Annealed and quenched fluctuations of transient random walks in random environment on $\mathbb{Z}$

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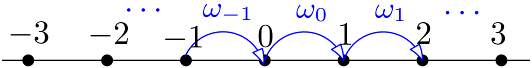
RWRE on $\mathbb{Z}$: definitions

Let μ be a law on $(0, 1)$.
Define an i.i.d. sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ with law μ ("environment").



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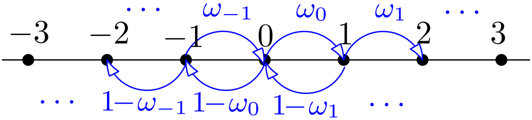

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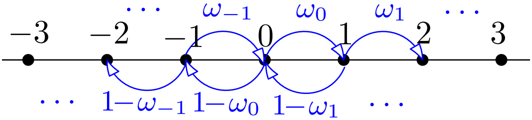

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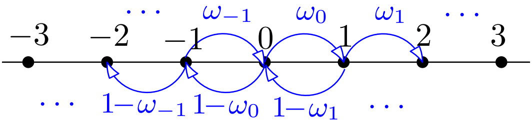

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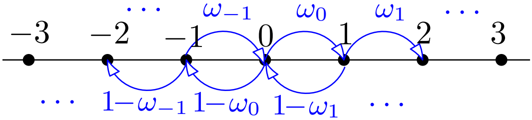

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Under $\mathbb{P}$, the RW "learns" about ω; transitions are *reinforced*.

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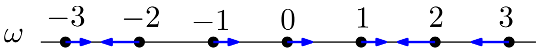

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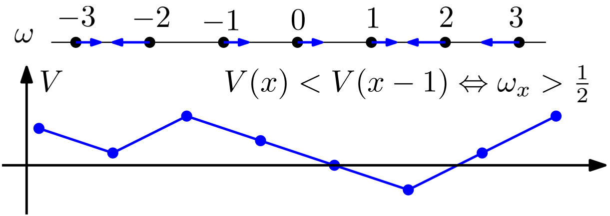

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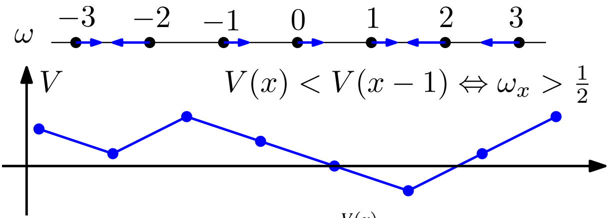

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i.e. $V(x) = \log \rho_1 + \cdots + \log \rho_x$ for $x \geq 0$, where $\rho_x = \dfrac{1 - \omega_x}{\omega_x}$

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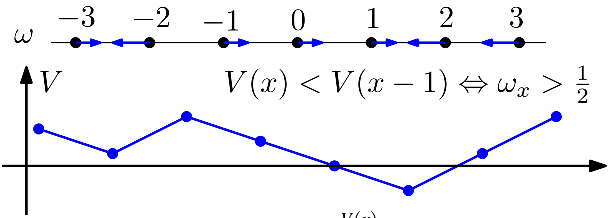

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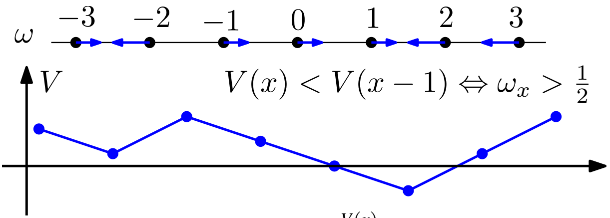

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Theorem (Solomon 1975)

- $\mathbb{P}$-a.s., $\begin{cases} X_n \to +\infty & \text{iff } \mathrm{E}[\log \rho_0] < 0 (\Leftrightarrow \mathrm{V(x)} \underset{\mathrm{x}\to\infty}{\longrightarrow} -\infty) \\ (X_n)_n \text{ recurrent} & \text{iff } \mathrm{E}[\log \rho_0] = 0 \end{cases}$

