WORKSHOP ON FACTORIZATION HOMOLOGY — TALK 1
HIGHER EILENBERG–STEENROD AXIOMS

DANIEL ROBERT-NICOUD

CONTENTS

The material presented here is extracted from Sections 1, 2.1–3, and 10 of [1].

1. Motivation: classical Eilenberg–Steenrod axioms

In the 40s, Eilenberg and Steenrod gave axioms that uniquely determine the homology theory of pairs of spaces. These axioms can be lifted to the chain level to give the following definition of a homology theory.

Definition 1.1. *A homology theory for spaces is a functor*

$$\mathcal{H} : \mathsf{Top} \longrightarrow \mathsf{Chain}(\mathbb{Z})$$

from topological spaces to differential graded abelian groups satisfying the following properties.

(1) *Homotopy invariance, i.e. $\mathcal{H}$ sends homotopies between continuous maps to homotopies between maps of chain complexes.*
(2) *The functor $\mathcal{H}$ is determined by its value on connected components, i.e. we want that the canonical maps*

$$\bigoplus_{i\in I}\mathcal{H}(X_i) \longrightarrow \mathcal{H}\left(\bigsqcup_{i\in I} X_i\right)$$

to be a weak equivalence. In other words, $\mathcal{H}$ is monoidal.
(3) *We want $\mathcal{H}$ to satisfy excision. Let*

$$Z \xrightarrow{i} X\ , \qquad Z \xrightarrow{j} Y$$

be inclusions of a closed subspace, and denote

$$\mathcal{H}(Z) \xrightarrow{i_*} \mathcal{H}(X)\ , \qquad \mathcal{H}(Z) \xrightarrow{j_*} \mathcal{H}(Y)\ .$$

We obtain a map

$$\mathcal{H}(Z) \xrightarrow{i_*-j_*} \mathcal{H}(X)\oplus\mathcal{H}(Y)\ ,$$

and the canonical map

$$\mathcal{H}(X)\oplus\mathcal{H}(Y) \longrightarrow \mathcal{H}(X\cup_Z Y)\ ,$$

whose composition with $i_* - j_*$ is zero. We require that the canonical map

$$\mathrm{cone}\left(\mathcal{H}(Z) \xrightarrow{i_*-j_*} \mathcal{H}(X)\oplus\mathcal{H}(Y)\right) \longrightarrow \mathcal{H}(X\cup_Z Y)$$

is a homotopy equivalence.

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Axiom (3) is not really excision in the classical sense, but rather Meyer–Vietoris, and it is known that (1)+(2)+(Meyer–Vietoris) is equivalent to (1)+(2)+(excision). Let's detail this a bit. If $f : A \to B$ is a chain map, then we have a triangle

$$A \xrightarrow{f} B \longrightarrow \mathrm{cone}(f) \longrightarrow sA ,$$

where sA is the suspension of A. Axiom (3) tells us that the homology of the cone of $i_* - j_*$ is the same as the homology of $X \cup_Z Y$. Taking the long sequence associated to the triangle of the cone and passing to homology, we recover Meyer–Vietoris.

It is a well-known result that the axioms above plus a choice of value for $\mathcal{H}(pt)$ determine a unique homology theory.

Theorem 1.2 (Eilenberg–Steenrod)**.** *Let G be an abelian group. Up to natural homotopy equivalences, there exists a unique homotopy theory*

$$\mathcal{H} : \mathsf{Top} \longrightarrow \mathsf{Chain}(\mathbb{Z})$$

such that $\mathcal{H}(pt) \simeq G$.

Of course, this theory is nothing other than the usual singular chains with coefficients in G. This result remains true if we take G to be an element of $\mathsf{Chain}(\mathbb{Z})$ instead of an abelian group (i.e. a complex concentrated in degree 0). Therefore, $\mathcal{H}$ gives a functor

$$\mathcal{H} : \mathsf{Top} \times \mathsf{Chain}(\mathbb{Z}) \longrightarrow \mathsf{Chain}(\mathbb{Z}) ,$$

which is well defined up to natural homotopy equivalences.

