Decorrelation estimates for a 1D random model in the localized regime

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Lattice Hamiltonian with off-diagonal disorder in dimension 1

Set $u = \{u(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we define

$$(H_\omega u)(n) = \omega_n(u(n) - u(n+1)) + \omega_{n-1}(u(n) - u(n-1))$$

$\{\omega_n\}_{n\in\mathbb{Z}}$: non-negative i.i.d. random variables with a bounded compactly supported density ρ;

essRan $\omega_n = [\alpha_0, \beta_0] \quad \forall n \in \mathbb{Z}$ where $\alpha_0, \beta_0 > 0$.

A few facts:

Non random spectrum: $\omega-$a.s., $\sigma(H_\omega) = \Sigma := [0, 4\beta_0]$.

Integrated density of states $N(E)$: $\omega-$a.s.,

$$N(E) := \lim_{|\Lambda|\to+\infty} \frac{\#\{\text{e.v.s of } H_\omega(\Lambda) \text{ smaller than E}\}}{|\Lambda|} \quad \forall E$$

where $H_\omega(\Lambda)$ is H_ω restricted on a "cube"$\Lambda \subset \mathbb{Z}$ with periodic boundary conditions.

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Two crucial inequalities

Wegner estimate (W):

$$\mathbb{P}(\mathrm{dist}(E, \sigma(H_\omega(\Lambda))) \leqslant \epsilon) \leq \frac{2\|s\rho(s)\|_\infty}{E-\epsilon}\epsilon|\Lambda|$$

for all cubes $\Lambda \subset \mathbb{Z}$ and $0 < \epsilon < E$.

Minami estimate (M):

$$\mathbb{P}\left(\#\{\sigma\left(H_\omega\left(\Lambda\right)\right) \cap J\} \geqslant 2\right) \leqslant C(|J||\Lambda|)^2/2a^2$$

for all $J = [a, b] \subset (0, +\infty)$, and $\Lambda \subset \mathbb{Z}$.

Remark: (W) et (M) do not hold at 0 (the bottom of the spectrum Σ).

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Localized regime

Localized regime: the region in Σ where H_ω is pure point and the corresponding eigenfunctions decay exponentially at infinity.

Theorem [Aizemann, Schenker, Friedrich & Hundertmark]

(Loc): Let $\Lambda = [-L, L]$ be a interval in $\mathbb{Z}$ and I be in the localized regime. Then, there exists $\nu > 0$ such that, for any $p > 0$, there exists $q > 0$ and $L_0 > 0$ such that, for $L \geqslant L_0$, with probability larger than $1 - L^{-p}$, if

1. $\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda)$ associated to an eigenvalue $E_{n,\omega} \in I$,
2. $x_{n,\omega} \in \Lambda$ is a maximum of $x \mapsto |\varphi_{n,\omega}(x)|$ in Λ,

then, for $x \in \Lambda_L$, one has

$$\boxed{|\varphi_{n,\omega}(x)| \leqslant L^q e^{-\nu|x - x_{n,\omega}|}}$$

The point $x_{n,\omega}$ is called a localization center for $\varphi_{n,\omega}$ or $E_{n,\omega}$.

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Local level statistics

Let $\Lambda = [-L, L]$ be a cube in $\mathbb{Z}$ and E a positive energy in the localized regime. Assume that $E_1(\omega, \Lambda) \leqslant E_2(\omega, \Lambda) \leqslant \cdots \leqslant E_{|\Lambda|}(\omega, \Lambda)$ are eigenvalues of $H_\omega(\Lambda)$.

Renormalized level at E:

$$\xi_n(E, \omega, \Lambda) = |\Lambda|\nu(E)(E_n(\omega, \Lambda) - E)$$

Point process:

$$\Sigma(\xi, E, \omega, \Lambda) = \sum_{n=1}^{|\Lambda|} \delta_{\xi_n}(E, \omega, \Lambda)(\xi)$$

Theorem [Germinet-Klopp'12, Miao'11]

- Pick $E > 0$ in the localized regime s.t. $\nu(E) > 0$.
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Local level statistics (next)

Consider two limits of $\Sigma(\xi, E, \omega, \Lambda), \Sigma(\xi, E', \omega, \Lambda)$ with $E \neq E'$.