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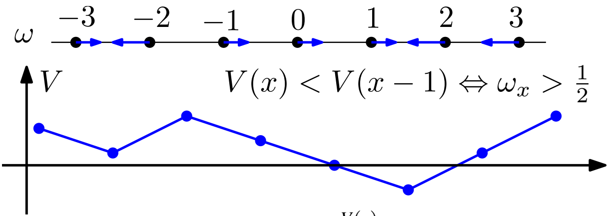

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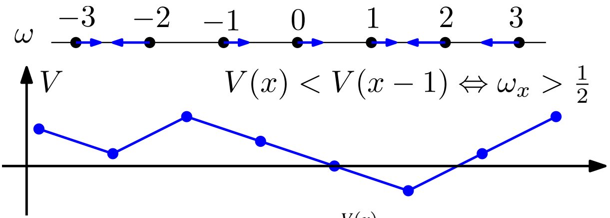

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- Assume $E[\log \rho_0] < 0$. $\mathbb{P}$-a.s., $\dfrac{X_n}{n} \to v$ where $\begin{cases} v > 0 & \text{if } \mathrm{E}[\rho_0] < 1 \\ v = 0 & \text{else.} \end{cases}$

Fluctuations: KKS theorem

Hypotheses

(a) $\exists\kappa > 0$ such that $E[\rho_0^\kappa] = 1$, and $E[\rho_0^\kappa(\log\rho_0)_+] < \infty$
(b) The law of $\log\rho_0$ is non-arithmetic.

Ex. For $\mu = B(a,b)$, $\kappa = b - a$.

Theorem (Kesten-Kozlov-Spitzer 1975)

Assume (a)-(b). Then, under $\mathbb{P}$,

- If $0 < \kappa < 1$, $\frac{X_n}{n^\kappa} \xrightarrow[n]{\text{(law)}} (A_\kappa \mathcal{S}_\kappa)^{-1/\kappa}$ $\qquad E[e^{it\mathcal{S}_\kappa}] = e^{-(-it)^\kappa}$
- If $1 < \kappa < 2$, $\frac{X_n - vn}{n^{1/\kappa}} \xrightarrow[n]{\text{(law)}} -v^{1+\frac{1}{\kappa}} A_\kappa \mathcal{S}_\kappa$ $\qquad E[e^{it\mathcal{S}_\kappa}] = e^{(-it)^\kappa}$
- If $\kappa > 2$, $\frac{X_n - vn}{\sqrt{n}} \xrightarrow[n]{\text{(law)}} \mathcal{N}(0,\sigma^2)$

where $A_\kappa > 0$ ($\mathcal{S}_\kappa$ is a totally asymmetric κ-stable r.v.)

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Main result

Theorem (Enriquez-Sabot-T.-Zindy 2010)

- For $0 < \kappa < 2$ ($\kappa \neq 1$),

$$A_\kappa = 2\left(\frac{\pi\kappa^2}{|\sin(\pi\kappa)|}(C_K)^2 E[\rho_0^\kappa \log \rho_0]\right)^{1/\kappa}$$

  where C_K is Kesten's renewal constant: $P(R > t) \sim C_K t^{-\kappa}$ with $R = 1 + \rho_1 + \rho_1\rho_2 + \cdots$.

