We prove the local well-posedness of the generalized Benjamin–Ono equations $\partial_t u + H^2 u + u^k \partial_x u = 0$, $k \geq 4$, in the scaling invariant spaces $\dot{H}^s(\mathbb{R})$ where $s_k = 1/2 - 1/k$. Our results also hold in the nonhomogeneous spaces $H^s(\mathbb{R})$. In the case $k = 3$, local well-posedness is obtained in $H^s(\mathbb{R})$, $s > 1/3$.

1 Introduction

In this article, we pursue our study of the Cauchy problem for the generalized Benjamin–Ono equations

$$\begin{cases}
\partial_t u + H^2 u + u^k \partial_x u = 0, & x, t \in \mathbb{R}, \\
u(t = 0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}$$

(gBO)

with $k$ an integer $\geq 3$ and with $H$ the Hilbert transform defined via the Fourier transform by

$$Hf = \mathcal{F}^{-1}(-i \text{ sgn}(\xi) \hat{f}(\xi)), \quad f \in S'(\mathbb{R}).$$

(1.1)
From the well-known formula
\[ \mathcal{H} f(x) = \frac{1}{\pi} \text{pv} \left( \frac{1}{x} \ast f \right)(x), \]
we see that \( \mathcal{H} f \) is real-valued, and consequently we look for real-valued solutions. In view of (1.1), and as noticed by Tao in [12], the Hilbert transform behaves roughly speaking like the constant \( \pm i \).

A remarkable feature of (gBO) is the following scaling invariance: if \( u(t, x) \) is a solution of the equation on \([-T, +T]\), then for any \( \lambda > 0 \), \( u_\lambda(t, x) = \lambda^{1/k} u(\lambda^2 t, \lambda x) \) also solves (gBO) on \([-\lambda^{-2} T, +\lambda^{-2} T]\) with initial data \( u_\lambda(0, x) \) and moreover
\[ \| u_\lambda(\cdot, 0) \|_{\dot{H}^s} = \lambda^{s+\frac{1}{2}-\frac{1}{k}} \| u(\cdot, 0) \|_{\dot{H}^s}. \]
Hence, the \( \dot{H}^s(\mathbb{R}) \) norm is invariant under the flow \( u \mapsto u_\lambda \) if and only if \( s = s_k = 1/2 - 1/k \) and thus we may expect well-posedness in \( \dot{H}^{s_k}(\mathbb{R}) \).

Let us now review the known results concerning the generalized Benjamin–Ono equations for \( k \geq 3 \). For a detailed discussion on the cases \( k = 1, 2 \), we refer the reader to [4, 9].

When \( k = 3 \), the local well-posedness is known in \( H^s(\mathbb{R}), s > 1/3 \) for small initial data [11] but only in \( H^s(\mathbb{R}), s > 3/4 \) for large initial data. In [13], we showed that (gBO) is \( C^4 \)-ill posed in \( H^s(\mathbb{R}), s < 1/3 \), in the sense that the flow map \( u_0 \mapsto u \) fails to be \( C^4 \). We prove here that well-posedness occurs in \( H^s(\mathbb{R}), s > 1/3 \), and without smallness assumption on the initial data.

Now consider the case \( k \geq 4 \). By compactness methods, the Cauchy problem (gBO) has been shown to be locally well posed in \( H^s(\mathbb{R}), s > 3/2 \) in [3]. Later, by means of a gauge transformation, Molinet and Ribaud [10] significantly improved this result and showed well-posedness in \( H^{1/2}(\mathbb{R}) \), for any \( k \geq 4 \). By a refinement of their method, we pushed down in [13] the well-posedness to \( H^s(\mathbb{R}), s > s_k \), but for higher order nonlinearities only (\( k \geq 12 \) in fact). On the other hand, in the particular case \( k = 4 \), Burq and Planchon [2] proved the local well-posedness in the critical space \( \dot{H}^{1/4}(\mathbb{R}) \).

In the context of small initial data, these results can be substantially improved. In this situation, in a sharp contrast with what occurs for \( k = 1 \), well-posedness can be obtained by a fixed point procedure. More precisely, in [10], it has been proved that in the case of small initial data, (gBO) is well posed in the critical space \( \dot{H}^{s_k}(\mathbb{R}) \). Previous results were derived in [11] and [8] where well-posedness was obtained in \( H^s(\mathbb{R}) \) for \( s > s_k \) and \( s > 3/4 \), respectively.

Inspired by the works of Burq–Planchon [2], we remove in this article the size restriction on the data and extend the well-posedness to \( \dot{H}^{s_k}(\mathbb{R}) \) for all \( k \geq 4 \). Our method
is flexible enough to get the desired bound in any $\dot{H}^s$-norm, leading in particular (with $s = 0$) to the well-posedness in the nonhomogeneous space $H^s(\mathbb{R})$. Also, a standard fixed point argument allows us to construct a unique solution in a subspace of $\dot{H}^s(\mathbb{R})$ with a continuous flow map $u_0 \mapsto u$. Recall that Biagioni and Linares [1] proved using solitary waves that this map cannot be uniformly continuous in $\dot{H}^s(\mathbb{R})$. In the case $s < s_k$, we also know that the solution map (if it exists) fails to be $C^{k+1}$ in $H^s(\mathbb{R})$, see [11].

