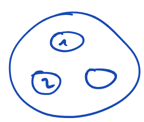
L'opérade des diagrammes de cordes parenthésés, le groupe GRT et les associateurs de Drinfeld

1. PaCD
2. GRT
3. Associateurs

corps de base $\mathbb{K}$ de caractéristique 0

1. même problème $\mathcal{D}_2 \overset{?}{\dashrightarrow} Op$ ("algébrique")



$\sim \mathrm{Conf}_n \mathbb{C}$

1ère réponse: [Hugo]

$\widehat{PaB} \in Op(Grpd)$: "utiliser le foncteur π_1" ⇝ Pb : point base

opérade — cat. sym monoïdale

2ème réponse: construction de Magnus: $G \xrightarrow{\text{monoidal}} \mathrm{gr}\, G := \bigoplus_{k \geq 1} \Gamma_k G / \Gamma_{k+1} G \otimes \mathbb{K}$

(grp, $\times$)

où $\Gamma_1 G := G$

$\Gamma_{k+1} G := [\Gamma_k G, G]$

$\underbrace{(\text{alg de Lie}, \oplus)}_{\text{cat sym mono}}$

Description de $\mathrm{gr}\, \pi_1(\mathrm{Conf}_n \mathbb{C})$

↳ alg de Lie d'holonomie $X \in Top$

$$\mathfrak{g}_X := \underbrace{\mathbb{L}\left(s^{-1} H_1(X), s^{-2}\, \mathrm{Im}\, \Delta\right)}_{\text{alg de Lie quadratique}}$$

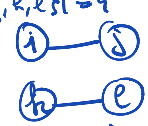
où $\Delta = {}^t\cup : H_2(X) \to H_1(X)^{\otimes 2}$

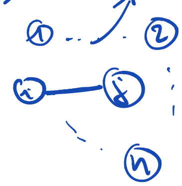
$\mathfrak{g}_{\mathcal{D}_2} \in Op(\text{alg de Lie})$

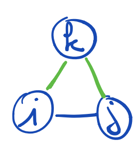
$\boxed{\text{Thm}}$ [Kohno '85]

$$\mathrm{gr}(\pi_1(\underset{\simeq\, \mathrm{Conf}_n \mathbb{C}}{\mathcal{D}_2(n)}) \otimes \mathbb{K} \cong \mathfrak{g}_{\mathcal{D}_2(n)} \cong \mathbb{L}\left(\{t_{ij}\}_{1 \leq i < j \leq n} \,;\, t^n_{ij} \wedge t^n_{kl} \ \& \ t^n_{ij} \wedge (t^n_{ik} + t^n_{jk})\right)$$

$|\{i,j,k,l\}| = 4$







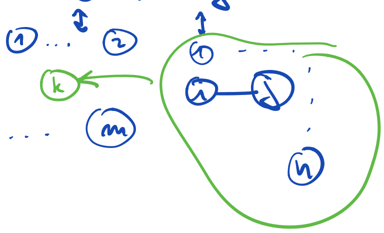
algèbres de Lie des tresses infinitésimales
aka
algèbres de Lie de Drinfeld-Kohno

$$= \underbrace{\mathfrak{t}^n}_{= H^\bullet(\mathrm{Conf}_n \mathbb{C})^!}$$

↳ Structure d'opérades:

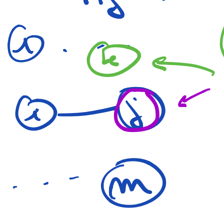
$$t^m \oplus t^n \xrightarrow{\text{ok}} t^{m+n-1}$$

$0 + t_{ij}$



$\longmapsto t_{i+k-1\; j+k-1}$

$t_{ij} + 0$



$\longmapsto t_{ij}$

$\longmapsto \sum_{1\leq l \leq n} t_{il}$

$\mathcal{U}$: alg enveloppante

$\xrightarrow{\text{sym monoï}}$ opérade dans la catégorie des alg de Hopf (cogèbres coassociatives)

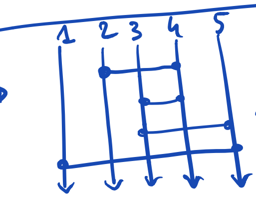
$$\hat{\mathcal{U}}(t^n) = \hat{A}\underbrace{\left(t_{ij}, [t_{ij}, t_{kl}] \;\&\; [t_{ij}, t_{ik}+t_{jk}]\right)}_{\text{alg asso quadratique}}$$

