NEARBY CYCLES OF AUTOMORPHIC ÉTALE SHEAVES

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Abstract. We show that the automorphic étale cohomology of a (possibly noncompact) PEL-type or Hodge-type Shimura variety in characteristic zero is canonically isomorphic to the cohomology of the associated nearby cycles over most of their mixed characteristics models constructed in the literature.

Contents

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1. Introduction

The étale cohomology of Shimura varieties are arguably one of the most centrally important geometric objects in modern algebraic number theory, because they provide natural grounds for relating the automorphic representations to Galois representations. To study any representation of the Galois group of number fields, it is important to understand its restriction to the various decomposition subgroups. In the case of the étale cohomology of Shimura varieties, such restrictions to decompositions groups can be analyzed by considering the cohomology of the associated nearby cycles over the *reductions* of Shimura varieties, namely the positive characteristic fibers of certainly naturally defined integral models, at least when the integral models in question are *proper*. The goal of this article is to show that this assumption of properness is redundant for most of the mixed characteristics models of Shimura varieties constructed in the literature. As byproducts, we obtain generalizations of many results previous known only in the proper case.

Let us explain our goal in more details. Suppose $\mathsf{X} \to \mathsf{S} = \mathrm{Spec}(R_0)$ is a model of a Shimura variety in mixed characteristics over the spectrum of a Henselian discrete valuation ring. Let $j : \eta = \mathrm{Spec}(K) \to \mathsf{S}$ (resp. $i : s = \mathrm{Spec}(k) \to \mathsf{S}$) denote the generic (resp. special) point of $\mathsf{S} = \mathrm{Spec}(R_0)$, with its structural morphism. Let $\bar{K}$ be an algebraic closure of K, let $\bar{R}_0$ denote the integral closure of R_0 in $\bar{K}$, with residue field $\bar{k}$ an algebraic closure of k, and let $\bar{j} : \bar{\eta} := \mathrm{Spec}(\bar{K}) \to \bar{\mathsf{S}} := \mathrm{Spec}(\bar{R}_0)$ (resp. $\bar{i} : \bar{s} := \mathrm{Spec}(\bar{k}) \to \bar{\mathsf{S}}$) denote the corresponding geometric point lifting i (resp. j). For simplicity, we shall denote by subscripts the pullbacks of various schemes over S to η, $\bar{\eta}$, s, or $\bar{s}$. Consider any rational prime number $\ell \neq p$. Then we have the (complex of) nearby cycles $R\Psi\mathbb{Q}_\ell := \bar{i}^* R\bar{j}_* \mathbb{Q}_\ell$ over $\mathsf{X}_{\bar{s}}$, which is nothing but the constant sheaf $\mathbb{Q}_\ell$ when the morphism $\mathsf{X} \to \mathsf{S}$ is *smooth* (see [3, XV, 2.1] and [13, XIII, 2.1.5]).

When the morphism $\mathsf{X} \to \mathsf{S}$ is *proper*, it is a consequence of the proper base change theorem (see [3, XII, 5.1]) that we have a canonical isomorphism

$$H^i_{\text{ét}}(\mathsf{X}_{\bar{\eta}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i_{\text{ét}}(\mathsf{X}_{\bar{s}}, R\Psi\mathbb{Q}_\ell) \tag{1.1}$$

of $\mathrm{Gal}(\bar{K}/K)$-modules, for each i. There are similar isomorphisms when we replace the coefficient sheaf $\mathbb{Q}_\ell$ with more general automorphic étale sheaves. As an immediate consequence, when $\mathsf{X} \to \mathsf{S}$ is smooth, $H^i_{\text{ét}}(\mathsf{X}_{\bar{\eta}}, \mathbb{Q}_\ell)$ and its analogues for more general automorphic étale sheaves are *unramified* as $\mathrm{Gal}(\bar{K}/K)$-modules. More generally, such isomorphisms allow us to study their left-hand sides by analyzing their right-hand sides, often using the geometry of $\mathsf{X}_{\bar{s}}$. They serve as the foundation of, for example, the important works [25], [46], [47], and [59], and hence of the subsequent works [61] and [60] based on them.

In fact, in the above-mentioned works, the analysis of the cohomology of nearby cycles were carried out without the assumption that the model $\mathsf{X} \to \mathsf{S}$ is proper. It is only in their initial steps—or final steps, depending on one's viewpoint—that they assume the existence of some isomorphisms as in (1.1), in order to relate their results to the étale cohomology in characteristic zero. (Such a relation to the cohomology in characteristic zero is crucial for the results to be useful.)

The goal of this article, as we repeat now again in more detail, is to show that isomorphisms as in (1.1), and their analogues for the compactly supported cohomology and for the intersection cohomology (of the associated minimal compactifications), exist for most constructions of mixed characteristics models in the literature, not just for the trivial coefficients but also for more general sheaves (which we call *automorphic sheaves*), without assuming that $\mathsf{X} \to \mathsf{S}$ is proper. (Certainly, $\mathsf{X} \to \mathsf{S}$ cannot be arbitrary—isomorphisms such as (1.1) can be destroyed by removing closed subschemes from the special fiber. We will show that the integral models we consider are natural in the sense that, intuitively speaking, there are no such missing subschemes.) Consequently, we will obtain almost for free several generalizations of the above-mentioned works to the nonproper case, without having to repeat their delicate arguments.

The existence of isomorphisms as in (1.1) is essentially known when $\mathsf{X} \to \mathsf{S}$ admits proper smooth compactifications whose boundaries are given by divisors with relative normal crossings, as in the case in [17, Ch. IV] of the Siegel moduli of principally polarized abelian schemes, or more generally as in the case in [37] (resp. [45]) of smooth integral models of PEL-type (resp. Hodge-type) Shimura varieties with hyperspecial levels at the residue characteristics (resp. odd residue characteristics), where nice toroidal compactifications are known to exist. In these cases, it follows from [13, XIII, 2.1.10] that there are also isomorphisms like (1.1) for the case of trivial coefficients (namely, $\mathbb{Q}_\ell$). Moreover, as explained in [17, Ch. VI, Sec. 6], it follows from the constructions of good toroidal compactifications of the Kuga families as in [17, Ch. VI] and [36] that there are also analogues of (1.1) for automorphic sheaves. (We warn the reader that the argument in [26, Sec. 7], for the usual and compactly supported cohomology in the setting of this paragraph, is unfortunately incomplete—the first step in the proof of [26, Lem. 7.1] should require some tameness assumption as in [13, XIII, 2.1.10]. To clarify the matters, we will also include these essentially known cases in our treatment.)

However, the *good reduction cases* (as they are often called) in the previous paragraph require the algebraic data defining the integral models of Shimura varieties to be unramified in the strongest possible sense. While these unramified cases are already very useful, we now also know good constructions of integral models of PEL-type and even Hodge-type Shimura varieties and their toroidal and minimal compactifications, in all ramified cases, thanks to the more recent developments in [65], [64], [45], [39], [42], and [41]. As we shall explain in later sections, the constructions we shall consider are good in a precise sense. Roughly speaking, étale locally, the natural inclusion from the integral model of the Shimura variety or Kuga family in question into its toroidal compactifications are direct products of some affine toroidal embeddings with the identity morphisms on some schemes (about which we know little), and it is only the factors of affine toroidal embeddings which matter for showing the compatibility between the formations of direct images (under the above natural inclusions) and of nearby cycles. Once we have such an étale local description of the toroidal boundary, the remaining arguments are straightforward, thanks to an idea due to Laumon (see [21, Lem. 7.1.4] and Remark 5.33).

As a special case, we have established [23, Conjecture 10.3] in all PEL-type cases for integral models at parahoric levels whose associated flat local models are known to be normal (including the cases considered in [50] and [51]; see Remark 6.15). Our results subsume some closely related results by Imai and Mieda in [28] for the supercuspidal parts of the cohomology, although we have learned from them that their assumptions in [28] can be much relaxed (see Remark 5.42 for more details).

Here is an outline of this article. In Section 2, we introduce the integral models of PEL-type and Hodge-type Shimura varieties that we consider, together with their toroidal and minimal compactifications, and summarize some of their important properties. (We note that none of the three PEL-type cases we consider is completely subsumed by the Hodge-type case. This is about the actual choices of integral models, but not about the classification in characteristic zero.) In Section 3, we define the automorphic sheaves we shall consider, first by using finite étale coverings of our Shimura varieties, and then by using the relative cohomology of certain Kuga families, which are isogenous to self-fiber products of the universal abelian schemes over some PEL-type Shimura varieties. In Section 4, we explain how to realize such Kuga families as some toroidal boundary strata of larger Shimura varieties, and realize their toroidal compactifications as closures of such strata. In Section 5, we review the definition and some basic properties of nearby cycles, and prove our main results of comparisons for the usual cohomology, the compactly supported cohomology, and the intersection cohomology. In Section 6, we explain some applications of such results, including the unipotency of inertial actions on the cohomology of PEL-type Shimura varieties at parahoric levels (for the flat integral models considered in [50] and [51] that are normal); and the above-mentioned generalizations of [46] and [47], and of (a slightly weaker form of) [59].

During the preparation of this article, we observed that the supports of nearby cycles over the good integral models of Shimura varieties we consider, even in the trivial coefficient case, enjoy some intriguing nice features near the toroidal and minimal boundary, which make it possible to talk about good toroidal and minimal compactifications of such supports. Moreover, the same can be said for several other kinds of subschemes over the integral models we consider. We shall pursue this topic in more detail in a forthcoming work.

We shall follow [37, Notation and Conventions] unless otherwise specified. While for practical reasons we cannot explain everything we need from the various constructions of toroidal and minimal compactifications we need, we recommend the reader to make use of the reasonably detailed indices and tables of contents in [37] and [38], when looking for the numerous definitions. For references to [37] and [36], the reader should also consult the errata available on the author's website for corrections to known errors and imprecisions.

2. Integral models with good compactifications

2.1. **The cases we consider.** Let $p > 0$ be a rational prime number.

Assumption 2.1. *Let $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ be a scheme over the spectrum of a discrete valuation ring R_0 of mixed characteristics $(0, p)$, which is the pullback of one the following integral models in the literature: (The various notations S_0, $\vec{\mathsf{S}}_0$, etc below are those in the works we cited, which we will freely use, but mostly only in proofs.)*

(Sm) *A smooth integral model $\mathsf{M}_{\mathcal{H}^\square} \to \mathsf{S}_0 = \operatorname{Spec}(\mathcal{O}_{F_0,(\square)})$ defined as a moduli of abelian schemes with PEL structures at a neat level $\mathcal{H}^\square \subset \mathrm{G}(\hat{\mathbb{Z}}^\square)$, as in* [37, Ch. 1, 2, and 7]*, with $p \in \square$ and $\mathcal{H} = \mathcal{H}^\square \times \prod_{q \in \square} \mathrm{G}(\mathbb{Z}_q)$. (When $\square = \{p\}$, it is shown in* [37, Prop. 1.4.3.4] *that the definition in* [37, Sec. 1.4.1] *by isomorphism classes agrees with the one in* [37, Sec. 1.4.2] *by $\mathbb{Z}_{(p)}^\times$-isogeny classes, which was Kottwitz's definition in* [33, Sec. 5].)

(Nm) *A flat integral model $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{S}}_0 = \operatorname{Spec}(\mathcal{O}_{F_0,(p)})$ of a moduli $\mathsf{M}_{\mathcal{H}} \to \mathsf{S}_0 = \operatorname{Spec}(F_0)$ at a neat level $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}})$ (essentially the same as above, but* *with* $\square = \emptyset$) *defined by taking normalizations over certain auxiliary good reduction models as in* [39, Sec. 6] (*which allow bad reductions due to arbitrarily high levels, ramifications, and collections of isogenies*). (*In this case, we also allow* F_0 *to be a finite extension of the reflex field, with* $\mathsf{M}_{\mathcal{H}}$ *etc replaced with their pullbacks.*) *For simplicity, we shall assume that, in the choice of the collection* $\{(L_j, \langle\,\cdot\,,\,\cdot\,\rangle_j)\}_{j\in J}$ *in* [39, Sec. 2], *we have* $(L_{j_0}, \langle\,\cdot\,,\,\cdot\,\rangle_{j_0}) = (p^{r_0}L, p^{-2r_0}\langle\,\cdot\,,\,\cdot\,\rangle)$ *for some* $j_0 \in J$ *and some* $r_0 \in \mathbb{Z}$.

(Spl) *A flat integral model* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_K)$ *of* $\mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} K \to \mathrm{Spec}(K)$ *defined by taking normalizations as in* [41, Sec. 2.4] *over the splitting models defined by Pappas–Rapoport as in* [50, Sec. 15]. (*By taking normalizations, we mean we also allow* $\mathcal{H}$ *to be arbitrarily higher levels, not just the same levels considered in* [50, Sec. 15].) *For simplicity, we shall assume that, in the choice of the collection* $\{(L_j, \langle\,\cdot\,,\,\cdot\,\rangle_j)\}_{j\in J}$ *in* [41, Choices 2.2.9], *we have* $(L_{j_0}, \langle\,\cdot\,,\,\cdot\,\rangle_{j_0}) = (p^{r_0}L, p^{-2r_0}\langle\,\cdot\,,\,\cdot\,\rangle)$ *for some* $j_0 \in J$ *and some* $r_0 \in \mathbb{Z}$.

(Hdg) *A flat integral model* $\mathscr{S}_K \to \mathrm{Spec}(\mathscr{O}_{E,(v)})$ *in the notation of* [45, Introduction] *at a neat level* K. *For consistency with the notation in other cases, we shall denote* K, E, *and* $\mathscr{S}_K$ *as* $\mathcal{H}$, F_0, *and* $\mathsf{M}_{\mathcal{H}}$, *respectively, in what follows. Essentially by construction, there exists some auxiliary good reduction Siegel moduli* $\mathsf{M}_{\mathcal{H}_{\mathrm{aux}}} \to \mathrm{Spec}(\mathbb{Z}_{(p)})$ *in Case* (Sm) *above, with a finite morphism* $\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}_{\mathrm{aux}}} \underset{\mathbb{Z}_{(p)}}{\otimes} \mathcal{O}_{F_0,(v)}$ *extending a closed immersion* $\mathsf{M}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \mathsf{M}_{\mathcal{H}_{\mathrm{aux}}} \underset{\mathbb{Z}}{\otimes} F_0$.

In all cases, there is some group functor G *over* $\mathrm{Spec}(\mathbb{Z})$, *and some reflex field* F_0.

- *In Cases* (Sm), (Nm), *and* (Spl), *the integral models are defined by* (*among other data*) *an integral PEL datum* $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ (*cf.* [37, Def. 1.2.1.3]), *which defines the group functor* G *as in* [37, Def. 1.2.1.6], *and the reflex field* F_0 *as in* [37, Def. 1.2.5.4]. *For technical reasons, we shall insist that* [37, Cond. 1.4.3.10] *is satisfied. In Cases* (Nm) *and* (Spl), *we allow the level* $\mathcal{H}$ *to be arbitrarily high at* p.
- *In Case* (Hdg), *we still have an integral PEL datum defining the auxiliary good reduction Siegel moduli* $\mathsf{M}_{\mathcal{H}_{\mathrm{aux}}}$, *which we abusively denote as* $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ (*with* $\mathcal{O} = \mathbb{Z}$, *without* aux *in the notation*), *which also defines a group functor* $\mathrm{G}_{\mathrm{aux}}$ *with an injective homomorphism* $\mathrm{G} \to \mathrm{G}_{\mathrm{aux}}$.

We shall say that we are in Case (Sm), (Nm), (Spl), or (Hdg) depending on the cases in Assumption 2.1 from where $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ is pulled back.

2.2. **Qualitative description of good compactifications.** The upshot is that the integral models considered in Assumption 2.1 are known to have good toroidal and minimal compactifications, constructed as in [37], [39], [42], [41], and [45]. Let us summarize some of their properties, which will be used later:

Proposition 2.2. *Let* $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ *be as above. Then there is a minimal compactification* $J_{\mathsf{X}^{\min}_{\mathcal{H}}} : \mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}^{\min}_{\mathcal{H}}$ *over* S, *together with a collection of toroidal compactifications* $J_{\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} : \mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *over* S, *labeled by certain compatible collections* Σ *of cone decompositions, satisfying the following properties:*

(1) *The structural morphism* $\mathsf{X}^{\min}_{\mathcal{H}} \to \mathsf{S}$ *is proper. For each* Σ, *there is a proper surjective structural morphism* $\oint_{\mathcal{H},\Sigma} : \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{X}^{\min}_{\mathcal{H}}$, *which is compatible with* $J_{\mathsf{X}^{\min}_{\mathcal{H}}}$ *and* $J_{\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ *in the sense that* $J_{\mathsf{X}^{\min}_{\mathcal{H}}} = \oint_{\mathcal{H},\Sigma} \circ J_{\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$.

(2) $\mathsf{X}_{\mathcal{H}}^{\min}$ *admits a stratification by locally closed subschemes* Z *flat over* S*, each of which is isomorphic to an analogue of* $\mathsf{X}_{\mathcal{H}}$ (*in Cases* (Sm)*,* (Nm)*, or* (Spl)) *or a finite quotient of it* (*in Case* (Hdg))*. Moreover, the incidence relation among the strata is preserved under pullback to fibers.*

(3) *Each* Σ *is a set* $\{\Sigma_{\mathsf{Z}}\}_{\mathsf{Z}}$ *of cone decompositions* Σ_{Z} *with the same index set as that of the strata of* $\mathsf{X}_{\mathcal{H}}^{\min}$*, which can be called the* **cusp labels** *for* $\mathsf{X}_{\mathcal{H}}$*. For simplicity, we shall suppress such cusp labels and denote the associated objects with the subscripts given by the strata* Z*.*

(4) *For each stratum* Z*, the cone decomposition* Σ_{Z} *is a cone decomposition of some* $\mathbf{P}$*, where* $\mathbf{P}$ *is the union of the interior* $\mathbf{P}^+$ *of a homogenous self-adjoint cone* (*see* [4, Ch. 2]) *and its rational boundary components, which is admissible with respect to some arithmetic group* Γ *acting on* $\mathbf{P}$ (*and hence also on* Σ_{Z})*.* (*For example, in the case of Siegel moduli, each* $\mathbf{P}^+$ *can be identified with the space of* $r \times r$ *symmetric positive definite pairings for some integer* r*, and* $\mathbf{P}$ *can be identified with the space of* $r \times r$ *symmetric positive semi-definite pairings with rational radicals.*) *Then* Σ_{Z} *has a subset* Σ_{Z}^+ *forming a cone decomposition of* $\mathbf{P}^+$*. If* τ *is a cone in* Σ_{Z} *that is not in* Σ_{Z}^+*, then there exists a stratum* Z' *of* $\mathsf{X}_{\mathcal{H}}^{\min}$ *whose closure in* $\mathsf{X}_{\mathcal{H}}^{\min}$ *contains* Z*, and a cone* τ' *in* $\Sigma_{\mathsf{Z}'}^+$*, whose* Γ'*-orbit is uniquely determined by the* Γ*-orbit of* τ (*where* Γ' *is the analogous arithmetic group acting on* $\Sigma_{\mathsf{Z}'}$*.*)

We may and we shall assume that Σ *is smooth and projective, and that, for each* Z *and* $\sigma \in \Sigma_{\mathsf{Z}}^+$*, its stabilizer* Γ_σ *in* Γ *is trivial.*

(5) *For each* Σ*, the associated* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *admits a stratification by locally closed subschemes* $\mathsf{Z}_{[\sigma]}$ *flat over* S*, labeled by the strata* Z *of* $\mathsf{X}_{\mathcal{H}}^{\min}$ *and the orbits* $[\sigma] \in \Sigma_{\mathsf{Z}}^+/\Gamma$*. The stratifications of* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *and* $\mathsf{X}_{\mathcal{H}}^{\min}$ *are compatible with each other in a precise sense: The preimage of a stratum* Z *of* $\mathsf{X}_{\mathcal{H}}^{\min}$ *is the* (*set-theoretic*) *disjoint union of the strata* $\mathsf{Z}_{[\sigma]}$ *of* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *with* $[\sigma] \in \Sigma_{\mathsf{Z}}^+/\Gamma$*. If* τ *is a face of a representative* σ *of* $[\sigma]$*, which is identified* (*as in* (4) *above*) *with the* Γ'*-orbit* $[\tau']$ *of some cone* τ' *in* $\Sigma_{\mathsf{Z}'}^+$*, where* Z' *is a stratum whose closure in* $\mathsf{X}_{\mathcal{H}}^{\min}$ *contains* Z*, then* $\mathsf{Z}_{[\sigma]}$ *is contained in the closure of* $\mathsf{Z}_{[\tau']}$*. Such an incidence relation among strata is preserved under pullback to fibers.*

(6) *For each stratum* Z *of* $\mathsf{X}_{\mathcal{H}}^{\min}$*, there is a proper surjective morphism* $C \to \mathsf{Z}$ *from a normal scheme which is flat over* S*, together with a morphism* $\Xi \to C$ *of schemes which is a torsor under the pullback of a split torus* E *with some character group* $\mathbf{S}$ *over* $\mathrm{Spec}(\mathbb{Z})$*, so that we have*

$$\Xi \cong \underline{\mathrm{Spec}}_{\mathscr{O}_C}\big(\bigoplus_{\ell \in \mathbf{S}} \Psi(\ell)\big)$$

for some invertible sheaves $\Psi(\ell)$*.* (*Each* $\Psi(\ell)$ *can be viewed as the subsheaf of* $(\Xi \to C)_* \mathscr{O}_\Xi$ *on which* E *acts via the character* $\ell \in \mathbf{S}$*.*)

(7) *For each* $\sigma \in \Sigma_{\mathsf{Z}}$*, consider*

$$\begin{aligned}
\sigma^\vee &:= \{\ell \in \mathbf{S} : \langle \ell, y\rangle \geq 0,\ \forall y \in \Sigma\},\\
\sigma_0^\vee &:= \{\ell \in \mathbf{S} : \langle \ell, y\rangle > 0,\ \forall y \in \Sigma\},\\
\sigma^\perp &:= \{\ell \in \mathbf{S} : \langle \ell, y\rangle = 0,\ \forall y \in \Sigma\} \cong \sigma^\vee/\sigma_0^\vee.
\end{aligned}$$

Then we have the affine toroidal embedding

$$\Xi \hookrightarrow \Xi(\sigma) := \underline{\mathrm{Spec}}_{\mathscr{O}_C}\big(\bigoplus_{\ell \in \sigma^\vee} \Psi(\ell)\big).$$

The scheme $\Xi(\sigma)$ *has a closed subscheme* Ξ_σ *defined by the ideal sheaf corresponding to* $\underset{\ell\in\sigma_0^\vee}{\oplus}\Psi(\ell)$, *so that* $\Xi_\sigma \cong \underline{\mathrm{Spec}}_{\mathscr{O}_C}(\underset{\ell\in\sigma^\perp}{\oplus}\Psi(\ell))$. *Then* $\Xi(\sigma)$ *admits a natural stratification by* Ξ_τ, *where* τ *are the faces of* σ *in* Σ_{Z}.

(8) *For each representative* $\sigma \in \Sigma_{\mathsf{Z}}^+$ *of an orbit* $[\sigma] \in \Sigma_{\mathsf{Z}}^+/\Gamma$, *let* $\mathfrak{X}_\sigma$ *denote the formal completion of* $\Xi(\sigma)$ *along* Ξ_σ, *and let* $(\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[\sigma]}}$ *denote the formal completion of* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *along* $\mathsf{Z}_{[\sigma]}$. *Then there is a canonical isomorphism* $\mathfrak{X}_\sigma \cong (\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[\sigma]}}$ *inducing a canonical isomorphism* $\Xi_\sigma \cong \mathsf{Z}_{[\sigma]}$.

