot \ldots \cdot v_s, w_1 \cdot \ldots \cdot w_t) \le N \le \mathsf{c}(H)\,.$$

Since the two factorizations $v_1 \cdot \ldots \cdot v_s$ and $w_1 \cdot \ldots \cdot w_t$ of b can be concatenated by a $\mathsf{c}_k(H)$-chain and since the original chain $z_0, \ldots, z_l$ cannot be refined, it follows that $t = \mathsf{d}(v_1 \cdot \ldots \cdot v_s, w_1 \cdot \ldots \cdot w_t) \le \mathsf{c}_k(H)$. Therefore, since $|z_{i-1}| \le |z| < |z'| \le |z_i|$, it follows that

$$d = |z'| - |z| \le |z_i| - |z_{i-1}| = t - s \le \mathsf{c}_k(H) - k\,.$$

Now suppose that $\mathsf{c}(H) \in \mathbb{N}$. By part 2, there is some minimal $m \in \mathbb{N}$ with $\mathsf{c}(H) = \mathsf{c}_m(H)$. Since $\mathsf{c}(H) > 0$, it follows that $m \ge 2$. Let $k \in \mathbb{N}_{\ge 2}$. If $k \ge m$, then $\mathsf{c}(H) = \mathsf{c}_m(H) = \mathsf{c}_k(H)$ and $\mathsf{c}_k(H) - k \le \mathsf{c}_m(H) - m$. Thus the assertion follows.

5. The inequalities are clear. Suppose that $p \in H$ is a prime element. Let $N \in \mathbb{N}$ and $a \in H$ with $\mathsf{c}(a) \ge N$ and $\min \mathsf{L}(a) \le k$. Then, for $t = k - \min \mathsf{L}(a)$, we have $\mathsf{L}(ap^t) = t + \mathsf{L}(a)$, $\min \mathsf{L}(ap^t) = k$ and $\mathsf{c}(ap^t) = \mathsf{c}(a) \ge N$. This implies that

$$\sup\{\mathsf{c}(a) \mid a \in H, \text{ with } k = \min \mathsf{L}(a)\} \ge \sup\{\mathsf{c}(a) \mid a \in H \text{ with } \min \mathsf{L}(a) \le k\}\,,$$

and thus equality holds in both inequalities.

6. Suppose that H is not factorial. We start with the left inequality. If $\mathsf{L}(uv) = \{2\}$ for all $u, v \in \mathcal{A}(H)$, then $\urcorner(H) = 0 \le \min\{\sup \Delta(H) + 2, \mathsf{c}_2(H)\}$. Let $u, v \in \mathcal{A}(H)$ with $\mathsf{L}(uv) = \{2, d_1, \ldots, d_l\}$ with $l \in \mathbb{N}$ and $2 < d_1 < \ldots < d_l$. Then $d_1 - 2 \in \Delta\big(\mathsf{L}(uv)\big) \subset \Delta(H)$, and thus we get $\urcorner(H) - 2 \le \sup \Delta(H)$. Let $z' = w_1 \cdot \ldots \cdot w_{d_1} \in \mathsf{Z}(uv)$ be a factorization of length d_1. Then, from the definition of d_1, we see $z = uv$ and z' cannot be concatenated by a d-chain with $d < d_1$. Thus $d_1 \le \mathsf{c}(uv) \le \mathsf{c}_2(H)$, and hence $\urcorner(H) \le \mathsf{c}_2(H)$.

To verify the right inequality, note that $\mathsf{c}_2(H) \le \mathsf{c}(H)$ follows from the definition. If $b \in H$ with $|\mathsf{Z}(b)| > 1$, then (3.1) shows that $2 + \sup \Delta\big(\mathsf{L}(b)\big) \le \mathsf{c}(b) \le \mathsf{c}(H)$, and therefore $2 + \sup \Delta(H) \le \mathsf{c}(H)$. $\square$

Corollary 4.3 will show that, for the Krull monoids under consideration, equality holds throughout (3.2). Obviously, such a result is far from being true in general. This becomes clear from the characterization of the catenary degree in terms of minimal relations, recently given by S. Chapman et al. in [8]. But we will demonstrate this by very explicit examples which also deal with the refinements $\mathsf c_k(H)$.

Examples 3.3.

1. Numerical monoids. The arithmetic of numerical monoids has been studied in detail in recent years (see [5, 6, 1, 7, 10, 9, 31] and the monograph [33]). The phenomena we are looking at here can already be observed in the most simple case where the numerical monoid has two generators.

Let $H = [\{d_1, d_2\}] \subset (\mathbb N_0, +)$ be a numerical monoid generated by integers d_1 and d_2, where $1 < d_1 < d_2$ and $\gcd(d_1, d_2) = 1$. Then $\mathcal A(H) = \{d_1, d_2\}$, and $d_1 d_2$ is the smallest element $a \in H$—with respect to the usual order in $(\mathbb N_0, \le)$—with $|\mathsf Z(a)| > 1$. Thus $\mathsf c_k(H) = 0$ for all $k < d_1$ (hence $\daleth(H) = 0$ if $d_1 > 2$), $\Delta(H) = \{d_2 - d_1\}$ and $\mathsf c_{d_1}(H) = d_2 = \mathsf c(H)$ (details of all this are worked out in [19, Example 3.1.6]). Thus, when $d_1 > 2$, the second two inequalities in Lemma 3.2.6 are strict.

2. Finitely primary monoids. A monoid H is called *finitely primary* if there exist $s, \alpha \in \mathbb N$ with the following properties:

> H is a submonoid of a factorial monoid $F = F^\times \times [p_1, \ldots, p_s]$ with s pairwise non-associated prime elements $p_1, \ldots, p_s$ satisfying
>
> $$H \setminus H^\times \subset p_1 \cdot \ldots \cdot p_s F \quad \text{and} \quad (p_1 \cdot \ldots \cdot p_s)^\alpha F \subset H\,.$$

The multiplicative monoid of every one-dimensional local noetherian domain R whose integral closure $\overline R$ is a finitely generated R-module is finitely primary ([19, Proposition 2.10.7]). Moreover, the monoid of invertible ideals of an order in a Dedekind domain is a product of a free monoid and a finite product of finitely primary monoids (see [19, Theorem 3.7.1]).

Let H be as above with $s \ge 2$. Then $3 \le \mathsf c(H) \le 2\alpha + 1$, $\min \mathsf L(a) \le 2\alpha$ for all $a \in H$, and hence $\sup\{\mathsf c(a) \mid a \in H \text{ with } k = \min \mathsf L(a)\} = 0$ for all $k > 2\alpha$ (see [19, Theorem 3.1.5]). This shows that the assumption in Lemma 3.2.5 requiring the existence of a prime element cannot be omitted. Concerning the inequalities in Lemma 3.2.6, equality throughout can hold (as in [19, Examples 3.1.8]) but does not hold necessarily, as the following example shows. Let $H \subset (\mathbb N_0^s, +)$, with $s \ge 3$, be the submonoid generated by

$$A = \{(m, 1, \ldots, 1), (1, m, 1, \ldots, 1), \ldots, (1, \ldots, 1, m) \mid m \in \mathbb N\}\,.$$

Then H is finitely primary with $A = \mathcal A(H)$ and $\daleth(H) = 0 < \mathsf c(H)$.

3. Finitely generated Krull monoids. Let G be an abelian group and $r, n \in \mathbb N_{\ge 3}$ with $n \ne r+1$. Let $e_1, \ldots, e_r \in G$ be independent elements with $\operatorname{ord}(e_i) = n$ for all $i \in [1, r]$, $e_0 = -(e_1 + \ldots + e_r)$ and $G_0 = \{e_0, \ldots, e_r\}$. Then $\mathcal B(G_0)$ is a finitely generated Krull monoid, $\Delta(G_0) = \{|n - r - 1|\}$, $\mathsf c(G_0) = \max\{n, r+1\}$ and

$$0 = \daleth(H) = \mathsf c_2(H) < 2 + \max \Delta(H) < \mathsf c(H)\,.$$

(see [19, Proposition 4.1.2]).

4. k-factorial monoids. An atomic monoid H is called k-factorial, where $k \in \mathbb N$, if every element $a \in H$ with $\min \mathsf L(a) \le k$ has unique factorization; k-factorial and, more generally, quasi-k-factorial monoids and domains have been studied in [2]. Clearly, if H is k-factorial but not $k+1$-factorial, then $0 = \mathsf c_k(H) < \mathsf c_{k+1}(H)$.

5. Half-factorial monoids. An atomic monoid H is called half-factorial if $\Delta(H) = \emptyset$ (cf. [19, Section 1.2]). Then, $\daleth(H) = 0$ and it follows that $\mathsf c_k(H) \le k$ for each $k \in \mathbb N$. Thus, by Lemma 3.2.3, we get that if $\mathsf c_k(H) > \mathsf c_{k-1}(H)$, then $\mathsf c_k(H) = k$. Without additional restriction on H, the set $K \subset \mathbb N_{\ge 2}$ of all k with $\mathsf c_k(H) > \mathsf c_{k-1}(H)$ can be essentially arbitrary; an obvious restriction is that it is finite for $\mathsf c(H)$ finite.

The arithmetic of Krull monoids is studied via transfer homomorphisms. We recall the required terminology and collect the results needed for the sequel.

Definition 3.4. A monoid homomorphism $\theta \colon H \to B$ is called a *transfer homomorphism* if it has the following properties:

- **(T 1)** $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.
- **(T 2)** If $u \in H$, $b, c \in B$ and $\theta(u) = bc$, then there exist $v, w \in H$ such that $u = vw$, $\theta(v) \simeq b$ and $\theta(w) \simeq c$.

Note that the second part of (T1) means precisely that units map to units and non-units map to non-units, while the first part means θ is surjective up to units. Every transfer homomorphism θ gives rise to a unique extension $\overline{\theta} \colon \mathsf Z(H) \to \mathsf Z(B)$ satisfying

$$\overline{\theta}(uH^\times) = \theta(u)B^\times \quad \text{for each} \quad u \in \mathcal A(H)\,.$$

For $a \in H$, we denote by $\mathsf c(a, \theta)$ the smallest $N \in \mathbb N_0 \cup \{\infty\}$ with the following property:

If z, $z' \in \mathsf Z_H(a)$ and $\overline{\theta}(z) = \overline{\theta}(z')$, then there exist some $k \in \mathbb N_0$ and factorizations $z = z_0, \ldots, z_k = z' \in \mathsf Z_H(a)$ such that $\overline{\theta}(z_i) = \overline{\theta}(z)$ and $\mathsf d(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$ (that is, z and z' can be concatenated by an N-chain in the fiber $\mathsf Z_H(a) \cap \overline{\theta}^{-1}(\overline{\theta}(z))$).