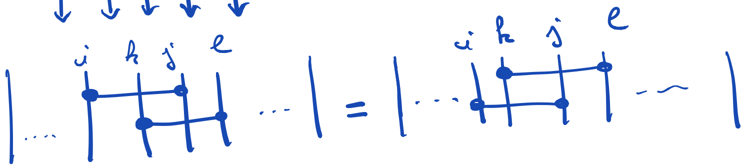
$\hat{T}$

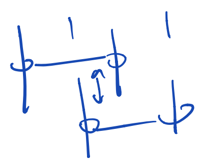
Opérade des diagrammes de cordes : $\boxed{\hat{CD}}$ eOp($\hat{cog}$)

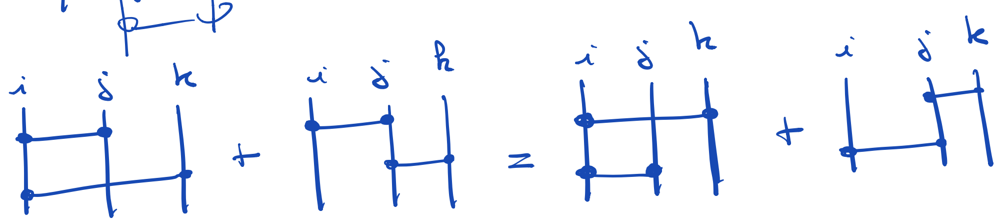
↕

Vassiliev: invariants des nœuds [Bar-Natan]

↳  $\longleftrightarrow$ $t_{15}\, t_{35}\, t_{34}\, t_{24}$ ⟵

* 



** 

Rappel $\widehat{\text{cogèbres}} \xrightarrow[\text{prouip}]{\text{Grp}}$ Grp (groupes)

$\downarrow \qquad\qquad \downarrow$

Cat($\widehat{\text{cog}}$) $\xrightarrow[]{\text{Grp}\ \text{prouip}}$ Grpd (groupoïdes)

Voir l'opérade $\hat{CD}$ eOp(Cat($\widehat{\text{cog}}$))

$\hat{CD}(n) \to \text{cat} \quad \downarrow \quad \hat{\mathcal{U}}(t^n)$

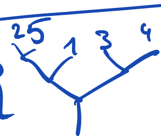
la catégorie des catégories enrichies en cogèbres

$\downarrow$ (C,Δ) $\longrightarrow \{x\in C \mid \Delta x = x\otimes x\}$

cat. objet $= *$

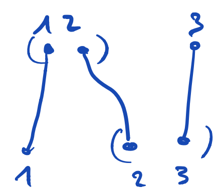
morph $= \mathrm{Morph}(*,*) = C$

Rappel: $\mathrm{Pa} := \underset{\text{générique}}{T(Y)} \in \mathrm{Op}(\mathrm{Set})$ (Magmas binaires)

$\mathrm{Pa}(n) = \Big\{$  $\leftrightarrow ((25)1)(34)$ parenthésage max $\Big\}$

$\rightsquigarrow \mathrm{Op}(\mathrm{Cat}(\mathrm{cog}))$: $\mathrm{Pa}(n)$: **objets**: parenthésages max

morph: $\mathrm{Hom}(\downarrow,\downarrow) := \mathbb{k}$.



Def: $\widehat{\mathrm{PaCD}} := \mathrm{Pa} \underset{\text{tenseur arité par arité}}{\otimes} \widehat{\mathrm{CD}}$

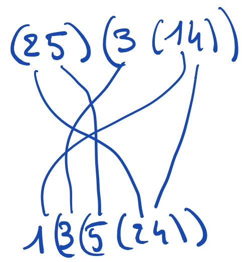
$\in \mathrm{Op}(\mathrm{Cat}(\widehat{\mathrm{cog}}))$

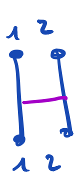
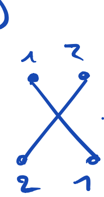
$\widehat{\mathrm{PaCD}}(n)$: une catégorie enrichie en cogèbres

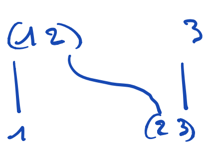
objets: parenthésages maximaux $(25)(3(14))$

morph: ceux de $\widehat{\mathrm{CD}}(n)$

$\hookrightarrow$ un élément de $\widehat{\mathrm{PaCD}}$:

 $\otimes\ \overline{t_{15}\ t_{35}\ t_{34}\ t_{24}}$

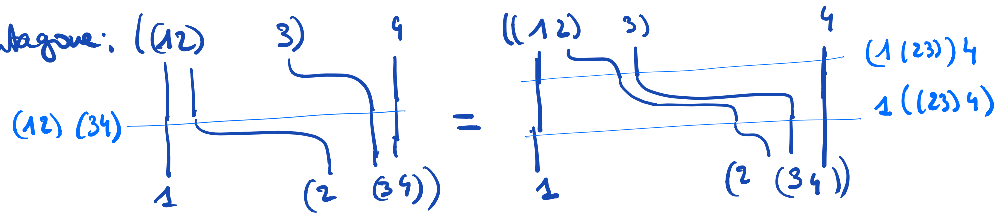
Ex: $H_{1,2} :=$  $t_{12} \in \mathrm{PaCD}(2)$; $\tau_{1,2} :=$  $\cdot 1 \in \mathrm{PaCD}(2)$

