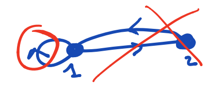
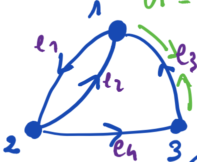
Graph Homology

$-0 = $  $= 0$

0

$e_1 \wedge e_2 \wedge e_3 \wedge e_4$

$d = \quad \overset{d}{\mapsto} \sum_i (-1)^i e_1 \wedge \dots \wedge de_i \wedge \dots \wedge e_4$



edge contraction

Graph: combinatorial object

Homology: boundary map — linear comb.

sgn S_n — $n = \#$vertices

$\rightarrow - - \leftarrow$

$\boxed{\text{field } k \text{ of char} = 0}$

---

① Kontsevich original approach '93 — iso of Hopf alg — "commutative graph cx"

$\boxed{\text{Thm}}$ $H_\bullet^{CE}\left(\mathbb{R}[p_1, p_2, \dots, q_1, q_2, \dots]_{\geq 1}, \{\,;\,\}\right) \cong \Lambda H_\bullet(\mathcal{G})$

∞ number

polynomial symplectic vector fields over $\mathbb{R}^{2\infty}$ which vanish at 0

Chevalley-Eilenberg homology

no cst term

Lie (Poisson) bracket

$$\{P, Q\} := \sum_{i=1}^{\infty} \frac{\partial P}{\partial p_i}\frac{\partial Q}{\partial q_i} - \frac{\partial P}{\partial q_i}\frac{\partial Q}{\partial p_i}$$

Rk: • LHS: ∞-dimensional — RHS: "simple" finite dimensional ?? .

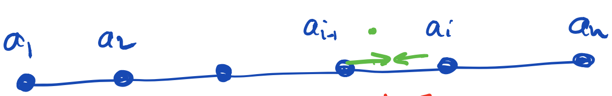
• stable result: with $p_1, \dots, p_n, q_1, \dots, q_n$

---

② Graph homology before Kontsevich

Bar construction [Cartan] : $BA : [a_1 | \dots | a_n] \overset{d}{\longmapsto} \sum_{i=1}^{n-1} (-1)^i [a_1 | \dots | a_i a_{i+1} | \dots | a_n]$

assoc alg



Bar construction for operads [Ginzburg-Kapranov '94]

$\boxed{\text{Thm}}$ [Loday-Quillen-Tsygan '83] A: unital associative alg

$$H_\bullet^{CE}(\mathfrak{gl}(A)) \underset{\text{Hopf alg}}{\cong} \Lambda HC_{\bullet-1}(A)$$

can be computed with the chain cx $A^{\otimes n+1} \circlearrowleft \mathbb{Z}/_{n+1}\mathbb{Z}$ cyclic group

Lie algebra of matrices of size ∞ with coefficients in A (0 nearly everywhere).

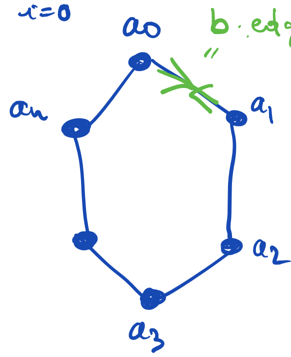
τ_{n+1} generator of $\mathbb{Z}/_{n+1}\mathbb{Z}$

$$A^{\otimes n+1}/_{\mathbb{Z}/_{n+1}\mathbb{Z}} \cong A^{\otimes n+1}/_{id-\tau_{n+1}}$$

$(a_0, a_1, \ldots, a_n) = \pm(a_1, \ldots, a_n, a_0)$

$$b(\;\;{}''\;\;) = \sum_{i=0}^{n-1}(-1)^i (a_0, \ldots, a_ia_{i+1}, \ldots, a_n) + (-1)^n (a_na_0, a_1, \ldots, a_{n-1})$$

$\xleftrightarrow{1-1}$



[Kontsevich]

Proof: $gl(A) = \mathrm{colim}_n\, gl_n(A)$

① $\underline{gl_n(k)} \hookrightarrow gl_n(A)$ ($sp(2n)$)

$\Rightarrow$ action on $\uparrow$

$\Rightarrow$ action on $H^{CE}_\bullet(gl(A))$

② Use classical theory of invariants [Weyl]

$$(gl^{\otimes n})_{gl} \cong k[S_n]$$

③ Apply Cartier-Milnor-Moore (Borel)