Then

$$\mathsf c(H, \theta) = \sup\{\mathsf c(a, \theta) \mid a \in H\} \in \mathbb N_0 \cup \{\infty\}$$

denotes the *catenary degree in the fibres.*

Lemma 3.5. *Let $\theta \colon H \to B$ be a transfer homomorphism of atomic monoids and $\overline{\theta} \colon \mathsf Z(H) \to \mathsf Z(B)$ its extension to the factorization monoids.*

1. *For every $a \in H$, we have $\mathsf L_H(a) = \mathsf L_B\big(\theta(a)\big)$. In particular, we have $\Delta(H) = \Delta(B)$ and $\daleth(H) = \daleth(B)$.*
2. *For every $a \in H$, we have $\mathsf c\big(\theta(a)\big) \le \mathsf c(a) \le \max\{\mathsf c\big(\theta(a)\big), \, \mathsf c(a, \theta)\}$.*
3. *For every $k \in \mathbb N$, we have*
$$\mathsf c_k(B) \le \mathsf c_k(H) \le \max\{\mathsf c_k(B), \mathsf c(H, \theta)\},$$
and hence
$$\mathsf c(B) \le \mathsf c(H) \le \max\{\mathsf c(B), \mathsf c(H, \theta)\}\,.$$

Proof. 1. and 2. See [19, Theorem 3.2.5].

3. Since, for every $a \in H$, we have $\mathsf L(a) = \mathsf L\big(\theta(a)\big)$, it follows that $\min \mathsf L(a) = \min \mathsf L\big(\theta(a)\big)$, and thus parts 1 and 2 imply both inequalities. $\square$

Lemma 3.6. *Let H be a Krull monoid, $\varphi \colon H \to F = \mathcal F(P)$ a cofinal divisor homomorphism, $G = \mathcal C(\varphi)$ its class group, and $G_P \subset G$ the set of classes containing prime divisors. Let $\widetilde{\boldsymbol\beta} \colon F \to \mathcal F(G_P)$ denoted the unique homomorphism defined by $\widetilde{\boldsymbol\beta}(p) = [p]$ for all $p \in P$.*

1. *The homomorphism $\boldsymbol\beta = \widetilde{\boldsymbol\beta} \circ \varphi \colon H \to \mathcal B(G_P)$ is a transfer homomorphism with $\mathsf c(H, \boldsymbol\beta) \le 2$.*
2. *For every $k \in \mathbb N$, we have*
$$\mathsf c_k(G_P) \le \mathsf c_k(H) \le \max\{\mathsf c_k(G_P), 2\},$$
and hence
$$\mathsf c(G_P) \le \mathsf c(H) \le \max\{\mathsf c(G_P), 2\}\,.$$
3. $\daleth(H) = \daleth(G_P) \le \mathsf D(G_P)$.

Proof. 1. This follows from [19, Theorem 3.4.10].

2. This follows from part 1 and Lemma 3.5.

3. Since β is a transfer homomorphism, we have $\daleth(H) = \daleth(G_P)$ by Lemma 3.5. In order to show that $\daleth(G_P) \le \mathsf D(G_P)$, let $U_1, U_2 \in \mathcal A(G_P)$. If $\mathsf D(G_P) = 1$, then $G_P = \{0\}$, $U = V = 0$ and $\daleth(G_P) = 0$. Suppose that $\mathsf D(G_P) \ge 2$ and consider a factorization $U_1U_2 = W_1 \cdot \ldots \cdot W_s$, where $s \in \mathbb N$ and $W_1, \ldots, W_s \in \mathcal A(G_P)$. It suffices to show that $s \le \mathsf D(G_P)$. For $i \in [1, s]$, we set $W_i = W_i^{(1)} W_i^{(2)}$ with $W_i^{(1)}, W_i^{(2)} \in \mathcal F(G_P)$ such that $U_1 = W_1^{(1)} \cdot \ldots \cdot W_s^{(1)}$ and $U_2 = W_1^{(2)} \cdot \ldots \cdot W_s^{(2)}$. If there are $i \in [1, s]$ and $j \in [1, 2]$, say $i = j = 1$, such that $W_i^{(j)} = W_1^{(1)} = 1$, then $W_1 = W_1^{(2)} \,|\, U_2$; hence $W_1 = U_2$, $W_2 = U_1$ and $s = 2 \le \mathsf D(G_P)$. Otherwise, we have $W_1^{(j)}, \ldots, W_s^{(j)} \in \mathcal F(G_P) \setminus \{1\}$, and hence $s \le \sum_{i=1}^s |W_i^{(j)}| = |U_j| \le \mathsf D(G_P)$. □

4. A structural result for the catenary degree

In Theorem 4.2 we obtain a structural result for the catenary degree. Since it is relevant for the discussion of this result, we start with a technical result.

Proposition 4.1. *Let G be an abelian group.*

1. *Let $G_0 = \{e_0, \ldots, e_r, -e_0, \ldots, -e_r\} \subset G$ be a subset with $e_1, \ldots, e_r \in G$ independent and $e_0 = k_1e_1 + \ldots + k_re_r$, where $k_i \in \mathbb N$ and $2k_i \le \operatorname{ord}(e_i)$ for all $i \in [1, r]$. If $\sum_{i=1}^r k_i \ne 1$, then $\daleth(G_0) \ge k_1 + \ldots + k_r + 1$.*
2. *Let $G_0 = \{-e, e\} \subset G$ be a subset with $3 \le \operatorname{ord}(e) < \infty$. Then $\daleth(G_0) \ge \operatorname{ord}(e)$.*
3. *Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $|G| \ge 3$ and $1 < n_1 \mid \ldots \mid n_r$, and let $(e_1, \ldots, e_r)$ be a basis of G with $\operatorname{ord}(e_i) = n_i$ for all $i \in [1, r]$. If $\{e_0, \ldots, e_r, -e_0, \ldots, -e_r\} \subset G_0 \subset G$, where $e_0 = \sum_{i=1}^r \lfloor \frac{n_i}{2} \rfloor e_i$, then $\daleth(G_0) \ge \max\{n_r,\, 1 + \sum_{i=1}^r \lfloor \frac{n_i}{2} \rfloor\}$.*

Proof. 1. If

$$A = e_0(-e_0) \prod_{i=1}^r e_i^{k_i}(-e_i)^{k_i},$$

then $\mathsf L(A) = \{2, k_1 + \ldots + k_r + 1\}$ (see [19, Lemma 6.4.1]). Thus, if $\sum_{i=1}^r k_i \ne 1$, the assertion follows by definition of $\daleth(G_0)$.

2. Let $n = \operatorname{ord}(e)$. Since $\mathsf L\big((-e)^n e^n\big) = \{2, n\}$, we get $\daleth(G_0) \ge n$.

3. Clear, by parts 1 and 2. □

Theorem 4.2. *Let H be a Krull monoid, $\varphi\colon H \to F = \mathcal F(P)$ a cofinal divisor homomorphism, $G = \mathcal C(\varphi)$ its class group, and $G_P \subset G$ the set of classes containing prime divisors. Then*

$$\mathsf c(H) \le \max\Big\{ \Big\lfloor \frac12 \mathsf D(G_P) + 1 \Big\rfloor,\ \daleth(G_P) \Big\}. \tag{4.1}$$

Proof. By Lemma 3.6, we have $\mathsf c(H) \le \max\{\mathsf c(G_P), 2\}$. If $\mathsf D(G_P) = 1$, then $G_P = \{0\}$, $G = [G_P] = \{0\}$, $H = F$ and $\mathsf c(H) = 0$. Thus we may suppose that $2 \le \mathsf D(G_P) < \infty$, and it is sufficient to show that

$$\mathsf c(G_P) \le d_0, \quad \text{where} \quad d_0 = \max\Big\{ \Big\lfloor \frac12 \mathsf D(G_P) + 1 \Big\rfloor,\ \daleth(G_P) \Big\}.$$

So we have to verify that, for $A \in \mathcal B(G_P^\bullet)$ and $z, z' \in \mathsf Z(A)$, there is a d_0-chain of factorizations between z and z'. Assuming this is false, consider a counter example $A \in \mathcal B(G_P^\bullet)$ such that $|A|$ is minimal, and for this A, consider a pair of factorizations $z, z' \in \mathsf Z(A)$ for which no d_0-chain between z and z' exists such that $|z| + |z'|$ is maximal (note $|A|$ is a trivial upper bound for the length of a factorization of A).

Note we may assume

$$\max\{|z|,\, |z'|\} \ge d_0 + 1 \ge \frac{1}{2}\mathsf D(G_P) + \frac{3}{2}, \tag{4.2}$$

else the chain z, z' is a d_0-chain between z and z', as desired. We continue with the following assertion.

A. Let

$$y = U_1 \cdot \ldots \cdot U_r \in \mathsf Z(A) \quad \text{and} \quad y' = V_1 \cdot \ldots \cdot V_s \in \mathsf Z(A)\,, \quad \text{where} \quad U_i,\, V_j \in \mathcal A(G_P)\,,$$

be two factorizations of A with $V_{j_1} | U_1 \cdot \ldots \cdot U_r U_{j_2}^{-1}$, for some $j_1 \in [1, s]$ and $j_2 \in [1, r]$. Then there is a d_0-chain of factorizations of A between y and y'.

Proof of **A**. We may assume $j_1 = 1$, $j_2 = r$, and we obtain a factorization

$$U_1 \cdot \ldots \cdot U_{r-1} = V_1 W_1 \cdot \ldots \cdot W_t\,,$$

where $W_1, \ldots, W_t \in \mathcal A(G_P)$. By the minimality of $|A|$, there is a d_0-chain of factorizations $y_0, \ldots, y_k$ between $y_0 = U_1 \cdot \ldots \cdot U_{r-1}$ and $y_k = V_1 W_1 \cdot \ldots \cdot W_t$, and there is a d_0-chain of factorizations $z_0, \ldots, z_l$ between $z_0 = W_1 \cdot \ldots \cdot W_t U_r$ and $z_l = V_2 \cdot \ldots \cdot V_s$. Then

$$y = y_0 U_r,\, y_1 U_r, \ldots, y_k U_r = V_1 z_0,\, V_1 z_1, \ldots, V_1 z_l = y'$$

is a d_0-chain between y and y'. □

We set $z = U_1 \cdot \ldots \cdot U_r$ and $z' = V_1 \cdot \ldots \cdot V_s$, where all U_i, $V_j \in \mathcal A(G_P)$, and without loss of generality we assume that $r \ge s$. Then, in view of (4.2) and $\mathsf D(G_P) \ge 2$, it follows that

$$r \ge d_0 + 1 \ge \frac{1}{2}\mathsf D(G_P) + \frac{3}{2} > 2. \tag{4.3}$$

Clearly, $s = 1$ would imply $r = 1$, and thus we get $s \ge 2$.

Suppose $\max \mathsf L(V_1V_2) \ge 3$. Then, by definition of $\daleth(G_P)$, there exists $y \in \mathsf Z(V_1V_2)$ with $3 \le |y| \le \daleth(G_P)$ and

$$\mathsf d(z', yV_3 \cdot \ldots \cdot V_s) = \mathsf d(V_1V_2, y) = |y| \le \daleth(G_P)\,. \tag{4.4}$$

But, since $|z| + |yV_3 \cdot \ldots \cdot V_s| > |z| + |z'|$, it follows, from the maximality of $|z| + |z'|$, that there is a d_0-chain of factorizations between $yV_3 \cdot \ldots \cdot V_s$ and z, and thus, in view of (4.4), a d_0-chain concatenating z' and z, a contradiction. So we may instead assume $\max \mathsf L(V_1V_2) = 2$.