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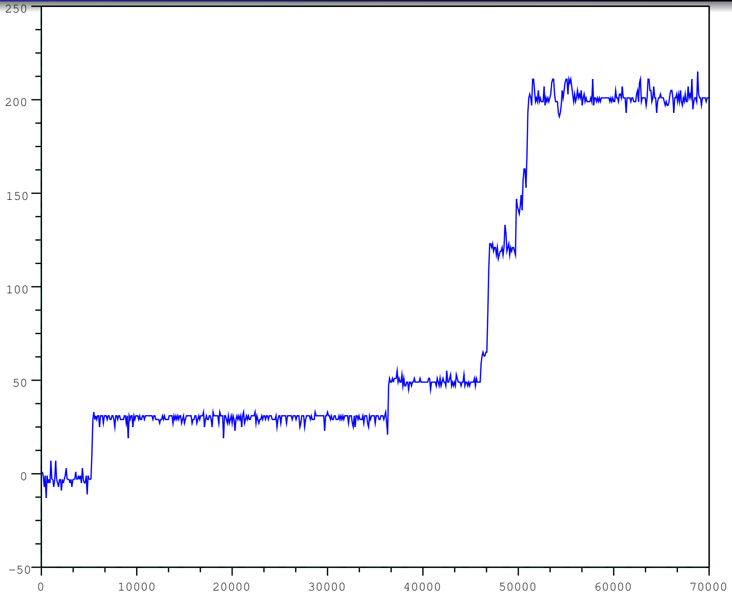
- Description of the *quenched* behaviour for $0 < \kappa < 2$

Sample trajectory $(0 < \kappa < 1)$



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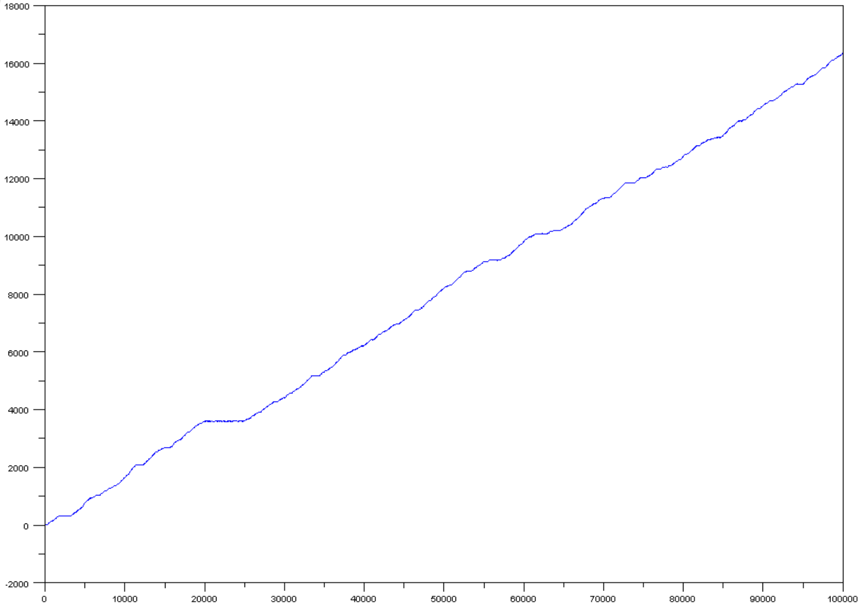

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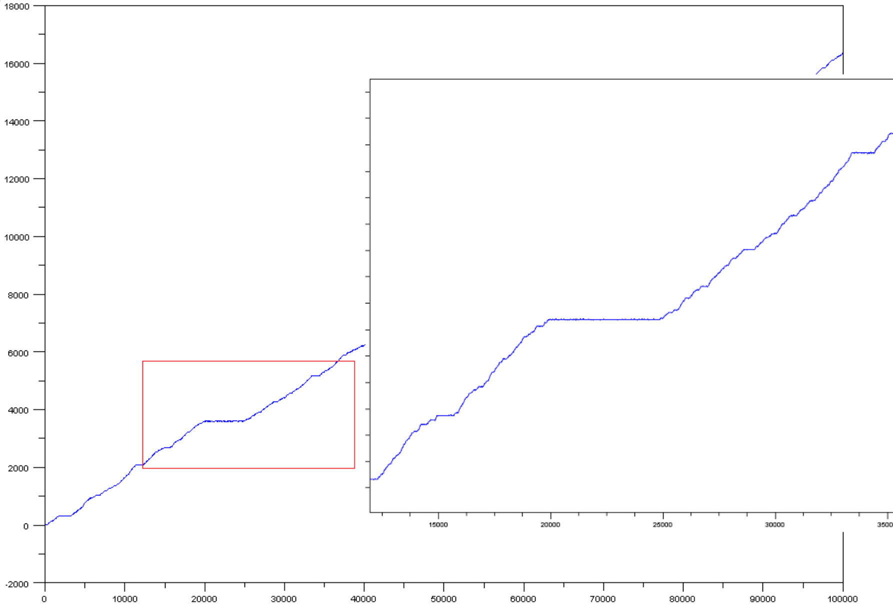
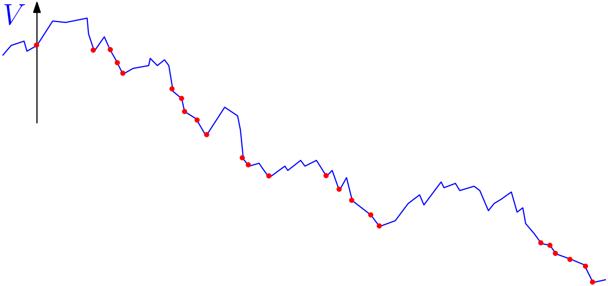