(9) *Let* x *be a point of* Ξ_σ, *which can be canonically identified with a point of* $\mathsf{Z}_{[\sigma]}$ *via the above isomorphism. Let us equip* $\Xi(\sigma)$ *with a coarser stratification induced by the* Γ*-orbits* $[\tau]$ *of* τ, *where* τ *are the faces of* σ. *Each such orbit* $[\tau]$ *can be identified with the* Γ'*-orbits* $[\tau']$ *of some cone* τ' *in* $\Sigma_{\mathsf{Z}'}^+$, *where* Z' *is a stratum whose closure in* $\mathsf{X}_{\mathcal{H}}^{\min}$ *contains* Z. *Then there exists an étale neighborhood* $\overline{U} \to \mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *of* x *and an étale morphism* $\overline{U} \to \Xi(\sigma)$ *such that the stratification of* $\overline{U}$ *induced by that of* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *coincides with the stratification of* $\overline{U}$ *induced by that of* $\Xi(\sigma)$, *in the sense that the preimage of the stratum* $\mathsf{Z}_{[\tau']}$ *of* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *coincides with the preimage of the* $[\tau]$*-stratum of* $\Xi(\sigma)$ *when* $[\tau]$ *determines* $[\tau']$ *as explained above; and such that the pullbacks of these étale morphisms to* $\mathsf{Z}_{[\sigma]}$ *and to* Ξ_σ *are both open immersions.* (*In particular,* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *and* $\Xi(\sigma)$, *equipped with their stratifications as explained above, are étale locally isomorphic at* x.)

The proof of Proposition 2.2 will be postponed until Section 2.3.

Lemma 2.3. *Zariski locally over* C, *the scheme* $\Xi(\sigma) \to C$ *is isomorphic to* $E(\sigma) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C \to C$.

Proof. This is because the invertible sheaves $\Psi(\ell)$ over C are Zariski locally trivial, and because the semigroup $\sigma^\vee$ is finitely generated (see [29, Ch. I, §1, Lem. 2]). ☐

Corollary 2.4. *Let* x *be any point of* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, *which we may assume to lie on some stratum* $\mathsf{Z}_{[\sigma]}$. *Let* σ *be any representative of* $[\sigma]$, *and let* $E \hookrightarrow E(\sigma)$ *and* E_σ *be the affine toroidal embedding and the closed* σ*-stratum of* $E(\sigma)$ *over* $\mathrm{Spec}(\mathbb{Z})$ (*defined analogously as in the case of* $\Xi \hookrightarrow \Xi(\sigma)$ *and* Ξ_σ, *but are simpler*). *Then there exists an étale neighborhood* $\overline{U} \to \mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *of* x *and an étale morphism* $\overline{U} \to E(\sigma) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C$ *such that the stratifications of* $\overline{U}$ *induced by that of* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *and by that of* $E(\sigma)$ *coincide with each other; and such that the pullbacks of these morphisms to* $\mathsf{Z}_{[\sigma]}$ *and to* $E_\sigma \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C$ *are both open immersions.*

Suppose τ *is a face of* σ. *Then the preimage of the stratum* $\mathsf{Z}_{[\tau']}$ *of* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *in* $\overline{U}$, *where* $[\tau']$ *is determined by* $[\tau]$ *as in property* (9) *of Proposition* 2.2, *is the preimage of the stratum* E_τ *of* $E(\sigma)$. *If we denote by* $\mathsf{Z}_{[\tau']}^{\mathrm{tor}}$ *the closure of* $\mathsf{Z}_{[\tau']}$ *in* $\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, *and by* $E_\tau(\sigma)$ *the closure of* E_τ *in* $E(\sigma)$, *then the above implies that, étale locally at* x, *the open immersion* $J_{\mathsf{Z}_{[\tau']}^{\mathrm{tor}}} : \mathsf{Z}_{[\tau']} \hookrightarrow \mathsf{Z}_{[\tau']}^{\mathrm{tor}}$ *can be identified with the product of the canonical open immersion* $J_{E_\tau(\sigma)} : E_\tau \hookrightarrow E_\tau(\sigma)$ *with the identify morphism on* C.

In particular, when $\tau = \{0\}$, *this means the preimage* U *of* X *in* $\overline{U}$ *coincides with the preimage of* E. *Moreover, étale locally at* x, *the open immersion* $J_{\mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}} : \mathsf{X} \hookrightarrow \mathsf{X}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *can be identified with the product of the canonical open immersion* $J_{E(\sigma)} : E \hookrightarrow E(\sigma)$ *with the identity morphism* Id_C *on* C.

Proof. The first paragraph is a consequence of Proposition 2.2, especially properties (7) and (9), and of Lemma 2.3. The second and third paragraphs are straightforward consequences of the first paragraph, and of the various definitions. □

Remark 2.5. Under Assumption 2.1, the results stated in Proposition 2.2 also incorporated some earlier constructions of integral models of toroidal and minimal compactifications as special cases, such as those in [17], [65], and [64] for the Siegel moduli with hyperspecial or parahoric levels at the residue characteristics.

Remark 2.6. Similar assertions can be made for the integral models of Hilbert moduli and their compactifications, including the cases of splitting models, considered in [54], [15], [58], and [57]. (Or one can consider the even older theories for modular curves.) We leave the precise statements of these cases to the interested readers.

2.3. **Existence of good compactifications.**

Proof of Proposition 2.2 *in Case* (Sm). In this case, $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ is the pullback of some $\mathsf{M}_{\mathcal{H}^\square} \to \mathsf{S}_0$ defined in [37, Ch. 1]. Then we can take $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{S}$ and $\mathsf{X}^{\min}_{\mathcal{H}} \to \mathsf{S}$ to be pullbacks of the toroidal and minimal compactifications $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^\square,\Sigma} \to \mathsf{S}_0$ and $\mathsf{M}^{\min}_{\mathcal{H}^\square} \to \mathsf{S}_0$ in [37, Thm. 6.4.1.1, 7.2.4.1, and 7.3.3.4], where the collections Σ can be any smooth and projective ones as in [37, Def. 6.3.3.4 and 7.3.1.3] (satisfying the [37, Cond. 6.2.5.25]). Then properties (2)–(8) follow from the statements there (where the last requirement in property (4) is satisfied by [37, Lem. 6.2.5.27]), and property (9) follows from the construction of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^\square,\Sigma}$ (with its stratification) by gluing good algebraic models as in [37, Sec. 6.3]. □

Remark 2.7. In Case (Sm), the isomorphism $\mathfrak{X}_\sigma \cong (\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^\wedge_{\mathsf{Z}_{[\sigma]}}$ in property (8) of Proposition 2.2 is the pullback under $\mathsf{S} \to \mathsf{S}_0$ of the canonical isomorphism $\mathfrak{X}_{\Phi_{\mathcal{H}^\square},\delta_{\mathcal{H}^\square},\sigma} \cong (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^\square,\Sigma})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}^\square},\delta_{\mathcal{H}^\square},\sigma)]}}$ in [37, Thm. 6.4.1.1(5)]. Since both $\Xi_{\Phi_{\mathcal{H}^\square},\delta_{\mathcal{H}^\square}}(\sigma)$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^\square,\Sigma}$ are separated and of finite type over the excellent Dedekind domain $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$, it follows from Artin's approximation (see [2, Thm. 1.12, and the proof of the corollaries in Sec. 2]) that there exists an étale neighborhood $\overline{U} \to \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ of x and an étale morphism $\overline{U} \to E(\sigma) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C$ such that pullbacks of these morphisms to Ξ_σ and to $\mathsf{Z}_{[\sigma]}$ are both open immersions. But this more abstract argument does not guarantee such isomorphisms to be compatible with stratifications. This is why we resort to the more involved construction in [37, Sec. 6.3], which is nevertheless also based on Artin's approximation.

Proof of Proposition 2.2 *in Case* (Nm). In this case, $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ is the pullback of some $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{S}}_0$ as in [39, Sec. 6]. Then we can take $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{S}$ and $\mathsf{X}^{\min}_{\mathcal{H}} \to \mathsf{S}$ to be pullbacks of the toroidal and minimal compactifications $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{S}}_0$ and $\vec{\mathsf{M}}^{\min}_{\mathcal{H}} \to \vec{\mathsf{S}}_0$ in [42, Thm. 6.1] and [39, Prop. 6.4], where Σ can be any collections which are projective and smooth, and satisfy [37, Cond. 6.2.5.25]. Then properties (2)–(8) follow from [42, Thm. 6.1] (and the references from there to various results in [39]) and [39, Thm. 12.1 and 12.6].

It remains to verify property (9). Given the isomorphism $\mathfrak{X}_\sigma = (\Xi(\sigma))^\wedge_{\Xi_\sigma} \cong (\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^\wedge_{\mathsf{Z}_{[\sigma]}}$ in property (8), which is the pullback of the isomorphism

$$\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} = (\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma))^\wedge_{\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} \xrightarrow{\sim} (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^\wedge_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$$

given by [42, Thm. 6.1(4)] (see also [39, (10.3) and Thm. 10.14]), since both $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ and $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ are separated and of finite type over the excellent Dedekind domain $\vec{\mathsf{S}}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, it follows from Artin's approximation (see [2, Thm. 1.12, and the proof of the corollaries in Sec. 2]) that, at each point x of $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$, there exist an étale neighborhood $\overline{U} \to \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ of x and an étale morphism $\overline{U} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ such that the pullbacks of these morphisms to $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ and to $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ are both open immersions. By also approximating in characteristic zero the various additional structures as in the proof of [37, Prop. 6.3.2.1] (cf. Remark 2.7), we can ensure in the above that the pullbacks of the stratifications of $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ and $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $\overline{U}$ are compatible with each other in characteristic zero. Since the strata of these stratifications are flat over $\vec{\mathsf{S}}_0$ (see [39, Cor. 10.15] and [42, Thm. 6.1(5)]), they are induced by their restrictions to the characteristic zero fiber, and hence they are also compatible with each other in mixed characteristics. Since the images of such $\overline{U} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ cover $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$, by considering their pullbacks under $\mathsf{S} \to \mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, property (9) follows. □

Proof of Proposition 2.2 *in Case* (Spl). In this case, $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ is the pullback of some $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_K)$ as in [41, Def. 2.4.5]. Then we can take $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{S}$ and $\mathsf{X}^{\mathrm{min}}_{\mathcal{H}} \to \mathsf{S}$ to be pullbacks of the toroidal and minimal compactifications $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \mathrm{Spec}(\mathcal{O}_K)$ and $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_K)$ in [41, Thm. 3.4.1 and 4.3.1], where Σ can be any compatible collections which are projective and smooth, and satisfy the mild [37, Cond. 6.2.5.25]. Then properties (2)–(8) follow from the statements there, and property (9) follows from the same argument as in the above proof of Proposition 2.2 in Case (Nm), using the analogous canonical isomorphism between formal completions in [41, Thm. 3.4.1(3)], the flatness of strata in [41, Thm. 3.4.1(2)], and Artin's approximation. (Note that the stratification of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ is the pullback of the one of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, by [41, Def. 3.1.8 and (3.1.9)], which is just a base change from F_0 to K in characteristic zero, by [41, Prop. 2.3.10].) □

Proof of Proposition 2.2 *in Case* (Hdg). In this case, $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ is the pullback of some $\mathscr{S}_K \to \mathrm{Spec}(\mathscr{O}_{E,(v)})$ as in [45]. Then we can take $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{S}$ and $\mathsf{X}^{\mathrm{min}}_{\mathcal{H}} \to \mathsf{S}$ to be pullbacks of the toroidal and minimal compactifications $\mathscr{S}^{\Sigma}_K \to \mathrm{Spec}(\mathscr{O}_{E,(v)})$ and $\mathscr{S}^{\mathrm{min}}_K \to \mathrm{Spec}(\mathscr{O}_{E,(v)})$ in [45, Thm. 4.1.5 and 5.2.11], where Σ is induced by some auxiliary choice $\widetilde{\Sigma}$ for $\mathcal{S}_{\mathcal{K}}$ as in [45, Sec. 4.1.4], or some refinement of it (see [45, Rem. 4.1.6]) which can be assumed to be projective and smooth. Hence, properties (2)–(8) follow from the statements there. By the proof of [45, Prop. 4.2.11], each stratum $\mathsf{Z}_{[\sigma]}$ of $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is open and closed in the preimage of a stratum of an auxiliary good reduction toroidal compactification in Case (Sm), and the isomorphism in (8) is compatible with the pullback of the corresponding isomorphism in Case (Sm). Therefore, property (9) follows from also approximating such open and closed subsets of the preimages in the argument in Case (Nm). □

3. Automorphic sheaves and their geometric constructions

3.1. **Construction using finite étale coverings.** Let $\ell > 0$ be a rational prime number. Let us fix the choice of an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

For simplicity of notation, let us assume the following:

(1) In Case (Sm), we have $\ell \not\in \Box$ and $\mathcal{H} = \mathcal{H}^\ell \mathcal{H}_\ell$ for some open compact subgroups $\mathcal{H}^\ell \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box\cup\{\ell\}})$ and $\mathcal{H}_\ell \subset \mathrm{G}(\mathbb{Z}_\ell)$.
(2) In Cases (Nm), (Spl), and (Hdg), we have $\mathcal{H} = \mathcal{H}^\ell \mathcal{H}_\ell$ for some open compact subgroups $\mathcal{H}^\ell \subset \mathrm{G}(\hat{\mathbb{Z}}^\ell)$ and $\mathcal{H}_\ell \subset \mathrm{G}(\mathbb{Z}_\ell)$.

For each integer $r > 0$, let $\mathcal{U}_\ell(\ell^r) := \ker(\mathrm{G}(\mathbb{Z}_\ell) \to \mathrm{G}(\mathbb{Z}/\ell^r\mathbb{Z}))$, and consider $\mathcal{H}(\ell^r) := \mathcal{H}^\ell \mathcal{U}_\ell(\ell^r)$, which is contained in $\mathcal{H}$ when r is sufficiently large. For such sufficiently large r, in all cases in Assumption 2.1, we have a finite cover $\mathsf{X}_{\mathcal{H}(\ell^r)} \to \mathsf{X}_{\mathcal{H}}$ which induces a finite Galois étale cover $\mathsf{X}_{\mathcal{H}(\ell^r)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ with Galois group $\mathcal{H}_\ell/\mathcal{U}_\ell(\ell^r)$, where $\mathsf{X}_{\mathcal{H}(\ell^r)}$ is defined as in the case of $\mathsf{X}_{\mathcal{H}}$ but with $\mathcal{H}$ replaced with its normal subgroup $\mathcal{H}(\ell^r)$. If $\ell \neq p$, then the finite cover $\mathsf{X}_{\mathcal{H}(\ell^r)} \to \mathsf{X}_{\mathcal{H}}$ is étale (and Galois) over all of S.

Consider any algebraic representation ξ of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ on a finite-dimensional vector space V_ξ over $\bar{\mathbb{Q}}_\ell$. By the general procedure explained in [33, Sec. 6] and [25, Sec. III.2], there is an associated lisse ℓ-adic étale sheaf $\mathcal{V}_\xi$ over $\mathsf{X} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. (Since $\mathsf{X} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is often not connected, the construction is not based on the consideration of representations of its étale fundamental group. Instead, it uses systems of possibly disconnected finite étale covers, such as the $\mathsf{X}_{\mathcal{H}(\ell^r)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ mentioned above.) If $\ell \neq p$, this lisse ℓ-adic étale sheaf over $\mathsf{X} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ extends over all of X, which we still abusively denote as $\mathcal{V}_\xi$.

Let us briefly spell out the procedure in our special case. As explained in [25, Sec. III.2], by the Baire category theorem (see, for example, [7, proof of Lem. 2.2.1.1] or [62, beginning of Sec. 2]), there exists a finite extension E of $\mathbb{Q}_\ell$ in $\bar{\mathbb{Q}}_\ell$, and an $\mathcal{O}_E$-lattice $V_{\xi,0}$ with a continuous action of $\mathrm{G}(\mathbb{Z}_\ell)$ (with respect to the ℓ-adic topology), such that $V_\xi \cong V_{\xi,0} \underset{\mathcal{O}_E}{\otimes} \bar{\mathbb{Q}}_\ell$ as continuous representations of $\mathrm{G}(\mathbb{Q}_\ell)$. For each $m > 0$, by the continuity of the action of $\mathrm{G}(\mathbb{Z}_\ell)$ on $V_{\xi,0}$, there exists an integer $r(m) > 0$ such that $\mathcal{H}(\ell^{r(m)}) \subset \mathcal{H}$ and $\mathcal{U}_\ell(\ell^{r(m)})$ acts trivially on $V_{\xi,0,\ell^m} := V_{\xi,0} \underset{\mathbb{Z}_\ell}{\otimes} (\mathbb{Z}/\ell^m\mathbb{Z})$. By abuse of notation, let us also denote by $\underline{V}_{\xi,0,\ell^m}$ the constant group scheme over $\mathrm{Spec}(\mathbb{Z})$, which carry an action of $\mathcal{H}_\ell/\mathcal{U}_\ell(\ell^{r(m)})$. Let us define $\mathcal{V}_{\xi,0,\ell^m}$ to be the torsion étale sheaf of sections over $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ of the contraction product $(\mathsf{X}_{\mathcal{H}(\ell^r)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \overset{\mathcal{H}_{\ell^{r(m)}}/\mathcal{U}_\ell(\ell^{r(m)})}{\times} \underline{V}_{\xi,0,\ell^r}$, and define the étale sheaves $\mathcal{V}_{\xi,0} := \varprojlim_m \mathcal{V}_{\xi,0,\ell^m}$ and $\mathcal{V}_\xi := \mathcal{V}_{\xi,0} \underset{\mathcal{O}_E}{\otimes} \bar{\mathbb{Q}}_\ell$ over $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ as usual. Then it is elementary (though tedious) to verify that such a construction is independent of the various choices, functorial in various natural senses, and allows one to define the Hecke actions on the cohomology groups if we take the limit over $\mathcal{H}$. We are admittedly being vague here, and we shall refer the readers to [33, Sec. 6] and [25, Sec. III.2] for more details. When $\ell \neq p$, the same construction defines the étale sheaf extensions of $\mathcal{V}_{\xi,0}$ and $\mathcal{V}_\xi$ to all of $\mathsf{X}_{\mathcal{H}}$. This construction certainly also works if we replace V_ξ with a continuous representations of $\mathrm{G}(\mathbb{Z}_\ell)$ on a (possibly torsion) finite $\mathbb{Z}_\ell$-module W_0, without reference to any representation over $\bar{\mathbb{Q}}_\ell$.

3.2. **Construction using Kuga families.** There is an alternative approach using the relative cohomology of Kuga families over the Shimura varieties, which is less systematic but crucially useful.

Consider the abelian scheme $\mathsf{f} : \mathsf{A} \to \mathsf{X}_{\mathcal{H}}$ defined as follows:

(1) In Case (Sm), it is the pullback of $A \to \mathsf{M}_{\mathcal{H}^{\square}}$, which is part of the tautological object $(A, \lambda, i, \alpha_{\mathcal{H}^{\square}}) \to \mathsf{M}_{\mathcal{H}^{\square}}$.

(2) In Case (Nm), it is the pullback of $\vec{A}_{j_0} \to \vec{\mathsf{M}}_{\mathcal{H}}$, which is part of the tautological objects $(\vec{A}_j, \vec{\lambda}_j, \vec{i}_j, \vec{\alpha}_{\mathcal{H}_j}) \to \vec{\mathsf{M}}_{\mathcal{H}}$ (with all $j \in J$) as in [39, Prop. 6.1], under the assumption that $(L_{j_0}, \langle\,\cdot\,,\,\cdot\,\rangle_{j_0}) = (L, \langle\,\cdot\,,\,\cdot\,\rangle)$ for some $j_0 \in J$,

(3) In Case (Spl), it is the pullback of $\vec{A}_{j_0} \to \vec{\mathsf{M}}_{\mathcal{H}}$ as in Case (Nm) above, via the composition of $\mathsf{X} \to \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ with the structural morphism $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \vec{\mathsf{M}}_{\mathcal{H}}$.

(4) In Case (Hdg), it is the pullback of $A_{\mathrm{aux}} \to \mathsf{M}_{\mathcal{H}_{\mathrm{aux}}}$, which is part of the tautological object over the auxiliary good reduction moduli $\mathsf{M}_{\mathcal{H}_{\mathrm{aux}}}$ (which we assumed to be in Case (Sm)), under the composition of canonical morphisms $\mathsf{X}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}_{\mathrm{aux}}}$ (see Assumption 2.1).