- Are they independent? That is, as $|\Lambda| \to +\infty$, do two above point processes converge weakly to two independent Poisson point processes?
- Yes for Anderson model:

Theorem (*For Anderson model*, [Klopp '11])

- Let $E \neq E'$ be two positive energies in the localized regime s.t. $\nu(E) > 0$, $\nu(E') > 0$.
- Then, for $U_+ \subset \mathbb{R}$ and $U_- \subset \mathbb{R}$ two compact intervals and $\{k_+, k_-\} \in \mathbb{N}^2$, one has

$$\mathbb{P}\left\{ \begin{array}{ll} \#\{j; \xi_j(E,\omega,\Lambda) \in U_+\} & = k_+ \\ \#\{j; \xi_j(E',\omega,\Lambda) \in U_-\} & = k_- \end{array} \right\} \xrightarrow[\Lambda \to \mathbb{Z}]{} e^{-|U_+|} \frac{|U_+|^{k_+}}{k_+!} e^{-|U_-|} \frac{|U_-|^{k_-}}{k_-!}$$

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Decorrelation estimate for the lattice Hamiltonian with off-diagonal disorder:

Theorem [P.'13]

- Pick $\alpha \in (0,1)$ and $E \neq E' > 0$ in the localized regime.
- For $\ell \approx L^\alpha$, we have

$$\mathbb{P}\left(\begin{Bmatrix} \sigma(H_\omega(\Lambda_\ell)) \cap (E + L^{-1}(-1,1)) \neq \emptyset \\ \sigma(H_\omega(\Lambda_\ell)) \cap (E' + L^{-1}(-1,1)) \neq \emptyset \end{Bmatrix}\right) \leqslant o\left(\frac{\ell}{L}\right)$$

Asymptotic independence:

Theorem [P.'13]

- Pick $n \geqslant 2$. Assume that $\{E_j\}_{1 \leqslant j \leqslant n}$ are in the localized regime s.t. $E_j > 0$, $E_j \neq E_k \; \forall j \neq k$ and $\nu(E_j) > 0$ for all $1 \leq j \leq n$.
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Key lemma to prove decorrelation estimates

Key lemma

- Pick $\Lambda = [-L, L] \in \mathbb{Z}$, $E \neq E' > 0$ in the localized regime and $\beta \in (1/2, 1)$.
- Denote by $\mathbb{P}^*$ the prob. of the following event (called $(*)$): There exists two simple eigenvalues of $H_\omega(\Lambda)$, say $E(\omega), E'(\omega)$ s.t.

$$\boxed{|E(\omega) - E| + |E'(\omega) - E'| \leqslant e^{-L^\beta}}$$

and

$$\boxed{\|\nabla_\omega E(\omega) - c^2 \nabla_\omega E'(\omega)\|_1 \leqslant e^{-L^\beta},\ c > 0}$$

- Then,

$$\boxed{\mathbb{P}^* \leqslant e^{-cL^{2\beta}}}$$

Proof of the key lemma

Let $u := u(\omega)$ and $v := v(\omega)$ be normalized eigenvectors associated to $E(\omega)$ and $E'(\omega)$.

$$\partial_{\omega_n} E(\omega) = (u(n) - u(n+1))^2 =: |Tu(n)|^2 \text{ for } n \in \Lambda$$

where $T : \ell^2(\Lambda) \longrightarrow \ell^2(\Lambda)$ is defined by

$$Tu(n) = u(n) - u(n+1) \text{ avec } u \in \ell^2(\Lambda)$$

Hence, if $\omega \in (*)$, we have

$$e^{-L^\beta} \geqslant \sum_n |Tu(n) - cTv(n)||Tu(n) + cTv(n)|$$

Therefore, there exists a partition $\Lambda = \mathcal{P} \cup \mathcal{Q}$, $\mathcal{P} \cap \mathcal{Q} = \emptyset$ s.t.

- for $n \in \mathcal{P}, |Tu(n) - cTv(n)| \leqslant e^{-L^\beta/2}$,
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- for $n \in \mathcal{P}, |Tu(n) - cTv(n)| \leqslant e^{-L^\beta/2}$,
- for $n \in \mathcal{Q}$, $|Tu(n) + cTv(n)| \leqslant e^{-L^\beta/2}$.