General scheme of proof

Excursions of V: $e_0 = 0$, $e_{n+1} = \inf\{k \geq e_n | V(k) \leq V(e_n)\}$.



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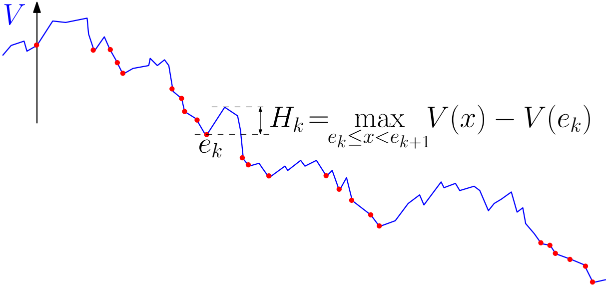

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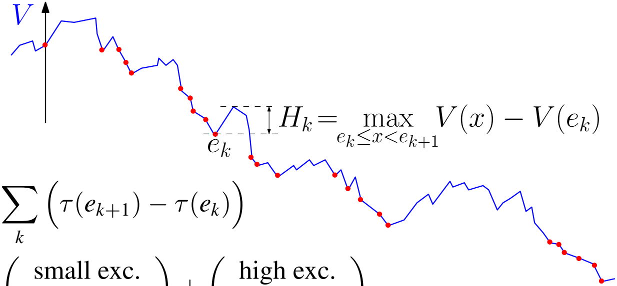


$$\tau(e_n) = \sum_k \Big(\tau(e_{k+1}) - \tau(e_k)\Big)$$

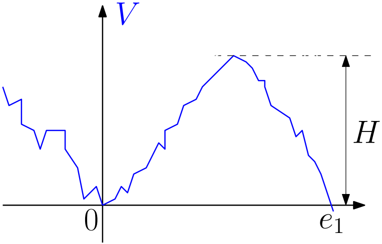
$$= \left(\begin{array}{c} \text{small exc.} \\ H < h_n \end{array} \right) + \left(\begin{array}{c} \text{high exc.} \\ H \geq h_n \end{array} \right)$$

- There are very few large excursions ($\Rightarrow$ crossing times almost i.i.d.)
- $\begin{cases} \text{Crossings of small excursions is } o(n^{1/\kappa}) \ (0 < \kappa < 1) \\ \text{Fluctuation of crossings of small excursions is } o(n^{1/\kappa}) \ (1 < \kappa < 2) \end{cases}$

Crossing time of a (high) excursion

Goal: estimate $\mathbb{P}(\tau(e_1) > t)$ as $t \to \infty$.

$$\mathbb{P}(\tau(e_1) > t) = \mathbb{P}(\tau(e_1) > t, H > \log t - \log\log t) + o(t^{-\kappa})$$



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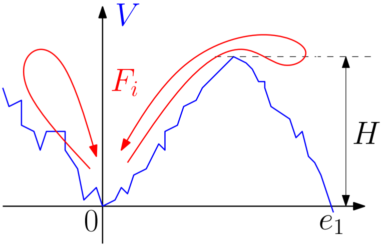

