The decorrelation estimate for a 1D tight-binding model in the localized regime

Trinh Tuan Phong, LAGA, University of Paris 13

Advisor: Prof. Frédéric Klopp, IMJ, University of Paris 6

The 1D discrete random Hamiltonian with off-diagonal disorder

We consider the following random Hamiltonian defined on $l^2(\mathbb{Z})$:

- $(H_\omega u)(n) = \omega_n(u_n - u_{n+1}) - \omega_{n-1}(u_{n-1} - u_n), \quad \forall n \in \mathbb{Z}.$
- $\omega := \{\omega_n\}_{n\in\mathbb{Z}}$: i.i.d random variables with a bounded, compactly supported density ρ.
- supp $\rho \in [\alpha_0, \beta_0]$ where $\beta_0 \geq \alpha_0 > 0$.

This model appears in the description of waves (light, acoustic waves, etc) which propagate through a disordered, discrete medium.
With this model, its integrated density of states (I.D.o.S.) can be defined as follows:
With $\mathbb{P}-$a.s. ω,

$$N(E) := \lim_{|\Lambda|\to+\infty} \frac{\{\text{eigenvalues of } H_\omega(\Lambda) \text{ smaller than E}\}}{|\Lambda|} \text{ for all } E$$

where $H_\omega(\Lambda)$ is operator H_ω restricted on a cube Λ in $\mathbb{Z}$ with periodic boundary condition (per.b.c.).
For example, if $\Lambda = [1, N]$ the finite-volume operator $H_\omega(\Lambda)$ with per.b.c. is the matrix of the following form:

$$\begin{pmatrix} \omega_N + \omega_1 & -\omega_1 & 0 & \dots & 0 & -\omega_N \\ -\omega_1 & \omega_1 + \omega_2 & -\omega_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \omega_{N-2} + \omega_{N-1} & -\omega_{N-1} \\ -\omega_N & 0 & 0 & \dots & -\omega_{N-1} & \omega_{N-1} + \omega_N \end{pmatrix}$$

Goal: we want to prove the decorrelation estimate for eigenvalues of this 1D tight binding model near two distinct energies in the localized regime. Consequently, the independence of the limits of two local level statistics at two distinct energies is obtained.

Two crucial inequalities

We state here a Wegner-type estimate and a Minami-type estimate for our model which are essentially important for proofs of Poisson statistics and decorrelation estimates as well.

(W) Wegner estimate i.e. for all $\Lambda \subset \mathbb{Z}$ and $0 < \epsilon < E$

$$\mathbb{P}(\text{dist}(E, \sigma(H_\omega(\Lambda))) \leqslant \epsilon) \leq \frac{2d\|s\rho(s)\|_\infty}{E - \epsilon}\epsilon|\Lambda|$$

for all $\Lambda \subset \mathbb{Z}$ and $0 < \epsilon < E$.
As a direct consequence of **(W)**, the I.D.o.S. $N(E)$ is absolutely continuous w.r.t. Lebesgue measure with a bounded derivative $\nu(E)$ called the density of states (D.o.S.) of H_ω.

(M) Minami estimate: For $J \subset \mathbb{R}$, we have

$$\mathbb{P}(\#\{\sigma(H_\omega(\Lambda)) \cap J\} \geqslant 2) \leqslant C|J|^2|\Lambda|^2$$

which implies that, with a prob. 1, all eigenvalues of the finite-volume operator $H_\omega(\Lambda)$ are simple.

The localized regime

An interval I is in the localized regime iff the spectrum of H_ω in I is pure point and correspond. eigenfunctions are exponentially decay.
Following is a precise version of the localized regime for finite-volume operator $H_\omega(\Lambda_L)$ with $\Lambda_L := [-L, L]$:

(Loc): There exists $\nu > 0$ s.t., for any $p > 0$, there exists $q > 0$ and $L_0 > 0$ s.t., for $L \geqslant L_0$, with proba. larger than $1 - L^{-p}$, if

- $\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda_L)$ associated to an energy $E_{n,\omega} \in I$,
- $x_{n,\omega} \in \Lambda_L$ is a maximum of $x \mapsto |\varphi_{n,\omega}(x)|$ in Λ_L,

then, for $x \in \Lambda_L$, one has

$$|\varphi_{n,\omega}(x)| \leqslant L^q e^{-\nu|x - x_{n,\omega}|}.$$

Local level statistics

- $E_1(\omega, \Lambda) \leqslant E_2(\omega, \Lambda) \leqslant \dots \leqslant E_{|\Lambda|}(\omega, \Lambda)$ eigenvalues of $H_\omega(\Lambda)$.
- Let E be an energy in the localized regime.
- Renormalized levels at E :
$$\xi_n(E, \omega, \Lambda) = |\Lambda|\nu(E)(E_n(\omega, \Lambda) - E).$$
- The local level statistics near E is the following point process:
$$\Sigma(\xi, E, \omega, \Lambda) = \sum_{n=1}^{|\Lambda|} \delta_{\xi_n}(E, \omega, \Lambda)(\xi)$$