As a result, if $s = 2$, then $V_1V_2 = A$ and $\mathsf L(A) = \{2\}$, contradicting $2 < r \in \mathsf L(A)$ (cf. (4.3)). Therefore we have $s \ge 3$.

We set $V_1 = V_1^{(1)} \cdot \ldots \cdot V_1^{(r)}$ and $V_2 = V_2^{(1)} \cdot \ldots \cdot V_2^{(r)}$, where $V_1^{(j)} V_2^{(j)} | U_j$ for all $j \in [1, r]$. In view of **A**, we see that each $V_1^{(i)}$ and $V_2^{(j)}$ is nontrivial. Thus (4.3) implies

$$|V_1V_2| \ge 2r \ge \mathsf D(G_P) + 3. \tag{4.5}$$

By the pigeonhole principle and in view of (4.3), there exists some $j \in [1, r]$, say $j = r$, such that

$$|V_1^{(r)} V_2^{(r)}| \le \frac{1}{r}|V_1V_2| \le \frac{2\mathsf D(G_P)}{r} < 4\,.$$

As a result, it follows in view of (4.5) that

$$|V_1^{(1)} \cdot \ldots \cdot V_1^{(r-1)} V_2^{(1)} \cdot \ldots \cdot V_2^{(r-1)}| \ge |V_1V_2| - 3 \ge \mathsf D(G_P). \tag{4.6}$$

Thus there exists a $W_1 \in \mathcal A(G_P)$ such that $W_1 \mid V_1^{(1)} \cdot \ldots \cdot V_1^{(r-1)} V_2^{(1)} \cdot \ldots \cdot V_2^{(r-1)}$.

Let $V_1V_2 = W_1 \cdot \ldots \cdot W_t$, where $W_2, \ldots, W_t \in \mathcal A(G_P)$. Since $s \ge 3$, we have $|V_1V_2| < |A|$. Thus, by the minimality of $|A|$, there is a d_0-chain of factorizations between V_1V_2 and $W_1 \cdot \ldots \cdot W_t$, and thus one between $z' = (V_1V_2)V_3 \cdot \ldots \cdot V_s$ and $(W_1 \cdot \ldots \cdot W_t)V_3 \cdot \ldots \cdot V_s$ as well. From the definitions of the $V_i^{(j)}$ and W_1, we have $W_1 \mid U_1 \cdot \ldots \cdot U_{r-1}$. Thus by **A** there is a d_0-chain of factorizations between $W_1 \cdot \ldots \cdot W_t V_3 \cdot \ldots \cdot V_s$ and $z = U_1 \cdot \ldots \cdot U_r$. Concatenating these two chains gives a d_0-chain of factorizations between z' and z, completing the proof. □

Corollary 4.3. *Let H be a Krull monoid, $\varphi\colon H \to F = \mathcal{F}(P)$ a cofinal divisor homomorphism, $G = \mathcal{C}(\varphi) \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ its class group, where $1 < n_1 | \ldots | n_r$ and $|G| \ge 3$, and $G_P \subset G$ the set of all classes containing prime divisors. Suppose that the following two conditions hold:*

(a) $\lfloor \frac{1}{2}\mathsf{D}(G_P) + 1 \rfloor \le \max\{n_r, 1 + \sum_{i=1}^r \lfloor \frac{n_i}{2} \rfloor\}$.

(b) *There is a basis $(e_1, \ldots, e_r)$ of G with $\operatorname{ord}(e_i) = n_i$, for all $i \in [1, r]$, such that $\{e_0, \ldots, e_r, -e_0, \ldots, -e_r\} \subset G_P$, where $e_0 = \sum_{i=1}^r \lfloor \frac{n_i}{2} \rfloor e_i$.*

Then

$$\daleth(H) = 2 + \max \Delta(H) = \mathsf{c}_2(H) = \mathsf{c}(H)\,.$$

Before giving the proof of the above corollary, we analyze the result and its assumptions.

Remark 4.4. Let all notation be as in Corollary 4.3.

1. Note that

$$1 + \sum_{i=1}^r \left\lfloor \frac{n_i}{2} \right\rfloor = 1 + \frac{\mathsf{r}_2(G) + \mathsf{d}^*(G)}{2}\,,$$

where $\mathsf{r}_2(G)$ denotes the 2-rank of G, i.e., the number of even n_is. Thus, if $\mathsf{D}(G) = \mathsf{d}^*(G) + 1$ (see the comments after (2.1) for some groups fulfilling this), then

$$\left\lfloor \frac{1}{2}\mathsf{D}(G) + 1 \right\rfloor \le 1 + \sum_{i=1}^r \left\lfloor \frac{n_i}{2} \right\rfloor,$$

and hence Condition (a) holds. Not much is known about groups G with $\mathsf{D}(G) > \mathsf{d}^*(G) + 1$ (see [22], [15, Theorem 3.3]). Note that groups of odd order with $\mathsf{D}(G) > \mathsf{d}^*(G) + 1$ yield examples of groups for which (a) fails, yet the simplest example of such a group we were able to find in the literature already has rank 8 (see [22, Theorem 5]).

2. In Examples 3.3, we pointed out that some assumption on G_P is needed in order to obtain the result $\daleth(H) = \mathsf{c}(H)$. Clearly, Condition (b) holds if every class contains a prime divisor. But since there are relevant Krull monoids with $G_P \ne G$ (for examples arising in the analytic theory of Krull monoids, we refer to [21, 26, 27]), we formulated our requirements on G_P as weak as possible, and we discuss two natural settings which enforce parts of Conditions (b) even if $G_P \ne G$.

(i) A Dedekind domain R is a quadratic extension of a principal ideal domain R' if $R' \subset R$ is a subring and R is a free R'-module of rank 2. If R is such a Dedekind domain, G its class group, and $G_P \subset G$ the set of classes containing prime divisors, then $G_P = -G_P$ and $[G_P] = G$. By a result of Leedham-Green [30], there exists, for every abelian group G, a Dedekind domain R which is a quadratic extension of a principal ideal domain and whose class group is isomorphic to G.

(ii) If $G_P \subset G$ are as in Corollary 4.3, then G_P is a generating set of G, and if $G \cong C_{p^k}^r$, where $p \in \mathbb{P}$ and $k, r \in \mathbb{N}$, then G_P contains a basis by [19, Lemma A.7].

3. Corollary 4.3 tells us that the catenary degree $\mathsf{c}(H)$ occurs as a distance of two factorizations of the following form

$$a = u_1 u_2 = v_1 \cdot \ldots \cdot v_{\mathsf{c}(H)}\,,$$

where $u_1, u_2, v_1, \ldots, v_{\mathsf{c}(H)} \in \mathcal{A}(H)$ and a has no factorization of length $j \in [3, \mathsf{c}(H) - 1]$. Of course, the catenary degree may also occur as a distance between factorizations which are not of the above form. In general, there are even elements a and integers $k \ge 3$ such that

$$\mathsf{c}(a) = \mathsf{c}(H)\,, \ \min \mathsf{L}(a) = k \quad \text{and} \quad \mathsf{c}(b) < \mathsf{c}(a) \tag{4.7}$$

for all proper divisors b of a. We provide a simple, explicit example.

Let $G = C_3 \oplus C_3$, (e_1, e_2) be a basis of G and $e_0 = -e_1 - e_2$. For $i \in [0, 2]$, let $U_i = e_i^3$ and let $V = e_0 e_1 e_2$. Then $A = V^3 \in \mathcal{B}(G)$, $\mathsf{Z}(A) = \{U_0 U_1 U_2, V^3\}$, $\mathsf{c}(A) = 3 = \mathsf{c}(G)$ (see Corollary 5.5) and $\mathsf{c}(B) = 0$ for all proper zero-sum subsequences B of A.

4. Let $\beta\colon H \to \mathcal{B}(G_P)$ be as in Lemma 3.6. Clearly, if $a \in H$ is such that $\mathsf{c}(a) = \mathsf{c}(H)$, then, using the notation of Remark 4.4.3, $a, \beta(a)$, u_1, u_2, $\beta(u_1)$ and $\beta(u_2)$ must be highly structured. On the opposite side of the spectrum, there is the following result: if $\operatorname{supp}\big(\beta(a)\big) \cup \{0\}$ is a subgroup of G, then $\mathsf{c}(a) \le 3$ (see [19, Theorem 7.6.8]), while (3.1) shows $\mathsf{c}(a) \ge 3$ whenever $|\mathsf{L}(a)| > 1$.

5. If H is factorial, in particular if $|G| = 1$, then $\daleth(H) = \mathsf{c}_2(H) = \mathsf{c}(H) = 0$ and $2 + \max\Delta(H) = 2$. If H is not factorial and $|G| = 2$, then $\daleth(H) = 0$ and $\mathsf{c}_2(H) = \mathsf{c}(H) = 2 + \max\Delta(H) = 2$.

Proof of Corollary 4.3. Lemma 3.2.6 and Theorem 4.2 imply that

$$\begin{aligned}\daleth(H) &\le \min\{2 + \max\Delta(H),\, \mathsf{c}_2(H)\} \le \max\{2 + \max\Delta(H),\, \mathsf{c}_2(H)\}\\ &\le \mathsf{c}(H) \le \max\Big\{\Big\lfloor \frac{1}{2}\mathsf{D}(G_P) + 1\Big\rfloor,\, \daleth(G_P)\Big\}.\end{aligned}$$

By assumption and by Proposition 4.1 and Lemma 3.6.3, it follows that

$$\Big\lfloor \frac{1}{2}\mathsf{D}(G_P) + 1\Big\rfloor \le \max\Big\{n_r,\, 1 + \sum_{i=1}^{r} \Big\lfloor \frac{n_i}{2}\Big\rfloor\Big\} \le \daleth(G_P) = \daleth(H),$$

and thus, in the above chain of inequalities, we indeed have equality throughout. $\square$

Corollary 4.5. *Let H be a Krull monoid, $\varphi\colon H \to F = \mathcal{F}(P)$ a cofinal divisor homomorphism, $G = \mathcal{C}(\varphi)$ its class group, $G_P \subset G$ the set of classes containing prime divisors, and suppose that $3 \le \mathsf{D}(G_P) < \infty$.*

1. *We have $\mathsf{c}(H) = \mathsf{D}(G_P)$ if and only if $\daleth(H) = \mathsf{D}(G_P)$.*
2. *If $\mathsf{c}(H) = \mathsf{D}(G)$, then $\mathsf{D}(G_P) = \mathsf{D}(G)$ and G is either cyclic or an elementary 2-group. If $G_P = -G_P$, then the converse implication holds as well.*

Proof. 1. By Theorem 4.2, (3.2) and Lemma 3.6.3, we have

$$\daleth(H) = \daleth(G_P) \le \mathsf{c}(H) \le \max\Big\{\Big\lfloor \frac{1}{2}\mathsf{D}(G_P) + 1\Big\rfloor,\, \daleth(G_P)\Big\} \le \mathsf{D}(G_P), \tag{4.8}$$

which we will also use for part 2. In view of $3 \le \mathsf{D}(G_P) < \infty$, we have $\lfloor \frac{1}{2}\mathsf{D}(G_P) + 1\rfloor < \mathsf{D}(G_P)$. Thus the assertion now directly follows from (4.8).