The goal is now to do something analogous in the derived setting, i.e. ∞-categories, and to replace $\mathsf{Chain}(\mathbb{Z})$ by some other (symmetric monoidal ∞-)category.

2. Recollection on E_∞-algebras

Let k be a commutative, unital ring, and denote by $\underline{\mathsf{Chain}}(k)$ the ∞-category of dg k-modules. One can recover it from the usual model structure on $\mathsf{Chain}(k)$. The derived tensor product $\otimes$ makes it into a symmetric monoidal ∞-category. If P, Q are two chain complexes and k is a field, then their mapping space is

$$\mathrm{Map}(P,Q)_n := \hom_{\mathsf{Chain}(k)}(P \otimes C_\bullet(\Delta^n), Q) .$$

The homotopy category of $\underline{\mathsf{Chain}}(k)$ is the usual derived category $D(k)$ of k-modules. The symmetric monoidal ∞-category $\underline{\mathsf{Chain}}(k)$ is enriched over itself, meaning that for every chain complexes P, Q there is a chain complex $\mathbb{R}\hom_k(P,Q)$ such that

$$\mathrm{Map}(R \otimes P, Q) \simeq \mathrm{Map}(R, \mathbb{R}\hom(P,Q)) .$$

In particular, the derived tensor product commutes with homotopy colimits.

The n-little cube operad Cube_n is the topological operad given by

$$\mathrm{Cube}_n(k) := \mathrm{Rect}\left(\bigsqcup_{i=1}^{k} (0,1)^n, (0,1)^n\right) ,$$

where the right hand side denotes the space of rectilinear embeddings of k disjoint copies of the unit n-cube into one unit n-cube. The operad structure is simply given by composition of embeddings. Taking chains, we obtain an operad over chain complexes $C_\bullet(\mathrm{Cube}_n)$.

Definition 2.1. *An E_n-algebra is an algebra over $C_\bullet(\mathrm{Cube}_n)$.*

The usual (Hinich) model structure on the category of E_n-algebras gives us an ∞-category of E_n-algebras. The symmetric monoidal structure of $\underline{\mathsf{Chain}}(k)$ lifts to a symmetric monoidal structure on this ∞-category of E_n-algebras, which is given by the (derived) tensor product on the underlying chain complexes.

Another point of view is the following. Let $\mathscr{O}$ be a topological operad, then we have an associated symmetric monoidal category with the non-empty, pointed finite sets as objects, and as morphisms $n_+ := \{0, \dots, n\} \to m_+ := \{0, \dots, m\}$ (where 0 is the basepoint) the disjoint union

$$\bigsqcup_{f : n_+ \to m_+} \prod_{i \in m_+} \mathscr{O}(|f^{-1}(i)|_+)$$

The tensor product is given by $n_+ \otimes m_+ := (n+m)_+$. We abuse notation and denote again by $\mathscr{O}$ the associated symmetric monoidal ∞-category.

Definition 2.2. *Let $(\mathscr{C}, \otimes)$ be a symmetric monoidal ∞-category. An $\mathscr{O}$-algebra in $\mathscr{C}$ is a symmetric monoidal ∞-functor from $\mathscr{O}$ to $\mathscr{C}$, an $\mathscr{O}$-coalgebra is a symmetric monoidal ∞-functor from $\mathscr{O}$ to $\mathscr{C}^{op}$.*

Example 2.3. *An E_n-algebra is an element of $\underline{\mathsf{Fun}}^{\otimes}(\mathrm{Cube}_n, \underline{\mathsf{Chain}}(k))$. A commutative dg algebra is an element of $\underline{\mathsf{Fun}}^{\otimes}(\underline{\mathsf{Fin}}_*, \underline{\mathsf{Chain}}(k))$, where $\underline{\mathsf{Fin}}_*$ is the symetric monoidal ∞-category of pointed finite sets.*

