NDR PAIRS, THE HEP, AND CW COMPLEXES

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1. Introduction

We recall some basic facts about the "homotopy extension property" (HEP). The treatment here was inspired by my reading of the texts of Hatcher and May.

2. Attaching cells

Let Y be a space. To "attach an n-cell" to Y, we give a map

$$f : S^{n-1} \to Y,$$

and we form the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ D^n \arrow[r] & Z = Y \cup_f D^n. \end{tikzcd}$$

Note that for some reason in this context by long tradition topologists use e^n instead of D^n to stand for the closed n-disk.

There are many variants of this construction. For example, we can attach lots of n-cells at once

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r, "f"] \arrow[d] & Y \arrow[d] \arrow[ddr] & \\ \coprod D^n \arrow[r] \arrow[drr] & Z \arrow[dr, dashed, "\exists!"] & \\ & & T. \end{tikzcd}$$

Note that, since the diagram is a pushout, we know how to map out of Z, and I have taken the liberty of displaying this in the preceding diagram.

3. CW complex

A CW complex is a pair of spaces (X, A) (this notation will mean $A \subseteq X$), together with an increasing sequence of subspaces $A = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$, with the following properties.

(1) X_n is obtained from X_{n-1} by attaching some n-cells: that is we have a pushout diagram of the form

$$\begin{tikzcd} \coprod_{\alpha \in I_n} S^{n-1} \arrow[r, "f"] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_{\alpha \in I_n} D^n \arrow[r] & X_n. \end{tikzcd}$$

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(2) $X = \operatorname{colim} X_i$.

Recall that the second statement means that to give a map out of X is equivalent to give a sequence of compatible maps out of the X_i. Diagramatically,

$$\begin{tikzcd} A \arrow[r] \arrow[drrr] & X_1 \arrow[r] \arrow[drr] & \cdots \arrow[r] & \operatorname{colim} X_i \arrow[d, dashed, "\exists!"] \\ & & & T. \end{tikzcd} \tag{3.1}$$

If A is a finite discrete space, we may omit it from the notation and simply say that X is a CW complex.

I have given the definition this way to emphasize that the construction tells us a lot about maps out of X. We shall see how useful this is in §6. This definition does *not* display an important fact about this topology, namely, that for K compact, the natural map

$$\operatorname{colim} \operatorname{map}(K, K_i) \to \operatorname{map}(K, X)$$

is an isomorphism.

4. The Homotopy Extension Property: results

Two important difficulties in working with homotopy equivalence is that, in very great generality, it is not preserved by quotients. Here are two examples: if $A \subset X$ and $A \sim *$, we do not in general know that $X \to X/A$ is a homotopy equivalence. If $A \subseteq X$ and we have two maps

$$f, g : A \to B,$$

then we can form the two pushouts

$$\begin{tikzcd} A \arrow[r, "{f,g}"] \arrow[d] & B \arrow[d] \\ X \arrow[r] & B \cup_f X, B \cup_g X. \end{tikzcd}$$

In general, $f \sim g$ does not imply $B \cup_f X \sim B \cup_g X$.

However, if the inclusion of A in X is nice enough, then these problems do not occur.

Theorem 4.1. *If (X, A) satisfies the Homotopy Extension Property (HEP) and $A \sim *$, then the projection $X \to X/A$ is a homotopy equivalence. If (X, A) satisfies the Homotopy Extension Property, and $f, g : A \to B$ are two maps, then*

$$B \cup_f X \sim B \cup_g X \operatorname{rel} B.$$

Proposition 4.2. *If (X, A) is a CW complex, then it satisfies the HEP, and so the conclusions of the preceding Theorem hold.*

In the next few sections, I shall show how to view the HEP as "mapping out of" problem, so that Proposition 4.2 follows easily from our description of CW complexes.

Example 1. Let $X = S^1$ and $A = S^1 \backslash N$, the complement of the North Pole. Then A is contractible, but X/A is a two-point space with one dense point. The only continuous maps $X/A \to X$ are constant, and so $X \to X/A$ cannot be a homotopy equivalence (unless S^1 is contractible, and we shall soon see that it is not.).

5. The Homotopy Exension Property: equivalent forms

Definition 5.1. We say that (X, A) has the *Homotopy Extension Property* if there is a solution (NOT NECESSARILY UNIQUE) to every mapping problem of the form

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d] & A \times I \arrow[d] \arrow[ddr, "h"] & \\ X \arrow[r, "i_0"] \arrow[drr, "f"'] & X \times I \arrow[dr, dashed, "\exists H"] & \\ & & T. \end{tikzcd} \tag{5.2}$$

In words, for every map $f : X \to T$ and every homotopy $h : A \times I \to T$ with $h_0 = f|_A$, there is a homotopy $H : X \times I$ extending h.