$\mathcal{H}$ com-cocom conn. Hopf alg

$$\cong S_\wedge(\mathrm{Prim}\,\mathcal{H}) \quad \square$$

where $sp(2n) = \mathbb{R}[p_1, \ldots, p_n, q_1, \ldots, q_n]_2$ ← monomials of weight 2

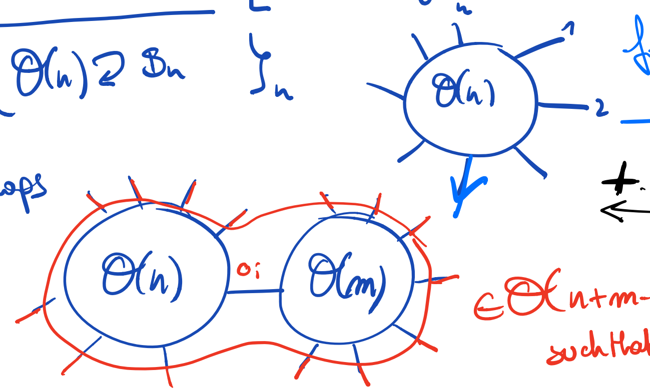
③ Generalisation of Kontsevich result [Conant-Vogtmann, '03]

Def: Cyclic operad $\{\mathcal{O}(n) \circlearrowleft S_n\}_n$

forget full

symm op.

composition maps



$\in \mathcal{O}(n+m-2)$ such that ....

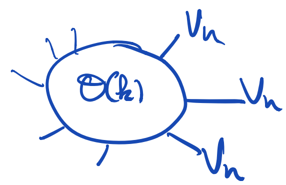
Construction: $V_n := k\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$: canonical symplectic manifold

$$\langle p_i, q_j \rangle = \delta_{ij} = -\langle q_j, p_i \rangle$$

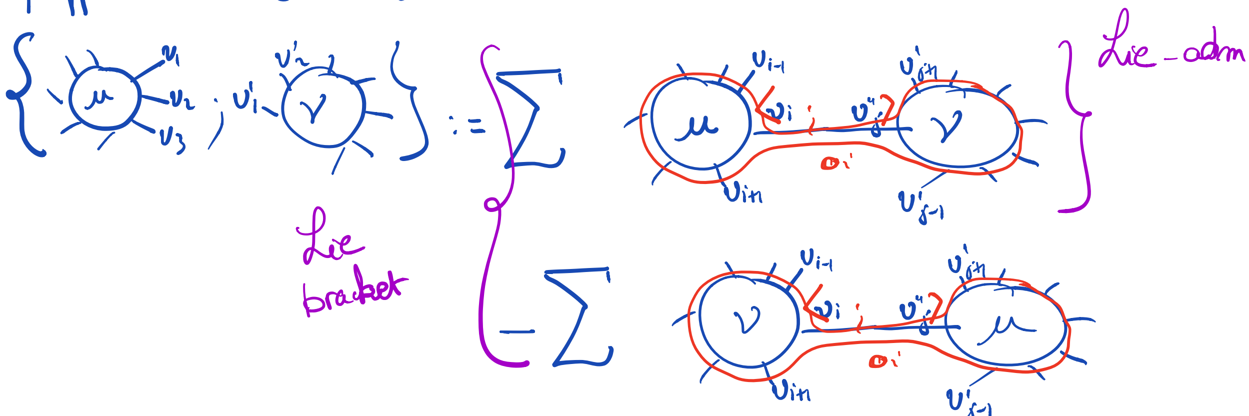
$$\langle p_i, p_j \rangle = 0 \qquad \langle q_i, q_j \rangle = 0$$

| "free $\mathcal{O}$ alg over V_n"