$\alpha_{1,2,3} :=$  $\cdot 1 \in \mathrm{PaCD}(3)$

Lemme: $H_{1,2}$, $\tau_{1,2}$ et $\alpha_{1,2}$ engendrent l'opérade $\widehat{PaCD}$

Relations $(\tau_{1,2})^{-1} = \tau_{2,1}$

pentagone:



$$\alpha_{1,2,34} \circ \alpha_{12,3,4} = \alpha_{2,3,4} \circ \alpha_{1,23,4} \circ \alpha_{1,2,3}$$

relations hexagonales:

$$\begin{tikzcd}
 & (12)3 \arrow[dl, "\tau"'] \arrow[dr, "\alpha"] & \\
(21)3 \arrow[d, "\alpha"'] & & 1(23) \arrow[d, "\tau"] \\
2(13) \arrow[dr, "\tau"'] & & (23)1 \arrow[dl, "\alpha"] \\
 & 2(31) &
\end{tikzcd}$$

$\underline{\underline{Z}}$ $\widehat{PaCD}$: on n'a pas toutes les relations

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② GRT

Rappel: $\widehat{GT} := \mathrm{Aut}^{+}_{Op(Grpd)}(\widehat{PaB})$ id sur les objets

version graduée

Def. $\widehat{GRT} := \overline{\mathrm{Aut}}^{+}_{Op(Grpd)}(\mathrm{Grp}\,\widehat{PaCD}) \cong \overline{\mathrm{Aut}}^{+}_{op(cat(cog))}(\widehat{PaCD})$

grap-like

qui préserve $\tau_{1,2}$

$\widehat{GRT}_1 :=$ $=$ $=$

qui préserve τ_{12} et $H_{1,2}$

$\boxed{\text{Thm}}$ [Drinfeld, Bar-Natan]

1) $\widehat{GRT_1} \simeq \{ \Phi(x,y) \in Grp(\underbrace{\hat{U}(\widehat{Lie}(x,y))}_{\hat{T}(x,y)}) \text{ tq } \Phi(x,y) = \Phi(y,x)^{-1}$

$\Phi = \varphi(x,y)$, $e^{\varphi} = 1 + x + \dots$, $\hat{T}(x,y) = e^{(\widehat{Lie}(x,y))}$

série en 2 variables qui vit dans $\widehat{Lie}(x,y)$

$x + 3y + \frac{1}{2}[x,y] + \dots$

$F(\hexagon)$

$\underbrace{\Phi(t_{12},t_{23})\Phi^{-1}(t_{13},t_{23})\Phi(t_{13},t_{12}) = 1}_{\in \hat{U}(t^3)}$

$\underbrace{\Phi(t_{13}+t_{23},t_{34})\Phi(t_{12},t_{23}+t_{24}) = \Phi(t_{12},t_{23})\Phi(t_{12}+t_{13},t_{24}+t_{34})\Phi(t_{23},t_{34})}_{\in \hat{U}(t^4)}$ $\}$ $\leftrightarrow F(\pentagon)$ vérifiée pareil

2) $\widehat{GRT} \cong \mathbb{k}^{\times} \ltimes \widehat{GRT_1}$

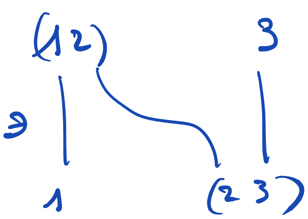
Démo: $\widehat{PaCD} \xrightarrow{F} {}^{op}\widehat{PaCD}$ automorphisme

Grp

$\tau_{12} \longmapsto \tau_{12}$

$H_{12} \longmapsto H_{12}$

$\alpha_{1,2,3} \longmapsto \widehat{PaCD}(3) \ni$



$\boxed{?} \in \hat{U}(t^3)$

$\cancel{f(c_3)} \cdot \Phi(t_{12},t_{23})$

$F(\alpha_{1,2,3})$

Lemme: $\left[\begin{array}{l} c_3 := t_{12}+t_{13}+t_{23} \text{ central dans } t^3 \\ t^3 \simeq \mathbb{k}c_3 \oplus \underbrace{Lie(t_{12},t_{23})}_{} \end{array} \right] \Rightarrow {}^{Grp}\hat{U}(t^3) \trianglelefteq \mathbb{k}[[c_3]] \otimes {}^{Grp}\hat{T}(t_{12},t_{23})$

