Higher Algebra with Operads

Bruno VALLETTE

University of Nice Sophia–Antipolis (Newton Institute, Cambridge)

British Mathematical Colloquium

March 27, 2013

Plan

Homotopy data and mixed complex structure

- **Homotopy data:** Deformation retract of chain complexes

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H) \qquad \boxed{\mathrm{id}_A - ip = d_A h + h d_A}\ .$$

- **Algebraic data:** $\Delta : A \to A, \quad d_A\Delta + \Delta d_A = 0, \quad \boxed{\Delta^2 = 0}$
  mixed complex $|\Delta| = 1$ (or bicomplex).

- **Transferred structure:** $\boxed{\delta_1 := p\Delta i}$

  Does δ_1 square to zero?

$$(\delta_1)^2 = p\Delta \underbrace{ip}_{\sim_h \ \mathrm{id}_A} \Delta i \neq 0 \text{ in general!}$$

Idea: Introduce $\boxed{\delta_2 := p\Delta h \Delta i}$

Then, $\boxed{\partial(\delta_2) = (\delta_1)^2}$ in $(\mathrm{Hom}(A, A), \partial := [d_A, -])$.

$\Longrightarrow$ δ_2 is a homotopy for the relation $(\delta_1)^2 = 0$.

Higher structure: multicomplex

Higher up, we consider: $\boxed{\delta_n := p(\Delta h)^{n-1}\Delta i}$, for $n \geq 1$.

Proposition

$$\boxed{\partial(\delta_n) = \sum_{k=1}^{n-1} \delta_k \delta_{n-k}} \quad \textit{in } (\mathrm{Hom}(A, A), \partial), \ \textit{for} \ \ n \geq 1 \ .$$

Definition (Multicomplex)

$(H, \delta_0 := -d_H, \delta_1, \delta_2, \ldots)$ graded vector space H endowed with a family of linear operators of degree $|\delta_n| = 2n - 1$ satisfying

$$\boxed{\sum_{k=0}^{n} \delta_k \delta_{n-k} = 0} \ , \quad \text{for} \ \ n \geq 0 \ .$$

Remark: A mixed complex = multicomplex s.t. $\delta_n = 0$, for $n \geq 2$.

Multicomplexes are homotopy stable

- Starting now from a multicomplex $(A, \Delta_0 = -d_A, \Delta_1, \Delta_2, \ldots)$
- Consider the transferred operators

$$\boxed{\delta_n := \sum_{k_1+\cdots+k_l=n} p\Delta_{k_1} h \Delta_{k_2} h \ldots h \Delta_{k_l} i}\ , \quad \text{for} \quad n \geq 1$$

Proposition

$$\boxed{\partial(\delta_n) = \sum_{k=1}^{n-1} \delta_k \delta_{n-k}} \quad \textit{in } (\mathrm{Hom}(A, A), \partial), \ \textit{for} \quad n \geq 1 \ .$$

$\Longrightarrow$ Again a multicomplex, **no need of further higher structure**.

Compatibility between Original and Transferred structures

$$\underbrace{(A, \Delta_0 = -d_A, \Delta_1, \Delta_2, \ldots)}_{\text{Original structure}} \xleftarrow{i} \underbrace{(H, \delta_0 = -d_H, \delta_1, \delta_2, \ldots)}_{\text{Transferred structure}}$$

- i chain map $\implies$ $\boxed{\Delta_0 i = i\delta_0}$
- **Question:** Does i commute with the Δ's and the δ's?

$$i\delta_1 = \underbrace{ip}_{\sim_h \, \mathrm{id}_A} \Delta_1 i \neq \Delta_1 i \quad \text{in general!}$$

- Define $\boxed{i_0 := i}$ and consider $\boxed{i_1 := h\Delta_1 i}$.

Then, $\boxed{\partial(i_1) = \Delta_1 i_0 - i_0\delta_1}$ in $(\mathrm{Hom}(H, A), \partial)$.

$\implies$ i_1 is a homotopy for the relation $\Delta_1 i_0 = i_0\delta_1$.