There are natural maps

$$pt = \mathrm{Cube}_0 \longrightarrow \mathrm{Cube}_1 \longrightarrow \mathrm{Cube}_2 \longrightarrow \cdots$$

given by taking the product of an n-cube with the whole interval $(0,1)$ in order to get an $(n+1)$-cube. The colimit of the diagram is denoted by Cube_∞, and in E_∞-algebra is an algebra over $C_\bullet(\mathrm{Cube}_\infty)$, or equivalently a symmetric monoidal functor from Cube_∞ to $\underline{\mathsf{Chain}}(k)$.

Example 2.4. *Let X be a topological space, then $C_\bullet(X)$ is an E_∞-coalgebra whose structure is induced by the diagonal map $X \to X \times X$.*

Question 2.5. *If k is a field of characteristic 0, are E_∞-coalgebras equivalent to cocommutative coalgebras?*

Again, the derived tensor product lifts to define a symmetric monoidal structure on the category of E_∞-algebras.

Proposition 2.6. *In the category of E_∞-algebras, the tensor product is a coproduct.*

3. Higher Eilenberg–Steenrod axioms

We will now give the axioms for a homology theory with values in $(\underline{\mathsf{Chain}}(k), \otimes)$ instead of $(\mathsf{Chain}(\mathbb{Z}), \oplus)$.

First notice that the homotopy commutative monoids in $(\underline{\mathsf{Chain}}(k), \otimes)$ are the E_∞-algebras.

Remark 3.1. *If k is a field of characteristic 0, then the ∞-category of E_∞-algebras over k is equivalent to the ∞-category of commutative dg algebras. Sometimes, it can help to think about it in this setting.*

We want functors starting from the symmetric monoidal ∞-category $(\underline{\mathsf{Top}}, \sqcup)$ and landing into the symmetric monoidal ∞-category $(\underline{\mathsf{Chain}}(k), \otimes)$. If X is a topological space, then the identity map induces a canonical map

$$X \sqcup X \xrightarrow{\mathrm{id}_X \sqcup \mathrm{id}_X} X\ ,$$

and thus, every space is canonically a commutative algebra object in $\underline{\mathsf{Top}}$. It follows that so must be the image of topological spaces under any symmetric monoidal functor.

Lemma 3.2. *Let $(\mathscr{C}, \otimes)$ be a symmetric monoidal ∞-category. Any symmetric monoidal functor $F : \underline{\mathsf{Top}} \to \mathscr{C}$ canonically lifts to a functor $\widetilde{F} : \underline{\mathsf{Top}} \to E_\infty\text{-}alg(\mathscr{C})$.*

The good definition of a homology theory is the following.

Definition 3.3. *A homology theory for spaces with values in the symmetric monoidal ∞-category $(\underline{\mathsf{Chain}}(k), \otimes)$ is an ∞-functor*

$$\mathcal{CH} : \underline{\mathsf{Top}} \times E_\infty\text{-}alg \longrightarrow E_\infty\text{-}alg\ ,$$

whose evaluation on an object (X, A) will be denoted by $CH_X(A)$, which satisfies the following axioms.

1. *There is a natural equivalence $CH_{pt}(A) \xrightarrow{\sim} A$.*
2. *The canonical maps*
$$\bigotimes_{i \in I} CH_{X_i}(A) \longrightarrow CH_{\bigsqcup_{i\in I} X_i}(A)$$
are weak equivalences.
3. *The functor $\mathcal{CH}$ commutes with homotopy pushouts of spaces, i.e. the canonical maps*
$$CH_X(A) \overset{\mathbb{L}}{\underset{CH_Z(A)}{\otimes}} CH_Y(A) \longrightarrow CH_{X \cup_Z Y}(A)$$
are weak equivalences.

The first axiom gives us the value of the homology theory on a point, the second one is a strong version of asking that $\mathcal{CH}$ is a symmetric monoidal ∞-functor, while the third one is analogous to excision.