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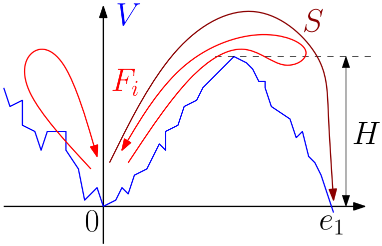

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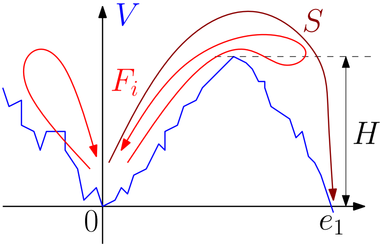

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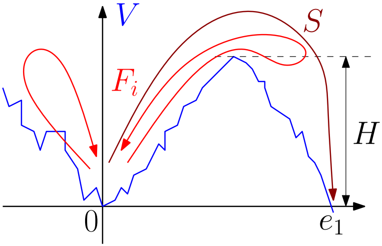

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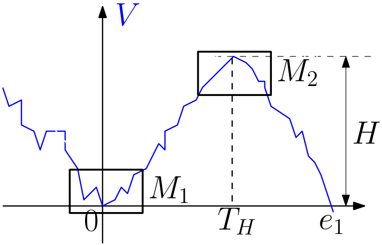

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$\mathbf{e} \sim \mathcal{E}(\mathbf{1})$,

$M_1 = \sum_{-\infty}^{T_H} e^{-V(x)}$, $M_2 = \sum_0^\infty e^{V(x)-H}$

- M_1, M_2, H are almost independent on $\{H > h\}$ with h large
- Property (Feller – Iglehart): $P(e^H > u) \sim C_I u^{-\kappa}$

Thus, $\mathbb{P}(\tau(e_1) > t) \sim Ct^{-\kappa}$ for some (rather explicit) C.

Interlude on heavy-tailed distributions

Let $T_1, T_2, \dots$ be i.i.d. r.v. ≥ 0 such that

$$P(T_i > t) \sim C t^{-\kappa}.$$

Then, if $0 < \kappa < 1$, $\quad \dfrac{T_1 + \cdots + T_n}{n^{1/\kappa}} \xrightarrow[n]{(law)} (C\,\Gamma(1-\kappa))^{1/\kappa} \mathcal{S}_\kappa$

and, if $1 < \kappa < 2$, $\quad \dfrac{T_1 + \cdots + T_n - nE[T_i]}{n^{1/\kappa}} \xrightarrow[n]{(law)} (-C\,\Gamma(1-\kappa))^{1/\kappa} \mathcal{S}_\kappa.$

For $\kappa > 2$, CLT.

Heavy-tail phenomenon:

- $\#\{1 \leq i \leq n : T_i \geq \varepsilon n^{1/\kappa}\} \xrightarrow[n]{(law)} \mathcal{P}(C\varepsilon^{-\kappa})$
- (for $0 < \kappa < 1$) $E\Big[\sum_{1 \leq i \leq n} T_i \mathbf{1}_{\{T_i < \varepsilon n^{1/\kappa}\}}\Big] \sim C\varepsilon^{1-\kappa} n^{1/\kappa}$

$\Rightarrow$ Up to an error of order $\varepsilon^{1-\kappa}$, $\frac{T_1+\cdots+T_n}{n^{1/\kappa}}$ is given by the $\mathcal{P}(C\varepsilon^{-\kappa})$ terms larger than $\varepsilon n^{1/\kappa}$.

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- (for $1 < \kappa < 2$) $\mathrm{Var}\Big(\sum_{1 \leq i \leq n} T_i \mathbf{1}_{\{T_i < \varepsilon n^{1/\kappa}\}}\Big) \sim C\varepsilon^{2-\kappa} n^{2/\kappa}$

$\Rightarrow$ Up to an error of order $\varepsilon^{1-\kappa/2}$, $\frac{T_1+\cdots+T_n-nE[T_i]}{n^{1/\kappa}}$ is given by the (centered) $\mathcal{P}(C\varepsilon^{-\kappa})$ terms larger than $\varepsilon n^{1/\kappa}$.