∞-morphisms of multicomplexes

Higher up, we consider:

$$\boxed{i_n := \sum_{k_1+\cdots+k_l=n} h\Delta_{k_1} h\Delta_{k_2} h \ldots h\Delta_{k_l} i}\ , \quad \text{for} \quad n \geq 1\ .$$

$$\Rightarrow \boxed{\partial(i_n) = \sum_{k=0}^{n-1} \Delta_{n-k} i_k - \sum_{k=0}^{n-1} i_k \delta_{n-k}} \text{ in } (\mathrm{Hom}(H, A), \partial), \text{ for } n \geq 1\ .$$

Definition (∞-morphism)

$i_\infty : (H, \delta_0 = -d_H, \delta_1, \delta_2, \ldots) \rightsquigarrow (A, \Delta_0 = -d_A, \Delta_1, \Delta_2, \ldots)$
collection of maps $\{i_n \colon H \to A\}_{n \geq 0}$ satisfying

$$\boxed{\sum_{k=0}^{n} \Delta_{n-k} i_k = \sum_{k=0}^{n} i_k \delta_{n-k}}\ , \quad \text{for} \quad n \geq 0\ .$$

The category ∞-mutlicomp

Proposition (Composite of ∞-morphisms)

$f : A \rightsquigarrow B$, $g : B \rightsquigarrow C$: two ∞-morphisms of multicomplexes.

$$\boxed{(gf)_n := \sum_{k=0}^{n} g_{n-k} f_k}\ , \quad \text{for} \quad n \geq 0\ ,$$

defines an associative and unital composite of ∞-morphisms.

Category: multicomplex with ∞-morphisms: ∞-multicomp.

Compact reformulation:

multicomplex $=$ square-zero element

$$\Delta(z) = \Delta_0 + \Delta_1 z + \Delta_2 z^2 + \cdots$$

in the algebra $\mathrm{End}_A[[z]]$,

∞-morphism $= i(z) \in \mathrm{Hom}(H, A)[[z]]$ s.t. $i(z)\delta(z) = \Delta(z)i(z)$,

composite $= g(z)f(z)$.

Homotopy Transfer Theorem for multicomplexes

∞-quasi-isomorphism: $i : H \overset{\sim}{\rightsquigarrow} A$ s.t. $i_0 : H \xrightarrow{\sim} A$ qi.

Theorem (HTT for multicomplexes, Lapin '01)

Given any deformation retract

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H) \qquad \mathrm{id}_A - ip = d_A h + h d_A$$

and any multicomplex structure on A, there exists a multicomplex structure on H such that i extends to an ∞-quasi-isomorphism.

Application 1: $\mathbb{K}$ field, (A, d, Δ) bicomplex, $(H(A,d), 0) = E^1$ deformation retract
$\Longrightarrow$ multicomplex structure on $H(A,d)$ =lift of the spectral sequence, i.e. $\delta^r \Rightarrow d^r$.

Application 2: Equivalence between the various definitions of cyclic homology [Loday-Quillen, Kassel].

Homotopy theory of mixed complexes

Definition (Homotopy category)

Localisation with respect to quasi-isomorphisms

$$\mathrm{Ho}(\text{mixed cx}) := \text{mixed cx}\,[qi^{-1}]$$

$$\mathrm{Hom}_{\mathrm{Ho}}(A,B) := \{A \to \bullet \text{``}\xleftarrow{\sim}\text{''} \bullet \to \bullet \cdots \bullet \text{''}\xleftarrow{\sim}\text{''} \bullet \to B\}/\sim$$

Theorem (?)

- *Every ∞-qi of multicomplexes admits a homotopy inverse.*
- $$\boxed{\mathrm{Ho}(\text{mixed cx}) \cong \infty\text{-mixed cx}/\sim_h}.$$

Proof.

[...] +Rectification:

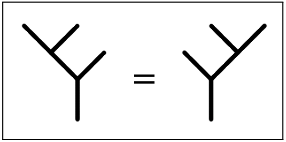
$$\exists\ Rect : \infty\text{-multicomp} \to \text{mixed cx, s.t. } H \overset{\sim}{\rightsquigarrow} Rect(H)\ .$$

□

Associative algebra and homotopy data

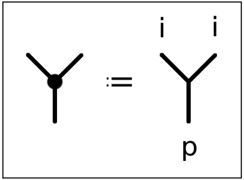
- **Initial structure:** an associative product on A

$$\nu : A^{\otimes 2} \to A, \quad \text{s.t.} \quad \nu(\nu(a,b),c) = \nu(a,\nu(b,c)) \ .$$