Theorem 3.4. *There is a unique such homology theory for spaces, which is given by derived Hochschild chains, i.e.*

$$CH_X(A) \simeq A \boxtimes X \ ,$$

where $\boxtimes$ is the tensor product of an E_∞-alg with a space, giving back and E_∞-alg. Moreover, given a functor $F : E_\infty$-alg $\to E_\infty$-alg, there is a unique homology theory satisfying axioms (2) and (3) and whose value on a point is $F(A)$, and it is given by $(X, A) \mapsto CH_X(F(A))$.

Remark 3.5. *The tensorization $\boxtimes$ appears as follows. Let $\mathscr{C}$ be an ordinary category with coproduct. Then $\mathscr{C}$ is tensored over* Sets *as follows: if $C \in \mathscr{C}$ and $S \in$ Sets, then*

$$C^S := \coprod_{s\in S} C \ .$$

Going higher, let $\mathscr{C}$ be an ∞-category, by which here we mean a quasicategory, and suppose that we have a realization functor

$$|-| : \mathsf{s}\mathscr{C} \longrightarrow \mathscr{C} \ .$$

Then $\mathscr{C}$ is enriched over sSets *as follows. Let $C \in \mathscr{C}$, and let $K_\bullet \in$ sSets. We have a simplicial object by considering the tensorization over* Sets *described above level by level, i.e.*

$$\left(C^{K_\bullet}\right)_p := C^{K_p}.$$

Define

$$C \boxtimes K_\bullet := \left|C^{K_\bullet}\right| \in \mathscr{C}$$

Later, we will see this in detail for the ∞-category of commutative dg algebras, where the realization is given by the Dold-Kan construction together with the shuffle product.

An immediate interesting corollaries of the theorem is the exponential rule:

$$CH_{X\times Y}(A) \simeq CH_X\big(CH_Y(A)\big)$$

in E_∞-alg.

A possible way to compute derived Hochschild homology is given by the following result.

Proposition 3.6. *Let $X \in \underline{\mathsf{Top}}$ and $A \in E_\infty$-alg. There is a natural equivalence (in $\underline{\mathsf{Chain}}(k)$)*

$$CH_X(A) \simeq C_\bullet(X) \overset{\mathbb{L}}{\underset{E_\infty}{\otimes}} A \ .$$

Moreover, if A is actually a commutative dg algebra, then

$$CH_X(A) \simeq C_\bullet(X) \overset{\mathbb{L}}{\underset{\underline{\mathsf{Fin}}_*}{\otimes}} A \ .$$

Here, the chains $C_\bullet(X)$ are seen as an E_∞-coalgebra with the structure induced by the diagonal map $\Delta : X \to X \times X$, i.e. a right module over the operad $E_\infty := C_\bullet(\mathrm{Cube}_\infty)$. The E_∞-algebra A is seen as a left module over E_∞, and the derived tensor product is given by

$$C_\bullet(X) \overset{\mathbb{L}}{\underset{E_\infty}{\otimes}} A := \mathrm{hocoeq}\left(\coprod_{f:\{0,\dots,q\}\to\{0,\dots,p\}} C_\bullet(X)^{\otimes p} \otimes E_\infty(p,q) \otimes A^{\otimes q} \rightrightarrows \coprod_n C_\bullet(X)^{\otimes n} \otimes A^{\otimes n} \right) .$$

Heuristically, this is a bar construction.

4. Higher Hochschild homology and cohomology

In order to have a higher Hochschild cohomology theory, one is led to consider pointed spaces instead of just topological spaces. The basepoint will have the role of the "base field" with respect to which we will dualize the chains. Let $X \in \underline{\mathsf{Top}}_*$ be a pointed space, and denote by $\tau : pt \to X$ the basepoint. Then τ induces a map

$$CH_A(\tau) : A \simeq CH_{pt}(A) \longrightarrow CH_X(A) \ ,$$

making $CH_X(A)$ into an A-module.