Lemma 3.1. *The general construction in Section* 3.1 *of étale sheaves over* $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *associates* $\mathrm{Hom}_{\mathbb{Z}}(L, R)$ *with the étale sheaf* $R^1(\mathsf{f} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_* R$*, where* R *can be either* $\bar{\mathbb{Q}}_\ell$*,* $\mathbb{Q}_\ell$*, or any finite (and possibly torsion)* $\mathbb{Z}_\ell$*-module. When* $\ell \neq p$*, the construction also works over all of* $\mathsf{X}_{\mathcal{H}}$ *and associates* $\mathrm{Hom}_{\mathbb{Z}}(L, R)$ *with the étale sheaf* $R^1\mathsf{f}_* R$*.*

Proof. It suffices to prove these in Cases (Sm), (Nm), and (Spl), because the construction in Case (Hdg) is the pullback of the analogous construction in Case (Sm). In all of these three cases, the abelian scheme $\mathsf{f} : \mathsf{A} \to \mathsf{X}_{\mathcal{H}}$ extends to some object $(\mathsf{A}, \lambda_{\mathsf{A}}, i_{\mathsf{A}}, \alpha_{\mathsf{A},\mathcal{H}^{\square}})$, where $\lambda_{\mathsf{A}} : \mathsf{A} \to \mathsf{A}^\vee$ is a polarization, $i_{\mathsf{A}} : \mathcal{O} \to \mathrm{End}_{\mathsf{A}}(\mathsf{A})$ is an $\mathcal{O}$-endomorphism structure for $(\mathsf{A}, \lambda_{\mathsf{A}})$ as in [37, Def. 1.3.3.1], and where $\alpha_{\mathsf{A},\mathcal{H}^{\square}}$ is a level-$\mathcal{H}^{\square}$ structure for $(\mathsf{A} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \lambda_{\mathsf{A}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, i_{\mathsf{A}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$, with $\square = \emptyset$ in Cases (Nm) and (Spl), such that $(\mathsf{A} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \lambda_{\mathsf{A}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, i_{\mathsf{A}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \alpha_{\mathsf{A},\mathcal{H}^{\square}})$ defines an object of $\mathsf{M}_{\mathcal{H}^{\square}}$ over $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ as in [37, Def. 1.4.1.4]. (In Case (Sm), this follows from the definition of $\mathsf{M}_{\mathcal{H}^{\square}}$ in [37, Sec. 1.4]. In Cases (Nm) and (Spl), this is because $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is isomorphic to a base change of some $\mathsf{M}_{\mathcal{H}}$ defined in [37, Sec. 1.4], and because $\mathsf{A} \to \mathsf{X}_{\mathcal{H}}$ is isomorphic to the pullback of $\vec{A}_{j_0} \to \vec{\mathsf{M}}_{\mathcal{H}}$.) For any geometric point $\bar{s} \to \mathsf{X}_{\mathcal{H}}$ of residue characteristic zero, $\alpha_{\mathsf{A},\mathcal{H}^{\square}}$ induces, in particular, a $\pi_1(\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \bar{s})$-invariant $\mathcal{H}_\ell$-orbit of isomorphisms $L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{T}_\ell\, \mathsf{A}_{\bar{s}}$ matching the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on L with the λ_{A}-Weil pairing on $\mathrm{T}_\ell\, \mathsf{A}_{\bar{s}}$ up to scalar multiples, compatible with the $\mathcal{O}$-module structures of L and of $\mathrm{T}_\ell\, \mathsf{A}_{\bar{s}}$ given by i_{A}. If $\ell \neq p$, then this isomorphism also extends to similar isomorphisms at geometric points $\bar{s} \to \mathsf{X}_{\mathcal{H}}$ of characteristic p. Hence, the lemma follows from the very definitions of the various objects. $\square$

Proposition 3.2. *Suppose* ξ *is an irreducible algebraic representation of* $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *on a finite-dimensional vector space* V_ξ *over* $\bar{\mathbb{Q}}_\ell$*. Then there exist* $t_\xi \in \mathbb{Z}$ *and* $n_\xi \in \mathbb{Z}_{\geq 0}$ *such that the associated étale sheaf* $\mathcal{V}_\xi$ *over* $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *is a direct summand of* $(R(\mathsf{f}^{\times n_\xi} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_* \bar{\mathbb{Q}}_\ell)(-t_\xi)$*, where* $\mathsf{f}^{\times n_\xi} : \mathsf{A}^{\times n_\xi} \to \mathsf{X}_{\mathcal{H}}$ *is the* n_ξ*-fold self-fiber-product of* $\mathsf{A} \to \mathsf{X}_{\mathcal{H}}$*, and where* $(-t_\xi)$ *denotes the Tate twist. If* $\ell \neq p$*, then* $\mathcal{V}_\xi$ *extends over all of* $\mathsf{X}_{\mathcal{H}}$ *as a direct summand of* $(R\mathsf{f}^{\times n_\xi}_* \bar{\mathbb{Q}}_\ell)(-t_\xi)$*. The integer* $m_\xi := n_\xi + 2t_\xi$ *depends only on the irreducible representation* ξ *(and the data defining* $\mathsf{X}_{\mathcal{H}}$*); and* $\mathcal{V}_\xi$ *is pointwise pure of weight* m_ξ*.*

Proof. By [37, Def. 1.2.1.6], and by the assumption that $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a subgroup of $\mathrm{G}_{\mathrm{aux}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ in Case (Hdg), $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a faithful representation of the reductive group $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. By [12, Prop. 3.1], the irreducible representation V_ξ is a direct summand of $(L \underset{\mathbb{Z}}{\otimes} \bar{\mathbb{Q}}_\ell)^{\otimes a_\xi} \otimes (L^\vee \underset{\mathbb{Z}}{\otimes} \bar{\mathbb{Q}}_\ell)^{\otimes b_\xi}$, for some integers $a_\xi, b_\xi \geq 0$, where the tensor products are over $\bar{\mathbb{Q}}_\ell$, and where $L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. Since $\langle \cdot\,, \cdot \rangle$ induces a perfect pairing $(L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_\ell) \times (L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(1)$ (where the Tate twist is formal, by tensor product with $\mathbb{Z}(1) := \ker(\exp : \mathbb{C} \to \mathbb{C}^\times)$), which is matched via the level structure (in the proof of Lemma 3.1) with the λ-Weil pairing $(\mathrm{V}_\ell \mathsf{A}_{\bar{s}}) \times (\mathrm{V}_\ell \mathsf{A}_{\bar{s}}) \to \mathrm{V}_\ell \mathbf{G}_{\mathrm{m},\bar{s}}$ up to scalar multiples at each geometric point $\bar{s} \to \mathsf{X}_{\mathcal{H}}$ of residue characteristic zero, it follows that $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_\ell \cong \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}_\ell)(1)$, which is matched with the isomorphism $\underline{\mathrm{Hom}}_{\mathbb{Q}_\ell}(R^1(\mathsf{f} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_* \mathbb{Q}_\ell, \mathbb{Q}_\ell) \cong R^1(\mathsf{f} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_* \mathbb{Q}_\ell(1)$ between étale sheaves. Hence the proposition follows from Lemma 3.1, from [11, Ex. 6.2.5(b) and Prop. 6.2.6], and from the fact that $(R^1(\mathsf{f} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_* \bar{\mathbb{Q}}_\ell)^{\otimes n_\xi}$ is a direct summand of $R^{n_\xi}(\mathsf{f}^{\times n_\xi} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_* \bar{\mathbb{Q}}_\ell \cong \wedge^{n_\xi} ((R^1(\mathsf{f} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_* \bar{\mathbb{Q}}_\ell)^{\oplus n_\xi})$, and $R^{n_\xi}(\mathsf{f}^{\times n_\xi} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_* \bar{\mathbb{Q}}_\ell$ (placed in degree n_ξ) is a direct summand of $R(\mathsf{f}^{\times n_\xi} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_* \bar{\mathbb{Q}}_\ell$ thanks to Lieberman's trick (cf. [43, Sec. 3.2]). □

Remark 3.3. While the proof of Proposition 3.2 is abstract, and so the integers t_ξ and n_ξ are not effective, by using Weyl's construction for representations of classical groups, we can write down explicit choices of t_ξ and n_ξ, depending on the highest weights of ξ. This is the approach taken in, for example, [25, Sec. III.2] and [47], and is spell out in precise detail in [43, Sec. 2, 3, and 4], the last of which also pinned down the optimal values of t_ξ and n_ξ in all PEL-type cases.

For torsion coefficients, we have the following subtler statements:

Proposition 3.4. *Suppose K_0 is a finite extension of $\mathbb{Q}$ in a fixed choice of algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, and suppose W_0 is a finite flat $\mathcal{O}_{K_0}$-module with an algebraic action of* G. *Suppose $W := W_0 \underset{\mathcal{O}_{K_0}}{\otimes} K_0$ is an irreducible representation of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} K_0$. Then there exist integers $t_W \in \mathbb{Z}$ and $n_W, c_W \in \mathbb{Z}_{\geq 0}$ (depending only on $\mathrm{G} \underset{\mathbb{Z}}{\otimes} K_0$, $L \underset{\mathbb{Z}}{\otimes} K_0$, and the weights of W) such that, as long as $\ell > c_W$, for each finite extension field K of the w-adic completion $K_{0,w}$ of K_0 at a place $w|\ell$ such that $\mathrm{G} \underset{\mathbb{Z}}{\otimes} K$ is split, and for each finite $\mathcal{O}_K$-module M, the étale sheaf $\mathcal{W}_{0,M}$ associated with $W_0 \underset{\mathcal{O}_{K_0}}{\otimes} M$ is a direct summand of $(R(\mathsf{f} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})_*^{\times n_W} M)(-t_W)$. If $\ell \neq p$, then $\mathcal{W}_{0,M}$ extends over all of $\mathsf{X}_{\mathcal{H}}$ as a direct summand of $(R\mathsf{f}_*^{\times n_W} M)(-t_W)$. In Cases* (Sm), (Nm), *or* (Spl), *the integers t_W, n_W, and c_W can be explicitly determined using only the PEL datum $(\mathcal{O}, \star, L, \langle \cdot\,, \cdot \rangle, h_0)$ and the weights of W.*

Proof. By [12, Prop. 3.1], there exists $t_W \in \mathbb{Z}$ and $a_W, b_W \in \mathbb{Z}_{\geq 0}$ such that $W = W_0 \underset{\mathcal{O}_{K_0}}{\otimes} K_0$ is a direct summand of $(L \underset{\mathbb{Z}}{\otimes} K_0)^{\otimes a_W} \otimes (L^\vee \underset{\mathbb{Z}}{\otimes} K_0)^{\otimes b_W}$. For all sufficiently large ℓ, and for each K as in the statement of the proposition, the weights of $W \underset{K_0}{\otimes} K$ and $(L \underset{\mathbb{Z}}{\otimes} K)^{\otimes a_W} \otimes (L^\vee \underset{\mathbb{Z}}{\otimes} K)^{\otimes b_W}$ are all ℓ-small, and there are no nontrivial extensions between admissible $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_K$-lattices in irreducible representations of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} K$ of such ℓ-small weights (see, for example, [53, Sec. 1] and the references there, which are applicable because $\mathrm{G} \underset{\mathbb{Z}}{\otimes} K$ is split). Hence, there exists some $c_W \in \mathbb{Z}_{\geq 0}$ (as in the statement of the proposition) such that $W_0 \underset{\mathcal{O}_{K_0}}{\otimes} \mathcal{O}_K$ is a direct summand of $(L \underset{\mathbb{Z}}{\otimes} \mathcal{O}_K)^{\otimes a_W} \otimes (L^\vee \underset{\mathbb{Z}}{\otimes} \mathcal{O}_K)^{\otimes b_W}$ whenever $\ell > c_W$ and K is as above, and we can conclude as in the proof of Proposition 3.2. In Cases (Sm), (Nm), or (Spl), since we are in PEL-type cases with residue characteristics prime to ℓ, we can explicitly determine the integers t_W, n_W, and c_W using only the PEL datum $(\mathcal{O}, \star, L, \langle \cdot\,, \cdot \rangle, h_0)$ and the weights of W, as in [43, Sec. 2, 3, and 4]. $\square$

4. Kuga families as toroidal boundary strata

4.1. **General statements.** In this section, we explain in Cases (Sm), (Nm), and (Spl) how to realize the scheme $\mathsf{A}^{\times n}$ (for any fixed choice of an integer $n \geq 0$) as a toroidal boundary stratum of a larger analogue of $\mathsf{X}_{\mathcal{H}}$, not just over $\mathsf{S} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ but over all of S, based on an argument in [36] (which is closely related to similar considerations in the context of mixed Shimura varieties in [52]). At the end of this section, we explain how the realization in Case (Sm) is also useful in Case (Hdg).

Proposition 4.1. *In Cases* (Sm)*,* (Nm)*, or* (Spl)*, let* $\mathsf{M} \to \mathsf{S}$ *and* $\mathsf{f} : \mathsf{A} \to \mathsf{X}_{\mathcal{H}}$ *be as above. Let* $n \geq 0$ *be an integer. Then there exist some* (*noncanonical*) $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}} \to \mathsf{S}$ (*depending on* n) *as in Assumption* 2.1 (*in which case we denote all associated objects below with a wide tilde*) *and some* $\widetilde{\Sigma}$ *as in Proposition* 2.2 *such that there exists a stratum* $\widetilde{\mathsf{Z}}$ *of* $\widetilde{\mathsf{X}}^{\min}_{\widetilde{\mathcal{H}}}$ *and a stratum* $\widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]}$ *of* $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ *such that there is an isomorphism* $\widetilde{\mathsf{Z}} \xrightarrow{\sim} \mathsf{M}$ *identifying the canonical morphism* $\widetilde{\oint}_{[\widetilde{\sigma}]} : \widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]} \to \widetilde{\mathsf{Z}}$ (*induced by the structural morphism* $\widetilde{\oint}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}} : \widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}} \to \widetilde{\mathsf{X}}^{\min}_{\widetilde{\mathcal{H}}}$) *with an abelian scheme over* M *that is* $\mathbb{Z}^\times_{(\ell)}$*-isogenous to the* n*-fold self-fiber product* $\mathsf{f}^{\times n} : A^{\times n} \to \mathsf{M}$ *of* $\mathsf{f} : A \to \mathsf{M}$.

Proposition 4.2. *In Proposition* 4.1*, given each* Σ *for* $\mathsf{X}_{\mathcal{H}}$*, there exist some choices of* $\widetilde{\Sigma}$ *and* $\widetilde{\sigma}$ *such that the canonical morphism* $\widetilde{\oint}_{[\widetilde{\sigma}]} : \widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]} \to \widetilde{\mathsf{Z}} \cong \mathsf{X}_{\mathcal{H}}$ *extends to a* (*proper surjective*) *morphism from the closure* $\widetilde{\mathsf{Z}}^{\mathrm{tor}}_{[\widetilde{\sigma}]}$ *of* $\widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]}$ *in* $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ *to* $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$.

4.2. **Constructions in Case** (Sm)**.** We shall follow the arguments in [36, Sec. 3A] and [38, Sec. 1.2.4] closely. (We will be rather brief in our explanations, but the proofs will still be rather lengthy because we need to introduce many definitions. They will be needed in the later proofs in Cases (Nm) and (Spl).)

Proof of Proposition 4.1 *in Case* (Sm). Let $(\mathcal{O}, \star, L, \langle \cdot\,, \cdot \rangle, h_0)$ be the integral PEL datum defining $\mathsf{M}_{\mathcal{H}}$. By assumption, there exists a maximal order $\mathcal{O}'$ in $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that the action of $\mathcal{O}$ on L extends to an action of $\mathcal{O}'$ on L. Consider

$$Q := \mathcal{O}^{\oplus n},$$
$$Q_{-2} := (\mathrm{Diff}^{-1}_{\mathcal{O}'/\mathbb{Z}}(1))^{\oplus n} \cong \mathrm{Hom}_{\mathcal{O}}(Q, \mathrm{Diff}^{-1}_{\mathcal{O}'/\mathbb{Z}}(1)) \subset Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Q}(1), \text{ and}$$
$$Q_0 := (\mathcal{O}')^{\oplus n} \cong \mathcal{O}' \cdot Q \subset Q \underset{\mathbb{Z}}{\otimes} \mathbb{Q},$$

where the $\mathcal{O}$-actions on $Q^\vee := \mathrm{Hom}_{\mathbb{Z}}(Q,\mathbb{Z}) \cong \mathrm{Hom}_{\mathcal{O}}(Q, \mathrm{Diff}^{-1}_{\mathcal{O}/\mathbb{Z}})$ and similar dual modules are induced by the right action of $\mathcal{O}^{\mathrm{op}}$ (and the anti-isomorphism $\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}^{\mathrm{op}}$), and consider the canonical pairing $\langle\,\cdot\,,\,\cdot\,\rangle_Q : Q_{-2} \times Q_0 \to \mathbb{Z}(1)$ given by $\langle (x_{-2}, x_0), (y_{-2}, y_0)\rangle_Q = \mathrm{Tr}_{(\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})/\mathbb{Q}}({}^t x_{-2}\, y_0 - {}^t x_0\, y_{-2})$.

Consider the integral PEL datum $(\mathcal{O}, \star, \widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim}, \widetilde{h})$ defined as follows:

- $\widetilde{L} := Q_{-2} \oplus L \oplus Q_0$.
- $\langle\,\cdot\,,\,\cdot\,\rangle^{\sim} : \widetilde{L} \times \widetilde{L} \to \mathbb{Z}(1)$ is defined (symbolically) by the matrix
$$\langle x, y\rangle^{\sim} := {}^t\begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \end{pmatrix} \begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle_Q \\ & \langle\,\cdot\,,\,\cdot\,\rangle & \\ -\,{}^t\langle\,\cdot\,,\,\cdot\,\rangle_Q & & \end{pmatrix} \begin{pmatrix} y_{-2} \\ y_{-1} \\ y_0 \end{pmatrix}.$$
- $\widetilde{h} : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(\widetilde{L} \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ is defined by
$$z = z_1 + \sqrt{-1}\, z_2$$
$$\mapsto \widetilde{h}(z) := \begin{pmatrix} z_1 \,\mathrm{Id}_{Q_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{R}} & & -z_2((2\pi\sqrt{-1}) \circ j_Q^{-1}) \\ & h(z) & \\ z_2(j_Q \circ (2\pi\sqrt{-1})^{-1}) & & z_1\,\mathrm{Id}_{Q_0 \underset{\mathbb{Z}}{\otimes} \mathbb{R}} \end{pmatrix},$$
where $2\pi\sqrt{-1} : \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}(1)$ and $(2\pi\sqrt{-1})^{-1} : \mathbb{Z}(1) \xrightarrow{\sim} \mathbb{Z}$ stand for the isomorphisms defined by the choice of $\sqrt{-1}$ in $\mathbb{C}$, and where $j_Q : (\mathrm{Diff}^{-1}_{\mathcal{O}'/\mathbb{Z}})^{\oplus n} \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{R} \xrightarrow{\sim} Q \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong (\mathcal{O}')^{\oplus n} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ can be identified with the identity morphism on $\mathcal{O}^{\oplus n} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$.

These define a group functor $\widetilde{\mathrm{G}}$ as in [37, Def. 1.2.1.6] (and the same reflex field F_0). By construction, there is a fully symplectic admissible filtration $\widetilde{\mathrm{Z}}$ on $\widetilde{L} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ (see [37, Def. 5.2.7.3]) induced by
$$\widetilde{\mathrm{Z}}_{-3} := 0 \subset \widetilde{\mathrm{Z}}_{-2} := Q_{-2} \subset \widetilde{\mathrm{Z}}_{-1} := Q_{-2} \oplus L \subset \widetilde{\mathrm{Z}}_0 := Q_{-2} \oplus L \oplus Q_0 = \widetilde{L} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square,$$
so that there are canonical isomorphisms $\mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-2} \cong Q_{-2} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$, $\mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-1} \cong L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$, and $\mathrm{Gr}^{\widetilde{\mathrm{Z}}}_0 \cong Q_0 \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ matching the pairings $\mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-2} \times \mathrm{Gr}^{\widetilde{\mathrm{Z}}}_0 \to \hat{\mathbb{Z}}^\square(1)$ and $\mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-1} \times \mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-1} \to \hat{\mathbb{Z}}^\square(1)$ induced by $\langle\,\cdot\,,\,\cdot\,\rangle^{\sim}$ with $\langle\,\cdot\,,\,\cdot\,\rangle_Q$ and $\langle\,\cdot\,,\,\cdot\,\rangle$, respectively. Let $\widetilde{X} := \mathrm{Hom}_{\mathcal{O}}(Q_{-2}, \mathrm{Diff}^{-1}_{\mathcal{O}/\mathbb{Z}}(1)) \cong (\mathcal{O}')^{\oplus n}$ and $\widetilde{Y} := Q_0 = (\mathcal{O}')^{\oplus n}$. Then the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_Q : Q_{-2} \times Q_0 \to \mathbb{Z}(1)$ induces a canonical embedding $\widetilde{\phi} : \widetilde{Y} \hookrightarrow \widetilde{X}$, which can be identified with the identity morphism on $(\mathcal{O}')^{\oplus n}$ in this case, and there are canonical isomorphisms $\widetilde{\varphi}_{-2} : \mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(\widetilde{X} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ and $\widetilde{\varphi}_0 : \mathrm{Gr}^{\widetilde{\mathrm{Z}}}_0 \xrightarrow{\sim} \widetilde{Y} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ (of $\hat{\mathbb{Z}}^\square$-modules). These data define a torus argument $\widetilde{\Phi} := (\widetilde{X}, \widetilde{Y}, \widetilde{\phi}, \widetilde{\varphi}_{-2}, \widetilde{\varphi}_0)$ for $\widetilde{\mathrm{Z}}$ as in [37, Def. 5.4.1.3]. Let $\widetilde{\delta} : \mathrm{Gr}^{\widetilde{\mathrm{Z}}} = \mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-2} \oplus \mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-1} \oplus \mathrm{Gr}^{\widetilde{\mathrm{Z}}}_0 \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ be the obvious splitting of $\widetilde{\mathrm{Z}}$ induced by the equality $Q_{-2} \oplus L \oplus Q_0 = \widetilde{L}$.

For any $\hat{\mathbb{Z}}^\square$-algebra R, let $\widetilde{\mathrm{P}}_{\widetilde{\mathrm{Z}}}(R)$ denote the subgroup of $\widetilde{\mathrm{G}}(R)$ consisting of elements g such that $g(\widetilde{\mathrm{Z}}_{-2} \underset{\hat{\mathbb{Z}}^\square}{\otimes} R) = \widetilde{\mathrm{Z}}_{-2} \underset{\hat{\mathbb{Z}}^\square}{\otimes} R$ and $g(\widetilde{\mathrm{Z}}_{-1} \underset{\hat{\mathbb{Z}}^\square}{\otimes} R) = \widetilde{\mathrm{Z}}_{-1} \underset{\hat{\mathbb{Z}}^\square}{\otimes} R$. Then we have a homomorphism $\mathrm{Gr}_{-1}^{\widetilde{\mathsf{Z}}} : \widetilde{\mathrm{P}}_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^{\Box}) \to \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ defined by taking graded pieces. Let $\widetilde{\mathrm{P}}'_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^{\Box})$ be the kernel of $\mathrm{Gr}_{-2}^{\widetilde{\mathsf{Z}}} \times \mathrm{Gr}_{0}^{\widetilde{\mathsf{Z}}}$, where the homomorphisms $\mathrm{Gr}_{-2}^{\widetilde{\mathsf{Z}}}$ and $\mathrm{Gr}_{0}^{\widetilde{\mathsf{Z}}}$ are defined analogously. Let $\widetilde{\mathcal{H}}^{\Box}$ be any *neat* open compact subgroup of $\widetilde{\mathrm{G}}(\hat{\mathbb{Z}}^{\Box})$ satisfying the following conditions:

(1) $\mathrm{Gr}_{-1}^{\widetilde{\mathsf{Z}}}(\widetilde{\mathcal{H}}^{\Box} \cap \widetilde{\mathrm{P}}'_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^{\Box})) = \mathrm{Gr}_{-1}^{\widetilde{\mathsf{Z}}}(\widetilde{\mathcal{H}}^{\Box} \cap \widetilde{\mathrm{P}}_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^{\Box})) = \mathcal{H}^{\Box}$. (Both equalities are conditions. Then $\mathcal{H}^{\Box}$ is a direct factor of $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}(\widetilde{\mathcal{H}}^{\Box} \cap \widetilde{\mathrm{P}}_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^{\Box}))$.)

(2) The splitting $\widetilde{\delta}$ defines a (group-theoretic) splitting of the surjection $\widetilde{\mathcal{H}}^{\Box} \cap \widetilde{\mathrm{P}}'_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^{\Box}) \twoheadrightarrow \mathcal{H}^{\Box}$ induced by $\mathrm{Gr}_{-1}^{\widetilde{\mathsf{Z}}}$.

(Such an $\widetilde{\mathcal{H}}^{\Box}$ exists because the pairing $\langle\,\cdot\,,\,\cdot\,\rangle^{\sim}$ is the direct sum of the pairings on $Q_{-2} \oplus Q_0$ and on L.) The data of $\mathcal{O}$, $(\widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim}, \widetilde{h})$, $\Box$, and $\widetilde{\mathcal{H}}^{\Box} \subset \widetilde{\mathrm{G}}(\hat{\mathbb{Z}}^{\Box})$ define a moduli problem $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ as in [37, Def. 1.4.1.4]. Let $\widetilde{\mathcal{H}} := \widetilde{\mathcal{H}}^{\Box} \times \prod_{q \in \Box} \widetilde{\mathrm{G}}(\mathbb{Z}_q)$.