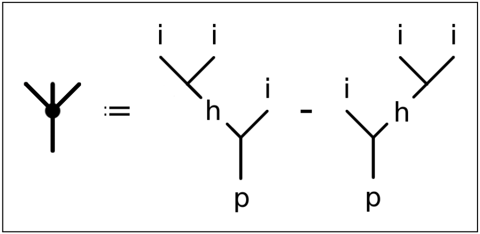
- **Transferred structure:** the binary product on H

$$\mu_2 := p\nu i^{\otimes 2} \ : \ H^{\otimes 2} \to H.$$

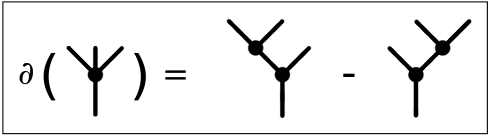


First homotopy for the associativity relation

- Is the transferred μ_2 associative? Anwser: in general, no!
- Introduce μ_3:



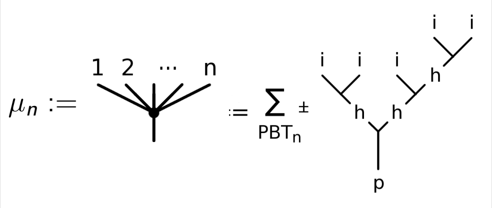
- In $\mathrm{Hom}(A^{\otimes 3}, A)$, it satisfies



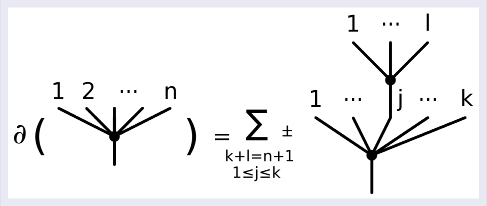
$\Longrightarrow$ μ_3 is a homotopy for the associativity relation of μ_2.

Higher structure

Higher up, in $\mathrm{Hom}(H^{\otimes n}, H)$, we consider:



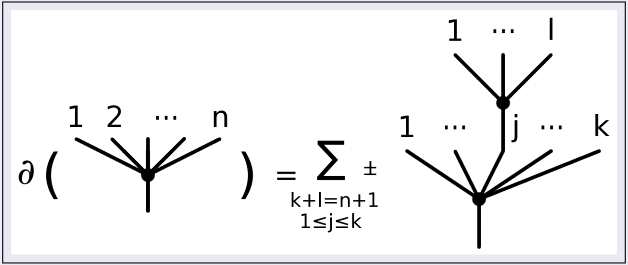
Proposition



A_∞-algebra

Definition (A_∞-algebra, Stasheff '63)

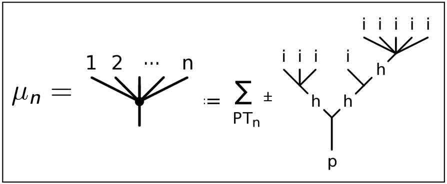
An A_∞-algebra is a chain complex $(H, d_H, \mu_2, \mu_3, \ldots)$ endowed with a family of multilinear maps of degree $|\mu_n| = n - 2$ satisfying



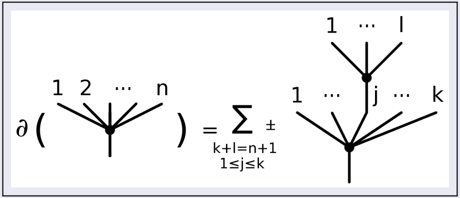
Remark: A dga algebra = A_∞-algebra s.t. $\mu_n = 0$, for $n \geq 3$.

A_∞-algebras are homotopy stable

- Starting from an A_∞-algebra $(A, d_A, \nu_2, \nu_3, \dots)$
- Consider



Proposition



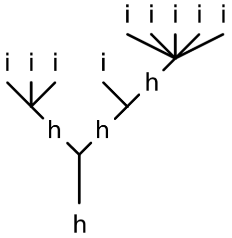
$\Longrightarrow$ Again an A_∞-algebra, **no need of further higher structure**.