2 Notations and Main Results

2.1 Notations

For $A$ and $B$ two positive numbers, we write $A \lesssim B$ if there exists $c > 0$ such that $A \leq cB$. Similarly, define $A \gtrsim B$, $A \sim B$ if $A \geq cB$ and $A \lesssim B \lesssim A$, respectively. When $A \leq cB$ with $c$ small enough, we write $A \ll B$. For any $f \in S'(\mathbb{R})$, we use $\hat{f}$ or $\tilde{f}$ to denote its Fourier transform. For $1 \leq p \leq \infty$, $L^p$ is the standard Lebesgue space and its space-time versions $L^p_x L^q_t$ and $L^q_T L^p_x (T > 0)$ are endowed with the norms

$$\|f\|_{L^p_x L^q_t} = \|\hat{f}\|_{L^q_t([-T, T])}$$

and

$$\|f\|_{L^p_T L^q_x} = \|\hat{f}\|_{L^p_x([-T, T])}.$$ 

The pseudo-differential operator $D^\alpha_x$ is defined by its Fourier symbol $|\xi|^\alpha$. We will denote by $P_+$ and $P_-$ the projection on, respectively, the positive and the negative spatial Fourier modes. Thus, one has

$$i\mathcal{H} = P_+ - P_-.$$

Let $\eta \in C_0^\infty(\mathbb{R})$, $\eta \geq 0$, supp $\eta \subset \{1/2 \leq |\xi| \leq 2\}$ with $\sum_{j=-\infty}^\infty \eta(2^{-j}\xi) = 1$ for $\xi \neq 0$. We set $p(\xi) = \sum_{j \leq -3} \eta(2^{-j}\xi)$ and consider, for all $j \in \mathbb{Z}$, the operator $Q_j$ defined by

$$Q_j(f) = \mathcal{F}^{-1}(\eta(2^{-j}\xi) \hat{f}(\xi)).$$

We adopt the following summation convention. Any summation of the form $r \lesssim j$, $r \gg j, \ldots$ is a sum over the $r \in \mathbb{Z}$ such that $2^r \lesssim 2^j \ldots$, thus for instance $\sum_{r \lesssim j} = \sum_{r, 2^r \lesssim 2^j}$. Next, we define the operators $Q_{\lesssim j} = \sum_{r \lesssim j} Q_r$, $Q_{\ll j} = \sum_{r \ll j} Q_r$, etc.

In what follows, we will use $\mathcal{S}_T$ to denote the space of the Schwartz functions on $\mathbb{R}^2$ restricted to $(-T, T) \times \mathbb{R}$. For $1 \leq p, q, r \leq \infty$ and $s \in \mathbb{R}$, let $\dot{B}_p^s r (L^q_T)$ be the closure of $\mathcal{S}_T$ for the norm

$$\|f\|_{\dot{B}_p^s r (L^q_T)} = \left( \sum_{j \in \mathbb{Z}} [2^{js} \|Q_j f\|_{L^q_p L^r_T}]^r \right)^{1/r} < \infty.$$
Finally, for $s \in \mathbb{R}$ and $\theta \in [0, 1]$, we define the solution space $\dot{S}^{s, \theta}_T$ (where lives our solution $u$) and the space $\dot{N}^{s, \theta}_T$ (where lives the nonlinear term $u^k \partial_x u$) by

$$\dot{S}^{s, \theta}_T = B^{s+\frac{3\theta-1}{4}}_4 \left( L^2_T \right), \quad \dot{N}^{s, \theta}_T = B^{s+\frac{1-3\theta}{4}}_4 \left( L^\infty_T \right).$$

We shall often work with the Besov space $\dot{N}^{s, 1}_T = B^{s-1/2}_4 (L^2_T)$, and we simplify the notation by setting $\dot{N}^{s, 1}_T = \dot{N}^s_T$.

### 2.2 Main results

We first state our well-posedness results in the case $k \geq 4$.

**Theorem 2.1.** Let $k \geq 4$ and $u_0 \in \dot{H}^s(\mathbb{R})$. There exist $T = T(u_0) > 0$ and a unique solution $u$ of (gBO) such that $u \in \dot{Z}_T$ with

$$\dot{Z}_T = C([\!-T, +T], \dot{H}^s(\mathbb{R})) \cap \dot{X}^k_T \cap L^k_x L^\infty_T.$$

Moreover, the flow map $u_0 \mapsto u$ is locally Lipschitz from $\dot{H}^s(\mathbb{R})$ to $\dot{Z}_T$.

In the nonhomogeneous case, one has the following result.