We now give two useful reformulations of this definition.

The first reformulation is based on the fact that a homotopy

$$h : A \times I \to T$$

can be viewed as a path

$$I \to \mathrm{map}(A, T) = T^A$$

in the mapping space, or as a function

$$\tilde{h} : A \to T^I$$

from A to the path space of T. Using this last point of view, we have the following.

Lemma 5.3. *A pair (X, A) satisfies the HEP if and only if there is a solution (NOT NECESSARILY UNIQUE) to every mapping problem of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r] \arrow[ur, dashed, "\exists"] & T. \end{tikzcd}$$

□

The second formulation uses the pushout diagram

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d] & A \times I \arrow[d] \\ X \arrow[r] & A \times I \cup_{A \times \{0\}} X. \end{tikzcd}$$

If all mapping problems (5.2) have a solution, then in particular we have a map

$$r : X \times I \to A \times I \cup_{A \times \{0\}} X$$

which is easily seen to be retract of the inclusion

$$A \times I \cup_{A \times \{0\}} X \to X \times I.$$

On the other hand, given a retraction r, the universal mapping property of the pushout $A \times I \cup_{A \times \{0\}} X$ implies that we have a solution to all mapping problems (5.2). Thus we have the following.

Lemma 5.4. *A pair (X, A) satisfies the HEP if and only if $A \times I \cup_{A \times \{0\}} X$ is a retract of $X \times I$.*

There is one more very useful characterization of pairs satisfying the HEP: they are NDR pairs. In the appendix, we define these and give a proof of the following.

Lemma 5.5. *A pair (X, A) satisfies the HEP if and only if it is an NDR pair.* □

In the appendix we use this Lemma to prove the

Lemma 5.6. *For $n \geq 0$, the pair (S^n, D^{n+1}) is an NDR pair, and so satisfies the HEP.* $\square$

6. CW complexes and the HEP

Now let's prove the following result, also proved in Hatcher's Chapter 0.

Proposition 6.1. *Let (X, A) be a CW complex. Then it satisfies the HEP.*

We prove this using the following Lemmas.

Lemma 6.2. *Suppose that (X_α, A_α) is a collection of HEP pairs. Then $(X, A) = (\coprod X_\alpha, \coprod A_\alpha)$ is an HEP pair.*

Proof. We must solve the mapping problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r] \arrow[ur, dashed, "\exists"] & T. \end{tikzcd} \tag{6.3}$$

But the solid arrows are equivalent to a collection of solid arrows in the diagrams

$$\begin{tikzcd} A_\alpha \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X_\alpha \arrow[r] \arrow[ur, dashed, "\exists"] & T, \end{tikzcd}$$

and each of these problems has a solution since (X_α, A_α) is an NDR pair. The defining property of the coproduct is that to give a mpa out of it is to give a map out of each summand. So the collection of solutions

$$X_\alpha \to T^I$$

in turn gives a solution to the mapping problem (6.3), as required. $\square$

Lemma 6.4. *Suppose that (X, A) is an HEP pair, $f : A \to B$ is a map, and Y is the pushout in the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y. \end{tikzcd}$$

Then (Y, B) is an HEP pair.

Proof. Consider the mapping problem

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r] & Y \arrow[r] \arrow[ur, dashed, "\exists"] & T. \end{tikzcd}$$

Examine the outer rectangle. The HEP for (X, A) gives the dotted arrow in the diagram below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r] \arrow[urr, dashed] & Y \arrow[r] & T. \end{tikzcd}$$

But now the pushout property of Y guarantees that we have the desired map. $\square$

Lemma 6.5. *Suppose that*

$$A = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X,$$

and $X = \operatorname{colim} X_i$. *If each* (X_i, X_{i-1}) *satisfies the HEP, then so does* (X, A).