Compatibility between Original and Transferred structures

$$\underbrace{(A, d_A, \nu_2, \nu_3, \ldots)}_{\text{Original structure}} \xleftarrow{i} \underbrace{(H, d_H, \mu_2, \mu_3, \ldots)}_{\text{Transferred structure}}$$

- i chain map $\Longrightarrow$ $\boxed{d_A i = i d_H}$
- **Question:** Does i commutes with the ν's and the μ's?
  Anwser: not in general!
- Define $\boxed{i_1 := i}$ and consider in $\mathrm{Hom}(H^{\otimes n}, A)$, for $n \geq 2$:

$$i_n := \sum_{\mathrm{PT_n}} \pm$$



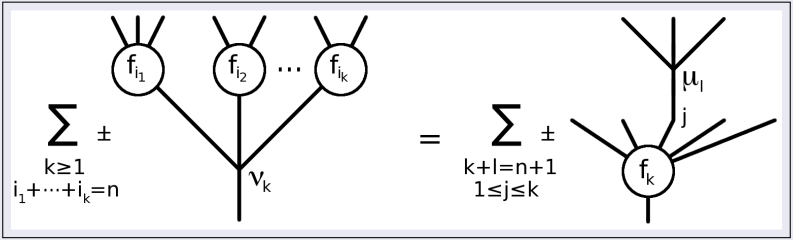
A_∞-morphism

Definition (A_∞-morphism)

$(H, d_H, \{\mu_n\}_{n\geq 2}) \rightsquigarrow (A,\ d_A,\ \{\nu_n\}_{n\geq 2})$ is a collection of linear maps

$$\{f_n\ :\ H^{\otimes n} \to A\}_{n\geq 1}$$

of degree $|f_n| = n-1$ satisfying



Example: The aforementioned $\{i_n\ :\ H^{\otimes n} \to A\}_{n\geq 1}$.

Homotopy Transfer Theorem for A_∞-algebras

∞-quasi-isomorphism: $i : H \overset{\sim}{\rightsquigarrow} A$ s.t. $i_0 : H \xrightarrow{\sim} A$ qi.

Theorem (HTT for A_∞-algebras, Kadeshvili '82)

Given any deformation retract

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H) \qquad \mathrm{id}_A - ip = d_A h + h d_A$$

and any A_∞-algebra structure on A, there exists an A_∞-algebra structure on H such that i extends to an ∞-quasi-isomorphism.

Application: $A = (C^\bullet_{\mathrm{Sing}}(X), \cup)$, transferred A_∞-algebra on $H^\bullet_{\mathrm{Sing}}(X)$= lifting of the (higher) Massey products.

The category ∞-A_∞-Alg

Compact reformulation:

A_∞-algebra $=$ square-zero coderivation in the coalgebra $T^c(sA)$,
A_∞-morphism $=$ morphism of dg coalgebras $T^c(sA) \to T^c(sB)$.
composite [?] $=$ *composite* of morphisms of dg coalgebras.

Category: A_∞-algebras with ∞-morphisms: ∞-A_∞-Alg.

$$
\begin{tikzcd}
\text{dga alg} \arrow[d, hook] & \\
\infty\text{-}A_\infty\text{-alg} \arrow[r, leftrightarrow, "\cong"] & \text{quasi-free dga coalg}
\end{tikzcd}
$$

Homotopy theory of dg associative algebras

Theorem (Munkholm '78, Lefèvre-Hasegawa '03)

- *Every ∞-qi of A_∞-algebras admits a homotopy inverse.*
- $\boxed{\mathrm{Ho}(\text{dga alg}) := \text{dga alg}\,[qi^{-1}] \cong \infty\text{-dga alg}/\sim_h}$.

Proof. Use

$$
\begin{tikzcd}
\text{dga alg} \arrow[r, shift right, "\mathrm{B}"'] \arrow[d, hook] & \text{(conil) dga coalg} \arrow[l, shift right, "\Omega"'] \\
\infty\text{-}A_\infty\text{-alg} \arrow[u, bend left, "\mathrm{Rect}"] \arrow[r, leftrightarrow, "\cong"] & \text{quasi-free dga coalg} \arrow[u, hook]
\end{tikzcd}
$$

$+$ [...] $+$ Rectification:

$$\exists\ \mathit{Rect} : \infty\text{-}A_\infty\text{-Alg} \to \text{dga alg, s.t. } H \overset{\sim}{\rightsquigarrow} \mathit{Rect}(H)$$

Exercise

Exercise: Consider your favorite category of algebras "of type $\mathcal{P}$" (eg. Lie algebras, associative algebras+unary operator Δ, etc.).