Definition 4.1. *Let A be an E_∞-algebra, and let M be a module over A.*

(1) *The derived Hochschild cochains of A with values in M over X are*

$$CH^X(A, M) := \mathbb{R}\hom_A(CH_X(A), M) \ .$$

(2) *The derived Hochschild chains of A with values in M over X are*

$$CH_X(A,M) \coloneqq M \overset{\mathbb{L}}{\underset{A}{\otimes}} CH_X(A) \ .$$

Here, $- \overset{\mathbb{L}}{\underset{A}{\otimes}} -$ denotes the lift of the derived tensor product of chain complexes to E_∞-modules over A. If k is a field and A is a commutative dg algebra over k, then it can be computed as a two-sided bar construction. Dually, $\mathbb{R}\hom_A(-,-)$ is the enriched mapping space. Both constructions are canonically two-sided E_∞-modules over A.

All of these constructions are well behaved, in the sense that they are functorial and that they respect the forgetful functors from modules to algebras in the way one would expect.

5. Explicit models for derived Hochschild chains

In this section, we specialize to the case where k is a field of characteristic 0. When A is a commutative dg algebra, then one can give explicit models for the derived Hochschild chains. In this section, all algebras are commutative dg algebras, and all spaces are simplicial sets (whose ∞-category is equivalent to the one of topological spaces).

Let A be a commutative dg algebra with differential $d : A \to A$ and multiplication $\mu : A \otimes A \to A$. Denote by n_+ the set $\{0, \dots, n\}$ as before, and define

$$CH_{n_+}(A) \coloneqq A^{\otimes n_+} \cong A^{\otimes(n+1)}.$$

If $f : m_+ \to n_+$ is any set map, define

$$f_* : A^{\otimes m_+} \longrightarrow A^{\otimes n_+}$$

by sending $a_0 \otimes \cdots \otimes a_m \in A \otimes m_+$ to

$$f_*(a_0 \otimes \cdots \otimes a_m) = (-1)^\epsilon b_0 \otimes \cdots \otimes b_n \ , \qquad \text{where} \qquad b_i \coloneqq \prod_{j \in f^{-1}(i)} a_j \ ,$$

and where $(-1)^\epsilon$ is the Koszul sign. This produces a functor from finite sets to commutative dg algebras. We extend it to any set Y by

$$Y \longmapsto CH_Y(A) \coloneqq \underset{\underline{\mathsf{Fin}} \ni K \to Y}{\operatorname{colim}} CH_K(A) \ .$$

Notice this is axiom (2) for discrete spaces. This again induces a functor from simplicial sets to simplicial commutative dg algebras, sending $Y_\bullet$ to $CH_{Y_\bullet}(A)$. Applying the Dold–Kan construction, we obtain the commutative dg algebra

$$\operatorname{Tot}\big(CH_{Y_\bullet}(A)\big) \ ,$$

where $\operatorname{Tot}(-)$ is the total complex, given by[1]

$$\operatorname{Tot}_n\big(CH_{Y_\bullet}(A)\big) \coloneqq s^n \bigoplus_{p+q=n} s^{-q} CH_{Y_p}(A)_q \cong \bigoplus_{p+q=n} s^p CH_{Y_p}(A)_q \ .$$

Here, s is a formal element of degree 1. The multiplication of this commutative dg algebra is the shuffle product, which is given in simplicial degree (p,q) by the composite

$$sh : CH_{Y_p}(A) \otimes CH_{Y_p}(A) \xrightarrow{sh^\times} CH_{Y_{p+q}}(A) \otimes CH_{Y_{p+q}}(A) \cong CH_{Y_{p+q}}(A \otimes A) \xrightarrow{CH_{Y_{p+q}}(\mu)} CH_{Y_{p+q}}(A) \ ,$$