Proof. Suppose given a mapping problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r] \arrow[ur, dashed, "\exists"] & T. \end{tikzcd}$$

If (X_1, A) satisfies the HEP, then we have a solution to the mapping problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r] \arrow[ur, dashed, "\exists"] & T. \end{tikzcd}$$

Suppose $n \geq 1$. Given a solution to the mapping problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X_n \arrow[r] \arrow[ur, dashed, "\exists"] & T, \end{tikzcd}$$

we obtain one for the mapping problem

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X_{n+1} \arrow[r] \arrow[ur, dashed, "\exists"] & T. \end{tikzcd}$$

By induction and using the mapping property (3.1) of X, we conclude that (X, A) has the HEP. $\square$

Proof of Proposition 6.1. This is now very easy. Lemmas 5.6 and 6.2 together imply that

$$\coprod S^{n-1} \to \coprod D^n$$

satisfies the HEP. For $n \geq 1$ we have a pushout diagram of the form

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod D^n \arrow[r] & X_n, \end{tikzcd}$$

so (X_n, X_{n-1}) satisfies the HEP. Since $X = \operatorname{colim} X_n$, Lemma 6.5 implies that (X, A) satisfies the HEP. $\square$

7. Applications

We can now easily prove Theorem 4.1. Consider first the case that (X, A) satisfies the HEP, and A is contractible. Let

$$c : A \to A^I$$

be a contraction: so $c(a)(0) = a$, while $c(a)(1) = *$. We can compose with the inclusion into X to get map

$$c : A \to X^I,$$

and becuase (X, A) satisfies the HEP, we have a solution to the mapping problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X^I \arrow[d] \\ X \arrow[r, equal] \arrow[ur, dashed, "h"] & X. \end{tikzcd}$$

Choose a solution. Let

$$\pi : X \to X/A$$

denote the projection. Then we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "c"] \arrow[d] & A^I \arrow[r, "\mathrm{ev}_1"] \arrow[d] & * \arrow[d] \\
X \arrow[r, "h"] \arrow[d, "\pi"'] & X^I \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"'] \\
X/A \arrow[r, "\bar{h}"] \arrow[urr, dashed, bend right, "s"'] & (X/A)^I \arrow[r, "\mathrm{ev}_1"] & X/A.
\end{tikzcd}$$

The dotted arrow s exists because, as you see by going clockwise from the top left, A maps to a single point in X under $h\,\mathrm{ev}_1$.

We must check that $s\pi \sim 1_X$ and $\pi s \sim 1_{X/A}$. For the first, the commutativity of the diagram means that

$$s\pi = \mathrm{ev}_1 h.$$

Now

$$X \xrightarrow{h} X^I \xrightarrow{\mathrm{ev}_0} X$$

is the identity, and so

$$s\pi \sim 1_X.$$

For the second, the commutativity of the diagram means that

$$\pi s = \mathrm{ev}_1 \bar{h}.$$

Since $\mathrm{ev}_0 h$ is the identity, we must have that

$$X/A \xrightarrow{\bar{h}} (X/A)^I \xrightarrow{\mathrm{ev}_0} X/A$$

is the identity as well. So $\pi s \sim 1_{X/A}$.

For the other part, I would say the same thing as Hatcher. Note that, as we say in Proposition A.6, if (X, A) satisfies the HEP, then

$$A \times I \cup X \times \{0\} \to X \times I$$

is a deformation retract. Let

$$d : X \times I \to X \times I$$

be such a deformation retract. So

$$d(x, 0) = x \tag{7.1}$$
$$d(a, t) = a \tag{7.2}$$
$$d(x, 1) \in A \times I \cup X \times \{0\}. \tag{7.3}$$

Let

$$H : A \times I \to B$$

be a homotopy from f to g, and consider

$$B \cup_H X \times I.$$

This contains

$$B \cup_f X \times \{0\}$$

and

$$B \cup_g X \times \{1\}$$

as subspaces. Because the deformation retract fixes A, it can be used to deform $B \cup_H X \times I$ to $B \cup_f X \times \{0\}$ or $B \cup_H X \times I$ to $B \cup_g X \times \{1\}$. Either deformation fixes B. $\square$

Appendix A. NDR pairs and the HEP

The characterizations of the HEP by mapping properties in the main text have the virtue that they are useful to work with in practice. It is very useful to have one more description: an HEP pair is an NDR pair. This material is classical; I have followed the treatment in May's *Concise introduction to algebraic topology.*

The idea is that the inclusion $i : A \to X$ will have good homotopical properties if the closure of A has an open neighborhood B, such that the inclusion $j : \bar{A} \to B$ is a deformation retract. This means that there is a map

$$r : B \to \bar{A}$$

such that

$$rj = 1_{\bar{A}}$$

and such that

$$jr \sim 1_B \operatorname{rel} \bar{A}. \tag{A.1}$$

Such a B was classically called a *collar* of A, and one said that A had a *collared neighborhood.* At some point, someone realized that it was useful to include the homotopy (A.1) as part of the data. The result is the notion of NDR pair.