2. We use that $[G_P] = G$. Furthermore, if $\mathsf{D}(G_P) = \mathsf{D}(G)$, it follows that $\Sigma(S) = G^\bullet$ for all zero-sum free sequences $S \in \mathcal{F}(G_P)$ with $|S| = \mathsf{D}(G_P) - 1$ (see [19, Proposition 5.1.4]). Obviously, this implies that $\langle \operatorname{supp}(U)\rangle = G$ for all $U \in \mathcal{A}(G_P)$ with $|U| = \mathsf{D}(G_P)$.

Suppose that $\mathsf{c}(H) = \mathsf{D}(G)$. Since $\mathsf{c}(H) \le \mathsf{D}(G_P) \le \mathsf{D}(G)$ (in view of (4.8)), it follows that $\mathsf{D}(G_P) = \mathsf{D}(G)$, and part 1 implies that $\daleth(H) = \mathsf{D}(G_P)$. Thus there exist $U, V \in \mathcal{A}(G_P)$ such that $\{2, \mathsf{D}(G)\} \subset \mathsf{L}(UV)$, and [19, Proposition 6.6.1] implies that $V = -U$ and $\mathsf{L}\big((-U)U\big) = \{2, \mathsf{D}(G)\}$ (since $\max \mathsf{L}((-U)U) \le \frac{|(-U)U|}{2} \le \mathsf{D}(G)$).

Assume to the contrary that G is neither cyclic nor an elementary 2-group. We show that there exists some $W \in \mathcal{A}(G_P)$ such that $W \mid (-U)U$ and $2 < |W| < \mathsf{D}(G)$. Clearly, W gives rise to a factorization $(-U)U = W W_2 \cdot \ldots \cdot W_k$ with $W_2, \ldots, W_k \in \mathcal{A}(G_P)$ and $2 < k < \mathsf{D}(G)$, a contradiction to $\mathsf{L}\big((-U)U\big) = \{2, \mathsf{D}(G)\}$.

Since $\langle \operatorname{supp}(U)\rangle = G$ (as noted above) is not an elementary 2-group, there exists some $g_0 \in \operatorname{supp}(U)$ with $\operatorname{ord}(g_0) > 2$, say $U = g_0^m g_1 \cdot \ldots \cdot g_l$ with $g_0 \notin \{g_1, \ldots, g_l\}$. Since $G = \langle \operatorname{supp}(U)\rangle$ is not cyclic, it follows that $l \ge 2$. Let $W' = (-g_0)^m g_1 \cdot \ldots \cdot g_l$. Then $W' \mid U(-U)$ and $|W'| = \mathsf{D}(G)$. Hence there exists some $W \in \mathcal{A}(G_P)$ with $W \mid W'$, and we proceed to show that $2 < |W| < \mathsf{D}(G)$, which will complete the proof. Since $U \in \mathcal{A}(G_P)$, we have $W \nmid g_1 \cdot \ldots \cdot g_l$, and thus $-g_0 \mid W$. Since $g_0 \notin \{g_1, \ldots, g_l\}$ and $g_0 \ne -g_0$, it follows that $W \ne g_0(-g_0)$, and thus $|W| > 2$.

Assume to the contrary that $|W| = \mathsf{D}(G)$. Then $W = W'$, and $\sigma(U) = \sigma(W') = 0$ implies $2mg_0 = 0$, and thus $m > 1$. We consider the sequence $S = g_0^m g_1 \cdot \ldots \cdot g_{l-1}$. Since $1 < m < \operatorname{ord}(g_0)$ and $2mg_0 = 0$, it follows that

$$0 \neq (m+1)g_0.$$

Since S is zero-sum free of length $|S| = \mathsf{D}(G) - 1$, we have $\Sigma(S) = G^\bullet$, and thus $0 \neq (m+1)g_0 \in \Sigma(S)$, say

$$(m+1)g_0 = sg_0 + \sum_{i \in I} g_i \quad \text{with} \quad s \in [0, m] \quad \text{and} \quad I \subset [1, l-1]\,.$$

If $s = 0$, then

$$0 = 2mg_0 = (m-1)g_0 + \sum_{i \in I} g_i \ \in \Sigma(S),$$

a contradiction. If $s \geq 1$, then it follows that

$$T = (-g_0)^{m+1-s} \prod_{i \in I} g_i$$

is a proper zero-sum subsequence of W, a contradiction to $W \in \mathcal{A}(G_P)$.

Suppose that $G_P = -G_P$ and $\mathsf{D}(G_P) = \mathsf{D}(G)$. Recall the comments after (2.1) concerning the value of $\mathsf{D}(G)$. First, we let G be an elementary 2-group. Then there is a $U = e_0 e_1 \cdot \ldots \cdot e_r \in \mathcal{A}(G_P)$ with $|U| = \mathsf{D}(G) = r + 1$. Thus, since $\langle \operatorname{supp}(U) \rangle = G$, and since a basis of an elementary 2-group is just a minimal (by inclusion) generating set, it follows that G_P contains the basis (say) $(e_1, \ldots, e_r)$ of G, and Proposition 4.1 and Lemma 3.6.3 imply that $\urcorner(H) = \urcorner(G_P) = \mathsf{D}(G_P) = \mathsf{D}(G) = r + 1$, whence $\mathsf{c}(H) = \mathsf{D}(G)$ follows from part 1. Second, let G be cyclic. If $U \in \mathcal{A}(G_P)$ with $|U| = \mathsf{D}(G_P) = \mathsf{D}(G)$, then $|U| = |G|$ and [19, Theorem 5.1.10] implies that $U = g^{|G|}$ for some $g \in G_P$ with $\operatorname{ord}(g) = |G|$. Hence $\mathsf{L}\big((-U)U\big) = \{2, |G|\}$, and now it follows from Lemma 3.6.3 that $|G| = \mathsf{D}(G_P) = \urcorner(G_P) = \urcorner(H)$, whence part 1 once more shows $\mathsf{c}(H) = \mathsf{D}(G) = \mathsf{D}(G_P)$. $\square$

5. An upper bound for the catenary degree

We apply our structural result on the catenary degree (Theorem 4.2) to obtain a new upper bound on the catenary degree (see Theorem 5.4) and a characterization result for Krull monoids with small catenary degree (see Corollary 5.6). We start with some technical results.

Lemma 5.1. *Let G be an abelian group and let $U, V \in \mathcal{F}(G^\bullet)$. Suppose that either $U, V \in \mathcal{A}(G)$ or that U and V are zero-sum free with $\sigma(UV) = 0$. Then $\max \mathsf{L}(UV) \leq \min\{|U|, |V|\}$. Moreover, if $\max \mathsf{L}(UV) = |U| \geq 3$, then $-\operatorname{supp}(U) \subset \Sigma(V)$.*

Proof. Let $UV = W_1 \cdot \ldots \cdot W_m$, where $m = \max \mathsf{L}(UV)$ and $W_1, \ldots, W_m \in \mathcal{A}(G)$. Let $U = U_1 \cdot \ldots \cdot U_m$ and $V = V_1 \cdot \ldots \cdot V_m$ with $W_i = U_i V_i$ for $i \in [1, m]$. If $U_i \neq 1$ and $V_i \neq 1$ for all $i \in [1, m]$, then $m \leq |U_1| + \ldots + |U_m| = |U|$ and likewise $m \leq |V|$. Moreover, if equality holds in the first bound, then $|U_i| = 1$ for $i \in [1, m]$, in which case each $V_i | V$ is a subsequence of V with $\sigma(V_i) = -\sigma(U_i) \in -\operatorname{supp}(U)$; since $\bigcup_{i=1}^m \{\sigma(U_i)\} = \operatorname{supp}(U)$, this means $-\operatorname{supp}(U) \subset \Sigma(V)$.

On the other hand, if there is some $j \in [1, m]$ such that $U_j = 1$ or $V_j = 1$, say $U_1 = 1$, then, since V contains no proper, nontrivial zero-sum subsequence, it follows that $W_1 = V_1 = V$, which, since U contains no proper, nontrivial zero-sum subsequence, implies $W_2 = U$. Hence, since $U, V \in \mathcal{F}(G^\bullet)$ with $\sigma(U) = \sigma(W_2) = 0 = \sigma(W_1) = \sigma(V)$ implies $|U|, |V| \geq 2$, we see that $m = 2 \leq \min\{|U|, |V|\}$. $\square$

Lemma 5.2. *Let G be an abelian group, $K \subset G$ a finite cyclic subgroup, and let $U, V \in \mathcal{A}(G)$ with $\max \mathsf{L}(UV) \geq 3$. If $\sum_{g \in K} \mathsf{v}_g(UV) \geq |K| + 1$ and there exists a nonzero $g_0 \in K$ such that $\mathsf{v}_{g_0}(U) > 0$ and $\mathsf{v}_{-g_0}(V) > 0$, then $\mathsf{L}(UV) \cap [3, |K|] \neq \emptyset$.*

Proof. Note $U, V \in \mathcal{A}(G)$ and $\max \mathsf{L}(UV) \ge 3$ imply $0 \notin \operatorname{supp}(UV)$. Moreover, note that if $\operatorname{supp}(U) \subset K$, then Lemma 5.1 implies that $\max \mathsf{L}(UV) \le |U| \le \mathsf{D}(K) = |K|$ (recall the comments after (2.1)), whence the assumption $\max \mathsf{L}(UV) \ge 3$ completes the proof. Therefore we may assume $\operatorname{supp}(U) \not\subset K$, and likewise that $\operatorname{supp}(V) \not\subset K$.

We factor $U = U_0 U'$ and $V = V_0 V'$ where U_0 and V_0 are subsequences of terms from K such that there exists some non-zero $g_0 \in K$ with $g_0 \mid U_0$ and $(-g_0) \mid V_0$, and $|U_0| + |V_0| = |K| + 1$. Note that by the assumption made above, both U_0 and V_0 are proper subsequences of U and V, respectively, and thus they are zero-sum free.

Let $U_0 = g_0 U_0'$ and $V_0 = (-g_0) V_0'$. Since U_0' and V_0' are both zero-sum free, we get (cf., e.g., [19, Proposition 5.1.4.4]) that $|\{0\} \cup \Sigma(U_0')| \ge |U_0'| + 1 = |U_0|$ and $|\{0\} \cup \Sigma(V_0')| \ge |V_0'| + 1 = |V_0|$. Since these sets are both subsets of K, the pigeonhole principle implies that

$$\big(g_0 + \big(\{0\} \cup \Sigma(U_0')\big)\big) \cap \big(\{0\} \cup \Sigma(V_0')\big) \neq \emptyset. \tag{5.1}$$

Let U_0'' and V_0'' denote (possibly trivial) subsequences of U_0' and V_0', respectively, such that $\sigma(V_0'') = g_0 + \sigma(U_0'') = \sigma(g_0 U_0'')$, whose existence is guaranteed by (5.1).