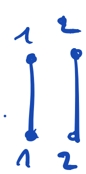
iso d'alg de Lie — abélienne — Lie libre

$\alpha_{1,2,3}$ inversible $\Rightarrow F(\alpha_{1,2,3})$ inversible $\Rightarrow f(c_3)$ inversible $f(c_3) = 1 + \dots$

Φ inversible

relation pentagone: $f^2 = f^3 \Rightarrow f = 1$

2) $\underset{\underset{PaCD(2)}{\cap}}{H_{12}} \xmapsto{F} PaCD(2) \ni$



$\boxed{} \in Grp(\underbrace{\hat{U}(t^2)}_{\hat{T}(t_{12})}) = e^{\lambda t_{12}}$, $\lambda \in \mathbb{k}^{\times}$

$t^2 = Lie(t_{12}) = \mathbb{k}t_{12}$

$\hookrightarrow GRT \trianglelefteq \underset{\underset{J}{H_{12}}}{\mathbb{k}^{\times}} \ltimes \underset{\underset{J}{\alpha_{123}}}{GRT_1}$

$\square$

$\underline{\text{Rk}}$: $\widehat{GRT_1}$ groupe prounipotent $= e^{\mathfrak{grt}_1}$ ← alg de Lie

$\parallel$ [Willwacher]

$\mathfrak{grt}_1 \cong H_0\big(\underbrace{\mathrm{Def}(\mathcal{L}ie \to GC)}\big)$

alg Lie de déformation du morphisme d'opérad $\mathcal{L}ie \to GC$

→ Déf des morphismes d'opérades modulaires $\overline{\mathcal{M}}_{g,n}$

③ Associateurs de Drinfeld

$\underline{\text{Def}}$:

$$\mathrm{Assoc} := \mathrm{Iso}^{+}_{\mathrm{Op}(\mathrm{Grpd})}\big(\widehat{\mathrm{PaB}}, \mathrm{Grp}(\widehat{\mathrm{PaCD}})\big)$$

$\boxed{\text{Thm}}$ $\mathrm{Assoc} \cong \Big\{ (\lambda, \Phi) \in \mathbb{K}^\times \times \exp(\widehat{\mathcal{L}ie}(x,y))$ tq $\Phi(x,y)^{-1} = \Phi(y,x)$

$\exp(\widehat{\mathcal{L}ie}(x,y)) = \mathrm{Grp}(\hat{T}(x,y))$

$$\left.\begin{aligned} &\Phi(t_{13}+t_{23}, t_{34})\,\Phi(t_{12}, t_{23}+t_{24}) = \\ &\underbrace{(t_{12},t_{23})\,\Phi(t_{12}+t_{13}, t_{24}+t_{34})\,\Phi(t_{23},t_{34})}_{\subset\ \hat{U}(t^4)} \end{aligned}\right\} \leftrightarrow F(\pentagon)$$

$$e^{\frac{\lambda}{2}t_{13}}\,\Phi(t_{12},t_{13})\,e^{\frac{\lambda}{2}t_{12}} = \Phi(t_{23},t_{13})\,e^{\frac{\lambda}{2}t_{12}}\,\Phi(t_{12},t_{23}) \leftrightarrow F(\hexagon)$$

$\underline{\text{Dem}}$: $\widehat{\mathrm{PaB}}$: générateurs $\tau, \alpha \xrightarrow{F} \widehat{\mathrm{PaCD}}$

$\tau \longmapsto$

$\alpha \longmapsto$ même argument $f(c_3)\,\Phi(x,y)$, $f(c_3) = 1$

$$\underset{\mathrm{PaB}(2)}{\tau} \overset{F}{\longmapsto} \mathrm{PaCD}(2) \ni \Big|\ \Big| \;\Box \in \mathrm{Grp}\big(\hat{U}(t^2)\big) = e^{\frac{\lambda}{2}t_{12}}$$

$\hat{T}(t_{12})$, $\lambda \in \mathbb{K}^\times$

$t^2 = \mathcal{L}ie(t_{12}) = \mathbb{K}\,t_{12}$

vérifier l'eqF($\pentagon$), F($\hexagon^{\pm}$) ⟹ morph d'op en groupoïdes

⊕ F iso.

□

<u>Exemples:</u> Φ_{KZ} : connexion de Knizhnik-Zamolodchikov

$\shortparallel$

séries en x, y avec coefficients multizetas.

Φ_{AT} $\longleftrightarrow$ théorie des nombres

$\longleftrightarrow$ questions de formalité : Kontsevich

Alekseev-Torossian

$\mathcal{D}_2$

quantification

par déformation

$\underbrace{\text{Alekseev-Torossian}}$ $\to$ Conj de Kashiwara-Vergne