- Find the good notions of $\mathcal{P}_\infty$-algebras and ∞-morphisms.
- Fill the diagram

$$
\begin{tikzcd}
\text{dg } \mathcal{P}\text{-alg} \arrow[r, harpoon, shift right, "???"'] \arrow[d, hook] & ??? \arrow[l, harpoon, shift right, "???"'] \\
\infty\text{-}\mathcal{P}_\infty\text{-alg } [?] \arrow[u, bend left, "\mathrm{Rect}"] \arrow[r, leftrightarrow, "\cong"] & ??? \arrow[u, hook]
\end{tikzcd}
$$

- to prove the Homotopy Transfer Theorem
- and the equivalence of categories

$$\boxed{\mathrm{Ho}(\text{dg } \mathcal{P}\text{-alg}) := \text{dg } \mathcal{P}\text{-alg}\,[qi^{-1}] \cong \infty\text{-dg } \mathcal{P}\text{-alg}/\sim_h}.$$

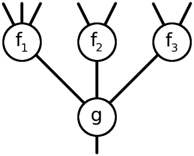
Plan

Operad

Multilinear Operations: $\mathrm{End}_A(n) := \mathrm{Hom}(A^{\otimes n}, A)$

Composition:

$$\begin{array}{rcl} \mathrm{End}_A(k) \otimes \mathrm{End}_A(i_1) \otimes \cdots \otimes \mathrm{End}_A(i_k) & \to & \mathrm{End}_A(i_1 + \cdots + i_k) \\ g \otimes f_1 \otimes \cdots \otimes f_k & \mapsto & g(f_1, \ldots, f_k) \end{array}$$



Definition (Operad)

- Collection: $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of $\mathbb{S}_n$-modules
- Composition: $\mathcal{P}_k \otimes \mathcal{P}_{i_1} \otimes \cdots \otimes \mathcal{P} i_k \to \mathcal{P}_{i_1+\cdots+i_k}$

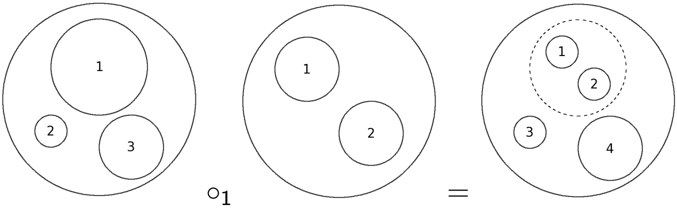
Examples of Operads

Definition (Algebra over an Operad)

Structure of $\mathcal{P}$-algebra on A: morphism of operads $\mathcal{P} \to \mathrm{End}_A$

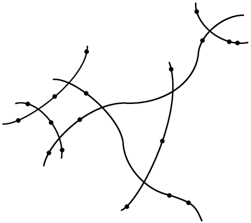
Examples:

- $D = T(\Delta)/(\Delta^2)$-algebras (modules) = mixed complexes.
- $As = \mathcal{T}$  -algebras = associative algebras.
- *Little discs* D_2*:* D_2-algebras $\cong$ double loop spaces $\Omega^2(X)$



Example in Geometry

Deligne–Mumford moduli space of stable curves: $\overline{\mathcal{M}}_{g,n+1}$



Definition (Frobenius manifold, aka Hypercommutative algebras)

Algebra over $H_\bullet(\overline{\mathcal{M}}_{0,n+1})$, i.e. $H_\bullet(\overline{\mathcal{M}}_{0,n+1}) \to \mathrm{End}_{H_\bullet(A)} \iff$ totally symmetric n-ary operation $(x_1, \ldots, x_n)$ of degree $2(n-2)$,

$$\sum_{S_1 \sqcup S_2 = \{1,\ldots,n\}} ((a, b, x_{S_1}), c, x_{S_2}) = \sum_{S_1 \sqcup S_2 = \{1,\ldots,n\}} \pm(a, (b, x_{S_1}, c), x_{S_2}) .$$