We set $W_1 = (g_0 U_0'')^{-1} U V_0''$ and $W_2 = V_0''^{-1} V (g_0 U_0'')$. Then, $UV = W_1 W_2$, and W_1 and W_2 are nontrivial zero-sum sequence; more precisely, $(-g_0) g_0 \mid W_2$ is a proper zero-sum subsequence (recall that by assumption U_0 and V_0 are proper subsequences of U and V, respectively). Since $\mathsf{L}(W_1) + \mathsf{L}(W_2) \subset \mathsf{L}(UV)$, and since by the above assertion $\min \mathsf{L}(W_1) \ge 1$ and $\min \mathsf{L}(W_2) \ge 2$, it suffices to assert that $\max \mathsf{L}(W_1) + \max \mathsf{L}(W_2) \le |K|$. Since, by Lemma 5.1, we have $\max \mathsf{L}(W_1) \le |V_0''| \le |V_0| - 1$ and $\max \mathsf{L}(W_2) \le |g_0 U_0''| \le |U_0|$, and since by assumption $|U_0| + |V_0| = |K| + 1$, this is the case. □

Lemma 5.3. *Let $t \in \mathbb{N}$ and $\alpha, \alpha_1, \ldots, \alpha_t \in \mathbb{R}$ with $\alpha_1 \ge \ldots \ge \alpha_t \ge 0$ and $\sum_{i=1}^{t} \alpha_i \ge \alpha \ge 0$. Then*

$$\prod_{i=1}^{t} (1 + x_i) \qquad \textit{is minimal}$$

over all $(x_1, \ldots, x_t) \in \mathbb{R}^t$ with $0 \le x_i \le \alpha_i$ and $\sum_{i=1}^{t} x_i = \alpha$ if

$$x_i = \alpha_i \textit{ for each } i \in [1, s] \quad \textit{and} \quad x_i = 0 \quad \textit{for each } i \in [s+2, t]$$

where $s \in [0, t]$ is maximal with $\sum_{i=1}^{s} \alpha_i \le \alpha$.

Proof. This is a simple calculus problem; for completeness, we include a short proof. We may assume $\alpha \neq 0$. By compactness and continuity, the existence of a minimum is clear. Let $\overline{x} = (x_1, \ldots, x_t)$ be a point where the minimum is attained. We note that for $x, y \in \mathbb{R}$ with $x \ge y \ge 0$ we have

$$(1 + x + \varepsilon)(1 + y - \varepsilon) < (1 + x)(1 + y) \tag{5.2}$$

for each $\varepsilon > 0$. Thus, it follows that $x_i \notin \{0, \alpha_i\}$ for at most one $i \in [1, t]$; if such an i exists we denote it by i_0, otherwise we denote by i_0 the maximal $i \in [1, t]$ with $x_i \neq 0$. Suppose that for $\overline{x}$ the value of α_{i_0} is maximal among all points where the minimum is attained. We observe that it suffices to assert that $x_j = \alpha_j$ for each j with $\alpha_j > \alpha_{i_0}$ and $x_j = 0$ for each j with $\alpha_j < \alpha_{i_0}$; in view of $x_i \in \{0, \alpha_i\}$ for $i \neq i_0$, we can then simply reorder the x_i for the i's with $\alpha_i = \alpha_{i_0}$ to get a point fulfilling the claimed conditions.

First, assume there exists some j with $\alpha_j > \alpha_{i_0}$ and $x_j \neq \alpha_j$, i.e., $x_j = 0$. Then, exchanging x_j and x_{i_0} (note $x_{i_0} \le \alpha_j$), yields a contradiction to the maximality of α_{i_0}.

Second, assume there exists some j with $\alpha_j < \alpha_{i_0}$ and $x_j \neq 0$, i.e., $x_j = \alpha_j > 0$. By definition of i_0, it follows that $0 < x_{i_0} < \alpha_{i_0}$. Thus, we can apply (5.2), in case $x_{i_0} < x_j$ first exchanging the two coordinates, to obtain a contradiction to the assumption that a minimum is attained in $\overline{x}$. □

Note that for $G \cong C_n^r$ the bound given by Theorem 5.4 is of the form $\daleth(H) \le \frac{5}{6}\mathsf{D}(G) + O_r(1)$. Thus, for n large relative to r this is an improvement on the bound $\daleth(H) \le \mathsf{D}(G)$.

Theorem 5.4. *Let H be a Krull monoid, $\varphi\colon H \to F = \mathcal{F}(P)$ a cofinal divisor homomorphism, $G = \mathcal{C}(\varphi)$ its class group, and $G_P \subset G$ the set of classes containing prime divisors. If* $\exp(G) = n$ *and* $\mathsf{r}(G) = r$, *then*

$$(5.3)\quad \daleth(H) \le \max\left\{n,\ \ \frac{2}{3}\mathsf{D}(G_P) + \frac{1}{3}\left\lfloor\left\lfloor\log_{\lfloor n/2\rfloor+1}|G|\right\rfloor \cdot \lfloor n/2\rfloor + |G|\cdot(\lfloor n/2\rfloor+1)^{-\lfloor\log_{\lfloor n/2\rfloor+1}|G|\rfloor}\right\rfloor\right\}$$
$$\le \max\left\{n,\ \ \frac{1}{3}\left(2\mathsf{D}(G_P) + \frac{1}{2}rn + 2^r\right)\right\}.$$

Proof. Since $\daleth(H) = \daleth(G_P)$ by Lemma 3.6.3, it suffices to show that $\daleth(G_P)$ satisfies the given bounds. Let $U,\ V \in \mathcal{A}(G_P)$ with $\max \mathsf{L}(UV) \ge 3$, and let

$$z = A_1 \cdot \ldots \cdot A_{r_1} B_1 \cdot \ldots \cdot B_{r_2} \in \mathsf{Z}(UV)\,,$$

where $A_i,\ B_j \in \mathcal{A}(G_P)$ with $|A_i| \ge 3$ and $|B_j| = 2$ for all $i \in [1, r_1]$ and all $j \in [1, r_2]$, be a factorization of UV of length $|z| = \min\big(\mathsf{L}(UV) \setminus \{2\}\big)$. Note $r_2 \ge 2$, else $|z| \le \frac{|UV|-2}{3} + 1 \le \frac{2\mathsf{D}(G_P)+1}{3}$, implying (5.3) as desired (the inequality between the two bounds in Theorem 5.4 will become apparent later in the proof). Our goal is to show $|z|$ is bounded above by (5.3). We set

$$S = B_2 \cdot \ldots \cdot B_{r_2} \in \mathcal{B}(G)\,.$$

Observe that, for every $i \in [2, r_2]$, B_i contains one term from $\operatorname{supp}(U)$ with the other from $\operatorname{supp}(V)$ (otherwise $\min\{|U|, |V|\} = 2$, contradicting $\max \mathsf{L}(UV) \ge 3$ in view of Lemma 5.1). Hence we can factor $S = S_U S_V$ so that $S_U = -S_V$ with $S_U | U$ and $S_V | V$. Let $\operatorname{supp}(S_U) = \{g_1, \ldots, g_s\}$ with the g_i distinct and indexed so that $\mathsf{v}_{g_1}(S_U) \ge \ldots \ge \mathsf{v}_{g_s}(S_U)$. If $\mathsf{v}_{g_1}(S_U) \ge (n+1)/2$, then

$$\sum_{g \in \langle g_1\rangle} \mathsf{v}_g(UV) \ge \mathsf{v}_{g_1}(S_U) + \mathsf{v}_{-g_1}(S_V) \ge n+1 \ge |\langle g_1\rangle| + 1\,,$$

and Lemma 5.2 implies that $|z| = \min\big(\mathsf{L}(UV) \setminus \{2\}\big) \in [3, n]$. Therefore we may assume $\mathsf{v}_{g_1}(S_U) \le \lfloor \frac{n}{2} \rfloor$.

Suppose

$$(5.4)\qquad |S_U| > \left\lfloor\log_{\lfloor n/2\rfloor+1}|G|\right\rfloor \cdot \lfloor n/2\rfloor + |G|\cdot(\lfloor n/2\rfloor+1)^{-\lfloor\log_{\lfloor n/2\rfloor+1}|G|\rfloor} - 1$$

or

$$(5.5)\qquad |S_U| > \frac{1}{2}nr + 2^r - 1.$$

Then Lemma 5.3 (applied with $\alpha = |S_U|$ and $\alpha_i = \lfloor n/2\rfloor$, and with $\alpha = |S_U|$, $\alpha_{r+1} = \max\{n/2,\, 2^r - 1\}$ and $\alpha_i = n/2$ for $i \ne r+1$, re-indexing the α_i if need be) along with $\mathsf{v}_{g_1}(S_U) \le \lfloor \frac{n}{2}\rfloor \le \frac{n}{2}$ implies that

$$(5.6)\qquad \prod_{i=1}^{s}\Big(\mathsf{v}_{g_i}(S_U) + 1\Big) > |G|.$$

Moreover, Lemma 5.3 also shows that the bound in (5.4) is at most the bound in (5.5).

Since each $g_i^{\mathsf{v}_{g_i}(S_U)}$ is zero-sum free, being a subsequence of the proper subsequence $S_U | U$, it follows that $\{0, g_i, 2g_i, \ldots, \mathsf{v}_{g_i}(S_U) g_i\}$ are $\mathsf{v}_{g_i}(S_U)+1$ distinct elements. Hence, in view of (5.6) and the pigeonhole principle, it follows that there exists $a_i,\ b_i \in [0, \mathsf{v}_{g_i}(S_U)]$, for $i \in [1, s]$, such that, letting

$$S_A = \prod_{i=1}^{s} g_i^{a_i} \in \mathcal{F}(G_P) \quad \text{and} \quad S_B = \prod_{i=1}^{s} g_i^{b_i} \in \mathcal{F}(G_P),$$

we have $\sigma(S_A) = \sigma(S_B)$ with $S_A \ne S_B$. Moreover, by replacing each a_i and b_i with $a_i - \min\{a_i, b_i\}$ and $b_i - \min\{a_i, b_i\}$, respectively, we may w.l.o.g. assume that

$$(5.7)\qquad a_i = 0 \quad \text{or} \quad b_i = 0$$

for each $i \in [1, s]$. By their definition and in view of (5.7), we have

$$S_A S_B \mid S_U \quad \text{and} \quad (-S_B)(-S_A) \mid (-S_U) = S_V .$$

From $S_A \neq S_B$, $\sigma(S_A) = \sigma(S_B)$ and $S_A | S_U$ with S_U a proper subsequence of $U \in \mathcal{A}(G_P)$, we conclude that $\sigma(S_A) = \sigma(S_B) \neq 0$, and thus both S_A and S_B are nontrivial. Since $\sigma(S_A) = \sigma(S_B)$, we have $\sigma(S_A(-S_B)) = 0$, and in view of (5.7), the g_i being distinct and $S_A|U$ and $S_B|U$ being zero-sum free, it follows that there is no 2-term zero-sum subsequence in $S_A(-S_B)$. Thus, letting $T = S_A(-S_B)$, recalling that

$$S_U S_V = S_U(-S_U) = S = B_2 \cdot \ldots \cdot B_{r_2},$$

and putting all the above conclusions of this paragraph together, we see that T is a nontrivial, zero-sum subsequence not divisible by a zero-sum sequence of length 2 such that $T(-T) \mid B_2 \cdot \ldots \cdot B_{r_2}$. However, this leads to factorizations $T(-T) = A_{r_1+1} \cdot \ldots \cdot A_{r_1'}$ and $S\big((-T)T\big)^{-1} = B_2' \cdot \ldots \cdot B_{r_2'}'$, where $A_i, B_j' \in \mathcal{A}(G_P)$ with $|A_i| \geq 3$ and $|B_j'| = 2$ for all $i \in [r_1+1, r_1']$ and all $j \in [2, r_2']$. But now the factorization

$$z' = A_1 \cdot \ldots \cdot A_{r_1} A_{r_1+1} \cdot \ldots \cdot A_{r_1'} B_1 B_2' \cdot \ldots \cdot B_{r_2'}' \in \mathsf{Z}(UV)$$

contradicts the minimality of $|z| = \min\big(\mathsf{L}(UV) \setminus \{2\}\big)$ (note $|z'| \geq r_1' + 1 \geq 3$ since $B_1 | z'$ and T and $-T$ were both nontrivial). So we may instead assume

$$|S_U| \leq \Big\lfloor \log_{\lfloor n/2 \rfloor + 1} |G| \Big\rfloor \cdot \lfloor n/2 \rfloor + |G| \cdot (\lfloor n/2 \rfloor + 1)^{-\lfloor \log_{\lfloor n/2 \rfloor + 1} |G| \rfloor} - 1 \leq \frac{1}{2} nr + 2^r - 1. \tag{5.8}$$