Next steps of the proof of the main result (KKS)

Let $h_n = \frac{1}{\kappa}\log n - \log\log n$ (hence $e^{h_n} = \varepsilon_n n^{1/\kappa}$, $\varepsilon_n = \frac{1}{\log n}$, cf. $\tau \simeq Me^H$)

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- Neglect (fluctuations of) crossing times of small excursions
- Ensure large excursions are way appart of each other w.h.p.
- Neglect time spent "backtracking" far away to the left of a high excursion before crossing it

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Let $h_n = \frac{1}{\kappa}\log n - \log\log n$ (hence $e^{h_n} = \varepsilon_n n^{1/\kappa}$, $\varepsilon_n = \frac{1}{\log n}$, cf. $\tau \simeq Me^H$)

$$\tau(e_n) = \sum_k \Big(\tau(e_{k+1}) - \tau(e_k)\Big) = \left(\begin{array}{c} \text{small exc.} \\ H < h_n \end{array} \right) + \left(\begin{array}{c} \text{high exc.} \\ H \geq h_n \end{array} \right)$$

- Neglect (fluctuations of) crossing times of small excursions
- Ensure large excursions are way appart of each other w.h.p.
- Neglect time spent “backtracking” far away to the left of a high excursion before crossing it
- Replace neglected parts by independent versions of them (which are negligible as well)

$\Rightarrow$ Reduction to i.i.d. copies of $\tau(e_1)$, hence the (annealed) limit theorem.

Consequences of the quenched description

$0 < \kappa < 1$

$\tau(e_n)$ is mainly given by a few terms $Me^H\mathbf{e}$ attached to deep valleys.
The time spent in-between is negligible in comparison to them.
$\Rightarrow$ **Localization** in a (random) deep valley. (cf. Enriquez-Sabot-Zindy)

$1 < \kappa < 2$

The fluctuations of $\tau(e_n)$ are mainly given by a few terms $Me^H(\mathbf{e}-1)$ attached to deep valleys.
$\Rightarrow$ **Practical interest** (explicit confidence intervals,...)
The fluctuations of $\tau(e_n)$ are almost a sum of i.i.d. terms like $Me^H(\mathbf{e}-1)$ (re-introducing independent small valleys).

$$\mathcal{L}\left(\frac{\tau(e_n)-E_\omega[\tau(e_n)]}{n^{1/\kappa}}\middle|\omega\right) \simeq \mathcal{L}\left(\sum_{i=1}^{n}\frac{\widehat{M}_i e^{\widehat{H}_i}}{n^{1/\kappa}}(\mathbf{e}_i-1)\middle|\widehat{M},\widehat{H}\right)$$

$\Rightarrow$ **Limit theorem** for the law of the quenched law of fluctuations

If $T_1, T_2, \ldots$ are i.i.d. with $P(T>t) \sim Ct^{-\kappa}$ and $0 < \kappa < 2$,

$$\left\{\frac{T_i}{n^{1/\kappa}}\middle| 1 \le i \le n\right\} \xrightarrow{(law)} \{\xi_i | i \ge 1\}$$

where ξ is a PPP of intensity $\lambda\kappa t^{-(\kappa+1)}dt$.

Consequences of the quenched description

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$\Rightarrow$ **Limit theorem** for the law of the quenched law of fluctuations

$$\mathcal{L}\left(\frac{\tau(e_n)-E_\omega[\tau(e_n)]}{n^{1/\kappa}}\middle|\omega\right) \xrightarrow[n]{(law,W_1)} \mathcal{L}\left(\sum_{i=1}^{\infty}\xi_i(\mathbf{e}_i-1)\middle|\xi\right)$$

where ξ is a PPP of intensity $\lambda\kappa t^{-(\kappa+1)}dt$. And W_1 is Wasserstein distance.