Homotopy algebra and operads

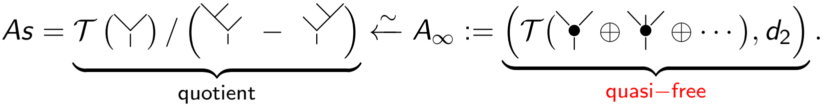
$$\begin{tikzcd} \text{operad } \mathcal{P} \arrow[d, dotted] & \mathcal{P}_\infty \arrow[l, "\sim"'] \arrow[d, dotted] : \text{ quasi-free replacement (cofibrant)} \\ \text{category of } \mathcal{P}\text{-algebras} \arrow[r, hook] & \text{category of homotopy } \mathcal{P}\text{-algebras} \end{tikzcd}$$

Examples:

- $\mathcal{P} = D$: D_∞-algebras $=$ multicomplexes

$$D = \underbrace{T(\Delta)/(\Delta^2)}_{\text{quotient}} \xleftarrow{\sim} D_\infty := \underbrace{\left(T(\delta \oplus \delta^2 \oplus \delta^3 \oplus \cdots), d_2\right)}_{\text{quasi-free}}.$$

- $\mathcal{P} = Ass$: Ass_∞-algebras $=$ A_∞-algebras



Koszul duality theory

$$\boxed{\mathcal{P}_\infty = \mathcal{T}(\text{operadic syzygies}) \xrightarrow{?\sim?} \mathcal{P}}$$

- **Quadratic presentation:** $\mathcal{P} = \mathcal{T}(V)/(R)$, where
$$R \subset \underbrace{\mathcal{T}^{(2)}(V)}_{\text{trees with 2 vertices}} .$$
- **Koszul dual cooperad:** quadratic cooperad $\mathcal{P}^{¡} := \mathcal{C}(sV, s^2R)$, i.e. defined by a (dual) universal property.
- **Candidate:** $\mathcal{P}_\infty = \Omega\mathcal{P}^{¡} = \mathcal{T}(\mathcal{P}^{¡}) \xrightarrow{?\sim?} \mathcal{P}$.
- **Criterion:** Quasi-isomorphism iff the Koszul complex $\mathcal{P} \circ_\kappa \mathcal{P}^{¡}$ is acyclic.
- **Examples:** D, Ass, Com, Lie, etc.

Operadic higher structure

For any Koszul operad $\mathcal{P}$

- $\exists$ a notion of composable ∞-morphisms: ∞-$\mathcal{P}_\infty$-Alg.

$\mathcal{P}_\infty$-algebra $=$ square-zero coderivation in the coalgebra $\mathcal{P}^{\mathrm{i}}(A)$,
∞-morphism $=$ morphism of dg coalgebras $\mathcal{P}^{\mathrm{i}}(A) \to \mathcal{P}^{\mathrm{i}}(B)$.

Theorem (HTT for P_∞-algebras, Galvez–Tonks-V.)

Given any deformation retract

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H) \qquad \mathrm{id}_A - ip = d_A h + h d_A$$

and any $\mathcal{P}_\infty$-algebra structure on A, there exists a $\mathcal{P}_\infty$-algebra structure on H such that i extends to an ∞-quasi-isomorphism.

"Application": [wheeled properads, Merkulov '10]
perturbation theory in QFT $=$ HTT for unimodular Lie bialgebras:
Feynman diagrams $=$ Graphs formulae for transferred structure.

Homotopy theory of dg $\mathcal{P}$-algebras

Theorem (V.)

- *Every ∞-qi of $\mathcal{P}_\infty$-algebras admits a homotopy inverse.*
- $\boxed{\mathrm{Ho}(\text{dg } \mathcal{P}\text{-alg}) := \text{dg } \mathcal{P}\text{-alg}\,[qi^{-1}] \cong \infty\text{-dg } \mathcal{P}\text{-alg}/\sim_h}$.

Proof. Use

$$\begin{tikzcd}
\text{dg } \mathcal{P}\text{-alg} \arrow[d, hook] \arrow[rr, shift right, "\mathrm{B}_\kappa"'] & & (\text{conil}) \text{ dg } \mathcal{P}^{¡}\text{-coalg} \arrow[ll, shift right, "\Omega_\kappa"'] \\
\infty\text{-}\mathcal{P}_\infty\text{-alg} \arrow[u, bend left, "\mathrm{Rect}"] \arrow[rr, leftrightarrow, "\cong"] & & \text{quasi-free } \mathcal{P}^{¡}\text{-coalg} \arrow[u, hook]
\end{tikzcd}$$