Now

$$\begin{aligned} |z| = r_1 + r_2 &\leq \frac{1}{3}|A_1 \cdot \ldots \cdot A_{r_1}| + \frac{1}{2}|B_1 \cdot \ldots \cdot B_{r_2}| \\ &= \frac{1}{3}(|UV| - 2|S_U| - 2) + \frac{1}{2}(2 + 2|S_U|) \leq \frac{1}{3}\Big(2\mathsf{D}(G_P) + |S_U| + 1\Big), \end{aligned}$$

which, together with (5.8), implies the assertion. □

As an added remark, note that the only reason to exclude the set B_1 from the definition of the sequences S and S_U was to ensure that $|z'| \geq 3$. However, if $r_1 \geq 1$, then $|z'| \geq 3$ holds even if B_1 is so included. Thus the bound in (5.3) could be improved by $-\frac{1}{3}$ in such case.

We state one more proposition—its proof will be postponed—and then we give the characterization of small catenary degrees.

Proposition 5.5. *Let $G = C_3 \oplus C_3 \oplus C_3$. Then $\daleth(G) = \mathsf{c}(G) = 4$.*

Corollary 5.6. *Let H be a Krull monoid with class group G and suppose that every class contains a prime divisor. Then $\daleth(H)$ is finite if and only if the catenary degree $\mathsf{c}(H)$ is finite if and only if G is finite. Moreover, we have*

1. *$\mathsf{c}(H) \leq 2$ if and only if $|G| \leq 2$.*
2. *$\mathsf{c}(H) = 3$ if and only if G is isomorphic to one of the following groups: $C_3, C_2 \oplus C_2$, or $C_3 \oplus C_3$.*
3. *$\mathsf{c}(H) = 4$ if and only if G is isomorphic to one of the following groups: C_4, $C_2 \oplus C_4$, $C_2 \oplus C_2 \oplus C_2$, or $C_3 \oplus C_3 \oplus C_3$.*

Proof. If G is finite, then $\mathsf{D}(G)$ is finite (see [19, Theorem 3.4.2]), and so Lemma 3.6.3 and Theorem 4.2 imply the finiteness of $\daleth(H)$ and of $\mathsf{c}(H)$. If G contains elements of arbitrarily large order, then the infinitude of $\daleth(G)$ follows by Proposition 4.1.2. And, if G contains an infinite independent set, the infinitude of $\daleth(G)$ follows by Proposition 4.1.1. In each case the infinitude of $\daleth(H)$ and $\mathsf{c}(H)$, thus follows by (3.2) and Lemma 3.6.3.

1. This part of the theorem is already known and included only for completeness. That $\mathsf{c}(H) \le 2$ implies $|G| \le 2$ can be found in [19, pp. 396], while $\mathsf{c}(H) \le \mathsf{D}(G) \le |G|$ follows from [19, Theorem 3.4.11 and Lemmas 5.7.2 and 5.7.4] and implies the other direction.

2. See [19, Corollary 6.4.9].

3. Recall the comment concerning the value of $\mathsf{D}(G)$ after (2.1). We may assume that G is finite. Note Proposition 4.1 implies $\mathsf{c}(G) \ge 4$ for each of the groups listed in part 3. As noted for part 1, we have $\mathsf{c}(G) \le \mathsf{D}(G) \le |G|$ in general. Thus $\mathsf{c}(C_4) \le 4$ and, since $\mathsf{D}(C_2 \oplus C_2 \oplus C_2) = 4$, $\mathsf{c}(C_2 \oplus C_2 \oplus C_2) \le 4$ as well. Moreover, Corollary 4.5 shows that $\mathsf{c}(C_2 \oplus C_4) \le \mathsf{D}(C_2 \oplus C_4) - 1 = 4$. Finally, $\mathsf{c}(C_3 \oplus C_3 \oplus C_3) \le 4$ follows by Proposition 5.5. Consequently, $\mathsf{c}(G) = 4$ for all of the groups listed in part 3.

In view of parts 1 and 2, it remains to show all other groups G not listed in Corollary 5.6 have $\mathsf{c}(G) \ge 5$. Set $\exp(G) = n$ and $\mathsf{r}(G) = r$. Now Proposition 4.1 shows that $\mathsf{c}(G) \ge 5$ whenever $n \ge 5$ or $r \ge 4$. This leaves only $C_4 \oplus C_4$, $C_4 \oplus C_4 \oplus C_4$, $C_2 \oplus C_4 \oplus C_4$ and $C_2 \oplus C_2 \oplus C_4$ for possible additional candidates for $\mathsf{c}(G) \le 4$. However, applying Proposition 4.1 to each one of these four groups shows $\mathsf{c}(G) \ge 5$ for each of them, completing the proof. □

The remainder of this section is devoted to the proof of Proposition 5.5, which requires some effort. Before going into details, we would like to illustrate that geometric and combinatorial questions in C_3^r have found much attention in the literature, and our investigations should be seen in the light of this background. The *Erdős-Ginzburg-Ziv constant* $\mathsf{s}(G)$ of a finite abelian group G is the smallest integer $l \in \mathbb{N}$ with the following property:

- Every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a zero-sum subsequence T of length $|T| = \exp(G)$.

If $r \in \mathbb{N}$ and φ is the maximal size of a cap in $\mathrm{AG}(r,3)$, then $\mathsf{s}(C_3^r) = 2\varphi + 1$ (see [12, Section 5]). The maximal size of caps in C_3^r has been studied in finite geometry for decades (see [13, 11, 32]; the precise values are only known for $r \le 6$). This shows the complexity of these combinatorial and geometric problems. Recently, Bhowmik and Schlage-Puchta determined the Davenport constant of $C_3 \oplus C_3 \oplus C_{3n}$. In these investigations, they needed a detailed analysis of the group $C_3 \oplus C_3 \oplus C_3$. Building on the above results for the Erdős–Ginzburg–Ziv constant $\mathsf{s}(G)$, in particular, using that $\mathsf{s}(C_3^3) = 19$, they determined the precise values of generalized Davenport constants in C_3^3 (see [4, Proposition 1], and [14] for more on generalized Davenport constants).

We need one more definition. For an abelian group G and a sequence $S \in \mathcal{F}(G)$ we denote

$$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G\} \in [0, |S|] \qquad \text{the } \textit{maximum of the multiplicities} \text{ of } S.$$

We give an explicit characterization of all minimal zero-sum sequences of maximal length over C_3^3. In particular, it can be seen that for this group the Olson constant and the Strong Davenport constant do not coincide (we do not want to go into these topics; the interested reader is referred to Section 10 in the survey article [16]).

Lemma 5.7. *Let $G = C_3 \oplus C_3 \oplus C_3$ and $U \in \mathcal{F}(G)$. Then the following statements are equivalent:*

(a) *$U \in \mathcal{A}(G)$ with $|U| = \mathsf{D}(G)$.*

(b) *There exist a basis (e_1, e_2, e_3) of G and $a_i, b_j \in [0,2]$ for $i \in [1,5]$ and $j \in [1,3]$ with $\sum_{i=1}^5 a_i \equiv \sum_{j=1}^3 b_j \equiv 1 \pmod 3$ such that*

$$U = e_1^2 \prod_{i=1}^{2} (a_i e_1 + e_2) \prod_{j=1}^{3} (a_{2+j} e_1 + b_j e_2 + e_3) \,.$$

In particular, $\mathsf{h}(U) = 2$ for each $U \in \mathcal{A}(G)$ with $|U| = \mathsf{D}(G)$.

Proof. Since $\mathsf{D}(G) = 7$ (see the comments by (2.1)) it is easily seen that statement (b) implies statement (a). Let $U \in \mathcal{A}(G)$ with $|U| = \mathsf{D}(G)$. First, we assert that $\mathsf{h}(U) = 2$ and, then, derive statement (b) as a direct consequence.

Since $\mathsf{h}(U) < \exp(G) = 3$, it suffices to show $\mathsf{h}(U) > 1$. Assume not. We pick some $e_1 \in \operatorname{supp}(U) \subset G^\bullet$. Let $G = \langle e_1 \rangle \oplus K$, where $K \cong C_3 \oplus C_3$ is a subgroup, and let $\phi : G \to K$ denote the projection (with respect to this direct sum decomposition). We set $V = e_1^{-1}U$. We observe that $\sigma(\phi(V)) = 0$.

We note that for each proper and nontrivial subsequence $S \mid V$ with $\sigma(\phi(S)) = 0$, we have that $e_1\sigma(S)$ is zero-sum free, that is

$$\sigma(S) = e_1. \tag{5.9}$$

In particular, we have $\max \mathsf{L}(\phi(V)) \le 2$ and, in combination with $\mathsf{h}(U) = 1$, we have $0 \nmid \phi(V)$.

We assert that $\mathsf{h}(\phi(V)) = 2$. First, assume $\mathsf{h}(\phi(V)) \ge 3$. This means that V has a subsequence $S' = \prod_{i=1}^{3}(a_ie_1 + g)$ with $g \in K$ and, since $\mathsf{h}(V) = 1$, we have $\{a_1e_1, a_2e_1, a_3e_1\} = \{0, e_1, 2e_1\}$ and $\sigma(S') = 0$, a contradiction. Second, assume $\mathsf{h}(\phi(V)) = 1$. Then, since $|\operatorname{supp}(\phi(V))| = 6$ and $|K^\bullet| = 8$, there exist $g, h \in K$ such that $(-g)g(-h)h \mid \phi(V)$, a contradiction to $\max \mathsf{L}(\phi(V)) = 2$.