where $sh^\times$ is explicitly given by

$$sh^\times(x \otimes y) \coloneqq \sum_{(\sigma,\theta) \in \mathrm{Sh}(p,q)} (-1)^{(\sigma,\theta)} s_{\sigma_1} \cdots s_{\sigma_p}(x) \otimes s_{\theta 1} \cdots s_{\theta_p}(y) \ ,$$

where (σ, θ) is a (p,q)-shuffle, i.e. a partition of $\{1, \dots, p+q\}$ into two disjoint subsets $\sigma = \{\sigma_1 < \sigma_2 < \cdots < \sigma_p\}$ and $\theta = \{\theta_1 < \cdots < \theta_q\}$. Here, s_i denotes the ith degeneracy map of $CH_{Y_\bullet}(A)$, which is a simplicial commutative dg algebra. The differential is given by

$$D\left(s^p \bigotimes_{j \in Y_p} a_j\right) \coloneqq \underbrace{(-1)^p s^p d_A \left(\bigotimes_{j \in Y_p} a_j\right)}_{d_1} + \underbrace{s^{p-1} \sum_{i=0}^{i} (-1)^i \partial_*^i \left(\bigotimes_{j \in Y_p} a_j\right)}_{d_2} .$$

[1] A remark on conventions: if A is a chain complex, then we denote by A_q the elements of degree q of A *seen as elements of degree q*. This is what other people might denote by $s^q A_q$.

Notice that d_1 and d_2 are actually differentials themselves, and they commute.

Remark 5.1. *One can alternatively take the normalized chains of $CH_{Y_\bullet}(A)$ instead of the total chain complex, which results in a smaller, but equivalent complex.*

Proposition 5.2. *The derived Hochschild chains of A over $Y_\bullet$ are given by the commutative dg algebra*

$$\big(\mathrm{Tot}\big(CH_{Y_\bullet}(A)\big), D, sh\big)$$

described above.

If $Y_\bullet$ is pointed, then one can define derived Hochschild cochains and derived Hochschild chains with value in a right module over A in a way analogous to the one seen above.

To conclude, we present some examples of explicit computations.

5.1. **The point.** The point $pt_\bullet$ is the discrete simplicial set given by a single p-simplex at every simplicial degree, and all faces and degeneracies are trivial. Thus $CH_{pt_p}(A) = A$ for all p, and we have

$$\mathrm{Tot}_n\big(CH_{pt_\bullet}(A)\big) = s^n \bigoplus_{p+q=n} s^{-q} CH_{pt_p}(A)_q = s^n \bigoplus_{q\leq n} s^{-q} A_q \ .$$

Fix $n \in \mathbb{Z}$, let $q \leq n$ and denote $p := n - q$. Take $a \in A_q$, then

$$d_1(s^p a) = (-1)^p s^p da = (-1)^p s^{n-1} s^{-(q-1)} da \ ,$$

and

$$d_2(s^p a) = \begin{cases} s^{p-1} a & \text{if } p \text{ even,} \\ 0 & \text{if } p \text{ odd.} \end{cases}$$

By taking homology with respect to d_2, we see that only a copy of A survives (the one with $p = 0$), and $d_1 = d_A$. Therefore, we have

$$\mathrm{Tot}\big(CH_{pt_\bullet}(A)\big) \simeq A$$

as expected. The product is easily seen to be given by

$$sh(s^{p_1} a_1 \otimes s^{p_2} a_2) = (-1)^{p_2|a_1|} s^{p_1+p_2} a_1 a_2 \ .$$

5.2. **The interval.** The interval $I_\bullet = \Delta[1]$ is the simplicial set given by

$$I_n := \{\underline{0}, \ldots, \underline{n+1}\} \ ,$$

where the elements are shorthand for the strings

$$\underline{i} = \underbrace{0\ldots0}_{i}\underbrace{1\ldots1}_{n-i} \in I_n \ .$$