The question of the limit of the above point process as $|\Lambda| \to +\infty$ has been studied by many mathematicians.
Following is an answer to the above question which implies the absence of energy levels repulsion, hence describes the localized regime of random systems:

Theorem

(Molchanov, Minami, Combes-Germinet-Klein, Germinet-Klopp) Assume that ***(W)***, ***(M)*** *and* ***(Loc)*** *hold true. Pick E an energy in the localized regime with $\nu(E) > 0$.*
Then, when $|\Lambda| \to +\infty$, the point process $\Sigma(\xi, E, \omega, \Lambda)$ converges weakly to a Poisson process.

The decorrelation estimate

A natural question:

- For $E \neq E'$, are the limits of $\Sigma(\xi, E, \omega, \Lambda), \Sigma(\xi, E', \omega, \Lambda)$ stochastically independent? I.e. as $|\Lambda| \to +\infty$, do the above point processes converge weakly to two independent Poisson processes?
- A positive answer to this question is given by F.Klopp [4] for the discrete Anderson model.

The key point in his proof is the inequality so-called the decorrelation estimate for Anderson model on $l^2(\mathbb{Z}^d)$.
That is exactly what we want to carry out here for the 1D lattice Hamiltonian with off-diagonal disorder. Following is our decorrelation estimate:

Theorem

For $\alpha \in (0, 1)$ and $\{E, E'\}$ in the localized regime s.t. $E \neq E'$, as $l \approx L^\alpha$, we get

$$\mathbb{P}\left(\begin{cases} \sigma(H_\omega(\Lambda_l)) \cap (E + L^{-1}(-1, 1)) \neq \emptyset \\ \sigma(H_\omega(\Lambda_l)) \cap (E' + L^{-1}(-1, 1)) \neq \emptyset \end{cases} \right) \leqslant C(l/L)^2 e^{(\log L)^\beta}.$$

The decorrelation estimate means that, up to a sub-polynomial error, the probability of obtaining simultaneously two eigenvalues near two distinct energies are bounded by the product of the estimates given by **(W)** for each of these two energies.
Roughly speaking, two eigenvalues of our model near two distinct energies, behave like two independent random variables.

Remark: To prove the decorrelation estimate for the present model, we follow the strategy in [4], but a different approach is needed as [4] exploits the diagonal structure of the potential of the Anderson model which can not be used for our case.

Asymptotic independence of two local level statistics

Thanks to the above decorrelation estimate, we obtain the independence of the limits of two local level statistics near two distinct energies:

Theorem (Theorem 1.2, [4])

Let $E \neq E'$, belong to localized regime s.t. $\nu(E) > 0,\ \nu(E') > 0$.
As $|\Lambda| \to +\infty$, two point processes $\Sigma(E, \omega, \Lambda)$, et $\Sigma(E', \omega, \Lambda)$ converge weakly to two independent Poisson processes i.e., for two disjoint intervals $U_+ \subset \mathbb{R}$ and $U_- \subset \mathbb{R}$ and $\{k_+, k_-\} \in \mathbb{N}^2$, we have

$$\mathbb{P}\begin{cases} \#\{j; \xi_j(E, \omega, \Lambda) \in U_+\} &= k_+ \\ \#\{j; \xi_j(E', \omega, \Lambda) \in U_-\} &= k_- \end{cases} \xrightarrow[\Lambda\to\mathbb{Z}^d]{} e^{-|U_+|}\frac{|U_+|^{k_+}}{k_+!} e^{-|U_-|}\frac{|U_-|^{k_-}}{k_-!}.$$

Moreover, instead of fixing two distinct energies, we can consider two sequences of energies which tend to each other and get a similar result as in the above theorem under some new assumptions:

Theorem (Theorem 1.12, [3])

Pick $E_0 \in I$ such that the density of states ν is continuous and positive at E_0.
Consider two sequences of energies, say $(E_\Lambda)_\Lambda$, $(E'_\Lambda)_\Lambda$ s.t.

- $E_\Lambda \xrightarrow[\Lambda\to\mathbb{Z}^d]{} E_0$ *and* $E'_\Lambda \xrightarrow[\Lambda\to\mathbb{Z}^d]{} E_0$,
- $|\Lambda||N(E_\Lambda) - N(E'_\Lambda)| \xrightarrow[\Lambda\to\mathbb{Z}^d]{} +\infty$.

Then, the point processes $\Xi(\xi, E_\Lambda, \omega, \Lambda)$ and $\Xi(\xi, E'_\Lambda, \omega, \Lambda)$ converge weakly respectively to two independent Poisson point processes in $\mathbb{R}$ with intensity Lebesgue measure.

Short references

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