**Theorem 2.2.** Let $k \geq 4$ and $u_0 \in H^s(\mathbb{R})$, $s \geq s_k$. There exist $T = T(u_0) > 0$ and a unique solution $u$ of (gBO) such that $u \in Z_T$ with

$$Z_T = C([\!-T, +T], H^s(\mathbb{R})) \cap X^k_T \cap L^k_x L^\infty_T.$$

Moreover, the flow map $u_0 \mapsto u$ is locally Lipschitz from $H^s(\mathbb{R})$ to $Z_T$.

**Remark 2.1.** We only obtain the local Lipschitz continuity of the map $u_0 \mapsto u$ in Theorems 2.1 and 2.2 in $\dot{H}^s(\mathbb{R})$ (resp. $H^s(\mathbb{R})$). As noticed in the Introduction, the solution map given by Theorem 2.1 is not uniformly continuous from $\dot{H}^s(\mathbb{R})$ to $C([\!-T, T], \dot{H}^s(\mathbb{R}))$. Moreover, when $s < s_k$, the flow map in Theorem 2.2 is no longer of class $C^{k+1}$ in $H^s(\mathbb{R})$. It is not clear whether the map given by Theorems 2.1 and 2.2 is $C^{k+1}$ or not.

**Remark 2.2.** The spaces $\dot{X}^k_T$ and $X^k_T$ will be defined in Section 3 and are directly related to the linear estimates for the linear Benjamin–Ono equation.

The main tools to prove Theorems 2.1 and 2.2 are the sharp Kato smoothing effect and the maximal in time inequality for the free solution $V(t)u_0$ where $V(t) = e^{it\mathcal{H}_0}$.
that for regular solutions, (gBO) is equivalent to its integral formulation

$$u(t) = V(t)u_0 \mp \int_0^t V(t - t')(u^k(t')\partial_x u(t')) dt'.$$

(2.1)

This equation can make sense when $u$ is a tempered distribution which lies in our resolution spaces $\dot{Z}_T$ (or $Z_T$). Throughout this article, we will call $(H^s)$ solution any tempered distribution $u \in \dot{Z}_T$ which satisfies (2.1) in the distributional sense. It is well known that if these solutions have sufficient smoothness and regularity, they coincide with classical solutions.

It is worth noticing that (gBO) provides a perfect balance between the derivative nonlinear term on one hand and the available linear estimates on the other hand. Heuristically, one may use (2.1) to write

$$\|D_x^{k+1/2} u\|_{L^\infty_T L^2} + \|u\|_{L^{k+1}_x L^\infty_T} \lesssim \|u_0\|_{\dot{H}^{s_k}} + \|D_x^{s_k-1/2} \partial_x (u^{k+1})\|_{L^1_T L^2_x}$$

$$\lesssim \|u_0\|_{\dot{B}^{s_k}_x} + \|D_x^{s_k+1/2} u\|_{L^{k+1}_x L^\infty_T} \|u\|_{L^k_T L^\infty_T}$$

and perform a fixed point procedure. Unfortunately, such an argument fails for several reasons:

- First, it is not clear whether the second inequality holds true or not. Indeed, we used the fractional Leibniz rule (see the Appendix in [7, 10]) at the end points $L^p$, $p = 1, \infty$. However, this inequality becomes true if one works in the associated Besov spaces $\dot{B}^{s_k+1/2}_x (L^2_T) \cap \dot{B}^{s_k+1}_x (L^\infty_T)$ and provides sharp well-posedness for small initial data; see [11], equation (48).

- The term $\|V(t)u_0\|_{L^k_T L^\infty_x}$ will be small only if $\|u_0\|_{\dot{H}^{s_k}}$ is small as well, even for small $T$. Nevertheless, as noticed in [2], if we consider instead the difference $V(t)u_0 - u_0$, then its $L^k_T L^\infty_T$-norm is small provided we restrict ourselves to a small interval $[-T, T]$ (see Lemma 3.4).

- In order to obtain a contraction factor, we also need to get a better share of the derivative in the nonlinear term. By a standard paraproduct decomposition, we see that the worst contribution in $\partial_x u^{k+1}$ is given by $\pi(u, u)$ where

$$\pi(f, g) = \sum_j \partial_x Q_j ((Q_{<j} f)^k Q_{\sim j} g).$$

The main idea is then to inject this term (or more precisely $\pi(u_0, u)$) in the linear part of the equation to get the variable coefficient Schrödinger-like equation

$$\partial_t u + H \partial_x^2 u + \pi(u_0, u) = f,$$

(2.2)
where \( f \) will be a well-behaved term. Linear estimates for equation (2.2) are obtained by the localized gauge transform

\[
\begin{align*}
  w_j = e^{\int_{-\infty}^{x} f_s(u_0) \, ds} P_j Q_j u, \quad j \in \mathbb{Z}.
\end{align*}
\]

Now we turn to the case \( k = 3 \). By similar considerations, we obtain the following result.