Take any projective smooth $\widetilde{\Sigma}$ for $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}^{\Box}}$, which defines a toroidal compactification $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}^{\Box}, \widetilde{\Sigma}}$ which is projective and smooth over S_0 by [37, Thm. 7.3.3.4]. Let $(\widetilde{\mathsf{Z}}, \widetilde{\Phi}, \widetilde{\delta})$ be as above, and let $(\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}} = (\widetilde{X}, \widetilde{Y}, \widetilde{\phi}, \widetilde{\varphi}_{-2, \widetilde{\mathcal{H}}^{\Box}}, \widetilde{\varphi}_{0, \widetilde{\mathcal{H}}^{\Box}}), \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}})$ be the induced triple at level $\widetilde{\mathcal{H}}^{\Box}$, inducing a cusp label $[(\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}})]$ at level $\widetilde{\mathcal{H}}^{\Box}$. Let $\widetilde{\sigma} \subset \mathbf{P}^{+}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}}$ be any *top-dimensional* nondegenerate rational polyhedral cone in the cone decomposition $\widetilde{\Sigma}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}}$ in $\widetilde{\Sigma}$. By [37, Thm. 6.4.1.1(2) and 7.2.4.1(5); see also the errata], the stratum $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\sigma})]}$ of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}^{\Box}}$ is a *zero-dimensional* torus bundle over the abelian scheme $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}}}$ over $\mathsf{M}_{\mathcal{H}^{\Box}}$. (We have canonical isomorphisms $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}})]} \cong \widetilde{\mathsf{M}}^{\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}}_{\widetilde{\mathcal{H}}^{\Box}} \cong \widetilde{\mathsf{M}}^{\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}^{\Box}}}_{\widetilde{\mathcal{H}}^{\Box}} \cong \mathsf{M}_{\mathcal{H}^{\Box}}$ because of the condition (1) above on the choice of $\widetilde{\mathcal{H}}^{\Box}$. The abelian scheme torsor $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}}} \to \widetilde{\mathsf{M}}^{\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}}_{\widetilde{\mathcal{H}}^{\Box}}$ is an abelian scheme because of the condition (2) above on the choice of $\widetilde{\mathcal{H}}^{\Box}$.) In other words, $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\sigma})]} \cong \widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}}}$. By the construction of $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}}}$ as in [37, Sec. 6.2.3–6.2.4], and by the same arguments in the proofs of [38, Lem. 8.1.3.2 and 8.1.3.5], it is $\mathbb{Z}^{\times}_{(\Box)}$-isogenous to the abelian scheme $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, A)^{\circ} \cong A^{\times n}$ over $\mathsf{M}_{\mathcal{H}^{\Box}}$. Since $\ell \notin \Box$, a $\mathbb{Z}^{\times}_{(\Box)}$-isogeny is a $\mathbb{Z}^{\times}_{(\ell)}$-isogeny.

Now it suffices to take $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}} \to \mathsf{S}$, $\widetilde{\mathsf{X}}^{\min}_{\widetilde{\mathcal{H}}} \to \mathsf{S}$, $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}, \widetilde{\Sigma}} \to \mathsf{S}$, and $\widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]} \to \widetilde{\mathsf{Z}}$ to be the pullbacks of $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}^{\Box}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, $\widetilde{\mathsf{M}}^{\min}_{\widetilde{\mathcal{H}}^{\Box}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}^{\Box}, \widetilde{\Sigma}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, and $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\sigma})]} \to \widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}^{\Box}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}^{\Box}})]}$, respectively, under $\mathsf{S} \to \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. □

Proof of Proposition 4.2 *in Case* (Sm). In this case, the proposition follows [36, Sec. 3B, the paragraph following Cond. 3.8]. □

4.3. **Constructions in Case** (Nm).

Proof of Proposition 4.1 *in Case* (Nm). Let us proceed with the same choices made in the proof of Proposition 4.1 in Case (Sm), with $\Box = \emptyset$. (We shall henceforth omit all the superscripts with $\Box$.) In this case, $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{S}}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ is defined with some choice of $\{(L_j, \langle\,\cdot\,,\,\cdot\,\rangle_j)\}_{j \in J}$, with $(L_{j_0}, \langle\,\cdot\,,\,\cdot\,\rangle_{j_0}) = (p^{r_0} L, p^{-2r_0} \langle\,\cdot\,,\,\cdot\,\rangle)$ for some $j_0 \in J$ and some $r_0 \in \mathbb{Z}$, which allows us to take $\mathsf{f} : \mathsf{A} \to \mathsf{X}_{\mathcal{H}}$ to be $\vec{A}_{j_0} \to \vec{\mathsf{M}}_{\mathcal{H}}$.

For each $j \in J$, let $(\widetilde{L}_j, \langle\,\cdot\,,\,\cdot\,\rangle_j^{\sim})$ be defined as in the case of $(\widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim})$, but with $\langle\,\cdot\,,\,\cdot\,\rangle_Q$ and $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ replaced with $r_j\langle\,\cdot\,,\,\cdot\,\rangle_Q$ and $(L_j, r'_j\langle\,\cdot\,,\,\cdot\,\rangle_j)$ for some sufficiently large integers $r_j, r'_j \geq 1$ such that $\langle\,\cdot\,,\,\cdot\,\rangle_j^{\sim}$ is induced by the restriction of $r_j\langle\,\cdot\,,\,\cdot\,\rangle^{\sim} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

For any $\hat{\mathbb{Z}}$-algebra R, let $\widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}}}(R)$ denote the subgroup of $\widetilde{\mathrm{P}}(R)$ consisting of elements g such that $g(\widetilde{\mathsf{Z}}_{-i} \underset{\hat{\mathbb{Z}}^{\Box}}{\otimes} R) \subset \widetilde{\mathsf{Z}}_{-i-1} \underset{\hat{\mathbb{Z}}^{\Box}}{\otimes} R$, for all i; let $\widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}},-2}(R)$ denote the subgroup of $\widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}}}(R)$ consisting of elements g such that $g(\widetilde{\mathsf{Z}}_{-i} \underset{\hat{\mathbb{Z}}^{\Box}}{\otimes} R) \subset \widetilde{\mathsf{Z}}_{-i-2} \underset{\hat{\mathbb{Z}}^{\Box}}{\otimes} R$, for all i; and let $\widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}},-1}(R) := \widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}}}(R)/\widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}},-2}(R)$.

Each element $g \in \widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}},-1}(\hat{\mathbb{Z}})$ is determined by its induced morphisms $g_{12} : \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1} = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \to \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-2} = Q_{-2} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} = (\mathrm{Diff}^{-1}_{\mathcal{O}'/\mathbb{Z}})^{\oplus n}$ and $g_{01} : \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{0} = (\mathcal{O}')^{\oplus n} \to \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1} = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ on the graded pieces, and g^m induces mg_{12} and mg_{01} for each integer m, which satisfy $mg_{12}(L_j \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) \subset (\mathrm{Diff}^{-1}_{\mathcal{O}'/\mathbb{Z}})^{\oplus n}$ and $mg_{01}((\mathcal{O}')^{\oplus n}) \subset L_j \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ for all $j \in J$ and all sufficiently large m. Consequently, up to replacing $\widetilde{\mathcal{H}}$ with an open compact subgroup, we may and we shall assume that it also satisfies the following additional requirements:

(3) $\widetilde{\mathcal{H}}$ stabilizes all the lattices $\widetilde{L}_j \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$.

(4) The subgroup $(\widetilde{\mathcal{H}} \cap \widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}))/(\widetilde{\mathcal{H}} \cap \widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}},-2}(\hat{\mathbb{Z}}))$ of $\widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}},-1}(\hat{\mathbb{Z}})$ is of the form $(\widetilde{\mathrm{U}}_{\widetilde{\mathsf{Z}},-1}(\hat{\mathbb{Z}}))^m$ for some integer $m \geq 1$ satisfying $mL \subset L_j$ for all $j \in J$.

These choices of $\{(\widetilde{L}_j, \langle\,\cdot\,,\,\cdot\,\rangle_j^{\sim})\}_{j\in J}$ and $\widetilde{\mathcal{H}}$ (and in particular the condition (3) above) allow us to define a moduli $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}} \to \mathrm{Spec}(F_0)$ as in [37, Def. 1.4.1.4] and an integral model $\vec{\widetilde{\mathsf{M}}}_{\widetilde{\mathcal{H}}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ as in [39, Prop. 6.1], and its minimal compactification $\widetilde{\mathsf{M}}^{\min}_{\widetilde{\mathcal{H}}} \to \mathrm{Spec}(F_0)$ has an integral model $\vec{\widetilde{\mathsf{M}}}^{\min}_{\widetilde{\mathcal{H}}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ as in [39, Prop. 6.4, and Thm. 12.1 and 12.16]. Moreover, for any projective smooth $\widetilde{\Sigma}$ for $\widetilde{\mathsf{M}}$, the toroidal compactification $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}} \to \mathrm{Spec}(F_0)$ as in [37, Thm. 6.4.1.1 and 7.3.3.4] extends to an integral model $\vec{\widetilde{\mathsf{M}}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ as in [42, Thm. 6.1]. Then pullbacks of $\vec{\widetilde{\mathsf{M}}}_{\widetilde{\mathcal{H}}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, $\vec{\widetilde{\mathsf{M}}}^{\min}_{\widetilde{\mathcal{H}}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, and $\vec{\widetilde{\mathsf{M}}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, for all projective smooth $\widetilde{\Sigma}$, define the models $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}} \to \mathsf{S}$, $\widetilde{\mathsf{X}}^{\min}_{\widetilde{\mathcal{H}}} \to \mathsf{S}$, and $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}} \to \mathsf{S}$, which satisfy the analogue of Proposition 2.2 for $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}}$ etc.

We claim that the pullback $\widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]} \to \widetilde{\mathsf{Z}}$ of $\vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]} \to \vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}})]}$ under $\mathsf{S} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ is isomorphic to the abelian scheme $\mathsf{f}^{\times n} : \mathsf{A}^{\times n} \to \mathsf{X}_{\mathcal{H}}$ we want.

Firstly, by the constructions of $\vec{\widetilde{\mathsf{M}}}^{\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}}$ and $\vec{\widetilde{\mathsf{M}}}^{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}}$ by taking normalizations (see [39, Prop. 7.4 and 8.1]), the canonical isomorphisms. $\widetilde{\mathsf{M}}^{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \widetilde{\mathsf{M}}^{\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(F_0)$ (because of the condition (1) on the choice of $\widetilde{\mathcal{H}}$ in the proof in Case (Sm)) induces canonical isomorphisms $\vec{\widetilde{\mathsf{M}}}^{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \vec{\widetilde{\mathsf{M}}}^{\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \vec{\mathsf{M}}_{\mathcal{H}}$ over $\mathrm{Spec}(\mathcal{O}_{F_0,(p)})$. Secondly, the condition (2) above on the choice of $\widetilde{\mathcal{H}}$ implies that the abelian scheme torsor $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}} \to \widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}^{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ over $\mathrm{Spec}(F_0)$ is an abelian scheme; moreover, the condition (4) above on the choice of $\widetilde{\mathcal{H}}$ implies that the canonical morphism $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, A)^\circ \cong A^{\times n}$ over $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}})]} \cong \widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}^{\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}}} \cong \mathsf{M}_{\mathcal{H}}$ (where $A \to \mathsf{M}_{\mathcal{H}}$ is the tautological abelian scheme) can be identified with $[m] : A^{\times n} \to A^{\times n}$ (the multiplication by m over $\mathsf{M}_{\mathcal{H}}$). Since $p^{r_0}L = L_{j_0}$ and $mL \subset L_j$, for all $j \in J$, there exists some integer $r_1 \geq 0$ (depending on r_0) such that $[mp^{r_1}] : A^{\times n} \to A^{\times n}$ extends to isogenies $\vec{A}_{j_0}^{\times n} \to \vec{A}_j^{\times n}$ over $\vec{\mathsf{M}}_{\mathcal{H}}$ for all $j \in J$, by [37, Cor. 1.3.5.4, Prop. 1.4.3.4, and Prop. 3.3.1.5] and the normality of $\vec{\mathsf{M}}_{\mathcal{H}}$. By the construction of $\vec{\widetilde{C}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}$ by taking normalization over the product of the auxiliary models indexed by $j \in J$ (see [39, Prop. 8.4]), and by Zariski's main theorem (see [22, III-1, 4.4.3, 4.4.11]), the canonical finite morphism $\vec{A}_{j_0} \to \prod\limits_{j\in J} \vec{A}_j^{\times n}$ (defined by the above isogenies) induces the composition of $[p^{r_1}] : \vec{A}_j^{\times n} \to \vec{A}_j^{\times n}$ and an isomorphism $\vec{A}_j^{\times n} \xrightarrow{\sim} \vec{\widetilde{C}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}$. Finally, since $\widetilde{\sigma}$ is top-dimensional, we have $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]} \cong \widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}$ and $\vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]} \cong \vec{\widetilde{C}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}$, which are compatible with their structural morphisms to $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}})]} \cong \mathsf{M}_{\mathcal{H}}$ and $\vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}})]} \cong \vec{\mathsf{M}}_{\mathcal{H}}$, respectively, by [39, Lem. 8.20 and Thm. 12.16] and [42, Thm. 6.1(5)]. Thus, by pulling back everything under $S \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, the above claim follows. □

Proof of Proposition 4.2 *in Case* (Nm). Let us proceed with the same setting of the proof of Proposition 4.1 in Case (Nm).

Since $\widetilde{\Sigma}$ satisfies [37, Cond. 6.2.5.25] by assumption, by the same argument as in the proof of [37, Lem. 6.2.5.27], the closure of each stratum of $\vec{\widetilde{\mathsf{M}}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ has no self-intersection. Therefore, by the étale local description spelled out in property (9) of Proposition 2.2 and in Corollary 2.4, and by the geometric normality of affine toroidal embeddings over their base schemes as explained in the proof of [39, Prop. 8.14], the closure of each stratum of $\vec{\widetilde{\mathsf{M}}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ is noetherian and normal. In particular, this is the case for the closure $\vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma}]}$ of $\vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma}]}$.

Following the same strategy as in [36, Sec. 3B], we would like to show that, for suitable choices of $\widetilde{\Sigma}$ and $\widetilde{\sigma}$, the noetherian normal scheme $\vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma}]}$ carries some object parameterized by $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, which satisfies the condition in [42, Thm. 6.1(6)]. Compared with the condition in [37, Thm. 6.4.1.1(6)], the main difference is that the condition in [42, Thm. 6.1(6)] also requires a collection of semi-abelian degenerations parameterized by $j \in J$, whose pullback to the generic point in characteristic zero is $\mathbb{Q}^\times$-isogenous to the tautological object parameterized by $\mathsf{M}_{\mathcal{H}}$. Without repeating (essentially verbatim) all the steps in [36, Sec. 3B], we shall at least explain how to create such a collection.

Consider the tautological semi-abelian objects $(\vec{\widetilde{G}}_j, \vec{\widetilde{\lambda}}_j, \vec{\widetilde{i}}_j, \vec{\widetilde{\alpha}}_{\widetilde{\mathcal{H}}}) \to \vec{\widetilde{\mathsf{M}}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$. By construction of $\vec{\widetilde{C}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}$, the pullbacks $\vec{G}_{j,\vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma}]}}$ of $\vec{G}_j$ to $\vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma}]}$ is an extension of the pullback of the abelian scheme $\vec{A}_j \to \mathsf{M}_{\mathcal{H}}$ by a split torus $\vec{\widetilde{T}}_{j,\vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}}$. Since $\vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ is noetherian and normal, by [37, Prop. 3.3.1.7], for each $j \in J$, the subtorus $\vec{\widetilde{T}}_{j,\vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}}$ of $\vec{\widetilde{G}}_{j,\vec{\widetilde{\mathsf{Z}}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}}$ extends to a subtorus $\vec{\widetilde{T}}_{j,\vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}}$ of the pullback $\vec{\widetilde{G}}_{j,\vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}}$ of $\vec{\widetilde{G}}_j$ to $\vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$, which allows us to define a quotient semi-abelian scheme $\overline{\vec{G}}_j := \vec{\widetilde{G}}_{j,\vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}} / \vec{\widetilde{T}}_{j,\vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}}$ over $\vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$. On the other hand, in characteristic zero, the pullback of the tautological object $(\widetilde{G},\widetilde{\lambda},\widetilde{i},\widetilde{\alpha}_{\widetilde{\mathcal{H}}}) \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ to the closure $\widetilde{\mathsf{Z}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ of $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ defines the semi-abelian scheme $\overline{G} := \widetilde{G}_{\widetilde{\mathsf{Z}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}} / \widetilde{T}_{\widetilde{\mathsf{Z}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}}$ over $\widetilde{\mathsf{Z}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$, exactly as in [36, Sec. 3B]. By construction, these quotient semi-abelian schemes $\overline{G}$ and $\{\overline{\vec{G}}_j\}_{j\in J}$ are $\mathbb{Q}^\times$-isogenous to each other over $\widetilde{\mathsf{Z}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]} \cong \vec{\widetilde{\mathsf{Z}}}^{\mathrm{tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. By the same argument as in [36, Sec. 3B], they also carry the additional PEL structures, providing the collection we need, and satisfy the condition in [42, Thm. 6.1(6)] as long as the same combinatorial [36, Cond. 3.8] is satisfied. Thus, the proposition follows. □

4.4. **Constructions in Case** (Spl)**.**

Proof of Proposition 4.1 *in Case* (Spl). Let us proceed with the setting of the proof of Proposition 4.1 in Case (Nm), so that $\vec{\widetilde{C}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}} \to \vec{\mathsf{M}}_{\mathcal{H}}$ is isomorphic to the abelian scheme $\vec{A}^{\times n}_{j_0} \to \vec{\mathsf{M}}_{\mathcal{H}}$, which is *smooth*. Since $\vec{\widetilde{C}}^{\mathrm{spl}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}$ is the normalization of $\vec{\widetilde{C}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}} \underset{\vec{\mathsf{M}}_{\mathcal{H}}}{\times} \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ (see [41, Def. 3.2.3 and Lem. 3.2.4]), and since this fiber product is already normal (by the smoothness of $\vec{\widetilde{C}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}} \to \vec{\mathsf{M}}_{\mathcal{H}}$ and the normality of $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$), the induced morphism $\vec{\widetilde{C}}^{\mathrm{spl}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}} \to \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ is isomorphic to the pullback of $\vec{A}^{\times n}_{j_0} \to \vec{\mathsf{M}}_{\mathcal{H}}$ under $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \vec{\mathsf{M}}_{\mathcal{H}}$. Again, since $\widetilde{\sigma}$ is top-dimensional, by [41, Prop. 3.2.11, Thm. 3.4.1(2), and Thm. 4.3.1(5)], the pullback $\widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]} \to \widetilde{\mathsf{Z}}$ of $\vec{\widetilde{\mathsf{Z}}}^{\mathrm{spl}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]} \to \vec{\widetilde{\mathsf{Z}}}^{\mathrm{spl}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}})]}$ under $\mathsf{S} \to \mathrm{Spec}(\mathcal{O}_K)$ is isomorphic to $\mathsf{f}^{\times n} : \mathsf{A}^{\times n} \to \mathsf{X}_{\mathcal{H}}$, as desired. □

Proof of Proposition 4.2 *in Case* (Spl). In this case, by the same argument as in the proof of Proposition 4.2 in Case (Nm), the pullbacks of the semi-abelian objects over $\vec{\widetilde{\mathsf{M}}}^{\mathrm{spl,tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ to the closure $\vec{\mathsf{Z}}^{\mathrm{spl,tor}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ of $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ in $\vec{\widetilde{\mathsf{M}}}^{\mathrm{spl,tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ define quotient objects which also carry the splitting structures given by the analogue of the recipe [41, (3.3.12)] (for $\vec{\widetilde{\mathsf{M}}}^{\mathrm{spl,tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ instead of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$). Hence, the condition in [41, Thm. 3.4.1(4)] applies to these quotient objects, which is satisfied under the same combinatorial [36, Cond. 3.8], by the same argument as in [36, Sec. 3B] (again). □

4.5. **Application to Case** (Hdg)**.** Let $\mathsf{X}_{\mathcal{H}_{\mathrm{aux}}} \to \mathsf{S}$ denote the pullback of $\mathsf{M}_{\mathcal{H}_{\mathrm{aux}}} \to \mathrm{Spec}(\mathbb{Z}_{(p)})$ under $\mathsf{S} \to \mathrm{Spec}(\mathbb{Z}_{(p)})$, and let $\mathsf{f}_{\mathrm{aux}} : \mathsf{A}_{\mathrm{aux}} \to \mathsf{X}_{\mathcal{H}_{\mathrm{aux}}}$ denote the pullback of $A \to \mathsf{M}_{\mathcal{H}_{\mathrm{aux}}}$ under the canonical morphism $\mathsf{X}_{\mathcal{H}_{\mathrm{aux}}} \to \mathsf{M}_{\mathcal{H}_{\mathrm{aux}}}$, so that $A \to \mathsf{X}_{\mathcal{H}}$ is the pullback of $A_{\mathrm{aux}} \to \mathsf{X}_{\mathcal{H}_{\mathrm{aux}}}$ under the canonical morphism $\mathsf{X}_{\mathcal{H}} \to \mathsf{X}_{\mathcal{H}_{\mathrm{aux}}}$ induced by $\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}_{\mathrm{aux}}} \underset{\mathbb{Z}_{(p)}}{\otimes} \mathcal{O}_{F_0,(v)}$.

Let $n \geq 0$ be an integer. Suppose that $\mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is induced by some $\mathsf{X}_{\mathcal{H}_{\mathrm{aux}}} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{aux}},\Sigma_{\mathrm{aux}}}$ (see the proof of Proposition 2.2 in Case (Hdg), in Section 2.3). By Propositions 4.1 and 4.2 (in Case (Sm)), there exists some integral model $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}} \to \mathsf{S}$ and some toroidal compactification $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ such that the structural morphism $\mathsf{f}^{\times n}_{\mathrm{aux}}$: $Y_{\mathrm{aux}} := \mathsf{A}^{\times n}_{\mathrm{aux}} \to \mathsf{X}_{\mathcal{H}_{\mathrm{aux}}}$ can be identified with the canonical morphism from some stratum $\widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]}$ of $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ to some stratum $\widetilde{\mathsf{Z}}$ of $\widetilde{\mathsf{X}}^{\mathrm{min}}_{\widetilde{\mathcal{H}}}$, and such that $\mathsf{f}^{\times n}_{\mathrm{aux}}$ (necessarily uniquely) extends to a morphism from the closure $\overline{Y}_{\mathrm{aux}} := \widetilde{\mathsf{Z}}^{\mathrm{tor}}_{[\widetilde{\sigma}]}$ of $\widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]}$ in $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ to some toroidal compactification $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{aux}},\Sigma_{\mathrm{aux}}}$.

Consider the canonical morphism $Y := \mathsf{A}^{\times n} \to Y_{\mathrm{aux}}$. Let $\overline{Y}$ denote the normalization of $\overline{Y}_{\mathrm{aux}}$ under the composition $Y \to Y_{\mathrm{aux}} \to \overline{Y}_{\mathrm{aux}}$ of canonical morphisms, which induces an open immersion $Y \hookrightarrow \overline{Y}$ because Y (being an abelian scheme over $\mathsf{X}_{\mathcal{H}}$) is normal. We have the following étale local description of $Y \hookrightarrow \overline{Y}$:

Proposition 4.3. *At each point x of $\overline{Y}$, there exist a scheme C_x of finite type over S; an affine toroidal embedding $E_x \hookrightarrow E_x(\tau_x)$, where E_x is a split torus over* $\mathrm{Spec}(\mathbb{Z})$ *and τ_x is some cone in the $\mathbb{R}$-dual of the character group of E_x; an étale neighborhood $x \to \overline{U} \to \overline{Y}$; and an étale morphism $\overline{U} \to E_x(\tau_x) \underset{\mathsf{S}}{\times} C_x$ such that $U := \overline{U} \underset{\overline{Y}}{\times} Y \cong \overline{U} \underset{E_x(\tau_x)}{\times} E_x$ (as open subschemes of $\overline{U}$).*

Proof. Without loss of generality, we may assume that $\mathsf{S} = \mathrm{Spec}(\mathcal{O}_{F_0,(v)})$. Suppose that x is mapped to the stratum $\mathsf{Z}_{[\breve{\tau}]}$ of $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$, which is above some stratum $\breve{\mathsf{Z}}$ (in the closure of $\widetilde{\mathsf{Z}} \cong \mathsf{X}_{\mathcal{H}_{\mathrm{aux}}}$) of $\widetilde{\mathsf{X}}^{\mathrm{min}}_{\widetilde{\mathcal{H}}}$. Suppose x is mapped to the stratum $\mathsf{Z}_{[\tau]}$ of $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, which is above some stratum Z of $\mathsf{X}^{\mathrm{min}}_{\mathcal{H}}$. Suppose that x is mapped to the stratum $\mathsf{Z}_{[\tau_{\mathrm{aux}}]}$ of $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{aux}},\Sigma_{\mathrm{aux}}}$, which is above some stratum $\mathsf{Z}_{\mathrm{aux}}$ of $\mathsf{X}^{\mathrm{min}}_{\mathcal{H}_{\mathrm{aux}}}$.