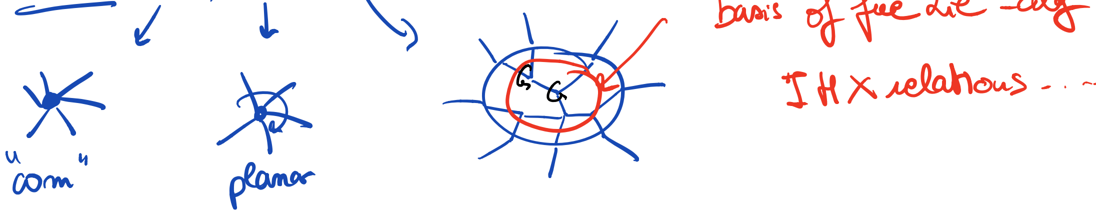
$$\Theta(V_n) := \bigoplus_{k\geq 2} \left(\Theta(k)\otimes V_n^{\otimes k}\right)_{S_k}$$



equipped with a generalized "Poisson" bracket



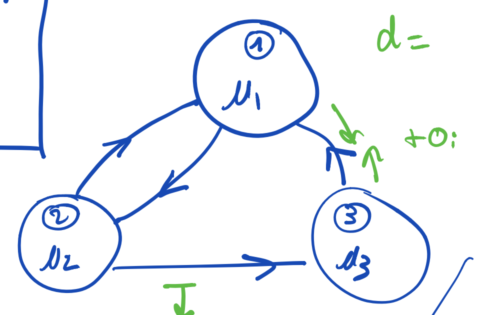
Ex: Com, Ass, Lie



$\boxed{\text{Thm}}$ [Conant-Vogtmann-Kontsevich]

$$H^{CE}_\bullet\left(\operatorname{colim}_n \Theta(V_n)\right) \underset{\text{Hopf alg}}{\cong} \Lambda H_\bullet(\mathcal{G}_c\Theta)$$

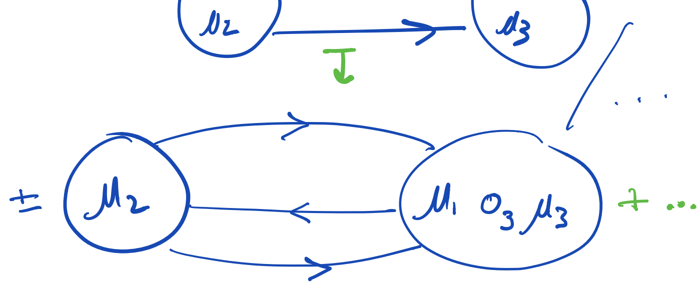
Complexe of connected graphs labelled by Θ



$\mu_1 \in \Theta(3)$

$\mu_2 \in \Theta(3)$

$\mu_3 \in \Theta(2)$



Rk: $H(\mathcal{G}_c \mathrm{Ass})$ "=" $H^\bullet$(moduli space of punctured surfaces)

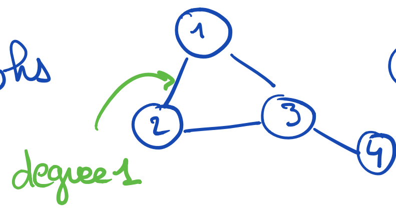
$H_\cdot(\mathfrak{g}_c\mathcal{L}ie) \longleftrightarrow H($outer automorphisms of free group$)$

④ Kontsevich graph operad

$|p_i|=0 \quad |q_i|=-1$ $\qquad H_\cdot(\mathcal{D}_2)$ little discs op

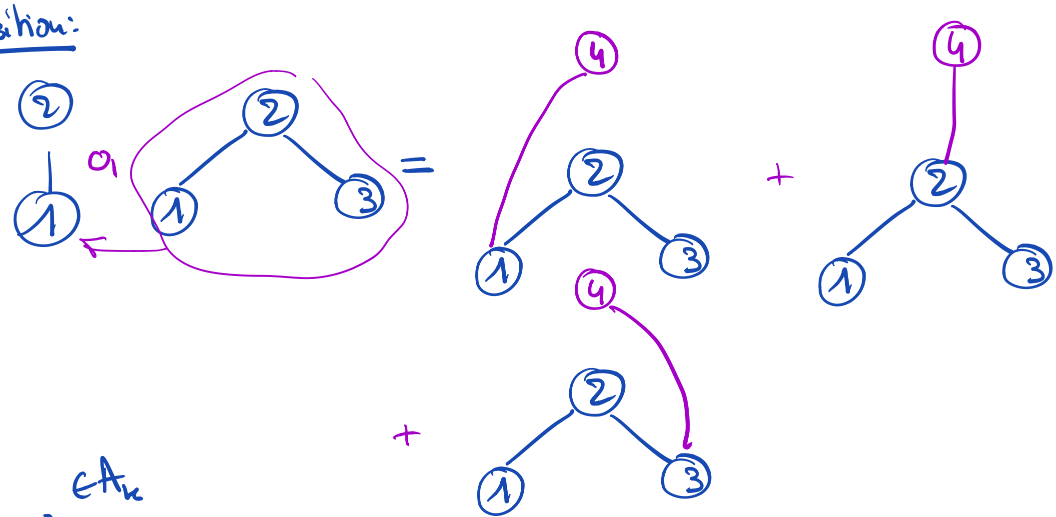
$$A_n := \left(k[p_1,\dots,p_n,q_1,\dots,q_n],\ \cdot\ ,\{\ ;\ \}\right) : \text{Gerstenhaber alg} = \text{Gerst\_alg}$$