Proof of the key lemma

Let $u := u(\omega)$ and $v := v(\omega)$ be normalized eigenvectors associated to $E(\omega)$ and $E'(\omega)$.

$$\boxed{\partial_{\omega_n} E(\omega) = (u(n) - u(n+1))^2 =: |Tu(n)|^2 \text{ for } n \in \Lambda}$$

where $T : \ell^2(\Lambda) \longrightarrow \ell^2(\Lambda)$ is defined by

$$\boxed{Tu(n) = u(n) - u(n+1) \text{ avec } u \in \ell^2(\Lambda)}$$

Hence, if $\omega \in (*)$, we have

$$\boxed{e^{-L^\beta} \geqslant \sum_n |Tu(n) - cTv(n)||Tu(n) + cTv(n)|}$$

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Proof of the key lemma (next)

"Lower bound": $J \subset \Lambda$ of length $O(L^{\beta})$ s.t.

$$\boxed{|u(n)|^2 + |u(n+1)|^2 \geqslant e^{-L^{\beta}/2} \text{ for } n \in J}$$

Decomposition:

$$\boxed{\mathcal{P} \cap J = \cup \mathcal{P}_j \text{ et } \mathcal{Q} \cap J = \cup \mathcal{Q}_j}$$

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Three steps to complete the proof of the key lemma

First step: Any $\mathcal{P}_j$ or $\mathcal{Q}_j$ can not contain more than four points.

Second step: From 4 consecutive points of J, we can form a 10×10 system of linear equations.

$$AU = b \text{ with } \|b\| \leq c_0 e^{-L^{\beta}/2} \text{ and } \|U\| \geq e^{-L^{\beta}/4}$$

where A is a 10×10 matrix and
$U := (u(n-2), \ldots, u(n+2), v(n-2), \ldots, v(n+2))^t$.

Observation:

$$|\det A| \leq M e^{-L^{\beta}/4} \text{ where } M \text{ depends } \alpha_0, \beta_0, E \text{ and } E'$$

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End of the proof

Restrictions on r.v.'s $\{\omega_n\}_{n\in\Lambda}$:

(i) $\left|\omega_n + \dfrac{E'-E}{4}\right| \leqslant Ce^{-L^\beta/8}$,

(ii) $\left|\omega_{n-1} + \dfrac{E'-E}{4}\right| \leqslant Ce^{-L^\beta/8}$,

(iii) $\left|\omega_{n-1}\omega_n - \dfrac{(E+E')^2}{4}\right| \leqslant Ce^{-L^\beta/4}$.

To sum up,

- R.v.'s $\{\omega_j\}_{j\in\Lambda}$ satisfy at least cL^β cond. de types (i)-(iii).
- ω_n are i.i.d. with a common bounded density + (i)-(iii) $\Longrightarrow$ For a given $\mathcal{P}$ and $\mathcal{Q}$, the event $(*)$ happens with a prob. at most $e^{-cL^{2\beta}}$.

Hence,

$$\boxed{\mathbb{P}^* \leqslant 2^L e^{-cL^{2\beta}} \leqslant e^{-\tilde{c}L^{2\beta}}}$$

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References

[1] Michael Aizenman, Jeffrey H.Schenker, Roland M. Friedrich, and Dirk Hundertmark. *Finite-volume fractional-moment criteria for Anderson localization*, Comm. Math. Phys., 224(1):219-253, 2001. Dedicated to Joel L. Lebowitz.

[2] Dong Miao, *Eigenvalue statistics for lattice Hamiltonian of off-diagonal disorder* , J. Stat. Phys (2011), 143: 509–522 DOI 10.1007/s10955-011-0190-2.

[3] Frédéric Klopp, *Decorrelation estimates for the eigenvalues of the discrete Anderson model in the localized regime*, Comm. Math. Phys. Vol. 303, pp. 233-260 (2011).

[4] Trinh Tuan Phong, *Decorrelation estimates for a 1D tight-binding model in the localized regime*, 31p., Annales Henri Poincaré, http://link.springer.com/article/10.1007%2Fs00023-013-0250-z

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