By (9) of Proposition 2.2 (see also the second paragraph of Corollary 2.4), étale locally at each point of $\mathsf{Z}_{[\widetilde{\tau}]}$, for each representative $\widetilde{\tau}$ of $[\widetilde{\tau}]$, and for some representative $\widetilde{\tau}'$ of $[\widetilde{\tau}']$, where $[\breve{\sigma}]$ determines $[\widetilde{\sigma}]$ as in the case of $[\tau]$ determining $[\tau']$ in (9) of Proposition 2.2, the canonical open immersion $Y_{\mathrm{aux}} = \mathsf{Z}_{[\widetilde{\sigma}]} \hookrightarrow \overline{Y}_{\mathrm{aux}} = \mathsf{Z}^{\mathrm{tor}}_{[\widetilde{\sigma}]}$ is isomorphic to the affine toroidal embedding $\breve{\Xi}_{\breve{\sigma}} \hookrightarrow \breve{\Xi}_{\breve{\sigma}}(\breve{\tau})$ over the $\breve{C}$ over $\breve{\mathsf{Z}}$, where $\breve{\Xi}_{\breve{\sigma}}(\breve{\tau})$ is the closure of $\breve{\Xi}_{\breve{\sigma}}$ in $\breve{\Xi}(\breve{\tau})$. (Such notation with $\breve{}$ instead of $\widetilde{}$ is to make it clear that $\breve{\tau} \in \Sigma^+_{\breve{\mathsf{Z}}}$ and $\widetilde{\sigma} \in \Sigma^+_{\widetilde{\mathsf{Z}}}$ cannot be directly compared; cf. [36, Sec. 2D].) Similarly, étale locally at each point of $\mathsf{Z}_{[\tau_{\mathrm{aux}}]}$, for each representative τ_{aux} of $[\tau_{\mathrm{aux}}]$, the canonical open immersion $\mathsf{X}_{\mathcal{H}_{\mathrm{aux}}} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{aux}},\Sigma_{\mathrm{aux}}}$ is isomorphic to the affine toroidal embedding $\Xi_{\mathrm{aux}} \hookrightarrow \Xi_{\mathrm{aux}}(\tau_{\mathrm{aux}})$ over the C_{aux} over $\mathsf{Z}_{\mathrm{aux}}$. As explained in [45, proof of Thm. 1, after Rem. 4.1.6], étale locally at each point of $\mathsf{Z}_{[\tau]}$, for each representative τ, the canonical open immersion $\mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is isomorphic to the affine toroidal embedding $\Xi \hookrightarrow \Xi(\tau)$ over the C over Z.

As explained in [36, Sec. 3B], we have $\breve{\mathsf{Z}} \cong \mathsf{Z}_{\mathrm{aux}}$, together with canonical morphisms $\breve{C} \to C_{\mathrm{aux}}$, $\breve{\Xi}_{\breve{\sigma}} \to \Xi_{\mathrm{aux}}$, and $\breve{\Xi}_{\breve{\sigma}}(\breve{\tau}) \to \Xi_{\mathrm{aux}}(\tau_{\mathrm{aux}})$. By [36, Lem. 4.9], $\breve{C} \to C_{\mathrm{aux}}$ is an abelian scheme torsor, and hence so is the pullback $C_x := C \underset{C_{\mathrm{aux}}}{\times} \breve{C} \to C$. Let $\Xi_x := \Xi \underset{\Xi_{\mathrm{aux}}}{\times} \breve{\Xi}_{\breve{\sigma}}$, which is a torsor over C_x under the subtorus $E_x := E \underset{E_{\mathrm{aux}}}{\times} \breve{E}_{\breve{\sigma}}$

of $\breve{E}_{\breve{\sigma}}$. By [24, Lem. 3.2], the normalization of $\breve{\Xi}_{\breve{\sigma}}(\breve{\tau})$ under the canonical morphism $\Xi_x \to \breve{\Xi}_{\breve{\sigma}}(\breve{\tau})$ is an affine toroidal embedding over C_x, which we can write as $\Xi_x \hookrightarrow \Xi_x(\tau_x)$ for some cone τ_x induced by $\breve{\tau}$.

Since the canonical morphism $\overline{Y}_{\mathrm{aux}} \to \mathsf{X}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{aux}},\Sigma_{\mathrm{aux}}}$ is induced by [37, Thm. 6.4.1.1(6)] (see [36, Sec. 3B]), whose proof is based on the good algebraic models (constructed by Artin's approximation), on which the proof of (9) of Proposition 2.2 in Case (Sm) (in Section 2.3) is also based, it follows that, étale locally at each point of $\mathsf{Z}_{[\breve{\tau}]}$, the canonical morphisms $Y_{\mathrm{aux}} \hookrightarrow \overline{Y}_{\mathrm{aux}}$ and $\mathsf{X}_{\mathcal{H}_{\mathrm{aux}}} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{aux}},\Sigma_{\mathrm{aux}}}$ are compatibly isomorphic to $\breve{\Xi}_{\breve{\sigma}} \hookrightarrow \breve{\Xi}_{\breve{\sigma}}(\breve{\tau})$ and $\Xi_{\mathrm{aux}} \hookrightarrow \Xi_{\mathrm{aux}}(\sigma_{\mathrm{aux}})$, respectively. By using the nested approximation in [67, Thm. 2.9] and [63, Thm. 11.4 and 11.5] instead of Artin's approximation, where the hypothesis (in the notation there) that $A_i \underset{A_{i-1}}{\otimes} B_{i-1}$ is noetherian is satisfied because the canonical isomorphism $\mathfrak{X}_\tau = (\Xi(\tau))^\wedge_{\Xi_\tau} \xrightarrow{\sim} (\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^\wedge_{\mathsf{Z}_{[\tau]}}$ in [45, Thm. 4.1.5(5); see the proof of Prop. 4.2.11] is induced by the canonical isomorphism $\mathfrak{X}_{\tau_{\mathrm{aux}}} = (\Xi_{\mathrm{aux}}(\tau_{\mathrm{aux}}))^\wedge_{\Xi_{\mathrm{aux},\tau_{\mathrm{aux}}}} \xrightarrow{\sim} (\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{aux}},\Sigma_{\mathrm{aux}}})^\wedge_{\mathsf{Z}_{[\tau_{\mathrm{aux}}]}}$, it follows that $Y \hookrightarrow \overline{Y}$ and $\mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ are compatibly isomorphic to $\Xi_x \hookrightarrow \Xi_x(\tau_x)$ and $\Xi \hookrightarrow \Xi(\tau)$, respectively, étale locally at x. Hence, the proposition follows. □

5. Nearby cycles and main comparisons

5.1. **Basic setup.** Suppose moreover that $\mathsf{S} = \mathrm{Spec}(R_0)$ is *Henselian*. Let $j : \eta = \mathrm{Spec}(K) \to \mathsf{S}$ (resp. $i : s = \mathrm{Spec}(k) \to \mathsf{S}$) denote the generic (resp. special) point of $\mathsf{S} = \mathrm{Spec}(R_0)$, with its structural morphism. Let $\bar{K}$ be an algebraic closure of K, let $\bar{R}_0$ denote the integral closure of R_0 in $\bar{K}$, with residue field $\bar{k}$ an algebraic closure of k, and let $\bar{j} : \bar{\eta} := \mathrm{Spec}(\bar{K}) \to \bar{\mathsf{S}} := \mathrm{Spec}(\bar{R}_0)$ (resp. $\bar{i} : \bar{s} := \mathrm{Spec}(\bar{k}) \to \bar{\mathsf{S}}$) denote the corresponding geometric point lifting i (resp. j). For simplicity, we shall denote by subscripts the pullbacks of various schemes over S to η, $\bar{\eta}$, s, or $\bar{s}$.

Consider any rational prime number $\ell \neq p$. Let Λ be either of $\mathbb{Z}/\ell^m\mathbb{Z}$ (for some integer $m \geq 1$), $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, $\bar{\mathbb{Q}}_\ell$, or a finite free module over these rings. (These are the coefficient rings accepted in, for example, [27, Sec. 3.1].) For simplicity, we shall also denote by Λ the constant étale sheaf with values in Λ. For each scheme X separated and of finite type over S, we denote by $D^b_c(X_\eta, \Lambda)$ the bounded derived category of Λ-étale constructible sheaves over X_η, and by $D^b_c(X_{\bar{s}} \times \bar{\eta}, \Lambda)$ the bounded derived category of Λ-étale constructible sheaves over $X_{\bar{s}}$ with compatible continuous $\mathrm{Gal}(\bar{K}/K)$-actions. (See [11, Sec. 1.1] and [16] when Λ is not torsion.) Then we have the functor of *nearby cycles*:

$$R\Psi_X : D^b_c(X_\eta, \Lambda) \to D^b_c(X_{\bar{s}} \times \bar{\eta}, \Lambda) : \mathcal{F} \mapsto \bar{i}^* R\bar{j}_*(\mathcal{F}_{\bar{\eta}}), \tag{5.1}$$

where $\mathcal{F}_{\bar{\eta}}$ denotes the pullback of $\mathcal{F}$ to $X_{\bar{\eta}}$. (See [13, XIII], [10, Th. finitude, Sec. 3], and [27, Sec. 4] for more details.)

Suppose we have a morphism $\varphi : X \to Y$ of schemes of finite type over S. Then we have the adjunction morphisms

$$R\Psi_Y \, R\varphi_{\eta,*}(\mathcal{F}) \to R\varphi_{\bar{s},*} \, R\Psi_X(\mathcal{F}), \tag{5.2}$$

and

$$R\varphi_{\bar{s},!} \, R\Psi_X(\mathcal{F}) \to R\Psi_Y \, R\varphi_{\eta,!}(\mathcal{F}), \tag{5.3}$$

which are isomorphisms when φ is *proper*, by the proper base change theorem (cf. [3, XII, 5.1] and [13, XIII, (2.1.7.1) and (2.1.7.3)]). In the opposite direction, we have the adjunction morphism

$$\varphi_{\bar{s}}^* \, R\Psi_Y(\mathcal{F}) \to R\Psi_X \, \varphi_\eta^*(\mathcal{F}), \tag{5.4}$$

which is an isomorphism when φ is *smooth*, by the smooth base change theorem (see [3, XVI, 1.2] and [13, XIII, (2.1.7.2)]). We will freely use these facts without repeating these references.

Lemma 5.5. *Let E be a split torus over S with character group $\mathbf{S}$, and let $J_{\overline{E}} : E \hookrightarrow \overline{E} = \bigcup_{\sigma \in \Sigma} E(\sigma)$ be a toroidal embedding over S defined by some rational polyhedral cone decomposition Σ of a cone in $\mathbf{S}_{\mathbb{R}}^\vee$. Then the adjunction morphisms*

$$R\Psi_{\overline{E}} \, RJ_{\overline{E}_\eta,*}(\Lambda_{E_\eta}) \to RJ_{\overline{E}_{\bar{s}},*} \, R\Psi_E(\Lambda_{E_\eta}) \tag{5.6}$$

and

$$J_{\overline{E}_{\bar{s}},!} \, R\Psi_E(\Lambda_{E_\eta}) \to R\Psi_{\overline{E}} \, J_{\overline{E}_\eta,!}(\Lambda_{E_\eta}) \tag{5.7}$$

are isomorphisms in $D_c^b(\overline{E}_{\bar{s}} \times \bar{\eta}, \Lambda)$.

Proof. Note that we have not assumed that Σ is *smooth*. Nevertheless, there exists a smooth refinement Σ' of Σ, so that the corresponding toroidal embedding $J_{\overline{E}'} : E \hookrightarrow \overline{E}' = \bigcup_{\sigma' \in \Sigma'} E(\sigma')$ is between smooth schemes over $\operatorname{Spec}(\mathbb{Z})$, and so that $\overline{E}' - E$ (with its reduced structure) is a simple normal crossings divisor flat over S. Therefore, by [13, XIII, 2.1.9], the adjunction morphisms

$$R\Psi_{\overline{E}'} \, RJ_{\overline{E}'_\eta,*}(\Lambda_{E_\eta}) \to RJ_{\overline{E}'_{\bar{s}},*} \, R\Psi_E(\Lambda_{E_\eta}) \tag{5.8}$$

and

$$J_{\overline{E}'_{\bar{s}},!} \, R\Psi_E(\Lambda_{E_\eta}) \to R\Psi_{\overline{E}'} \, J_{\overline{E}'_\eta,!}(\Lambda_{E_\eta}) \tag{5.9}$$

are isomorphisms. By [29, Ch. I, §2, Thm. 8], there is a canonical proper morphism $\varphi : \overline{E}' \to \overline{E}$ satisfying $J_{\overline{E}} = \varphi \circ J_{\overline{E}'}$. It induces canonical isomorphisms $R\Psi_{\overline{E}} \, RJ_{\overline{E}_\eta,*}(\Lambda_{E_\eta}) \cong R\Psi_{\overline{E}} \, R\varphi_{\eta,*} \, RJ_{\overline{E}'_\eta,*}(\Lambda_{E_\eta}) \xrightarrow{\sim} R\varphi_{\bar{s},*} \, R\Psi_{\overline{E}'} \, RJ_{\overline{E}'_\eta,*}(\Lambda_{E_\eta})$ and $RJ_{\overline{E}_{\bar{s}},*} \, R\Psi_E(\Lambda_{E_\eta}) \cong R\varphi_{\bar{s},*} \, RJ_{\overline{E}'_{\bar{s}},*} \, R\Psi_E(\Lambda_{E_\eta})$, which are compatible with each other under the adjunction morphisms (5.6) and (5.8). It also induces canonical isomorphisms $J_{\overline{E}_{\bar{s}},!} \, R\Psi_E(\Lambda_{E_\eta}) \cong R\varphi_{\bar{s},*} \, RJ_{\overline{E}'_{\bar{s}},!} \, R\Psi_E(\Lambda_{E_\eta})$ and $R\Psi_{\overline{E}} \, J_{\overline{E}_\eta,!}(\Lambda_{E_\eta}) \cong R\Psi_{\overline{E}} \, R\varphi_{\eta,*} \, J_{\overline{E}'_\eta,!}(\Lambda_{E_\eta}) \xrightarrow{\sim} R\varphi_{\bar{s},*} \, R\Psi_{\overline{E}'} \, J_{\overline{E}'_\eta,!}(\Lambda_{E_\eta})$, which are compatible with each other under the adjunction morphisms (5.7) and (5.9). Thus, we can conclude because (5.8) and (5.9) are isomorphisms. □

Lemma 5.10. *Let $J_{\overline{Y}} : Y \hookrightarrow \overline{Y}$ be an open immersion between schemes separated and of finite type over S, which satisfies the following condition: For each geometric point $x \to \overline{Y}$, there exist a scheme C of finite type over S, a toroidal embedding $J_{\overline{E}} : E \hookrightarrow \overline{E}$ as in Lemma 5.5 (which defines the adjunction morphisms (5.6) and (5.7)), an étale neighborhood $x \to \overline{U} \to \overline{Y}$, and an étale morphism $\overline{U} \to \overline{E} \underset{\mathsf{S}}{\times} C$ such that $U := \overline{U} \underset{\overline{Y}}{\times} Y \cong \overline{U} \underset{\overline{E}}{\times} E$ (as open subschemes of $\overline{U}$). Then the adjunction morphisms*

$$R\Psi_{\overline{Y}} \, RJ_{\overline{Y}_\eta,*}(\Lambda_{Y_\eta}) \to RJ_{\overline{Y}_{\bar{s}},*} \, R\Psi_Y(\Lambda_{Y_\eta}) \tag{5.11}$$

and

$$J_{\overline{Y}_{\bar{s}},!}\, R\Psi_Y(\Lambda_{Y_\eta}) \to R\Psi_{\overline{Y}}\, J_{\overline{Y}_\eta,!}(\Lambda_{Y_\eta}) \tag{5.12}$$

are isomorphisms in $D_c^b(\overline{Y}_{\bar{s}} \times \bar{\eta}, \Lambda)$.

Proof. Note that $Y_\eta \to \eta$ might not be smooth. Since the assertion of the lemma can be verified étale locally over $\overline{Y}$, it suffices to show that the adjunction morphisms

$$R\Psi_{\overline{E} \underset{\mathsf{S}}{\times} C}\, RJ_{(\overline{E} \underset{\mathsf{S}}{\times} C)_\eta,*}(\Lambda_{(E \underset{\mathsf{S}}{\times} C)_\eta}) \to RJ_{(\overline{E} \underset{\mathsf{S}}{\times} C)_{\bar{s}},*}\, R\Psi_{E \underset{\mathsf{S}}{\times} C}(\Lambda_{(E \underset{\mathsf{S}}{\times} C)_\eta}) \tag{5.13}$$

and

$$J_{(\overline{E} \underset{\mathsf{S}}{\times} C)_{\bar{s}},!}\, R\Psi_{E \underset{\mathsf{S}}{\times} C}(\Lambda_{(E \underset{\mathsf{S}}{\times} C)_\eta}) \to R\Psi_{\overline{E} \underset{\mathsf{S}}{\times} C}\, J_{(\overline{E} \underset{\mathsf{S}}{\times} C)_\eta,!}(\Lambda_{(E \underset{\mathsf{S}}{\times} C)_\eta}) \tag{5.14}$$

are isomorphisms, where $J_{\overline{E} \underset{\mathsf{S}}{\times} C} = J_{\overline{E}} \underset{\mathsf{S}}{\times} \mathrm{Id}_C : E \underset{\mathsf{S}}{\times} C \hookrightarrow \overline{E} \underset{\mathsf{S}}{\times} C$ is the canonical open immersion. By the Künneth isomorphisms as in [3, XVII, 5.4.3] and [5, Sec. 4.2.7], and by Gabber's theorem (see [27, Thm. 4.7]) on nearby cycles over products of schemes of finite type over S, we have canonical isomorphisms

$$\begin{aligned}(R\Psi_{\overline{E}}\, RJ_{\overline{E}_\eta,*}(\Lambda_{E_\eta})) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_C(\Lambda_{C_\eta})) &\xrightarrow{\sim} R\Psi_{\overline{E} \underset{\mathsf{S}}{\times} C}((RJ_{\overline{E}_\eta,*}(\Lambda_{E_\eta})) \overset{L}{\underset{\eta}{\boxtimes}} \Lambda_{C_\eta})\\ &\xrightarrow{\sim} R\Psi_{\overline{E} \underset{\mathsf{S}}{\times} C}\, RJ_{(\overline{E} \underset{\mathsf{S}}{\times} C)_\eta}(\Lambda_{(E \underset{\mathsf{S}}{\times} C)_\eta}),\end{aligned}$$

$$\begin{aligned}(RJ_{\overline{E}_{\bar{s}},*}\, R\Psi_E(\Lambda_{E_\eta})) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_C(\Lambda_{C_\eta})) &\xrightarrow{\sim} RJ_{(\overline{E} \underset{\mathsf{S}}{\times} C)_{\bar{s}},*}((R\Psi_E(\Lambda_{E_\eta})) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_C(\Lambda_{C_\eta})))\\ &\xrightarrow{\sim} RJ_{(\overline{E} \underset{\mathsf{S}}{\times} C)_{\bar{s}},*}\, R\Psi_{E \underset{\mathsf{S}}{\times} C}(\Lambda_{(E \underset{\mathsf{S}}{\times} C)_\eta}),\end{aligned}$$

$$\begin{aligned}(J_{\overline{E}_{\bar{s}},!}\, R\Psi_E(\Lambda_{E_\eta})) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_C(\Lambda_{C_\eta})) &\xrightarrow{\sim} J_{(\overline{E} \underset{\mathsf{S}}{\times} C)_{\bar{s}},!}((R\Psi_E(\Lambda_{E_\eta})) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_C(\Lambda_{C_\eta})))\\ &\xrightarrow{\sim} J_{(\overline{E} \underset{\mathsf{S}}{\times} C)_{\bar{s}},!}\, R\Psi_{E \underset{\mathsf{S}}{\times} C}(\Lambda_{(E \underset{\mathsf{S}}{\times} C)_\eta}),\end{aligned}$$

and

$$\begin{aligned}(R\Psi_{\overline{E}}\, J_{\overline{E},!}(\Lambda_{E_\eta})) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_C(\Lambda_{C_\eta})) &\xrightarrow{\sim} R\Psi_{\overline{E} \underset{\mathsf{S}}{\times} C}((J_{\overline{E}_\eta,!}(\Lambda_{E_\eta})) \overset{L}{\underset{\eta}{\boxtimes}} \Lambda_{C_\eta})\\ &\xrightarrow{\sim} R\Psi_{\overline{E} \underset{\mathsf{S}}{\times} C}\, J_{(\overline{E} \underset{\mathsf{S}}{\times} C)_\eta,!}(\Lambda_{(E \underset{\mathsf{S}}{\times} C)_\eta}),\end{aligned}$$

where the former (resp. latter) two isomorphisms are compatible with each other under the adjunction morphisms (5.6) and (5.13) (resp. (5.7) and (5.14)) (and the identity morphism on $R\Psi_C(\Lambda_{C_\eta})$). Hence, (5.13) and (5.14) are isomorphisms because (5.6) and (5.7) are, by Lemma 5.5. □

5.2. **Compatibility with good compactifications.** Let S, i, j, $\bar{i}$, $\bar{j}$, etc be as in Section 5.1. Consider any $\mathsf{X}_\mathcal{H} \to \mathsf{S}$ as in Assumption 2.1, with toroidal and minimal compactifications $J_{\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} : \mathsf{X}_\mathcal{H} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ and $J_{\mathsf{X}^{\min}_\mathcal{H}} : \mathsf{X}_\mathcal{H} \hookrightarrow \mathsf{X}^{\min}_\mathcal{H}$ over S, as in Proposition 2.2. Consider any étale sheaf $\mathcal{V}$ that is either of the form $\mathcal{V}_\xi$ as in Proposition 3.2, associated with some irreducible representation ξ of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ on a finite-dimensional vector space V_ξ over $\bar{\mathbb{Q}}_\ell$, in which case we take $\Lambda = \bar{\mathbb{Q}}_\ell$; or of the form $\mathcal{W}_{0,M}$ as in Proposition 3.4, where M is finite free over either $\mathbb{Z}_\ell$ or $\mathbb{Z}/\ell^m\mathbb{Z}$ (for some integer $m \geq 1$), in which case we take $\Lambda = M$, such that $\ell > c_W$.

Theorem 5.15. *The adjunction morphisms*

$$R\Psi_{\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} RJ_{(\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_\eta,*}(\mathcal{V}) \to RJ_{(\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_{\bar{s}},*} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}) \tag{5.16}$$

and

$$J_{(\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_{\bar{s}},!} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}) \to R\Psi_{\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} J_{(\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_\eta,!}(\mathcal{V}) \tag{5.17}$$

are isomorphisms in $D^b_c((\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_{\bar{s}} \times \bar{\eta}, \Lambda)$.