**Theorem 2.3.** Let \( k = 3 \) and \( u_0 \in H^s(\mathbb{R}), s > 1/3 \). There exist \( T = T(u_0) > 0 \) and a unique solution \( u \) of (gBO) such that \( u \in Z_T \) with

\[
Z_T = C([−T, +T], H^s(\mathbb{R})) \cap X_T^\infty \cap L_x^3 L_T^\infty.
\]

Moreover, the flow map \( u_0 \mapsto u \) is locally Lipschitz from \( H^s(\mathbb{R}) \) to \( Z_T \).

This article is organized as follows. In Section 3, we recall some sharp estimates related with the linear operator \( V(t) \), and we derive linear estimates for equation (2.2). Section 4 is devoted to the case \( k \geq 4 \). Finally, we prove Theorem 2.3 in Section 5.

### 3 Linear Estimates

#### 3.1 Estimates for the linear BO equation

In this section, we recall the well-known linear estimates for the Benjamin–Ono equation. Note that all results stated here hold as well for the Schrödinger operator \( S(t) = e^{i t \Delta} \).

The following lemma summarizes the main estimates related to the group \( V(t) \).

**Lemma 3.1.** Let \( \varphi \in S(\mathbb{R}) \), then

\[
\begin{align*}
  &\| V(t) \varphi \|_{L_x^2 L_T^2} \lesssim \| \varphi \|_{L^2}, \\
  &\| D_x^{1/2} V(t) \varphi \|_{L_x^2 L_T^2} \lesssim \| \varphi \|_{L^2}, \\
  &\| D_x^{-1/4} V(t) \varphi \|_{L_x^2 L_T^2} \lesssim \| \varphi \|_{L^2}.
\end{align*}
\]

Moreover, if \( T \leq 1 \) and \( j \geq 0 \),

\[
\begin{align*}
  &\| Q_{\leq 0} V(t) \varphi \|_{L_x^2 L_T^\infty} \lesssim \| Q_{\leq 0} \varphi \|_{L^2} \quad (3.4) \\
  &2^{-j/2} \| Q_j V(t) \varphi \|_{L_x^2 L_T^\infty} \lesssim \| Q_j \varphi \|_{L^2}.
\end{align*}
\]
Definition 3.1 ([10], Definition 2.2). A triplet \((\alpha, p, q) \in \mathbb{R} \times [2, \infty]^2\) is said to be 1-admissible if \((\alpha, p, q) = (1/2, \infty, 2)\) or
\[
4 \leq p < \infty, \quad 2 < q \leq \infty, \quad \frac{2}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad \alpha = \frac{1}{p} + \frac{2}{q} - \frac{1}{2}.
\] (3.6)

By Sobolev embedding and interpolation between estimates (3.2) and (3.3), we obtain the following result.

Proposition 3.1 ([10], Proposition 2.3). If \((\alpha, p, q)\) is 1-admissible, then for all \(\varphi \in S(\mathbb{R})\),
\[
\|D_x^\alpha V(t)\varphi\|_{L^p_x L^q_T} \lesssim \|\varphi\|_{L^2}.
\] (3.7)

Now we define our resolution spaces.

Definition 3.2. Let \(k \geq 4\) and \(s \in \mathbb{R}\) be fixed. For \(0 < \varepsilon \ll 1\), we define the spaces \(\dot{X}^s_T = \dot{S}_s^{s, \varepsilon} \cap S_T^{s, 1}\) endowed with the norm
\[
\|u\|_{\dot{X}^s_T} = \|u\|_{\dot{S}_s^{s, \varepsilon}} + \|u\|_{S_T^{s, 1}}.
\]

At this stage it is important to remark that \(\dot{X}^s_T\) does not contain any \(L^\infty_T\) component. As a consequence, for each \(u \in \dot{X}^s_T\) and \(\eta > 0\) fixed, we can choose \(T = T(u)\) such that \(\|u\|_{\dot{X}^s_T} < \eta\). On the other hand, it is clear from the definition of \(\dot{B}_p^s(L^q_T)\) that \(S_T\) is dense in \(\dot{X}^s_T\).

We next state the \(L^p_x L^q_T\) and \(L^q_T L^p_x\) estimates for the linear operator \(f \mapsto \int_0^t V(t - t') f(t') dt\).

Lemma 3.2 ([10], Proposition 2.7). Let \(\alpha \in \mathbb{R}\), and \(2 < p, q \leq \infty\) such that for all \(\varphi \in S(\mathbb{R})\),
\[
\|D_x^\alpha V(t)\varphi\|_{L^p_x L^q_T} \lesssim \|\varphi\|_{L^2}.
\] Then for all \(f \in S(\mathbb{R}^2)\),
\[
\|D_x^{1/2} \int_0^t V(t - t') f(t') dt'\|_{L^p_x L^q_T} \lesssim \|f\|_{L^1_x L^{q/2}_T}, \quad (3.8)
\]
\[
\|D_x^{\alpha + 1/2} \int_0^t V(t - t') f(t') dt'\|_{L^p_x L^q_T} \lesssim \|f\|_{L^1_x L^{q/2}_T}. \quad (3.9)
\]

Similarly, if
\[
\|D_x^\alpha V(t)\varphi\|_{L^p_x L^q_T} \lesssim \|\varphi\|_{H^s}.
\]
for any $\varphi \in \mathcal{S}(\mathbb{R})$, then

$$
\left\| D_x^{\alpha + 1/2} \int_0^t V(t - t') f(t') \, dt' \right\|_{L_x^p L_t^q} \lesssim \| (D_x)^{s} f \|_{L_x^1 L_t^{\frac{q}{s}}}.
$$

(3.10)

We shall also need the following Besov version of Lemma 3.2.