$+$ Model Category on (conil) dg $\mathcal{P}^{¡}$-coalg: $\underline{\text{we} \subsetneq \text{qi}}$

$+$ Rectification:

$$\exists\ Rect : \infty\text{-}\mathcal{P}_\infty\text{-Alg} \to \text{dg } \mathcal{P}\text{-alg, s.t. } H \overset{\sim}{\rightsquigarrow} Rect(H)$$

Plan

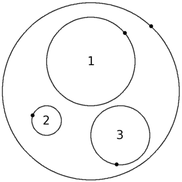
Batalin–Vilkovisky algebras

Definition (Batalin–Vilkovisky algebra)

Graded commutative algebra $(A, d_A, \cdot)$ endowed with a linear operator $\Delta^2 = 0$, $d_A\Delta + d_A\Delta = 0$, of order 2:

$$\Delta(abc) = \Delta(ab)c + \Delta(bc)a + \Delta(ca)b - \Delta(a)bc - \Delta(b)ca - \Delta(c)ab\,.$$

Examples : $H_\bullet(TCFT)$, $\mathbb{H}_\bullet(\mathcal{L}X)$ (string topology), Dolbeault complex of Calabi-Yau manifolds, the bar construction BA, etc.
Operadic topological interpretation: $H_\bullet(fD_2) = BV$.



Homotopy BV-algebras

Theorem (Galvez–Tonks–V.)

The inhomogeneous Koszul duality theory provides us with a quasi-free resolution $BV_\infty := \Omega BV^{i} \xrightarrow{\sim} BV$.

Proof. Problem:

$BV \cong \mathcal{T}(\cdot, \Delta)/$(homogeneous quadratic and cubical relations)

Solution: Introduce

$$[\text{-},\text{-}] \ := \ \Delta \circ (\text{-} \cdot \text{-}) \ - \ (\Delta(\text{-}) \cdot \text{-}) \ - \ (\text{-} \cdot \Delta(\text{-}))$$

a degree 1 Lie bracket $\Longrightarrow$ new presentation of the operad BV:

$$BV \ \cong \ \mathcal{T}(\cdot, \Delta, [\, ,\,])/(\text{inhomogeneous quadratic relations}) \ .$$

$\square$

Application: BV_∞-algebras & ∞-morphisms.

Corollary: HTT & Ho(dg BV-alg) $\cong$ ∞-dg BV-alg$/ \sim_h$.

Applications in Mathematical Physics

Application: Lian–Zuckerman conjecture for Topological Vertex Operator Algebra.

Theorem (Lian–Zuckerman '93)

$$H^{\bullet}_{BRST}(TVOA) : BV\text{-}algebra.$$

Theorem (Lian–Zuckerman conjecture, Galvez–Tonks–V)

$C^{\bullet}_{BRST}(TVOA) = TVOA$: **explicit** *BV_∞-algebra, which lifts the Lian–Zuckerman operations.*

Remarks:

- Lian–Zuckerman conjecture similar to the Deligne conjecture.
- Conjecture: some converse should be true, i.e. $BV_\infty \cong TVOA$.

Application in Geometry

Theorem (Barannikov–Kontsevich–Manin)

$(A, d, \cdot, \Delta)$ dg BV-algebra satisfying the $d\Delta$-lemma

$$\ker d \cap \ker \Delta \cap (\mathrm{Im} d + \mathrm{Im} \Delta) = \mathrm{Im}(d\Delta) = \mathrm{Im}(\Delta d)$$

$\Longrightarrow H_\bullet(A, d)$ carries a Frobenius manifold structure, which extends the transferred commutative product.

Application: B-side of the Mirror Symmetry Conjecture.

Question: Application of the HTT for BV_∞-algebras???

$$BV^{¡} \cong T^c(\delta) \otimes Com_1^* \circ Lie^* \stackrel{???}{\longleftrightarrow} H_\bullet(\overline{\mathcal{M}}_{0,n+1}),$$ so, not yet!

Topological interpretation: homotopy trivialization of S^1

Conjecture: [Costello–Kontsevich] $fD_2 \,/_h\, S^1 \cong \overline{\mathcal{M}}_{0,n+1}$.

Theorem (Drummond-Cole – V.)

Minimal model of BV : $\mathcal{T}\big(T^c(\delta) \oplus H^{\bullet+1}(\mathcal{M}_{0,n+1})\big) \xrightarrow{\sim} BV$.