Proof. Let $t = t_\xi$ and $n = n_\xi$ if $\mathcal{V} = \mathcal{V}_\xi$ as in Proposition 3.2, in which case we take $\Lambda = \bar{\mathbb{Q}}_\ell$; or let $t = t_W$ and $n = n_W$ if $\mathcal{V} = \mathcal{W}_{0,M}$ as in Proposition 3.4, with $\ell > c_W$ there, in which case we take $\Lambda = M$.

In Cases (Sm), (Nm), or (Spl), let $\widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]} \to \widetilde{\mathsf{Z}}$ be as in Proposition 4.1, which realizes the abelian scheme $\mathsf{f}^{\times n} : \mathsf{A}^{\times n} \to \mathsf{X}_{\mathcal{H}}$ up to $\mathbb{Z}^\times_{(\ell)}$-isogeny. Let $Y := \widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]}$; and let $\overline{Y} := \widetilde{\mathsf{Z}}^{\mathrm{tor}}_{[\widetilde{\sigma}]}$, the closure of $\widetilde{\mathsf{Z}}_{[\widetilde{\sigma}]}$ in $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$. Let $J_{\overline{Y}} : Y \hookrightarrow \overline{Y}$ denote the canonical open immersion. By Proposition 4.2, up to modifying the choices of $\widetilde{\Sigma}$ and $\widetilde{\sigma}$, we may assume that the induced morphism $h : Y \to \mathsf{X}_{\mathcal{H}}$ extends to a (necessarily proper surjective) morphism $\overline{h} : \overline{Y} \to \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. In Case (Hdg), we simply take $Y \hookrightarrow \overline{Y}$ as in the paragraph preceding Proposition 4.3, with canonical morphisms $h : Y \to \mathsf{X}_{\mathcal{H}}$ and $\overline{h} : \overline{Y} \to \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. In all cases, we know that $\mathcal{V}$ is a direct summand of $Rh_*(\Lambda_Y)(-t)[n]$, and therefore $RJ_{(\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_\eta,*}(\mathcal{V})$ and $J_{(\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_\eta,!}(\mathcal{V})$ are direct summands of $R\overline{h}_{\eta,*} RJ_{\overline{Y}_\eta,*}(\Lambda_{Y_\eta})(-t)[n]$ and $R\overline{h}_{\eta,*} J_{\overline{Y}_\eta,!}(\Lambda_{Y_\eta})(-t)[n]$, respectively.

By the proper base change theorem (applied to $\overline{h}$), and by Tate twisting and shifting, it suffices to note that the adjunction morphisms

$$R\Psi_{\overline{Y}} RJ_{\overline{Y}_\eta,*}(\Lambda_{Y_\eta}) \to RJ_{\overline{Y}_{\bar{s}},*} R\Psi_Y(\Lambda_{Y_\eta}) \tag{5.18}$$

and

$$J_{\overline{Y}_{\bar{s}},!} R\Psi_Y(\Lambda_{Y_\eta}) \to R\Psi_{\overline{Y}} J_{\overline{Y}_\eta,!}(\Lambda_{Y_\eta}) \tag{5.19}$$

are isomorphisms: In Cases (Sm), (Nm), and (Spl), this follows from Lemma 5.10 and the second paragraph of Corollary 2.4 (applied to the strata of $\widetilde{\mathsf{X}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$). In Case (Hdg), this follows from Lemma 5.10 and Proposition 4.3. □

Corollary 5.20. *We have canonical isomorphisms*

$$R\Gamma((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}) \xrightarrow{\sim} R\Gamma((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})) \tag{5.21}$$

and

$$R\Gamma_c((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})) \xrightarrow{\sim} R\Gamma_c((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}), \tag{5.22}$$

which are compatible with their natural continuous $\mathrm{Gal}(\bar{K}/K)$*-actions.*

Proof. Since $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{S}$ is proper, by the proper base change theorem, these follow from the two canonical isomorphisms (5.16) and (5.17) in $D^b_c((\mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_{\bar{s}} \times \bar{\eta}, \Lambda)$. □

Corollary 5.23. *The adjunction morphisms*

$$R\Psi_{\mathsf{X}^{\min}_{\mathcal{H}}} RJ_{(\mathsf{X}^{\min}_{\mathcal{H}})_\eta,*}(\mathcal{V}) \to RJ_{(\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{s}},*} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}) \tag{5.24}$$

and

$$J_{(\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{s}},!} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}) \to R\Psi_{\mathsf{X}^{\min}_{\mathcal{H}}} J_{(\mathsf{X}^{\min}_{\mathcal{H}})_\eta,!}(\mathcal{V}) \tag{5.25}$$

are isomorphisms in $D^b_c((\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{s}} \times \bar{\eta}, \Lambda)$.

Proof. Since the structural morphism $\oint_{\mathcal{H},\Sigma} : \mathsf{X}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{X}^{\mathrm{min}}_{\mathcal{H}}$ is proper (for any choice of Σ), this follows from Theorem 5.15 and from the proper base change theorem. $\square$

Let $d := \dim((\mathsf{X}_{\mathcal{H}})_\eta)$. Suppose $\Lambda = \mathbb{Q}_\ell$ or $\bar{\mathbb{Q}}_\ell$. Then $\Lambda[d]$ is a perverse sheaf on $(\mathsf{X}_{\mathcal{H}})_\eta$, and we can consider its middle perversity extension $J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_\eta,!*}(\Lambda[d])$. By [27, Cor. 4.5], $R\Psi_{\mathsf{X}_{\mathcal{H}}}(\Lambda[d])$ is a perverse sheaf on $(\mathsf{X}_{\mathcal{H}})_{\bar{s}}$. The same are true for $\mathcal{V}_\xi[d]$, for each $\mathcal{V}_\xi$ as in Proposition 3.2 (with $\ell \neq p$ and $\Lambda = \bar{\mathbb{Q}}_\ell$).

Theorem 5.26. *We have a canonical isomorphism*

$$R\Psi_{\mathsf{X}^{\mathrm{min}}_{\mathcal{H}}} J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_\eta,!*}(\mathcal{V}_\xi[d]) \xrightarrow{\sim} J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}},!*} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}_\xi[d]) \tag{5.27}$$

in the category of perverse sheaves over $(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}}$ *with compatible continuous* $\mathrm{Gal}(\bar{K}/K)$*-actions.*

Proof. Let us denote by ${}^p\mathcal{H}^0$ the zeroth perverse cohomology of a $\bar{\mathbb{Q}}_\ell$-sheaf, which is a perverse sheaf. By definition, we have

$$J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_?,!*} = \mathrm{Im}({}^p\mathcal{H}^0 \circ J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_?,!} \to {}^p\mathcal{H}^0 \circ RJ_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_?,*}),$$

for $? = \eta$ or $\bar{s}$, where the image is taken in the abelian category of perverse sheaves. Therefore, it suffices to show that there are canonical isomorphisms

$${}^p\mathcal{H}^0(J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}},!} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}_\xi[d])) \xrightarrow{\sim} R\Psi_{\mathsf{X}^{\mathrm{min}}_{\mathcal{H}}} {}^p\mathcal{H}^0(J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_\eta,!}(\mathcal{V}_\xi[d])) \tag{5.28}$$

and

$$R\Psi_{\mathsf{X}^{\mathrm{min}}_{\mathcal{H}}} {}^p\mathcal{H}^0(RJ_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_\eta,*}(\mathcal{V}_\xi[d])) \xrightarrow{\sim} {}^p\mathcal{H}^0(RJ_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}},*} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}_\xi[d])) \tag{5.29}$$

which are compatible with the canonical morphisms induced by the canonical morphisms $J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_?,!} \to RJ_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_?,*}$ of functors, for $? = \eta$ and $\bar{s}$. By [27, Cor. 4.5] again, we have a canonical isomorphism ${}^p\mathcal{H}^0 \circ R\Psi \cong R\Psi \circ {}^p\mathcal{H}^0$ of functors, because the functor $R\Psi$ is t-exact for the middle perversity. Therefore, by applying the functor ${}^p\mathcal{H}^0$ to (5.25) and (5.24), we obtain the desired isomorphisms (5.28) and (5.29). $\square$

Remark 5.30. Theorem 5.26 is interesting already in Case (Sm), when there is no level at p and when everything is unramified in the strongest sense. Then we have

$$R\Psi_{\mathsf{X}^{\mathrm{min}}_{\mathcal{H}}} J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_\eta,!*}(\mathcal{V}_\xi[d]) \xrightarrow{\sim} J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}},!*}(\mathcal{V}_\xi[d])$$

over $(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}}$ by the smoothness of $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ (see [3, XV, 2.1] and [13, XIII, 2.1.5]), where we abusively denote by the same symbols the extension of $\mathcal{V}_\xi[d]$ over all of $\mathsf{X}_{\mathcal{H}}$ and its pullback to $(\mathsf{X}_{\mathcal{H}})_{\bar{s}}$. Intuitively (but imprecisely), while the minimal compactification $\mathsf{X}^{\mathrm{min}}_{\mathcal{H}} \to \mathsf{S}$ is not smooth, it has "constant singularities" when moving between the geometric fibers: On one hand, $J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_\eta,!*}(\mathcal{V}_\xi[d])$ and $J_{(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}},!*}(\mathcal{V}_\xi[d])$ take care of the singularities of the fibers of $\mathsf{X}^{\mathrm{min}}_{\mathcal{H}} \to \mathsf{S}$ over η and $\bar{s}$, respectively. On the other hand, $R\Psi_{\mathsf{X}^{\mathrm{min}}_{\mathcal{H}}}$ takes care of the "bad reduction" of $\mathsf{X}^{\mathrm{min}}_{\mathcal{H}} \to \mathsf{S}$.

Corollary 5.31. *We have a canonical isomorphism*

$$R\Gamma((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})_{\bar{\eta}}, J_{(\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})_{\bar{\eta}},!*}(\mathcal{V}_\xi[d])) \xrightarrow{\sim} R\Gamma((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}}, J_{(\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}},!*} R\Psi(\mathcal{V}_\xi[d])), \tag{5.32}$$

which is compatible with the natural continuous $\mathrm{Gal}(\bar{K}/K)$*-actions.*

Proof. Since $\mathsf{X}^{\mathrm{min}}_{\mathcal{H}} \to \mathsf{S}$ is proper, by the proper base change theorem, these follow from the canonical isomorphism (5.27) in the category of perverse sheaves over $(\mathsf{X}^{\mathrm{min}}_{\mathcal{H}})_{\bar{s}}$ with compatible continuous $\mathrm{Gal}(\bar{K}/K)$-actions. $\square$

Remark 5.33. Our strategies thus far are essentially the same ones as in the special case of Siegel moduli at parahoric levels in [66, Sec. 4], which was inspired by [21, Prop. 7.1.1], based on the crucial [21, Lem. 7.1.4] due to Laumon. With our better knowledge today, we can extend them to all cases considered in Assumption 2.1.

Remark 5.34. It should be possible to avoid the use of Propositions 4.1 and 4.2, and hence also the assertions about stratifications in property (9) of Proposition 2.2 and in Corollary 2.4, by requiring instead that the approximations match the automorphic étale sheaves at torsion levels, which still preserves the filtrations induced by the actions of the parabolic subgroups associated with the boundary strata, whose graded pieces descend to C. (Such an idea is perhaps more appealing in Case (Hdg).) We shall leave the details to the interested readers.

Remark 5.35. Let us conclude by briefly explaining why the isomorphisms (5.16), (5.17), and (5.27) are compatible with Hecke actions. While there are many different ways of defining them, the essential setup is as follows: Let $\mathcal{H}$ and $\mathcal{H}'$ be two neat open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}})$, and let $g \in \mathrm{G}(\mathbb{A}^\infty)$ be an element such that $g\mathcal{H}g^{-1} \subset \mathcal{H}'$ and such that there are a proper morphism $[g] : \mathsf{X}_{\mathcal{H}} \to \mathsf{X}_{\mathcal{H}'}$ and a morphism

$$[g]^*\mathcal{V} \to \mathcal{V} \tag{5.36}$$

over $\mathsf{X}_{\mathcal{H}}$, where we abusively denote the analogous sheaf $\mathcal{V}$ over $\mathsf{X}_{\mathcal{H}'}$ by the same symbols. In Cases (Sm) and (Hdg), due to the restrictions on the levels at p, and hence also on the p-part of g, this proper morphism $[g]$ is necessarily finite. In Cases (Nm) and (Spl), it is also finite if $\mathsf{X}_{\mathcal{H}}$ and $\mathsf{X}_{\mathcal{H}'}$ are defined by the same collection J as in [39, Sec. 2], and if g stabilizes J. (Otherwise, we need to introduce different choices of J in these cases, and allow the morphism to be proper in positive characteristics.) By definition, (5.36) induces by adjunction a morphism

$$\mathcal{V} \to R[g]_*\,[g]^*\mathcal{V} \to R[g]_*\mathcal{V} \tag{5.37}$$

over $\mathsf{X}_{\mathcal{H}'}$. Since $[g]$ is proper, this induces a morphism

$$R\Psi_{\mathsf{X}_{\mathcal{H}'}}(\mathcal{V}) \to R\Psi_{\mathsf{X}_{\mathcal{H}'}}\, R[g]_{\eta,*}(\mathcal{V}) \xrightarrow{\sim} R[g]_{\bar{s},*}\, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}) \tag{5.38}$$

over $(\mathsf{X}_{\mathcal{H}'})_{\bar{s}}$. Hence, there are commutative diagrams

$$\begin{tikzcd} R\Gamma((\mathsf{X}_{\mathcal{H}'})_{\bar{\eta}}, \mathcal{V}) \arrow[r] \arrow[d] & R\Gamma((\mathsf{X}_{\mathcal{H}'})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}'}}(\mathcal{V})) \arrow[d] \\ R\Gamma((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}) \arrow[r] & R\Gamma((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})) \end{tikzcd} \tag{5.39}$$

and

$$\begin{tikzcd} R\Gamma_c((\mathsf{X}_{\mathcal{H}'})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}'}}(\mathcal{V})) \arrow[r] \arrow[d] & R\Gamma_c((\mathsf{X}_{\mathcal{H}'})_{\bar{\eta}}, \mathcal{V}) \arrow[d] \\ R\Gamma_c((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})) \arrow[r] & R\Gamma_c((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}) \end{tikzcd} \tag{5.40}$$

in which the vertical morphisms are induced by (5.36) and (5.38), which are compatible with each other under the canonical morphisms. Consequently, the isomorphisms (5.21) and (5.22) in Corollary 5.20 are also compatible with Hecke actions.

Remark 5.41. The compatibility with the actions of Hecke *correspondences* on cohomology requires more explanation. Suppose $\mathcal{H}$ and $\mathcal{H}'$ are as above, satisfying moreover the condition that $\mathcal{H} \to \mathcal{H}'$, so that there is also a proper morphism $[1] : \mathsf{X}_{\mathcal{H}} \to \mathsf{X}_{\mathcal{H}'}$. The two morphisms $[1], [g] : \mathsf{X}_{\mathcal{H}} \to \mathsf{X}_{\mathcal{H}'}$ extend to proper morphisms $[g] : \mathsf{X}_{\mathcal{H}} \to \mathsf{X}_{\mathcal{H}'}$ between integral models of minimal and toroidal compactifications (at the expense of using different J's and, in general, different collections of cone decompositions), by [39, Prop. 13.15], [42, Prop. 7.3], [41, Prop. 2.4.17], and [45, Sec. 4.1.12 and 5.2.12]. (The upshot is that the compatibility between stratifications ensures that the Hecke correspondence realized by integral models at higher level stays proper over the integral models at the original level, both before and after compactifications.) For definitions requiring finite morphisms even in positive characteristics, we have such morphisms between the integral models of Shimura varieties when g stabilizes the same collection J at levels $\mathcal{H}$ and $\mathcal{H}'$, which also extend to finite morphisms between the integral models of minimal compactifications. Consequently, we can define the actions of Hecke correspondences on the cohomology (for both the usual and compactly supported cohomology of the Shimura varieties, and the associated nearby cycles over their integral models) by the general arguments in [20, Lem. 1.3.1] and [18, Sec. 6.1.7], which are compatible with the canonically defined isomorphisms (5.21) and (5.22) in Corollary 5.20.

Remark 5.42. For the special cases of bad reductions which arise only because of higher levels at p above a hyperspecial one, our results subsume the closely related results by Imai and Mieda, matching the *supercuspidal parts* of the two sides of (1.1) (and their analogues), in their current form in [28, Thm. 4.2, and Rem. 4.3 and 4.4]. Nevertheless, their method based on the consideration of adic spaces is quite flexible and of some independent interest. We have learned from them that, still for comparing the supercuspidal parts, their condition can be much relaxed. This is because they can make use of morphisms to a good reduction Siegel moduli rather than to integral models associated with the same Shimura datum.

6. Applications

6.1. **Notation system.** In this section, unless otherwise stated, we shall consider mainly Cases (Sm), (Nm), and (Spl). In these cases, we shall consider the following notation system, which might differ from those in the works we cite (including our own ones). Consider a fixed choice of an integral PEL datum $(\mathcal{O}, \star, L, \langle \cdot, \cdot \rangle, h_0)$ as in [37, Def. 1.4.1.1], which defines $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ in any of the three cases in Assumption 2.1, which are based on the definition of moduli problems of abelian schemes with additional PEL structures *up to isomorphism* as in [37, Sec. 1.4.1].

For many applications in the literature, which are based on moduli problems of abelian schemes with additional structures up to $\mathbb{Z}_{(p)}^\times$-isogeny in [33], it is only the base extension $(\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}, \star, L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}, \langle \cdot, \cdot \rangle \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}, h_0)$ that matters, or even the one with $\mathbb{Z}_{(p)}$ replaced with $\mathbb{Z}_p$. We note that the condition for p to be *good* as in [37, Def. 1.4.1.1], which determines whether we can be in Case (Sm), can be verified using only such base extensions of the integral PEL datum to $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$.

As in [37, (1.2.5.1)], the polarization $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ defines a decomposition $L \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong V_0 \oplus V_0^c$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules, where c denotes the complex conjugation, and where $h_0(z)$ acts on V_0 and V_0^c as $1 \otimes z$ and $1 \otimes z^c$, respectively. This is a Hodge decomposition with V_0 and V_0^c denoting the parts of weights $(-1,0)$ and $(0,-1)$, respectively. This induces a cocharacter $\mu : \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} \mathbb{C} \to \mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ which sends $z \in \mathbb{C}^\times$ to the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-module automorphism of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ acting as z on V_0 and as 1 on V_0^c, whose $(\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C})$-conjugacy class $[\mu]$ is well defined and has a field of definition the subfield F_0 of $\mathbb{C}$. (This does not require V_0 to have a model over F_0.)

Let $\bar{\mathbb{Q}}_p$ denote a fixed choice of an algebraic closure of $\mathbb{Q}_p$, and let $F_0 \to \bar{\mathbb{Q}}_p$ be any fixed choice of an algebra homomorphism, which determines, in particular, a p-adic place v of F_0. By abuse of language, we can also talk about the corresponding $(\mathrm{G} \underset{\mathbb{Z}}{\otimes} \bar{\mathbb{Q}}_p)$-conjugacy class $[\mu]$ of cocharacters $\mu : \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} \bar{\mathbb{Q}}_p \to \mathrm{G} \underset{\mathbb{Z}}{\otimes} \bar{\mathbb{Q}}_p$. The pair $(\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p, [\mu])$ can be viewed as a *local Shimura datum* at p—indeed, this is the viewpoint taken in many works on local models.

Consider $\mathcal{O}_{F_0,v}$, the v-adic completion of the ring $\mathcal{O}_{F_0}$ of integers in the reflex field F_0. Let $K := \mathrm{Frac}(\mathcal{O}_{F_0,v})$, let $\bar{K} := \bar{\mathbb{Q}}_p$, and let K^{ur} denote the maximal unramified extension of K in $\bar{K}$. Let $\Gamma_K := \mathrm{Gal}(\bar{K}/K)$ and $\Gamma_{K^{\mathrm{ur}}} := \mathrm{Gal}(\bar{K}/K^{\mathrm{ur}})$. Then we also have the inertia group $I_K := \ker(\Gamma_K \to \Gamma_{K^{\mathrm{ur}}})$ and the Weil group W_K in Γ_K. We shall also denote $\mathcal{O}_{F_0,v}$ by $\mathcal{O}_K$ (for simplicity); the ring of integers in K^{ur} by $\mathcal{O}_{K^{\mathrm{ur}}}$; and the residue fields of $\mathcal{O}_K$ and $\mathcal{O}_{K^{\mathrm{ur}}}$ by k and $\bar{k}$, respectively.

Let F denote the center of the semisimple algebra $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, which is a finite product of totally real or CM fields. Suppose that $\widetilde{K}$ is the smallest (finite) extension of K in $\bar{K}$ which contains all the images $\tau(F)$, where $\tau : F \to \bar{K}$ runs through all possible algebra homomorphisms, and over which $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ splits. (Then this is an acceptable choice of K in [41, Sec. 2.3]. Please do not confuse the notation K there with the K here.) Then we similarly define $\Gamma_{\widetilde{K}}$, $I_{\widetilde{K}}$, and $W_{\widetilde{K}}$. We denote the ring of integers in $\widetilde{K}$ by $\mathcal{O}_{\widetilde{K}}$, and denote its residue field by $\widetilde{k}$.

Throughout this section, we fix the choice of a prime $\ell > 0$ different from p, and assume that the requirements in Section 3.1 are satisfied by our choices of $\mathcal{H}$. Unless otherwise specified, $\mathcal{V}$ will denote an étale sheaf of the form $\mathcal{V}_\xi$ or $\mathcal{W}_{0,M}$ as in the beginning of Section 5.2. Since $\ell \neq p$, we may and we shall assume that $\mathcal{V}$ is not just defined over $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, but also over all of $\mathsf{X}_{\mathcal{H}}$ in mixed characteristics $(0,p)$. For simplicity, we shall often denote the pullbacks of $\mathcal{V}$ by the same symbol.