operad of all operations on A_n $\longrightarrow$ $\boxed{\mathrm{Gra}}$: graphs



(5)—$(6) \in \mathrm{Gra}(6)$

Composition:



$\mathrm{Gra}(n) \ni \gamma$

$$\gamma(P_1,\dots,P_k) = \text{product}\Big(\prod \{P_i,P_j\}\Big) \in A_k \qquad \text{o—o} \text{ "=" } \{\ ;\ \}$$

(i)—$(j) \in \gamma$

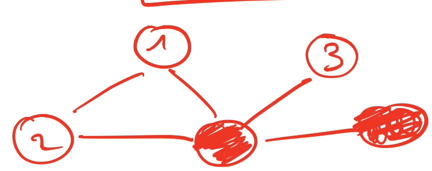
$\mathrm{Com} \longrightarrow \mathrm{Gerst} \longrightarrow \mathrm{Gra}$; $\ \bullet \longmapsto (1)\ (2)$

$\mathrm{sLie} \longrightarrow \mathrm{Gerst}$, $\ \mathrm{sLie} \longrightarrow \mathrm{Gra}$; $\ \{\ ;\ \} \longmapsto (1)$—

Twisting procedure for operads [Willwacher]

$$\mathrm{TwGra} := \Big(\underbrace{\mathrm{Gra} \vee \bullet^{\text{arity } 0}}_{\text{op coproduct}},\ d\Big)$$

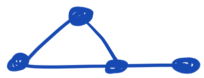
$(1)\!\bullet$—$\bullet$

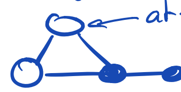


$\mathrm{TwGra\_alg}$ "=" $\mathrm{Gra\text{-}alg}$ + MC elements

MC elements $\% \{i\}$ $\quad$ $\exists$ 1 —$\bullet \in MC(Gra(1))$

$$TwGra = TwGra(0) \oplus \underbrace{TwGra}_{\text{arity} \geq 1} \overset{\sim}{=} Graphs \quad : \text{dg operad}$$





at least one

$\boxed{\text{Thm}}$ [Lambrechts-Volic, Kontsevich, Willwacher] Formality of little discs

$$\begin{matrix} Gerst & \xleftarrow{\sim} & Graphs & \xrightarrow{\sim} & \Omega^*_{PA}(FM_2) \\ \| & & & & \\ H_\bullet(\mathcal{D}_2) & & & & \end{matrix}$$

Willwacher: $Def(s\,Lie \longrightarrow Gra) \cong \underbrace{(Hom_{\mathfrak{S}}(Lie^*, Gra), [\,,\,], \partial)}_{\text{convolution Lie alg}}$

$$\uparrow\wr$$

$$\Omega\, Lie^* = \mathcal{T}(s^{-1} Lie^*)$$

$$\cong TwGra(0)$$

$\cup$ sub dg Lie alg made up of connected $\geq$ trivalent graphs

Kontsevich graph cx : GC

$\boxed{\text{Thm}}$ [Willwacher]

$$\boxed{H^0(GC) \cong \underset{\text{Lie alg}}{grt_1}}$$

Theory of twisting procedure $\Longrightarrow$ $\underset{\cup\, GC}{TwGra(0)} = Def(sLie \to Gra) \circlearrowright TwGra$ acts by derivations $\quad TwGra \underset{\wr}{\ } Ger$

$$\underset{\text{Lie alg}}{grt_1} \xrightarrow{\text{morph of}} H_0(Der(Gerst_\infty))$$

$\boxed{\text{Thm}}$ [Willwacher]

$$\boxed{grt := grt_1 \rtimes k \cong H_0(Der(Gerst_\infty))}$$

$$\Downarrow$$

$$H_0 Aut(\mathcal{D}_2^{\mathbb{Q}}) \cong GT$$