**Lemma 3.3.** Let $k \geq 4$. For all $f \in \mathcal{S}(\mathbb{R}^2)$,

$$
\left\| \int_0^t V(t - t') f(t') \, dt' \right\|_{L_x^k L_t^\infty} \lesssim \| f \|_{\dot{N}^{s_k,1}}.
$$

**Proof.** Note that the triplets $(1/2, \infty, 2)$ and $(-s_k, k, \infty)$ are both 1-admissible. In particular, we deduce

$$
\left\| \int_{-T}^T D_x^{1/2} V(-t') h(t') \, dt' \right\|_{L_x^2} \lesssim \| h \|_{L_x^1 L_t^{\infty}}, \quad \forall h \in \mathcal{S}(\mathbb{R}^2),
$$

which is the dual estimate of (3.7) for $(\alpha, p, q) = (1/2, \infty, 2)$. Since $L^2 = \dot{B}_{2}^{0,2}$, we infer

$$
\left\| \int_{-T}^T D_x^{1/2} V(-t') h(t') \, dt' \right\|_{L_x^2} \lesssim \| h \|_{\dot{B}_{1}^{0,2}(L_x^\infty)}, \quad \forall h \in \mathcal{S}(\mathbb{R}^2).
$$

The usual $TT^*$ argument provides

$$
\left\| \int_{-T}^T V(t - t') f(t') \, dt' \right\|_{L_x^k L_t^\infty} \lesssim \| f \|_{\dot{B}_{1}^{-s_k,2}(L_x^\infty)}.
$$

We can conclude with the Christ–Kiselev lemma for reversed norms (Theorem B in [2]).

3.2 Linear estimates for equation (2.2)

Here and hereafter we only consider the case $k \geq 4$. The special case $k = 3$ will be discussed in Section 5.

The next lemma will be crucial in the proof of our main results.

**Lemma 3.4.** Let $k \geq 4$ and $u_0 \in \dot{H}^{s_k}$. For any $\eta > 0$, there exists $T = T(u_0)$ such that

$$
\| V(t) u_0 - u_0 \|_{L_x^k L_t^\infty} < \eta.
$$
Proof. Let \( N > 0 \) to be chosen later. By Sobolev embedding theorem together with (3.7), we get

\[
\|V(t)u_0 - u_0\|_{L_t^k L_x^\infty} \lesssim \sum_{|j| < N} \|Q_j(V(t)u_0 - u_0)\|_{L_t^k L_x^\infty} + \left( \sum_{|j| > N} \|Q_j u_0\|_{H^2}^2 \right)^{1/2}.
\]

Note that \( v = V(t)u_0 - u_0 \) solves the equation

\[
\partial_t v + \mathcal{H} \partial_x^2 v = -\mathcal{H} \partial_x^2 u_0
\]

with zero initial data. Thus, \( V(t)u_0 - u_0 = \int_0^t V(t - t') \mathcal{H} \partial_x^2 u_0 \, dt' \) and

\[
\sum_{|j| < N} \|Q_j(V(t)u_0 - u_0)\|_{L_t^k L_x^\infty} \lesssim \sum_{|j| < N} 2^{2j} \left\| \int_0^t V(t') Q_j u_0 \, dt' \right\|_{L_t^k L_x^\infty}
\]

\[
\lesssim T \sum_{|j| < N} 2^{2j} \|V(t)Q_j u_0\|_{L_t^k L_x^\infty}
\]

\[
\lesssim T 2^{2N} \|u_0\|_{H^2}.
\]

It suffices now to choose sufficiently large \( N \) and then \( T \) small enough.

Let us now turn back to the nonlinear (gBO) equation. We note that the sign of the nonlinearity is irrelevant in the study of the local problem, and we choose for convenience the plus sign.

To avoid any technical difficulties, we only consider smooth functions \( u \in S_T \). Using standard paraproduct rearrangements, we can rewrite the nonlinear term in (gBO) as follows:

\[
\partial_x Q_j(u^{k+1}) = \partial_x Q_j \left( \lim_{r \to \infty} (Q_{<r} u)^{k+1} \right)
\]

\[
= \partial_x Q_j \left( \sum_{-\infty}^{\infty} (Q_{<r+1} u)^{k+1} - (Q_{<r} u)^{k+1} \right)
\]

\[
= \partial_x Q_j \left( \sum_{-\infty}^{\infty} Q_r u (Q_{<r} u)^k \right)
\]

\[
= \partial_x Q_j \left( \sum_{r < j} Q_r u (Q_{<r} u)^k \right) + \partial_x Q_j \left( \sum_{r > j} (Q_{<r} u)^2 (Q_{<r} u)^{k-1} \right)
\]

\[
= \partial_x Q_j ((Q_{<j} u)^k Q_{<j} u) - g_j,
\]

where

\[
g_j = -\partial_x Q_j \left( \sum_{r > j} (Q_{<r} u)^2 (Q_{<r} u)^{k-1} \right).
\]
Note that while deriving our estimates, we will have to choose the implicit constant in (3.11) to be dependent on the data \( u_0 \).