Definition A.2. Let X be a space, and let $A \subset X$ be a subspace. The pair (X, A) is an *NDR pair* if there are maps

$$\begin{aligned} u :& X \to I \\ d :& X \times I \to X \end{aligned}$$

such that

$$\begin{aligned} A &= u^{-1}(0) & & \\ d(x, 0) &= x & & x \in X \\ d(a, t) &= a & & a \in A \\ d(x, 1) &\in A & & u(x) < 1. \end{aligned}$$

The maps u, d might be called *NDR data* for (X, A), or *present* (X, A) *as an NDR pair.* We say that (X, A) is a *DR* pair if moreover we can take $u(x) < 1$ for all x.

Here's one justification for the terminology. Let

$$B = u^{-1}([0, 1)).$$

So B is an open neighborhood of A in X.

Lemma A.3. *The map d induces a deformation retraction of the inclusion of A in B.* □

Lemma A.4. *The inclusion of* 0 *in* $I = [0, 1]$ *is a DR pair.* □

Proof. Take

$$d(x, t) = tx$$

and

$$u(x) = x/2.$$

□

Lemma A.5. *If* (X, A) *and* (Y, B) *are NDR pairs, then so is*

$$(Z, C) = (X \times Y, X \times B \cup A \times Y).$$

If either (X, A) *or* (Y, B) *is a DR pair, then so is* (Z, C).

Proof. Suppose u and d are NDR data for (X, A), and (v, e) are NDR data for (Y, B). Let $w : X \times Y \to I$ be given by

$$w(x, y) = \min(u(x), v(y)),$$

and let $f : X \times Y \times I \to X \times Y$ be given by

$$f(x, y, t) = \begin{cases} (x, y) & \text{if } v(x) = u(y) = 0 \\ (d(x, t), e(y, tu(x)/v(y))) & \text{if } v(y) \geq u(x) \\ (d(x, tv(y)/u(x)), e(y, t)) \text{ if } u(x) \geq v(y). \end{cases}$$

Notice that if $v(y) \geq u(x)$ and $v(y) = 0$, then $u(x) = 0$; and if $u(x) \geq v(y)$ and $u(x) = 0$, then $v(y) = 0$, so f is continuous, and $w^{-1}(0) = C$. It is easy to check that this pair functions is NDR data for (Z, C), and DR data in the indicated cases. □

The relevance to the HEP is the following.

Proposition A.6. *The following are equivalent, about a pair (X, A).*

(1) (X, A) *is an NDR pair.*
(2) $(X \times I, X \times \{0\} \cup A \times I)$ *is a DR pair.*
(3) $X \times \{0\} \cup A \times I$ *is retract of* $X \times I$*.*
(4) (X, A) *satisfies the HEP.*

Proof. Lemma A.4 and Lemma A.5 prove that 1 implies 2, which certainly imlies 3. Lemma 5.4 shows that 3 and 4 are equivalent. It remains to show that 3 implies 1. Suppose given a retraction

$$r : X \times I \to X \times \{0\} \cup A \times I.$$

Let $\pi_1 : X \times I \to X$ and $\pi_2 : X \times I \to I$ be the projections. Let

$$u(x) = \sup\{t - \pi_2 r(x, t) | t \in I\},$$

and let

$$d(x, t) = \pi_1 r(x, t).$$

First, $u^{-1}(0) = A$, since if $t - \pi_2 r(x, t) = 0$ and $t > 0$ then

$$r(x, t) \in A \times I$$

so $x \in A$. Second,

$$d(x, 0) = \pi_1 r(x, 0) = x,$$

since r is a retraction. Third,

$$d(a, t) = \pi_1 r(a, t) = a,$$

again since r is a retraction. Finally, if $u(x) < 1$ then $\pi_2 r(x, 1) > 0$ so $r(x, 1) \in A \times I$, and $d(x, 1) \in A$. □

We conclude with the following example.

Lemma A.7. (D^n, S^{n-1}) *is an NDR pair, and so satisfies the HEP.*

Proof. Let

$$u(x) = \begin{cases} 2(1 - \|x\|) & \|x\| \geq 1/2 \\ 1 & \|x\| < 1/2. \end{cases}$$

Let

$$s : \mathbb{R} \to \mathbb{R}$$

be a given by the formula

$$s(r) = \begin{cases} 0 & r \leq \frac{1}{4} \\ 4(r - \frac{1}{4}) & 1/4 < r \leq \frac{1}{2}. \\ 1 & \frac{1}{2} < r. \end{cases}$$

Let

$$d(x,t) = s(\|x\|)t\frac{x}{\|x\|} + (1 - s(\|x\|)t)x$$

for $x \neq 0$, and $d(0,t) = 0$. Then u and d have the desired properties. $\square$