Here, 0 and 1 are the two 0-simplices, while $\underline{1} = 01 \in I_1$ is the non-degenerate 1-simplex. There are no other non-degenerate simplices. The ith face map acts by eliminating the $(i+1)$th character in the string. Therefore, we have

$$\partial^i \underline{j} = \begin{cases} \underline{j-1} & \text{if } i \leq j \ , \\ \underline{j} & \text{if } i > j \ . \end{cases}$$

The total complex is given by

$$\mathrm{Tot}_n\big(CH_{I_\bullet}(A)\big) = s^n \bigoplus_{\substack{p\geq 0 \\ p+q_0+\cdots+q_{p+1}=n}} s^{-q_0} A_{q_0} \otimes \cdots \otimes s^{-q_{n+1}} A_{q_{n+1}} \ .$$

Let $n \geq 0, p \geq 0$ and $q_0, \ldots, q_{n+1} \in \mathbb{Z}$ be such that $p + q_0 + \cdots + q_{p+1} = n$. Take $a_i \in A_{q_i}$ for all i. Then

$$d_1(s^p a_0 \otimes \cdots \otimes a_{p+1}) = (-1)^p s^p \sum_{k=0}^{p+1} (-1)^{\sum_{0\leq j<k} |a_j|} a_0 \otimes \cdots \otimes da_k \otimes \cdots \otimes a_{p+1} \ ,$$

and

$$d_2(s^p a_0 \otimes \cdots \otimes a_{p+1}) = s^{p-1} \sum_{i=0}^{p} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{p+1} \ .$$

The shuffle product of $s^p a_0 \otimes \cdots \otimes a_{p+1}$ and $s^q b_0 \otimes \cdots \otimes b_{q+1}$ is easily seen to be the sum of all possible ways to shuffle the a_is and the b_js (with Koszul signs).

5.3. **The circle.** We model the circle as a simplicial set $S^1_\bullet$ by taking a single 0-cell and gluing both ends of a 1-cell to it. We have

$$S^1_n = \{\underline{0}, x_1, x_2, \ldots, x_n\}\ ,$$

where we think to x_i as the string

$$x_i = \underbrace{0\ldots0}_{i-1}(00)\underbrace{0\ldots0}_{n-i-1}\ .$$

The bracketed zeros (00) represent the 1-cell, while the normal zeros represent the 0-cell. The ith face acts by removing the $(i+1)$-th character and then removing the brackets if they enclose a single 0. Therefore, we have

$$\partial^i(\underline{0}) = \underline{0}\ ,\qquad \text{and}\qquad \partial^i(x_j) = \begin{cases} x_{j-1} & \text{if } i \leq j-2\ ,\\ \underline{0} & \text{if } i = j-1, j\ ,\\ x_j & \text{if } i \geq j+1\ . \end{cases}$$

The simplicial chain complex $CH_{S^1_\bullet}(A)$ is given by

$$CH_{S^1_p}(A) = A \otimes A^{\otimes p},$$

where the first copy of A corresponds to $\underline{0}$, and the others to the x_js. The total chain complex resembles the one of the interval (with one less copy of A each time). The part d_1 of the differential is as usual, while the other piece is given (up to the suspending therm s^p, which we omit) by

$$\begin{aligned} d_2(a_0 \otimes \cdots \otimes a_p) = {} & a_0a_1 \otimes a_2 \otimes \cdots \otimes a_p + \\ & + \sum_{i=1}^{p-1} (-1)^i a_0 a_i a_{i+1} \otimes a_2 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_{i+2} \otimes \cdots \otimes a_p \\ & + (-1)^p a_0 a_p \otimes a_1 \otimes \cdots \otimes a_{p-1}\ . \end{aligned}$$

References

[1] Gregory Ginot, *Notes on factorization algebras, factorization homology and applications*, arXiv:1307.5213

Laboratoire Analyse, Géométrie et Applications, Université Paris 13, Sorbonne Paris Cité, 99 Avenue Jean Baptiste Clément, 93430 Villetaneuse, France

E-mail address: robert-nicoud@math.univ-paris13.fr