Application: New notion of BV_∞-algebras.

Homotopy trivialization of the circle $\Longleftrightarrow$ trivial action of $T^c(\delta)$

$$H_\bullet(\overline{\mathcal{M}}_{0,n+1})^{¡} = H^{\bullet+1}(\mathcal{M}_{0,n+1}) \text{ \& Koszul} \qquad \text{[Getzler '95]}$$

Solution of the conjecture over $\mathbb{Q}$

$$BV_\infty \,/_h\, \Delta = \underbrace{\mathcal{T}(H^{\bullet+1}(\mathcal{M}_{0,n+1}))}_{\text{homotopy Frobenius manifold}} \xrightarrow{\sim} \underbrace{H_\bullet(\overline{\mathcal{M}}_{0,n+1})}_{\text{Frobenius manifold}} .$$

HTT for homotopy BV-algebras with Δ trivialization

[BKM]: $(A, d, \cdot, \Delta)$ dg BV-algebra satisfying the $d\Delta$-lemma $\Longrightarrow$ $H_\bullet(A, d)$ carries a Frobenius manifold structure.

Theorem (Drummond-Cole – V.)

$(A, d, \cdot, \Delta)$ dg BV-algebra satisfying the Hodge–de Rham condition $\Longrightarrow$ $H_\bullet(A, d)$ carries a homotopy Frobenius manifold structure , which extends the Frobenius manifold structure and

$$Rect(H_\bullet(A), d) \sim (A, d, \cdot, \Delta) \text{ in } \mathrm{Ho}(dg\ BV\text{-}alg)$$

$$\begin{tikzcd} H_\bullet(\overline{\mathcal{M}}_{0,n+1}) \arrow[r, "{[BKM]}"] & \mathrm{End}_{H_\bullet(A)} \\ H^{\bullet+1}(\mathcal{M}_{0,n+1}) \arrow[u, dotted, "\kappa"] \arrow[ur, dotted, "{[DCV]}"'] & \end{tikzcd}$$

De Rham cohomology of Poisson manifolds

Theorem (Koszul '85)

(M, π) Poisson manifold $\Longrightarrow$ De Rham complex

$$(\Omega^\bullet M, d_{DR}, \wedge, \Delta := [i_\pi, d_{DR}]): \textit{ BV-algebra.}$$

Theorem (Merkulov '98): M symplectic manifold satisfying the Hard Lefschetz condition $\Longrightarrow$ $H^\bullet_{DR}(M)$: Frobenius manifold.

Theorem (Dotsenko–Shadrin–V.)

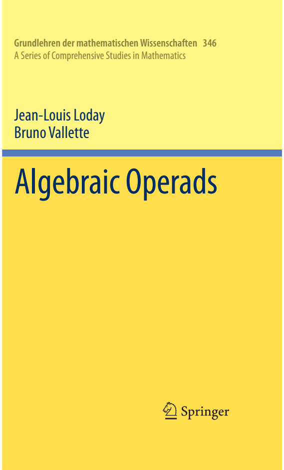
For any Poisson manifold $M \Longrightarrow H^\bullet_{DR}(M)$: homotopy Frobenius manifold , s.t.

$$Rect\left(H_\bullet^{DR}(M)\right) \sim (\Omega^\bullet M, d_{DR}, \wedge, \Delta) \textit{ in } \mathrm{Ho}(dg\ BV\text{-}alg).$$

Generalization: (M, π, E) Jacobi manifold (eg contact),

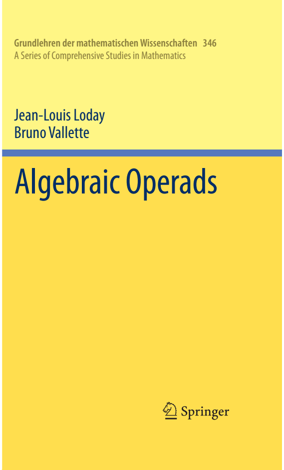
$(\Omega^\bullet \mathcal{M}, d_{DR}, \wedge, \Delta_1 := [i_\pi, d_{DR}], \Delta_2 := i_\pi i_E)$: BV_∞-algebra.

http://math.unice.fr/∼brunov/Operads.html



Thank you!

http://math.unice.fr/∼brunov/Operads.pdf



Thank you!