Now we set
\[
\pi(f, g) = \sum_j \partial_x Q_j ((Q_{< j} f) k Q_{< j} g),
\]
so that (gBO) reads
\[
\partial_t u + \mathcal{H} \partial_x^2 u + \pi(u, u) = g(t, x),
\]
with
\[
g = \sum_j g_j.
\]
Setting now
\[
f = \pi(u_0, u) - \pi(u, u) + g,
\]
we see that (gBO) is equivalent to
\[
\begin{cases}
\partial_t u + \mathcal{H} \partial_x^2 u + \pi(u_0, u) = f(t, x), \\
u(0, x) = u_0(x).
\end{cases}
\tag{3.12}
\]
We intend to solve (gBO) by a fixed point procedure on the Duhamel formulation of (3.12):
\[
u(t) = U(t)u_0 - \int_0^t U(t - t') f(t') dt',
\tag{3.13}
\]
where \( U(t)\phi \) is solution to
\[
\partial_t u + \mathcal{H} \partial_x^2 u + \pi(u_0, u) = 0, \quad u(0) = \phi.
\]
It is worth noticing that \( U(t) \) depends on the data \( u_0 \). The equivalence between (3.12) and (3.13) is clear for regular solutions \( u \in S_T \) since the linear part is time-translation invariant.

Setting \( u_j = Q_j u \) and \( f_j = Q_j f \), we get from (3.12) that
\[
\begin{align*}
\partial_t u_j + \mathcal{H} \partial_x^2 u_j + \partial_x((u_{0, < j})^k u_j) &= \partial_x[Q_j, (u_{L, < j})^k] u_{< j} - \partial_x[Q_j, (u_{L, < j})^k] u_{< j} + f_j \\
&= \partial_x[Q_j, (u_{L, < j})^k] u_{< j} + f_j
\end{align*}
\tag{3.14}
\]
where \( u_L = V(t)u_0 \) is the solution of the free (BO) equation, and where
\[
[Q_j, \phi] \theta = Q_j(\phi \theta) - \phi Q_j \theta
\]
is the commutator between $Q_j$ and $\varphi$. We will denote by $R_j$ the right-hand side of (3.14). Now apply the Riesz projector on the positive frequencies to $u_j$ and set $v_j = P_+ u_j$, where this leads to

$$i \partial_t v_j + \partial_x^2 v_j + i \partial_x ((u_{0, < j})^k v_j) = iP_+ R_j.$$  

With $b_{< j} = \frac{1}{2} (u_{0, < j})^k$, we obtain

$$i \partial_t v_j + (\partial_x + i b_{< j})^2 v_j = g_j$$  

(3.15)

with

$$g_j = -i \partial_x b_{< j} v_j - b_{< j}^2 v_j + i P_+ R_j.$$  

(3.16)

**Lemma 3.5.** Let $v_j \in S_T$ be a solution to (3.15) with initial data $v_{0,j} \in \dot{H}^{s_0} \cap \dot{H}^s$. Then there exists a positive nondecreasing polynomial function $p_k$ such that

$$\|v_j\|_{\dot{X}_T^s} \leq p_k(\|u_0\|_{\dot{H}^{s_0}})(\|v_{0,j}\|_{\dot{H}^s} + \|g_j\|_{\dot{X}_T^s}).$$

**Proof.** Since $b_{< j}$ can be rewritten as a convolution product with $\sum_{r < j} \eta(2^{-r} \cdot)$, $b_{< j}$ is a real-valued function. Following Tao [12], we can take its antiderivative and define the localized gauge transform $w_j$ by

$$w_j = e^{i \int x b_{< j} v_j}.$$  

Then we easily check that $w_j$ solves

$$i \partial_t w_j + \partial_x^2 w_j = e^{i \int x b_{< j}} g_j.$$  

From the well-known linear estimates on the Schrödinger equation (Lemmas 3.1 and 3.2), we infer

$$\|\partial_x w_j\|_{L^\infty L^2_T} \lesssim \|e^{i \int x b_{< j}} v_{0,j}\|_{\dot{H}^{1/2}} + \|g_j\|_{L^1_T L^2_x}.$$  

Since $\partial_x w_j = e^{i \int x b_{< j}} (\partial_x v_j + i b_{< j} v_j)$, we have

$$\|\partial_x w_j\|_{L^\infty L^2_T} \lesssim \|\partial_x w_j\|_{L^\infty L^2_T} + \|b_{< j} v_j\|_{L^\infty L^2_T} \lesssim \|\partial_x w_j\|_{L^\infty L^2_T} + 2^{-j} \|b_{< j}\|_{L^\infty} \|\partial_x v_j\|_{L^\infty L^2_T}.$$  