So, let $g_1g_2 \mid V$ with $\phi(g_1) = \phi(g_2)$, and denote this element by e_2. Further, let $e_3 \in K$ such that $G = \langle e_1, e_2, e_3 \rangle$ and let $\phi' : G \to \langle e_3 \rangle$ denote the projection (with respect to this basis). If there exists a subsequence $T \mid (g_1g_2)^{-1}V$ with $\sigma(\phi(T)) = -e_2$, then $\sigma(g_1T)$ and $\sigma(g_2T)$ are distinct elements of $\langle e_1 \rangle$, a contradiction to (5.9). So, $-e_2 \notin \Sigma(\phi((g_1g_2)^{-1}V))$, which in view of $\mathsf{h}(\phi(V)) < 3$ and $0 \nmid \phi(V)$, implies that $\operatorname{supp}(\phi((g_1g_2)^{-1}V)) \cap \langle e_2 \rangle = \emptyset$. Since $\sigma(\phi'((g_1g_2)^{-1}V)) = 0$, it follows that $\phi'((g_1g_2)^{-1}V) = e_3^2(-e_3)^2$. Let $V = g_1g_2h_1h_2f_1f_2$ such that $\phi'(h_i) = e_3$ and $\phi'(f_i) = -e_3$ for $i \in [1, 2]$. We note that $\phi(h_1+f_1)\phi(h_2+f_2) = 0e_2$, the only sequence of length two over $\langle e_2 \rangle$ that has sum e_2 yet does not have $-e_2$ as a subsum. Likewise, $\phi(h_1+f_2)\phi(h_2+f_1) = 0e_2$. Thus $\phi(h_1+f_1) = \phi(h_1+f_2)$ or $\phi(h_1+f_1) = \phi(h_2+f_1)$ that is $\phi(f_1) = \phi(f_2)$ or $\phi(h_1) = \phi(h_2)$. By symmetry, we may assume $\phi(h_1) = \phi(h_2)$. Let $j \in [1, 2]$ such that $\phi(h_1 + f_j) = e_2$. Then $\sigma(h_if_jg_1g_2) \in \langle e_1 \rangle$ for $i \in [1, 2]$, yet $\sigma(h_1f_jg_1g_2) \ne \sigma(h_2f_jg_1g_2)$, as h_1 and h_2 are distinct by the assumption $\mathsf{h}(U) = 1$. This contradicts (5.9) and completes the argument.

It remains to obtain the more explicit characterization of U. Let $U = e_1^2W$ for some suitable $e_1 \in G^\bullet$, and let K and ϕ as above. Similarly to (5.9), we see that $\phi(W)$ is a minimal zero-sum sequence over $K \cong C_3^2$. Since $\phi(W)$ has length $5 = \mathsf{D}(C_3^2)$, it follows that $\phi(W) = e_2^2 \prod_{j=1}^{3}(b_je_2 + e_3)$ for independent (e_2, e_3) and $b_j \in [0, 2]$ with $\sum_{j=1}^{3} b_j \equiv 1 \pmod 3$ (cf., e.g., [19, Example 5.8.8]). Since $\sigma(W) = e_1$, the claim follows. $\square$

Proof of Proposition 5.5. Let $G = C_3 \oplus C_3 \oplus C_3$. Recall that $\mathsf{D}(G) = 7$ (see the comments by (2.1)). Thus it suffices to prove $\daleth(G) \le 4$, since then combing with Proposition 4.1.3 and Corollary 4.3 yields

$$4 \le \daleth(G) = \mathsf{c}(G) \le 4\,.$$

Suppose by contradiction that $\daleth(G) \ge 5$. Consider a counter example $U, V \in \mathcal{A}(G)$ with $\max \mathsf{L}(UV) > 4$ and $\mathsf{L}(UV) \cap [3, 4] = \emptyset$ such that $|U| + |V|$ is maximal. Since $\max \mathsf{L}(UV) \ge 5$ and thus by Lemma 5.1 $\min\{|U|, |V|\} \ge 5$, and since $\max\{|U|, |V|\} \le \mathsf{D}(G) = 7$, we know $|U| + |V| \in [10, 14]$. Let $w = W_1 \cdot \ldots \cdot W_t \in \mathsf{Z}(UV)$, where $t \ge 5$ and $W_i \in \mathcal{A}(G)$ for $i \in [1, t]$, be a factorization of UV of length at least 5.

Note that, for some $j \in [1, t]$, say $j = 1$, we must have $W_1 = (-g)g$, where $g \in G^\bullet$, since otherwise

$$|w| \le \lfloor \frac{|UV|}{3} \rfloor \le \lfloor \frac{14}{3} \rfloor = 4\,,$$

a contradiction. Since $g(-g)$ divides neither U nor V, we may assume that $U = gU'$ and $V = (-g)V'$, where U', $V' \in \mathcal{F}(G)$ are both zero-sum free.

CASE 1: We have $g \notin \Sigma(U')$ or $-g \notin \Sigma(V')$, say $g \notin \Sigma(U')$.

Then, since $-2g = g$ and $U = gU' \in \mathcal{A}(G)$, we have $(-g)^2U' \in \mathcal{A}(G)$. Since $W_1 = (-g)g$, then letting $W_1' = g^{-1}W_1(-g)^2 = (-g)^3$ and $W_i' = W_i$ for $i \in [2, t]$, we see that $w' = W_1' \cdot \ldots \cdot W_t' \in \mathsf{Z}(G)$ is a factorization of $((-g)^2U')V$ with $|w'| = t = |w| \geq 5$. As a consequence, $\max \mathsf{L}((-g)^2U'V) \geq 5$, whence the maximality of $|U| + |V|$ ensures that $((-g)^2U')V$ has a factorization

$$z = A_1 \cdot \ldots \cdot A_r \in \mathsf{Z}\big((-g)^2U'V\big)$$

with $r \in [3,4]$, where $A_i \in \mathcal{A}(G)$ for $i \in [1,r]$. Note, since $-g|V$, that $\mathsf{v}_{-g}((-g)^2U'V) \geq 3$.

If $(-g)^2|A_j$ for some $j \in [1,r]$, then, letting $A'_j = A_j(-g)^{-2}g$ and $A'_i = A_i$ for $i \neq j$, gives a factorization $z' = A'_1 \cdot \ldots \cdot A'_r \in \mathsf{Z}(G)$ of UV with $r \in [3,4]$ and $A'_i \in \mathcal{A}(G)$ for $i \in [1,r]$, contradicting that $\mathsf{L}(UV) \cap [3,4] = \emptyset$. Therefore we may assume

$$\mathsf{v}_{-g}(A_i) \leq 1 \quad \text{for all } i \in [1,r]. \tag{5.10}$$

As a result, since $\mathsf{v}_{-g}((-g)^2U'V) \geq 3$, we see that at least three A_i contain $-g$, say w.l.o.g. A_1, A_2 and A_3 with

$$|A_1| \leq |A_2| \leq |A_3|. \tag{5.11}$$

For $i,\, j \in [1,3]$ distinct, we set

$$B_{i,j} = (-g)^{-2}A_iA_jg \in \mathcal{B}(G)\,.$$

Note that there is no 2-term zero-sum subsequence of $B_{i,j}$ which contains g as otherwise $\mathsf{v}_{-g}(A_iA_j) \geq 3$, contradicting (5.10). Consequently,

$$\max \mathsf{L}(B_{i,j}) \leq 1 + \lfloor \frac{|B_{i,j}| - 3}{2} \rfloor\,. \tag{5.12}$$

CASE 1.1: $r = 3$.

Suppose $|A_i| + |A_j| = 9$ for distinct $i,\, j \in [1,3]$. Then $|B_{i,j}| = 8 > \mathsf{D}(G)$, whence $\min \mathsf{L}(B_{i,j}) \geq 2$, while (5.12) implies $\max \mathsf{L}(B_{i,j}) \leq 3$; thus letting $z_B \in \mathsf{Z}(B_{i,j})$ be any factorization of $B_{i,j}$, we see that $z' = z_BA_k \in \mathsf{Z}(UV)$, where $\{i,j,k\} = \{1,2,3\}$, is a factorization of UV with $|z| \in [3,4]$, contradicting $\mathsf{L}(UV) \cap [3,4] = \emptyset$. So we may instead assume

$$|A_i| + |A_j| \neq 9 \quad \text{for all distinct } i,\, j \in [1,3]. \tag{5.13}$$

Suppose $-g \in \Sigma((-g)^{-1}A_i)$ for some $i \in [1,3]$. Then, since $\sigma((-g)^{-1}A_i) = g = -2g$, we can write

$$A_i = (-g)S_1S_2$$

with $S_1,\, S_2 \in \mathcal{F}(G)$ and $\sigma(S_1) = \sigma(S_2) = -g$. Let $\{i,j,k\} = \{1,2,3\}$ and $\{x,y\} = \{1,2\}$. Lemma 5.1 implies $gS_x \in \mathcal{A}(G)$ and

$$(-g)^{-1}A_jS_y \in \mathcal{B}(G) \quad \text{with} \quad \max \mathsf{L}\big((-g)^{-1}A_jS_y\big) \leq \min\{|(-g)^{-1}A_j|,\, |S_y|\}\,. \tag{5.14}$$

Noting that $\Big((-g)^{-1}A_jS_y\Big)\big(gS_x\big)A_k = UV$ and letting $z_B \in \mathsf{Z}\big((-g)^{-1}A_jS_y)\big)$ be any factorization of $(-g)^{-1}A_jS_y$, we see that the factorization $z' = z_B\big(gS_x\big)A_k \in \mathsf{Z}(UV)$ will contradict $\mathsf{L}(UV) \cap [3,4] = \emptyset$ unless $|z_B| \geq 3$. Thus (5.14) implies $|S_y| \geq 3$ and $|(-g)^{-1}A_j| \geq 3$. Since $y \in \{1,2\}$ and $j \in \{1,2,3\} \setminus \{i\}$ are arbitrary, this implies first that $|S_1|,\, |S_2| \geq 3$, whence $|A_i| \geq 7$, and second that $|A_j|,\, |A_k| \geq 4$ for $j,\, k \neq i$. Combining these estimates, we find that $15 \leq |A_1| + |A_2| + |A_3| = |((-g)^2U')V| \leq 2\mathsf{D}(G) = 14$, a contradiction. So we conclude that

$$-g \notin \Sigma((-g)^{-1}A_i) \quad \text{for all } i \in [1,3]\,. \tag{5.15}$$

Suppose $|A_2| \leq 4$. Let $z_B \in \mathsf{Z}(B_{1,3})$ be a factorization of $B_{1,3} = \big((-g)^{-1}A_1\big)\big((-g)^{-1}A_3g\big)$. In view of (5.15), we see that $(-g)^{-1}A_3g$ is zero-sum free, whence Lemma 5.1 and (5.11) imply $|z_B| \leq |(-g)^{-1}A_1| < |A_2| \leq 4$. Thus $z' = z_BA_2 \in \mathsf{Z}(UV)$ is a factorization of UV with $|z'| \leq 4$, whence $\mathsf{L}(UV) \cap [3,4] = \emptyset$ implies $|z'| = 2$ and $|z_B| = 1$, that is, $B_{1,3} \in \mathcal{A}(G)$ is an atom. Consequently, $g^{-1}B_{1,3} = (-g)^{-2}A_1A_3$ is zero-sum free. Hence, noting that

$$UV((-g)g)^{-1} = \Big((-g)^{-2}A_1A_3\Big)\Big((-g)^{-1}A_2\Big)\,.$$

we see that Lemma 5.1 implies

$$\max \mathsf{L}\big(UV((-g)g)^{-1})\big) \le |(-g)^{-1}A_2| < |A_2| \le 4\,.$$

which contradicts that $W\big((-g)g\big)^{-1} = W_2 \cdot \ldots \cdot W_t \in \mathsf{Z}\big(UV((-g)g)^{-1})\big)$ is a factorization of length $t-1 = |W|-1 \ge 4$. So we can instead assume $|A_2| \ge 5$.