6.2. **Unipotency of inertial actions.** Our first application is unsurprising and considered known in the folklore, but has not been documented in the literature:

Theorem 6.1. *Suppose we are in Case* (Sm), *with* $\mathsf{S} = \mathrm{Spec}(\mathcal{O}_K) = \mathrm{Spec}(\mathcal{O}_{F_0,v})$. *Suppose $\mathcal{V}$ is any étale sheaf over* $\mathsf{X} \to \mathsf{S}$ *as above. For each i, we have the following canonical isomorphisms of Γ_K-modules: for the usual cohomology,*

$$H^i_{\text{ét}}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}) \xrightarrow{\sim} H^i_{\text{ét}}((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, \mathcal{V}); \tag{6.2}$$

for the compactly supported cohomology,

$$H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, \mathcal{V}) \xrightarrow{\sim} H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}); \tag{6.3}$$

and, when $\mathcal{V}$ is of the form $\mathcal{V}_\xi$, for the intersection cohomology

$$H^i_{\text{ét}}((\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{\eta}}, (J_{(\mathsf{M}^{\min}_{\mathcal{H}})_{\bar{\eta}},!*}(\mathcal{V}_\xi[d]))[-d]) \xrightarrow{\sim} H^i_{\text{ét}}((\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{s}}, (J_{(\mathsf{M}^{\min}_{\mathcal{H}})_{\bar{s}},!*}(\mathcal{V}_\xi[d]))[-d]) \tag{6.4}$$

(of the minimal compactification), where $d = \dim((\mathsf{X}_{\mathcal{H}})_\eta)$*. In particular, these* Γ_K*-modules are* **unramified** *(namely,* I_K *acts trivially on them). If* m_ξ *is the integer such that* $\mathcal{V}_\xi$ *is pointwise pure of weight* m_ξ *as in Proposition* 3.2*, then both sides of* (6.2) *(resp.*(6.3)*) are mixed of weights* $\geq i + m_\xi$ *(resp.* $\leq i + m_\xi$*), and both sides of* (6.4) *are pure of weight* $i + m_\xi$*.*

Proof. With the choices $\bar{\eta} = \mathrm{Spec}(\bar{K})$ and $\bar{s} = \mathrm{Spec}(\bar{k})$ of geometric points above the generic and special points $\eta = \mathrm{Spec}(K)$ and $s = \mathrm{Spec}(k)$ of $\mathsf{S} = \mathrm{Spec}(\mathcal{O}_K)$, these follow from Corollary 5.20, from Remark 5.30 and Corollary 5.31, and from [11, Cor. 3.3.4 and 3.3.5, and Prop. 6.2.6] and [5, Cor. 5.3.2]. $\square$

Corollary 6.5. *The analogous cohomology groups for* $\mathsf{M}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \bar{F}_0$*, where* $\bar{F}_0$ *is an algebraic closure of* F_0*, are unramified as representations of* $\mathrm{Gal}(\bar{F}_0/F_0)$ *at the places above* $\Box$*, and at any places above a rational prime* $q \neq \ell$ *such that* q *is good as in* [37, Def. 1.4.1.1] *and such that* $\mathcal{H}$ *is maximal at* q *in the sense that* $\mathcal{H} = \mathcal{H}^q \mathcal{H}_q$ *for some open compact subgroup* $\mathcal{H}^q \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box \cup \{q\}})$ *and* $\mathcal{H}_q = \mathrm{G}(\mathbb{Z}_q)$*. (The étale sheaves and their cohomology groups are also defined over the localizations of* $\mathrm{Spec}(\mathcal{O}_{F_0,(q)})$*, for all* q *as above, and they are compatible with each others over* $\mathrm{Spec}(F_0)$*.)*

Proof. For each q as above, by [37, Lem. 1.4.4.2], we may and we shall replace $\Box$ with $\Box \cup \{q\}$, and assume that $p = q \in \Box$. Hence, the corollary follows from Theorem 6.1 because, by [10, Arcata, V, Cor. 3.3], any base change from $\bar{F}_0$ to $\bar{K}$ induces an isomorphism between the étale cohomology groups (which are equivariant with respect to the induced homomorphism $\mathrm{Gal}(\bar{F}_0/F_0) \to \Gamma_K = \mathrm{Gal}(\bar{K}/K)$). $\square$

Remark 6.6*.* In the special case of Siegel moduli in [17], this is recorded in [17, Ch. VI, Sec. 6, Prop. 6.1]. Indeed, our use of Kuga families and their good toroidal compactifications is based on the idea outlined there.

By [30, Thm. 2.3.8], and by the same argument of the proof of Theorem 6.1:

Theorem 6.7. *Suppose we are in Case* (Hdg)*, with* $K := \mathcal{O}_{F_0,v}$*, the* v*-adic completion of* $\mathcal{O}_{F_0}$ *at a place* $v|p$*, and with* $\mathcal{H} = \mathcal{H}^p \mathcal{H}_p$*, where* $\mathcal{H}_p$ *is a hyperspecial maximal open compact subgroup of* $\mathrm{G}(\mathbb{Q}_p)$*. If* $p = 2$*, suppose moreover that* [30, (2.3.4)] *holds. Then the analogues of the assertions of Theorem* 6.1 *also hold here.*

Next, let us turn to the more interesting examples of integral models with parahoric levels at p, or more particularly those with local models considered by [50] and [51] that are known to agree with the normalizations of the naive integral models in [56] (or more precisely the normalizations of the images of the generic fibers in the naive integral models). There are two stages of such a theory, both important for our purpose: one first constructs certain nice "local models", which can be defined by certain linear algebraic data, and shows that the nearby cycles over such "local models" have certain good behavior; then one shows that these nice "local models" are indeed local models for some integral models of Shimura varieties, in the sense that the latter is up to smooth morphisms isomorphic to the former. (However, while there is a rich literature in the former state, the corresponding latter stage has not always been carried out.) We shall record several instances where we have useful information for both stages, which are covered by Cases (Nm) and (Spl) in Assumption 2.1 (so that our results apply).

Following [51], assume for simplicity that $p > 2$ and that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is a maximal order in $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ (stable under $\star$). Suppose $\mathscr{L}$ is a (periodic and self-dual) *multichain* of $(\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p)$-lattices in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$, as in [56, Def. 3.4] and [41, Sec. 2.1].

On one hand, as explained in [56, 3.2] and [41, Choices 2.2.9], there exists a finite subset $\mathscr{L}_J = \{\Lambda_j\}_{j \in J}$ of $\mathscr{L}$ such that an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-lattice Λ in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ belongs to $\mathscr{L}$ if and only if there exist some $r \in \mathbb{Z}$ and $j \in J$ such that $\Lambda = p^r \Lambda_j$, and there exists a collection $\{(1, L_j, \langle\,\cdot\,,\,\cdot\,\rangle_j)\}_{j \in J}$ (with the same index set) for the consideration in [39, Sec. 2] such that $\Lambda_j = L_j \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$, such that $L_j \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p$, and such that $L_{j_0} = p^{r_0} L$ for some $j_0 \in J$. Let $\mathcal{H}$ be any open compact subgroup of $\mathrm{G}(\mathbb{A}^\infty)$ such that its image $\mathcal{H}^p$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}^p)$ is a *neat* (see [37, Def. 1.4.1.8]) open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$, in which case $\mathcal{H}$ is also neat, and such that the image $\mathcal{H}_p$ of $\mathcal{H}$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\mathbb{Z}_p)$ is the connected stabilizer (i.e., the identity component of the stabilizer) of the multichain $\mathscr{L}$ (cf. [41, Def. 2.1.10 and Choices 2.2.10]) (which is then a parahoric subgroup of $\mathrm{G}(\mathbb{Q}_p)$), so that the collection $\{(1, L_j, \langle\,\cdot\,,\,\cdot\,\rangle_j)\}_{j \in J}$ defines a flat integral model $\vec{\mathsf{M}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ as in [39, Prop. 6.1].

On the other hand, as in [51, Sec. 8.2.4], suppose that $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ is connected and splits over a tamely ramified extension of $\mathbb{Q}_p$. (This is the case, for example, when $\widetilde{K}$ is tamely ramified over $\mathbb{Q}_p$. Since p is odd, it also satisfies the condition $p \nmid \pi_1(\mathrm{G}(\mathbb{Q}_p)_{\mathrm{der}})$ there.) Then the local Shimura datum $(\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p, [\mu])$ (up to a sign convention) and the parahoric subgroup $\mathcal{H}_p$ of $\mathrm{G}(\mathbb{Q}_p)$ define a local model M^{loc}, which is normal (by [51, Thm. 1.1; see also the explanations in Rem. 8.2 and 8.3]). Therefore, by [41, Prop. 2.2.11 and Rem. 2.4.13], the pullback $\mathsf{X}_{\mathcal{H}} \to \mathsf{S} = \mathrm{Spec}(\mathcal{O}_K) = \mathrm{Spec}(\mathcal{O}_{F_0,v})$ of the above $\vec{\mathsf{M}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, which fits into Case (Nm) of Assumption 2.1, is isomorphic to the flat integral model in [51, Thm. 1.2; see also the explanation in Sec. 8.2.5] (with the $\mathcal{O}_K = \mathcal{O}_{F_0,v}$ here denoted $\mathcal{O}_{\mathbf{E}_{\mathfrak{P}}}$ there).

Theorem 6.8. *Let $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ be defined as above, in Case* (Nm)*. For each i, we have the following canonical isomorphisms of Γ_K-modules: for the usual cohomology,*

$$H^i_{\text{ét}}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}) \xrightarrow{\sim} H^i_{\text{ét}}((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})); \tag{6.9}$$

for the compactly supported cohomology,

$$H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})) \xrightarrow{\sim} H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}); \tag{6.10}$$

and for the intersection cohomology

$$H^i_{\text{ét}}((\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{\eta}}, (J_{(\mathsf{M}^{\min}_{\mathcal{H}})_{\bar{\eta}},!*}(\mathcal{V}_\xi[d]))[-d]) \xrightarrow{\sim} H^i_{\text{ét}}((\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{s}}, (J_{(\mathsf{M}^{\min}_{\mathcal{H}})_{\bar{s}},!*} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}_\xi[d]))[-d]) \tag{6.11}$$

(*of the minimal compactification*)*, where $d = \dim((\mathsf{X}_{\mathcal{H}})_\eta)$. Moreover, the restrictions of the Γ_K-actions of these modules to $I_{\widetilde{K}}$* (*but not I_K*) *are all* **unipotent***, and even* **trivial** *when $\mathcal{H}_p$ is a* **very special** *subgroup of* $\mathrm{G}(\mathbb{Q}_p)$ (*see* [51, Sec. 10.3.2]).

Proof. With the choices $\bar{\eta} = \mathrm{Spec}(\bar{K})$ and $\bar{s} = \mathrm{Spec}(\bar{k})$ of geometric points above the generic point $\eta = \mathrm{Spec}(K)$ and special point $s = \mathrm{Spec}(k)$ of $\mathsf{S} = \mathrm{Spec}(\mathcal{O}_K)$, these follow from Corollaries 5.20 and 5.31, from the fact that the M^{loc} above is a local model for $\mathsf{X}_{\mathcal{H}}$ (which means the latter is up to smooth morphisms isomorphic to the former), and from [51, Thm. 1.4, and more detailed results in Sec. 10.3]. $\square$

Remark 6.12. The assumption that $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ splits over a tamely ramified extension of $\mathbb{Q}_p$ can be relaxed, at least when $p \geq 5$, thanks to [44], as soon as the local models defined in [44] is shown to be provide local models for any $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ in Case (Nm) in Assumption 2.1. This possibility is now known in the folklore, although we did not spell it out only because such a link was not explicitly provided in [44].

More generally, we can still define a flat integral model $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{S}}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ as in [39, Sec. 6] and [41, Choices 2.2.10], with the assumptions that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is a maximal order in $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ (stable under $\star$) and that the image $\mathcal{H}_p$ of $\mathcal{H}$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\mathbb{Z}_p)$ is the connected stabilizer of $\mathscr{L}$, but without the assumptions that $p > 2$ and that $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ is connected and splits over a tamely ramified extension of $\mathbb{Q}_p$. By making the choices as in [41, Choices 2.3.1], we can define the corresponding splitting models $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_{\widetilde{K}})$ as in [41, Sec. 2.4] (with the K there given by the $\widetilde{K}$ here), which coincides with the normalizations of the integral models introduced in [50, Sec. 15]. Then we take $\mathsf{X}_{\mathcal{H}} \to \mathsf{S} = \mathrm{Spec}(\mathcal{O}_{\widetilde{K}})$ to be this $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_{\widetilde{K}})$, which fits into Case (Spl) of Assumption 2.1.

Theorem 6.13. *With the choice of $\mathsf{X}_{\mathcal{H}} \to \mathsf{S} = \mathrm{Spec}(\mathcal{O}_{\widetilde{K}})$ as above in Case* (Spl)*, the isomorphisms in Theorem* 6.8*, which are now for $\Gamma_{\widetilde{K}}$-modules instead of Γ_K-modules, also exist. Moreover, the analogous assertions concerning the unipotency and triviality of the restrictions of the $\Gamma_{\widetilde{K}}$-actions of these modules to $I_{\widetilde{K}}$ are valid if our context fits into one of the following cases:*

(1) $\vec{\mathsf{M}}_{\mathcal{H}} \to \mathsf{S}$ *is defined as in the paragraphs preceding Theorem* 6.8*.*
(2) *The context of* [50, Part I]*, where $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ is* (*up to center*) *of the form $\mathrm{Res}_{K'/\mathbb{Q}_p} \mathrm{GL}_d$ for some finite extension K' of $\mathbb{Q}_p$.*
(3) *The context of* [50, Part II]*, where $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ is* (*up to center*) *of the form $\mathrm{Res}_{K'/\mathbb{Q}_p} \mathrm{GSp}_{2g}$ for some finite extension K' of $\mathbb{Q}_p$.*

Proof. With the choices $\bar{\eta} = \mathrm{Spec}(\bar{K})$ and $\bar{s} = \mathrm{Spec}(\bar{k})$ of geometric points above the generic point $\eta = \mathrm{Spec}(\widetilde{K})$ and special point $s = \mathrm{Spec}(\widetilde{k})$ of $\mathsf{S} = \mathrm{Spec}(\mathcal{O}_{\widetilde{K}})$, the isomorphisms as in Theorem 6.8, which are now for $\Gamma_{\widetilde{K}}$-modules instead of Γ_K-modules, follow from Corollaries 5.20 and 5.31. As for the restrictions of the $\Gamma_{\widetilde{K}}$-actions of these modules to $I_{\widetilde{K}}$, since the left-hand sides of the isomorphisms as in Theorem 6.8 have the same restrictions to $I_{\widetilde{K}}$ as those of the original ones in Theorem 6.8, the case (1) is nothing but a repetition of Theorem 6.8; the case (2) follows from [50, Thm. 1.31 and Rem. 13.2(a)] (see also [49, Rem. 7.4]); and the case (3) follows from [50, Rem. 13.2(b)]. (In cases (2) and (3), the splitting models were involved in the proofs in [50], but not explicitly mentioned in the conclusions.) $\square$

Remark 6.14. The two cases (2) and (3) in Theorem 6.13 cover, for example, the two cases spelled out in [39, Lem. 14.6 and 14.7].

Remark 6.15. The isomorphism (6.10) in Theorem 6.8 and its analogue in Theorem 6.13 established [23, Conjecture 10.3] for all integral models of PEL-type Shimura varieties (with parahoric levels at p) considered in [51] and [50].

Remark 6.16. As soon as we have analogues of the results we cited from [51] and [50] in Case (Hdg), and also the constructions of their splitting models (and their toroidal and minimal compactifications) in that context, the analogues of Theorem 6.8 and 6.13 can be proved by exactly the same arguments.

6.3. **Mantovan's formula.** Let us follow the setting in [46] and [47]. Assume that p is a good prime for an integral PEL datum $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ as in [37, Def. 1.4.1.1], and consider the trivial collection $J = \{j_0\}$ with $\{(g_{j_0}, L_{j_0}, \langle\,\cdot\,,\,\cdot\,\rangle_{j_0}\} = \{(1, L, \langle\,\cdot\,,\,\cdot\,\rangle\}$, as in [39, Ex. 2.3]. For simplicity, assume that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is simple and involves no factor of type D, in the sense of [37, Def. 1.2.1.15].

Let $\mathcal{H}$ be any neat open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}})$, let $\mathcal{H}^p$ denote the image of $\mathcal{H}$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}^p)$, and let $\mathcal{H}_0 := \mathcal{H}^p\mathrm{G}(\mathbb{Z}_p)$. Since p is a good prime for $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$, we have a good reduction integral model $\mathsf{M}_{\mathcal{H}^p} \to \operatorname{Spec}(\mathcal{O}_{F_0,(p)})$ as in [37, Sec. 1.4.1]. Since $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is simple, by [37, Prop. 1.4.4.3] and [33, Sec. 8], the canonical morphism $\mathsf{M}_{\mathcal{H}_0} \to \mathsf{M}_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is an isomorphism. Since the schemes $\vec{\mathsf{M}}_{\mathcal{H}_0}$ and $\vec{\mathsf{M}}_{\mathcal{H}}$ over $\vec{\mathsf{S}}_0 = \operatorname{Spec}(\mathcal{O}_{F_0,(p)})$ in [39, Prop. 6.1] are independent of the choices of auxiliary models, by taking $\mathsf{M}_{\mathcal{H}^p}$ as an auxiliary good reduction model, we have $\vec{\mathsf{M}}_{\mathcal{H}_0} \cong \mathsf{M}_{\mathcal{H}^p}$, and we can take $\vec{\mathsf{M}}_{\mathcal{H}}$ to be the normalization of $\vec{\mathsf{M}}_{\mathcal{H}_0}$ under the composition $\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}_0} \to \vec{\mathsf{M}}_{\mathcal{H}_0} \cong \mathsf{M}_{\mathcal{H}^p}$ of canonical morphisms.

Let us take S to be $\operatorname{Spec}(\mathcal{O}_K) = \operatorname{Spec}(\mathcal{O}_{F_0,v})$ as in Section 6.1, and take $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ to be the pullback of $\vec{\mathsf{M}}_{\mathcal{H}} \to \operatorname{Spec}(\mathcal{O}_{F_0,(p)})$, as in Case (Nm) in Assumption 2.1. In this case, the p is unramified in the linear algebraic data, except that the level at p might be higher than $\mathrm{G}(\mathbb{Z}_p)$.

Given the local datum $(\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p, [\mu])$, there is a partially ordered finite subset $\mathcal{B}$ of $B(\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p)$ (see [31], [55], and [34, Sec. 6]), whose elements parameterize, roughly speaking, quasi-isogeny classes of Barsotti–Tate groups over $\bar{k}$ with quasi-polarized and endomorphism structures of the type defined by $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ and by the conjugacy class $[\mu]$ determined by h_0. By classifying the quasi-isogeny classes of Barsotti–Tate groups associated with the pullbacks of the tautological object $(A, \lambda, i, \alpha_{\mathcal{H}^p}) \to \mathsf{M}_{\mathcal{H}^p}$ to the geometric points of $(\mathsf{M}_{\mathcal{H}^p})_s$, we can write $(\mathsf{X}_{\mathcal{H}_0})_s \cong (\mathsf{M}_{\mathcal{H}^p})_s$ as a (set-theoretic) disjoint union of locally closed subschemes $(\mathsf{X}_{\mathcal{H}_0})_s^b$, labeled by elements $b \in \mathcal{B}$. (This is often called the *Newton stratification*, because $\underset{b' \leq b}{\cup} (\mathsf{X}_{\mathcal{H}_0})_s^{b'}$ is closed in $(\mathsf{X}_{\mathcal{H}_0})_s$, for each $b \in \mathcal{B}$. However, in general, the closure of $(\mathsf{X}_{\mathcal{H}_0})_s^b$ might be strictly smaller than this union. To avoid confusion, we shall avoid the terminology of stratifications in the remainder of this subsection.) Then $(\mathsf{X}_{\mathcal{H}})_s$ is the (set-theoretic) disjoint union of the (locally closed) preimages $(\mathsf{X}_{\mathcal{H}})_s^b$ of $(\mathsf{X}_{\mathcal{H}_0})_s^b$ under the canonical morphism $(\mathsf{X}_{\mathcal{H}})_s \to (\mathsf{X}_{\mathcal{H}_0})_s$, and $\underset{b' \leq b}{\cup} (\mathsf{X}_{\mathcal{H}})_s^{b'}$ is closed for each $b \in \mathcal{B}$. As $\mathcal{H}$ varies among neat open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}})$, such stratifications are respected by the Hecke actions of $\mathrm{G}(\mathbb{A}^{\infty,p}) \times \mathrm{G}(\mathbb{Q}_p)^+$, where $\mathrm{G}(\mathbb{Q}_p)^+$ is the sub-monoid of $\mathrm{G}(\mathbb{Q}_p)$ generated by $\mathrm{G}(\mathbb{Z}_p)$ and the scalar multiplication by p. We shall replace s with $\bar{s}$ when denoting their pullbacks to $\bar{s}$.

For each $b \in \mathcal{B}$, there is a formal scheme $\mathcal{M}_0^b$ over $\operatorname{Spf}(\mathcal{O}_{K^{\mathrm{ur}}})$ as in [56, Def. 3.21, Cor. 3.40, and onwards], which carries an action of a group $J_b(\mathbb{Q}_p)$, where J_b is an algebraic group over $\mathbb{Q}_p$ associated with b. Let $\mathcal{M}_0^{b,\mathrm{rig}}$ denote the rigid analytic generic fiber of $\mathcal{M}_0^b$ over K^{ur}. As in [56, 5.32 and onwards], one can also define coverings $\mathcal{M}_{\mathcal{H}_p}^{b,\mathrm{rig}}$ of $\mathcal{M}_0^{b,\mathrm{rig}}$ parameterized by open compact subgroups $\mathcal{H}_p \subset \mathrm{G}(\mathbb{Z}_p)$. (These are the *Rapoport–Zink spaces.*) Consider the étale cohomology of each $\mathcal{M}_{\mathcal{H}_p}^{b,\mathrm{rig}}$ (following, for example, [6]), and consider the limit

$$H^i_{\text{ét}}(\mathcal{M}^{b,\mathrm{rig}}, \bar{\mathbb{Q}}_\ell) := \varinjlim_{\mathcal{H}_p \subset \mathrm{G}(\mathbb{Z}_p)} H^i_{\text{ét},c}(\mathcal{M}_{\mathcal{H}_p}^{b,\mathrm{rig}} \underset{K^{\mathrm{ur}}}{\otimes} \bar{K}, \bar{\mathbb{Q}}_\ell) \tag{6.17}$$

for each i, where the notation of $\mathcal{M}^{b,\mathrm{rig}}$ is only symbolic, which is a smooth/smooth/continous representation of $\mathrm{G}(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_K$. As explained in [47, Sec. 2.4.2], there is a well-defined functor (called the *Mantovan functor*)

$$\mathcal{E}_b : \mathrm{Groth}(J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(\mathrm{G}(\mathbb{Q}_p) \times W_K) \tag{6.18}$$

from the Grothendieck group of admissible representations of $J_b(\mathbb{Q}_p)$ to the Grothendieck group of admissible/continuous representations of $\mathrm{G}(\mathbb{Q}_p) \times W_K$, which assigns to each admissible virtual representation ρ of $J_b(\mathbb{Q}_p)$ the admissible virtual representation

$$\mathcal{E}_b(\rho) := \sum_{i,j} (-1)^{i+j} [\mathrm{Ext}^i_{J_b(\mathbb{Q}_p)\text{-smooth}}(H^i_{\text{ét},c}(\mathcal{M}^{b,\mathrm{rig}}, \bar{\mathbb{Q}}_\ell), \rho)(-\dim(\mathcal{M}^{b,\mathrm{rig}}))] \tag{6.19}$$

of $\mathrm{G}(\mathbb{Q}_p) \times W_K$, where $(-\dim(\mathcal{M}^{b,\mathrm{rig}}))$ denotes the Tate twist. This extends to a functor

$$\mathcal{E}_b : \mathrm{Groth}(\mathrm{G}(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(\mathrm{G}(\mathbb{A}^\infty) \times W_K) \tag{6.20}$$

by combining the above functor with the identity functor on $\mathrm{Groth}(\mathrm{G}(\mathbb{A}^{\infty,p}))$, the Grothendieck group of admissible representations of $\mathrm{G}(\mathbb{A}^{\infty,p})$.

For each $b \in \mathcal{B}$ such that $(\mathsf{X}_{\mathcal{H}_0})^b_{\bar{s}}$ is nonempty, and for any fixed choice of a closed point x of $(\mathsf{X}_{\mathcal{H}_0})^b_{\bar{s}}$, there is a *central leaf* $\mathsf{C}^b_{\mathcal{H}^p}$ passing through x, which is a reduced closed subscheme of $(\mathsf{X}_{\mathcal{H}_0})^b_{\bar{s}}$, which contains x and is smooth over $\bar{s}$. There is a tower of schemes $\mathsf{Ig}^b_{\mathcal{H}^p,m} \to \mathsf{C}^b_{\mathcal{H}^p}$ (the *Igusa varieties*) parameterized by integers $m \geq 1$. (See [46, Sec. 4] and [47, Sec. 2.5.2–2.5.4] for more details.)