On the other hand, we can make $2^{-j} \|(u_{0, < j})^k\|_{L^\infty}$ as small as desired by choosing the implicit constant $J = J(u_0)$ (which comes from (3.11)) in $u_{0, < j}$ large enough:

$$2^{-j} \|(u_{0, < j})^k\|_{L^\infty} \lesssim 2^{-j} 2^{j - J} \|u_0\|_{L^k}^k \lesssim c(u_0) 2^{-j} \ll 1.$$
We also need to bound the contribution with (3.17):

$$\| \partial_x v_j \|_{L^\infty_x L^2_t} \lesssim \| \epsilon^{j \beta < j} v_{0j} \|_{H^{1/2}} + \| g_j \|_{L^1_x L^2_t}.$$  

We now use the fractional Leibniz rule (Theorem A.12 in [7]) and Bernstein and Sobolev inequalities to estimate the first term in the right-hand side,

$$\| \epsilon^{j \beta < j} v_{0j} \|_{H^{1/2}} \lesssim \| \epsilon^{j \beta < j} \|_{L^\infty} \| v_{0j} \|_{H^{1/2}} + \| D_x^{1/2} \epsilon^{j \beta < j} \|_{L^1_x} \| v_{0j} \|_{L^{1/2-1}} \lesssim \| v_{0j} \|_{H^{1/2}} + \| (u_{0, < j})^k \|_{L^{1/2-1}} \| v_{0j} \|_{L^{1/2-1}} \lesssim \| v_{0j} \|_{H^{1/2}} + \| (2^{j/2k} \cdot e_j)^k \|_{L^2} 2^{j} \| v_{0j} \|_{L^2} \lesssim (1 + \| u_0 \|_{L^k}) \| v_{0j} \|_{H^{1/2}},$$

where we used that $|\partial_x \epsilon^{j \beta < j}| = \frac{1}{2} |u_{0, < j}|^k$ in the second estimate. This leads to

$$\| \partial_x v_j \|_{L^\infty_x L^2_t} \lesssim (1 + \| u_0 \|_{L^k}) \| v_{0j} \|_{H^{1/2}} + \| g_j \|_{B^{1/2}_{1,2}(L^2_t)}.$$  

(3.17)

Since $v_j, g_j$ as well as $v_{0j}$ are frequency localized, we get

$$\| v_j \|_{B^{1/2}_{1,2}(L^2_t)} \lesssim (1 + \| u_0 \|_{L^k}) \| v_{0j} \|_{H^{1/2}} + \| g_j \|_{B^{1/2}_{1,2}(L^2_t)}.$$  

We also need $L^\infty_x L^\infty_t$-norm estimates. Our equation can be rewritten as

$$i \partial_t v_j + \partial_x^2 v_j = g_j + h_j$$

with

$$h_j = b_{< j}^2 v_j - i \partial_x (b_{< j} v_j) - ib_{< j} \partial_x v_j.$$  

Thus, we get from Lemmas 3.1 and 3.2 that

$$\| v_j \|_{B^{s-1/2}_{1,2}(L^\infty_t)} \lesssim \| v_{0j} \|_{H^s} + \| g_j \|_{B^{s-1/2}_{1,2}(L^2_t)} + \| h_j \|_{B^{s-1/2}_{1,2}(L^2_t)}.$$  

We bound the $h_j$ contribution with (3.17):

$$\| b_{< j}^2 v_j \|_{B^{s-1/2}_{1,2}(L^2_t)} \lesssim 2^{j(s-1/2)} \| b_{< j} \|_{L^1} \| v_j \|_{L^\infty_t L^2_x} \lesssim (2^{-j/2} \| b_{< j} \|_{L^2})^2 \| v_j \|_{B^{s+1/2}_{1,2}(L^2_t)} \lesssim \| b \|_{L^1} (1 + \| u_0 \|_{L^k}) \| v_{0j} \|_{H^s} + \| g_j \|_{B^{s-1/2}_{1,2}(L^2_t)},$$

and similarly

$$\| \partial_x (b_{< j} v_j) + b_{< j} \partial_x v_j \|_{B^{s-1/2}_{1,2}(L^2_t)} \lesssim 2^{j(s+1/2)} \| b_{< j} \|_{L^1} \| v_j \|_{L^\infty_t L^2_x} \lesssim \| b \|_{L^1} (1 + \| u_0 \|_{L^k}) \| v_{0j} \|_{H^s} + \| g_j \|_{B^{s-1/2}_{1,2}(L^2_t)}.$$
Therefore,

\[
\|v_j\|_{H^{s-1/4,2}(\mathbb{R}^3)} \lesssim p_k(\|u_0\|_{H^k})\|v_{0,j}\|_{H^s} + \|g_j\|_{H^{s-1/2,2}(\mathbb{R}^3)}
\]  

(3.18)

and the claim follows by interpolation between (3.18) and (3.17). 