Observe that

$$\operatorname{supp}((-g)^{-1}A_i) \cap \langle g\rangle = \emptyset \quad \text{for } i \in [1,3], \tag{5.16}$$

since otherwise $\mathsf{v}_{-g}(A_i) \ge 2$ or $\mathsf{v}_g(UV) \ge 2$—the first contradicts (5.10), while the the second contradicts the supposition of CASE 1 that $g \notin \Sigma(U')$ as $g \nmid V$. From (5.16), we see that $|A_1| \ge 3$, which, combined with $5 \le |A_2| \le |A_3|$ and $|A_1|+|A_2|+|A_3| = |((-g)^2U')V| \le 2\mathsf{D}(G) = 14$, implies that

$$(|A_1|,|A_2|,|A_3|) \in \{(3,5,5),(3,5,6),(4,5,5)\}.$$

Thus, in view of (5.13), we conclude that $|A_1| = 3$ and $|A_2| = |A_3| = 5$.

Since $|B_{1,j}| = 7$, for $j \in \{2,3\}$, it follows from (5.12) that

$$B_{1,j} \in \mathcal{A}(G) \quad \text{for } j \in \{2,3\} \tag{5.17}$$

is an atom as otherwise $z' = z_BA_k \in \mathsf{Z}(UV)$, where $z_B \in \mathsf{Z}(B_{1,j})$ and $\{1,j,k\} = \{1,2,3\}$, will contradict $\mathsf{L}(UV) \cap [3,4] = \emptyset$. Since $|B_{2,3}| = 9 > \mathsf{D}(G)$, it follows from (5.12) that $z' = z_BA_1 \in \mathsf{Z}(UV)$, for some $z_B \in \mathsf{Z}(B_{2,3})$, will contradict $\mathsf{L}(UV) \cap [3,4] = \emptyset$ unless all $z_B \in \mathsf{Z}(B_{2,3})$ have $|z_B| = 4$. Consequently, since there is no 2-term zero-sum containing g in $B_{2,3} = (-g)^{-2}A_2A_3g$ (recall the argument used to prove (5.12)), we conclude that $A_2A_3 = (-g)Xa(-g)(-X)b$ for some $X = x_1x_2x_3 \in \mathcal{F}(G)$ and $a,\, b \in G$ with

$$a+b = -g.$$

Thus, in view of (5.15), we find that w.l.o.g.

$$A_2 = (-g)Xa \quad \text{and} \quad A_3 = (-g)(-X)b.$$

If $a = b$, then $2a = a+b = -g$ implies $a = g$, in contradiction to (5.16). Therefore $a \ne b$.

Let

$$A_1 = (-g)Y \quad \text{with} \quad Y = y_1y_2 \in \mathcal{F}(G).$$

In view of (5.17), (5.16) and Lemma 5.7, we see that there are terms $a' \in \operatorname{supp}(YXa) = \operatorname{supp}(B_{1,2}g^{-1})$ and $b' \in \operatorname{supp}(Y(-X)b) = \operatorname{supp}(B_{1,3}g^{-1})$ with

$$\mathsf{v}_{a'}(YXa) \ge 2 \quad \text{and} \quad \mathsf{v}_{b'}(Y(-X)b) \ge 2.$$

If $y_1 = y_2$, then $2y_1 = y_1+y_2 = g$ (in view of $A_1 = (-g)y_1y_2$), in contradiction to (5.16); if $x_i = x_j$ for i and j distinct, then $x_i^2(-x_i)^2|X(-X)$, so that $x_i^2(-x_i)^2|UV$ is subsequence of 4 terms all from $\langle x_i\rangle$, whence Lemma 5.2 implies UV has a factorization of length 3, contradicting $\mathsf{L}(UV) \cap [3,4] = \emptyset$; and if $y_i = x_j$ or $y_i = -x_j$ for some $i \in [1,2]$ and $j \in [1,3]$, then the 2-term zero-sum $y_i(-x_j)$ or y_ix_i divides $B_{1,3}$ or $B_{1,2}$, respectively, contradicting (5.17). Consequently, $\mathsf{v}_{a'}(YXa) \ge 2$ and $\mathsf{v}_{b'}(Y(-X)b) \ge 2$ force $a' = a$ and $b' = b$. Moreover, since $a \ne b$, we have $ab|XY(-X)$. Since $a+b = -g$, we have $a^2b^2(-g) \in \mathcal{B}(G)$. However, noting that there is no 2-term zero-sum subsequence of the length 5 zero-sum $a^2b^2(-g)$, we actually have $C = a^2b^2(-g) \in \mathcal{A}(G)$. Note that $UV = g(-g)YX(-X)ab$ and $C|UV$ (in view of $ab|XY(-X)$). Let $z_B \in \mathsf{Z}(UVC^{-1})$. Since $|UVC^{-1}| = |A_1|+|A_2|+|A_3|-1-|C| = 7$, we have $|z_B| \le 3$, while clearly UVC^{-1} contains some 2-term zero-sum subsequence from $X(-X)$, so that $|z_B| \ge 2$. As a result, the factorization $z' = z_BC \in \mathsf{Z}(UV)$ contradicts that $\mathsf{L}(UV) \cap [3,4] = \emptyset$, completing the subcase.

CASE 1.2: $r = 4$.

If $-g \in \operatorname{supp}(A_4)$ as well, then we may w.l.o.g. assume $|A_1| \le |A_2| \le |A_3| \le |A_4|$, in which case $|(-g)^2U'V| = |A_1|+|A_2|+|A_3|+|A_4| \le 2\mathsf{D}(G) = 14$ implies $|B_{1,2}| \le 5$. Thus $z' = z_BA_3A_4 \in \mathsf{Z}(UV)$, where $z_B \in \mathsf{Z}(B_{1,2})$, contradicts $\mathsf{L}(UV) \cap [3,4] = \emptyset$ in view of (5.12). Therefore we may assume $-g \notin \operatorname{supp}(A_4)$. Consequently, in view of (5.10) and the definition of the A_i, we find that $-g \notin \operatorname{supp}(V')$.

Since $|A_4| \geq 2$, we see that $|(-g)^2U'V| = |A_1|+|A_2|+|A_3|+|A_4| \leq 2\mathsf{D}(G) = 14$ implies $|B_{1,2}| \leq 7$, with equality only possible if $|(-g)^2U'V| = 14$. However, if $|B_{1,2}| \leq 6$, then $z' = z_B A_3 A_4 \in \mathsf{Z}(UV)$, where $z_B \in \mathsf{Z}(B_{1,2})$, contradicts $\mathsf{L}(UV) \cap [3,4] = \emptyset$ in view of (5.12). Therefore we indeed see that $|B_{1,2}| = 7$ and $|(-g)^2U'V| = 14$. As a result, since $(-g)^2U' \in \mathcal{A}(G)$ implies $|U| + 1 = |(-g)^2U'| \leq \mathsf{D}(G) = 7$, and since $|V| \leq \mathsf{D}(G) = 7$ as well, it follows that $|V| = 7$ and $|U| = 6$.

Since $|V| = 7 = \mathsf{D}(G)$, it follows that $-g \in \Sigma(V') = G^\bullet$. Thus, since $\sigma(V') = g = 2(-g)$, we see that we can write $V' = S_1S_2$ with S_1, $S_2 \in \mathcal{F}(G)$ and $\sigma(S_1) = \sigma(S_2) = -g$, and w.l.o.g. assume $|S_1| \leq |S_2|$. Then, since $|V'| = 6$ and $-g \notin \operatorname{supp}(V')$, we infer that $2 \leq |S_1| \leq 3$.

But now consider $g^{-1}U(-g)S_1 \in \mathcal{B}(G)$ and $((-g)S_1)^{-1}Vg \in \mathcal{B}(G)$. By Lemma 5.1,

$$\big(((-g)S_1)^{-1}V\big)g \in \mathcal{A}(G)$$

is an atom. Let

$$z_B \in \mathsf{Z}\big(g^{-1}U(-g)S_1\big)\,.$$

Since $|g^{-1}U(-g)S_1| = |U| + |S_1| \geq |U| + 2 = 8 > \mathsf{D}(G)$, we have $|z_B| \geq 2$. Since $g \notin \Sigma(U') = \Sigma(g^{-1}U)$ by the supposition of CASE 1, Lemma 5.1 implies $|z_B| < |(-g)S_1| \leq 4$. Thus $z' = \big(((-g)S_1)^{-1}V\big)z_B \in \mathsf{Z}(UV)$ has $|z'| \in [3,4]$, contradicting $\mathsf{L}(UV) \cap [3,4] = \emptyset$ and completing CASE 1.

CASE 2: We have $g \in \Sigma(U')$ and $-g \in \Sigma(V')$.

Then, since $\sigma(U') = -g = 2g$ and $\sigma(V') = g = 2(-g)$, we can write $U' = S_1S_2$ and $V' = T_1T_2$ with S_1, S_2, T_1, $T_2 \in \mathcal{F}(G)$, $\sigma(S_1) = \sigma(S_2) = g$ and $\sigma(T_1) = \sigma(T_2) = -g$. Let $i \in \{1,2\}$ and $j \in \{1,2\}$. Note that

$$gT_{3-j} \in \mathcal{A}(G) \quad \text{and} \quad (-g)S_{3-i} \in \mathcal{A}(G)$$

by Lemma 5.1. Also, $S_iT_j \in \mathcal{B}(G)$ and, for $z_B \in \mathsf{Z}(S_iT_j)$, Lemma 5.1 implies

$$|z_B| \leq \min\{|S_i|,\, |T_j|\}\,. \tag{5.18}$$

Now $z' = (gT_{3-j})\big((-g)S_{3-i}\big)z_B \in \mathsf{Z}(UV)$ will contradict $\mathsf{L}(UV) \cap [3,4] = \emptyset$ unless $|z_B| \geq 3$, in which case (5.18) implies $|S_i| \geq 3$ and $|T_j| \geq 3$. Since i and j were arbitrary, this implies $|S_i|$, $|T_j| \geq 3$ for all $i,j \in \{1,2\}$. Hence, since $|U| = 1 + |S_1| + |S_2| \leq \mathsf{D}(G) = 7$, we see that $|S_1| = |S_2| = 3$, and likewise $|T_1| = |T_2| = 3$. Thus we must have $|z_B| = 3$ for all choices of $i,j \in \{1,2\}$, which is only possible if $S_i = -T_j$ for all choices of $i,j \in \{1,2\}$. However, this implies $U = -V$ and, moreover, that $\mathsf{v}_{-x}(T_i) \geq 1$ for $i \in [1,2]$ and $x \in \operatorname{supp}(S_1S_2)$. Consequently, letting $x \in \operatorname{supp}(S_1S_2)$, we see that $\mathsf{v}_{-x}(V) \geq 2$, whence $U = -V$ implies $\mathsf{v}_x(U) \geq 2$. Thus $x^2(-x)^2 \,|\, UV$ is a subsequence of 4 terms all from $\langle x \rangle$, whence Lemma 5.2 implies UV has a factorization of length 3, contradicting $\mathsf{L}(UV) \cap [3,4] = \emptyset$ and completing CASE 2 and the proof. $\square$

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