For each irreducible algebraic representation ξ of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ on a finite-dimensional vector space V_ξ over $\bar{\mathbb{Q}}_\ell$, which defines an étale sheaf $\mathcal{V}_\xi$ over $\mathsf{X}_{\mathcal{H}_0}$ (because $\ell \neq p$) as in Proposition 3.2, consider the limits

$$H^i_{\text{ét},c}(\mathsf{X}_{\bar{\eta}}, \mathcal{V}_\xi) := \varinjlim_{\mathcal{H}} H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}_\xi) \tag{6.21}$$

and

$$H^i_{\text{ét},c}(\mathsf{X}_{\bar{s}}, R\Psi_{\mathsf{X}}(\mathcal{V}_\xi)) := \varinjlim_{\mathcal{H}} H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}_\xi)) \tag{6.22}$$

for each i, where the notations of $\mathsf{X}_{\bar{\eta}}$, $\mathsf{X}_{\bar{s}}$, and $R\Psi_{\mathsf{X}}$ are only symbolic, and where the limits are over neat open compact subgroups $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}})$, which are admissible/continuous representations of $\mathrm{G}(\mathbb{A}^{\infty,p}) \times \mathrm{G}(\mathbb{Q}_p)^+ \times W_K$.

For each $b \in \mathcal{B}$, consider the limit

$$H^i_{\text{ét},c}(\mathsf{X}^b_{\bar{s}}, (R\Psi_{\mathsf{X}}(\mathcal{V}_\xi))|_{\mathsf{X}^b_{\bar{s}}}) := \varinjlim_{\mathcal{H}} H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})^b_{\bar{s}}, (R_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}_\xi))|_{(\mathsf{X}_{\mathcal{H}})^b_{\bar{s}}}) \tag{6.23}$$

for each i, where the notations of $\mathsf{X}_{\bar{s}}^{b}$ and $(R\Psi_{\mathsf{X}}\mathcal{V}_{\xi})|_{\mathsf{X}_{\bar{s}}^{b}}$ are only symbolic, and where the limits are over neat open compact subgroups $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}})$, which is an admissible/continuous representation of $\mathrm{G}(\mathbb{A}^{\infty,p}) \times \mathrm{G}(\mathbb{Q}_p)^{+} \times W_K$. Since $(\mathsf{X}_{\mathcal{H}})_s$ is the disjoint union of the locally closed subschemes $(\mathsf{X}_{\mathcal{H}})_s^{b}$ as b runs through all elements in $\mathcal{B}$, and since $\underset{b' \leq b}{\cup} (\mathsf{X}_{\mathcal{H}})_s^{b'}$ is closed for each $b \in \mathcal{B}$, we have the equality

$$\sum_i (-1)^i \, [H^i_{\text{ét},c}(\mathsf{X}_{\bar{s}}, R\Psi_{\mathsf{X}}(\mathcal{V}_{\xi}))] = \sum_{b \in \mathcal{B}} \sum_i (-1)^i \, [H^i_{\text{ét},c}(\mathsf{X}_{\bar{s}}^{b}, (R\Psi_{\mathsf{X}}(\mathcal{V}_{\xi}))|_{\mathsf{X}_{\bar{s}}^{b}})] \tag{6.24}$$

between virtual representations of $\mathrm{G}(\mathbb{A}^{\infty,p}) \times \mathrm{G}(\mathbb{Q}_p)^{+} \times W_K$.

For each $b \in \mathcal{B}$, let us denote by the same symbols the pullbacks of $\mathcal{V}_{\xi}$ to $\mathsf{Ig}_{b,x,m}$ under the composition $\mathsf{Ig}^{b}_{\mathcal{H}^p,m} \to \mathsf{C}^{b}_{\mathcal{H}^p} \to (\mathsf{X}_{\mathcal{H}_0})_{\bar{s}} \to \mathsf{X}_{\mathcal{H}_0}$ of canonical morphisms, for each $m \geq 1$. Consider the limit

$$H^i_{\text{ét},c}(\mathsf{Ig}^{b}, \mathcal{V}_{\xi}) := \varinjlim_{\mathcal{H}^p,m} \; H^i_{\text{ét},c}(\mathsf{Ig}^{b}_{\mathcal{H}^p,m}, \mathcal{V}_{\xi}), \tag{6.25}$$

for each i, where the notation of Ig^{b} is only symbolic, and where the limit is over neat open compact subgroups $\mathcal{H}^p$ of $\mathrm{G}(\hat{\mathbb{Z}}^p)$ and integers $m \geq 1$, which is an admissible representation of $\mathrm{G}(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ (see [46, Sec. 4, Prop. 7] and [47, Sec. 2.5.5]).

We can finally state our reformulation of Mantovan's formula:

Theorem 6.26. *With the setting as above, for each $b \in \mathcal{B}$, we have an equality*

$$\sum_i (-1)^i \, [H^i_{\text{ét},c}(\mathsf{X}_{\bar{s}}^{b}, (R\Psi_{\mathsf{X}}(\mathcal{V}_{\xi}))|_{\mathsf{X}_{\bar{s}}^{b}})] = \sum_j (-1)^j \, \mathcal{E}_b([H^j_{\text{ét},c}(\mathsf{Ig}^{b}, \mathcal{V}_{\xi})]) \tag{6.27}$$

between virtual representations of $\mathrm{G}(\mathbb{A}^{\infty,p}) \times \mathrm{G}(\mathbb{Q}_p)^{+} \times W_K$.

Remark 6.28. Theorem 6.26 is not exactly what Mantovan proved. In [46] and [47], the integral models at levels $\mathcal{H}$ higher than $\mathcal{H}_0$, which we shall denote by $\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}}$, are defined by introducing Drinfeld level structures at p, which is generally different from our definition in Case (Nm) by taking normalizations as in [39, Prop. 6.1].

Proof of Theorem 6.26. For each $\mathcal{H}$, let us denote by $\pi^{\mathrm{Dr}}_{\mathcal{H}} : \mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}} \to \mathsf{X}_{\mathcal{H}_0}$ the forgetful morphism, which is finite by [46, Sec. 6, Prop. 15]. For each $b \in \mathcal{B}$, let us denote by $(\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}})^{b}_{s}$ the preimage of $(\mathsf{X}_{\mathcal{H}_0})^{b}_{s}$ in $(\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}})_s$. We shall replace s with $\bar{s}$ when denoting their pullbacks to $\bar{s}$. By the proper base change theorem (see [3, XII, 5.1]), the adjunction morphism

$$R\Psi_{\mathsf{X}_{\mathcal{H}_0}} \, R(\pi^{\mathrm{Dr}}_{\mathcal{H}})_{\eta,*}(\mathcal{V}_{\xi}) \to R(\pi^{\mathrm{Dr}}_{\mathcal{H}})_{\bar{s},*} \, R\Psi_{\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}}}(\mathcal{V}_{\xi}) \tag{6.29}$$

is an isomorphism. On the other hand, let us denote by $\pi_{\mathcal{H}} : \mathsf{X}_{\mathcal{H}} \to \mathsf{X}_{\mathcal{H}_0}$ the canonical finite morphism. Then, similarly, the adjunction morphism

$$R\Psi_{\mathsf{X}_{\mathcal{H}_0}} \, R(\pi_{\mathcal{H}})_{\eta,*}(\mathcal{V}_{\xi}) \to R(\pi_{\mathcal{H}})_{\bar{s},*} \, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}_{\xi}) \tag{6.30}$$

is also an isomorphism. Since $(\pi^{\mathrm{Dr}}_{\mathcal{H}})_{\eta}$ can be identified with $(\pi_{\mathcal{H}})_{\eta}$ under the canonical identifications $(\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}})_{\eta} \cong \mathsf{M}_{\mathcal{H}} \cong (\mathsf{X}_{\mathcal{H}})_{\eta}$, the left-hand sides of (6.29) and (6.30) are canonically isomorphic to each other, and we have a canonical isomorphism

$$R(\pi^{\mathrm{Dr}}_{\mathcal{H}})_{\bar{s},*} \, R\Psi_{\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}}}(\mathcal{V}_{\xi}) \cong R(\pi_{\mathcal{H}})_{\bar{s},*} \, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}_{\xi}).$$

By the proper base change theorem again, this induces a canonical isomorphism

$$H^i_{\text{ét},c}((\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}})^{b}_{\bar{s}}, (R\Psi_{\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}}}(\mathcal{V}_{\xi}))|_{(\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}})^{b}_{\bar{s}}}) \cong H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})^{b}_{\bar{s}}, (R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}_{\xi}))|_{(\mathsf{X}_{\mathcal{H}})^{b}_{\bar{s}}}) \tag{6.31}$$

for each $b \in \mathcal{B}$ and each i, by taking global sections over $(\mathsf{X}_{\mathcal{H}_0})^b_{\bar{s}}$. By the same explanation as in Remark 5.35, as $\mathcal{H}$ varies among neat open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}})$, the isomorphisms given by (6.31) are compatible with the Hecke actions of $\mathrm{G}(\mathbb{A}^{\infty,p}) \times \mathrm{G}(\mathbb{Q}_p)^+$. Therefore, for each $b \in \mathcal{B}$ and each i, the limit of $H^i_{\text{ét},c}((\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}})^b_{\bar{s}}, (R\Psi_{\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}}}(\mathcal{V}_\xi))|_{(\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}})^b_{\bar{s}}})$ is canonically isomorphic to (6.23), as representations of $\mathrm{G}(\mathbb{A}^{\infty,p}) \times \mathrm{G}(\mathbb{Q}_p)^+ \times W_K$. Thus, the first identity in [47, Thm. 3.1] implies the identity (6.27), as desired. □

Without requiring the morphism $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ to be proper as in the case of [46, Sec. 8, Thm. 22] and [47, Thm. 3.1], we can deduce from Theorem 6.26 the following:

Theorem 6.32. *With the setting as above, we have an equality*

$$\sum_i (-1)^i \, [H^i_{\text{ét},c}(\mathsf{X}_{\bar{\eta}}, \mathcal{V}_\xi)] = \sum_{b\in\mathcal{B}} \sum_j (-1)^j \, \mathcal{E}_b([H^j_{\text{ét},c}(\mathsf{Ig}^b, \mathcal{V}_\xi)]) \tag{6.33}$$

between virtual representations of $\mathrm{G}(\mathbb{A}^\infty) \times W_K$.

Proof. By Corollary 5.20 and Remark 5.35, the identities (6.24) and (6.27) imply the identity (6.33) between virtual representations of $\mathrm{G}(\mathbb{A}^{\infty,p}) \times \mathrm{G}(\mathbb{Q}_p)^+ \times W_K$, which extends to an identity between virtual representations of $\mathrm{G}(\mathbb{A}^\infty) \times W_K$, by the same explanation as in [46, Sec. 8]. □

Remark 6.34. Mantovan and Moonen announced in 2008 their joint work on the construction of toroidal compactifications for integral models $\mathsf{X}^{\mathrm{Dr}}_{\mathcal{H}}$ with Drinfeld level structures (as in Remark 6.28), and mentioned as an application the generalization of Mantovan's formula to the nonproper case by analyzing the contribution of the boundary. Nevertheless, our statement of Theorem 6.32 contains no boundary terms, and our proof of it uses only the known results in [46] and [47] (and the comparison results in Section 5.2).

Remark 6.35. It is possible to generalize Mantovan's formula to some cases where p is not a good prime. See, for example, the explanation in [61, Sec. 5.2]. (The earlier work [25], up to suitable reformulation, can be considered another example.) Our method should also allow one to remove the properness assumption in such cases, although we admit that this possibility is not very interesting.

6.4. **Scholze's formula.** Let us follow the setting of [59, Sec. 5]. Consider any integral PEL datum $(\mathcal{O}, \star, L, \langle \cdot\,, \cdot \rangle, h_0)$ satisfying the following properties:

(1) $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$ is maximal (and stable under $\star$) in $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.
(2) $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is simple, so that its center F is a field, and splits as an F-algebra.
(3) All places of $F^+ := F^{\star=\mathrm{Id}}$ above p are unramified in F.
(4) $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves no factor of type D, in the sense of [37, Def. 1.2.1.15].

Consider the trivial collection $J = \{j_0\}$ with $\{(g_{j_0}, L_{j_0}, \langle \cdot\,, \cdot \rangle_{j_0}\} = \{(1, L, \langle \cdot\,, \cdot \rangle\}$, as in [39, Ex. 2.3]. Let $\mathcal{H}$ be any neat open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}})$, let $\mathcal{H}^p$ denote the image of $\mathcal{H}$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}^p)$, and let $\mathcal{H}_0 = \mathcal{H}^p\mathrm{G}(\mathbb{Z}_p)$.

Let us take S to be $\mathrm{Spec}(\mathcal{O}_K) = \mathrm{Spec}(\mathcal{O}_{F_0,v})$ as in Section 6.1, and take $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ to be the pullback of $\vec{\mathsf{M}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, as in Case (Nm) in Assumption 2.1. While this is very similar to the setting of Section 6.3, we do not assume that p is good as in [37, Def. 1.4.1.1]. Although we no longer have the moduli

$\mathsf{M}_{\mathcal{H}^p} \to \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ as in [37, Sec. 1.4.1], we can still define a naive one over $\mathsf{S} = \mathrm{Spec}(\mathcal{O}_K)$, which we denote by $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p} \to \mathsf{S}$, as in [59, Def. 5.1]. In general, we cannot claim that $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ is flat. Nevertheless, given the above properties (3) and (4), as explained in [51, Sec. 8.2.5 (a)], the closure $\mathsf{M}_{\mathcal{H}^p}$ of $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ in $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ is already normal. Hence, by [39, Prop. 6.1 and its proof], $\mathsf{X}_{\mathcal{H}_0}$ is canonically isomorphic to $\mathsf{M}_{\mathcal{H}^p}$ over S. (This will suffice for our purpose, because the nearby cycles defined by $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ are necessarily supported on the closed subscheme $(\mathsf{M}_{\mathcal{H}^p})_{\bar{s}}$ of $(\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p})_{\bar{s}}$.)

Let ξ be an irreducible algebraic representation of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ on a finite-dimensional vector space V_ξ over $\bar{\mathbb{Q}}_\ell$, which defines an étale sheaf $\mathcal{V}_\xi$ over $\mathsf{X}_{\mathcal{H}_0}$ (because $\ell \neq p$) as in Proposition 3.2. Let us abusively define $H^i_{\text{ét},c}(\mathsf{X}_{\bar{\eta}}, \mathcal{V}_\xi)$ as in (6.21), for each i. By [60, Prop. 5.5], this is compatible with the definition in the next paragraph there, which carries commuting actions of Γ_K, $\mathrm{G}(\mathbb{Z}_p)$, and $\mathrm{G}(\mathbb{A}^{\infty,p})$ needed below.

The following is [59, Thm. 5.7] when $\mathsf{X}_{\mathcal{H}}$ (or equivalently $\mathsf{X}_{\mathcal{H}_0}$) is proper over S, with the best possible lower bound $j_0(f^p) = 1$ (which is then independent of f^p):

Theorem 6.36. *With the setting as above, suppose $f^p \in C^\infty_c(\mathrm{G}(\mathbb{A}^{\infty,p}))$. Then there exists a positive integer $j_0(f^p)$ such that, for all integers $j \geq j_0(f^p)$, all element $\tau \in W_K$ that is mapped to the j-th power of the geometric Frobenius* $\mathrm{Frob} \in \mathrm{Gal}(\bar{k}/k)$, *and all $h \in C^\infty_c(\mathrm{G}(\mathbb{Z}_p))$, we have the following formula based on the Langlands–Kottwitz method and a test function $\phi_{\tau,h}$ introduced by Scholze:*

$$\begin{aligned} &\sum_i (-1)^i \ \mathrm{tr}(\tau \times h f^p | H^i_{\text{ét},c}(\mathsf{X}_{\bar{\eta}}, \mathcal{V}_\xi)) \\ &= \ker^1(\mathbb{Q}, \mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \sum_{\substack{(\gamma_0;\gamma,\delta):\\ \alpha(\gamma_0;\gamma,\delta)=1}} c(\gamma_0;\gamma,\delta)\, O_\gamma(f^p)\, TO_{\delta\sigma}(\phi_{\tau,h}) \ \mathrm{tr}(\xi(\gamma_0)), \end{aligned} \tag{6.37}$$

where:

1. $\ker^1(\mathbb{Q}, \mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ *is the locally trivial elements in the Galois cohomology group* $H^1(\mathbb{Q}, \mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$, *as in* [33, p. 393].
2. *The sum at the right-hand side runs over a complete set of representatives of degree j Kottwitz triples $(\gamma_0;\gamma,\delta)$ as in* [59, Def. 5.6] *and* [60, Def. 2.1] *with invariant $\alpha(\gamma_0;\gamma,\delta) = 1$, where $\alpha(\gamma_0;\gamma,\delta)$ is the invariant constructed in* [32, Sec. 2]. (*For the readability of the remaining statements, let us mention that $\gamma_0 \in \mathrm{G}(\mathbb{Q})$, $\gamma \in \mathrm{G}(\mathbb{A}^{\infty,p})$, and $\delta \in \mathrm{G}(\mathbb{Q}_{p^r})$.*)
3. *The Haar measures on $\mathrm{G}(\mathbb{Q}_p)$ and $\mathrm{G}(\mathbb{Q}_{p^r})$ are normalized such that $\mathrm{G}(\mathbb{Z}_p)$ and $\mathrm{G}(\mathbb{Z}_{p^r})$ have volume 1, where $p^r = (\#k)^j$, and where $\mathbb{Z}_{p^r}$ and $\mathbb{Q}_{p^r}$ are the unique unramified extensions of $\mathbb{Z}_p$ and $\mathbb{Q}_p$, respectively, whose residue field has p^r elements. We shall denote by σ the automorphisms of $\mathbb{Z}_{p^r}$ and of $\mathbb{Q}_{p^r}$ inducing the p-th power automorphism of the residue field $\mathbb{F}_{p^r}$*
4. *$c(\gamma_0;\gamma,\delta)$ is the volume factor defined in* [32, p. 172].
5. *$O_\gamma(f^p) = \int_{\mathrm{G}(\mathbb{A}^{\infty,p})_\gamma \backslash \mathrm{G}(\mathbb{A}^{\infty,p})} f^p(x^{-1}\gamma x)\, d\bar{x}$ is the orbital integral, where $\mathrm{G}(\mathbb{A}^{\infty,p})_\gamma$ is the centralizer of γ in $\mathrm{G}(\mathbb{A}^{\infty,p})$.*
6. *$\phi_{\tau,h}$ is the test function in $C^\infty_c(\mathrm{G}(\mathbb{Q}_{p^r}))$ introduced in* [59, Def. 4.1].

(7) $TO_{\delta\sigma}(\phi_{\tau,h}) = \int_{(\mathrm{G}(\mathbb{Q}_{p^r})_\delta)^\sigma\backslash \mathrm{G}(\mathbb{Q}_{p^r})} \phi_{\tau,h}(y^{-1}\delta\sigma(y))\, d\bar{y}$ *is the twisted orbital integral, where* $(\mathrm{G}(\mathbb{Q}_{p^r})_\delta)^\sigma$ *is the* σ*-invariants in the centralizer* $\mathrm{G}(\mathbb{Q}_{p^r})_\delta$ *of* δ *in* $\mathrm{G}(\mathbb{Q}_{p^r})$.

(8) $\xi(\gamma_0) \in \mathrm{End}_{\bar{\mathbb{Q}}_\ell}(V_\xi)$ *and its trace* $\mathrm{tr}(\xi(\gamma_0))$ *are defined by the algebraic representation* ξ *of* $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *on the finite-dimensional vector space* V_ξ *over* $\bar{\mathbb{Q}}_\ell$.

Remark 6.38. Most of the terms above, except for the test function $\phi_{\tau,h}$, were introduced by Kottwitz (see [32] and related works). The crucial test function $\phi_{\tau,h}$ introduced by Scholze in [59, Def. 4.1] is defined using the cohomology of certain deformation spaces (of Barsotti–Tate groups with additional structures) constructed in [59, Sec. 3], which depends only on the data at p and is local in nature. On the contrary, the properness of $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$ depends on the existence of rational parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (see the discussions in [37, Sec. 5.3.3], [35, Sec. 4.2], and [40]), which is global in nature. This convinced us that it is reasonable to consider the generalization of [59, Thm. 5.7] to the nonproper case.

Proof of Theorem 6.36. The arguments in [59, Sec. 6] makes no reference to the properness of $\mathsf{X}_{\mathcal{H}} \to \mathsf{S}$. Hence, our main task is to generalize the arguments in [59, Sec. 7]. Even in mixed characteristics, we have the analogue of the commutative diagram in the beginning of [59, Sec. 7] for the minimal compactifications, consisting of finite morphisms (possibly highly ramified at the boundary). (Since $g \in \mathrm{G}(\mathbb{Z}_p)$, we do not have to introduce any other choices of J.) Hence, by Corollary 5.20, by the proper base change theorem (see [3, XII, 5.1]), by Remark 5.41 (and its references to [20, Lem. 1.3.1] and [18, Sec. 6.1.7]), and by the same argument as in [59, Sec. 7], $\mathrm{tr}(\tau \times hf^p | H^i_{\text{ét},c}(\mathsf{X}_{\bar{\eta}}, \mathcal{V}_\xi))$ is equal to the trace of the action of a correspondence on the associated cohomology of nearby cycles over $\bar{s}$.

We need this latter trace to be computable by some Lefschetz–Verdier trace formula (over nonproper schemes). The upshot is that this correspondence can be defined using only the finite étale morphisms $[1], [g^p] : (\mathsf{X}_{\mathcal{H}_1})_{\bar{s}} \to (\mathsf{X}_{\mathcal{H}_0})_{\bar{s}}$ and the geometric Frobenius correspondence on $(\mathsf{X}_{\mathcal{H}_0})_{\bar{s}}$ induced by $\mathrm{Frob} \in \mathrm{Gal}(\bar{k}/k)$, where $\mathcal{H}_1 := (\mathcal{H}^p \cap (g_p)^{-1}\mathcal{H}^p(g_p))\mathrm{G}(\mathbb{Z}_p)$, together with a more complicated correspondence between the sheaves of nearby cycles (defined by pushforwards from higher levels at p, using also τ and h). We obtain the analogue of [59, Thm. 7.1] in our context (for $j \geq j_0(f^p)$) by also referring to [69, Thm. 2.3.2 (b)], with the bound $j_0(f^p)$ here given by the ramification bound d in [69, Thm. 2.3.2 (c)] determined by $X = (\mathsf{X}^{\min}_{\mathcal{H}_0})_{\bar{s}}$, $U = (\mathsf{X}_{\mathcal{H}_0})_{\bar{s}}$, and $(c_1, c_2) = ([1], [g^p]) : C = (\mathsf{X}^{\min}_{\mathcal{H}_1})_{\bar{s}} \to X \underset{\bar{s}}{\times} X$. (In the proper case in [59, Thm. 7.1], there is no need to introduce compactifications, and one can take $j_0(f^p) = d = 1$ because the morphism $[g^p] : C \to X$ is étale.)

Once the analogue of [59, Thm. 7.1] is known (for $j \geq j_0(f^p)$), we can conclude with the same arguments in the last paragraph of [59, Sec. 7], which are pointwise in nature over $(\mathsf{X}_{\mathcal{H}_0})_{\bar{s}}$ and do not require $\mathsf{X}_{\mathcal{H}_0} \to \mathsf{S}$ to be proper. □

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