We are now ready to prove the main linear estimate on (3.12).

**Proposition 3.2.** Let \( u \in S_T \) be a solution of (3.12) with initial data \( u_0 \in \dot{H}^s \cap \dot{H}^k, s \in \mathbb{R} \). Then there exist \( T = T(u_0) > 0 \) and a positive nondecreasing polynomial function \( p_k \) such that on \([-T,+T],\)

\[
\|u\|_{X_T^{s}} \leq p_k(\|u_0\|_{H^k})(\|u_0\|_{H^s} + \|f\|_{X_T^s}).
\]

**Proof.** It is well known that \( P_+ \) is an unbounded operator on \( L^1 \). However, working in dyadic annulus, it is clear that \( P_+ Q_j \) is continuous on \( L^1 \) for any \( j \in \mathbb{Z} \). Furthermore, using that \( |P_+ u_j| = |P_- u_j| \) (since \( u \) is real) and Lemma 3.5, we infer by (3.14)

\[
\|u_j\|_{X_T^s} \lesssim \|v_j\|_{X_T^s} \lesssim p_k(\|u_0\|_{H^k})(\|Q_j u_0\|_{H^s} + \|f_j\|_{X_T^s} + \|\partial_x (u_{0,j})^k v_j\|_{X_T^{s}} + \|(u_{0,j})^{2k} v_j\|_{X_T^{s}} + \|\partial_x (Q_j, (u_{L,j})^k u_{-j})\|_{X_T^{s}} + \|\partial_x (Q_j, (u_{L,j})^k u_{-j})\|_{X_T^{s}})
\]

\[
= p_k(\|u_0\|_{H^k})(\|Q_j u_0\|_{H^s} + \|f_j\|_{X_T^s} + A + B + C + D).
\]

We bound \( A \) by

\[
A \lesssim 2^{j(s-1/2)}\|\partial_x (u_{0,j})^k v_j\|_{L^1 L^2_T} \lesssim 2^{-j} \|\partial_x (u_{0,j})^k\|_{L^1} 2^{j(s+1/2)} \|v_j\|_{L^\infty L^2_T} 
\]

\[
\lesssim 2^{-j} \|\partial_x (u_{0,j})^k\|_{L^1} \|v_j\|_{X_T^s}.
\]

As previously, \( 2^{-j} \|\partial_x (u_{0,j})^k\|_{L^1} \) can be made as small as needed by choosing the implicit constant \( J = J(u_0) \) in \( u_{0,j} \) large enough:

\[
2^{-j} \|\partial_x (u_{0,j-1})^k\|_{L^1} \lesssim 2^{-j} 2^{-j} \|u_0\|_{L^k}^k \lesssim c(u_0) 2^{-J} \ll 1.
\]

One proceeds similarly for \( B \):

\[
B \lesssim 2^{j(s-1/2)} \|(u_{0,j})^{2k} v_j\|_{L^1 L^2_T} \lesssim 2^{-j} \|(u_{0,j})^{k}\|_{L^\infty} \|u_0\|_{L^k}^k 2^{j(s+1/2)} \|v_j\|_{L^\infty L^2_T} \ll \|v_j\|_{X_T^s}.
\]
Now we estimate $C$. By the commutator lemma (Lemma 2.4 in [2]), we get

$$C \lesssim 2^{j(s-1/2)}\|\partial_x [Q_j, (u_L, \ll_j)]^k - (u_{0, \ll_j})^k\|_{L^1_T L^2_x} - 2^{j(s+1/2)}\|\partial_x (u_L, \ll_j)^k\|_{L^1_T L^2_x} - \|u_L - u_0\|_{L^1_T L^2_x} \left(\|u_L\|_{L^1_T L^2_x}^{k-1} + \|u_0\|_{L^1_T L^2_x}^{k-1}\right) \|u_j\|_{X^1_T},$$

thanks to Lemma 3.4. Finally, we deal with term $D$. By similar considerations, we get

$$D \lesssim 2^{j(s-1/2)}\|\partial_x [Q_j, (u_L, \ll_j)]^k\|_{L^1_T L^2_x} - 2^{j(s-1/2)}2^{-j\|\partial_x (u_L, \ll_j)^k\|_{L^2_T L^2_x} - \|u_L - u_0\|_{L^1_T L^2_x} \left(\|u_L\|_{L^1_T L^2_x}^{k-1} + \|u_0\|_{L^1_T L^2_x}^{k-1}\right) \|u_j\|_{X^1_T}.$$  

Since the triplet $(\frac{3k}{4} - s, (\frac{k}{2} - \frac{3}{4})^{-1}, \frac{3}{2})$ is 1-admissible, for any $\eta > 0$, we can choose $T > 0$ small enough such that

$$\|D^{3/4}_x u_L\|_{L^{1-s} L^2_x} < \eta.$$  

Gathering all these estimates, we infer

$$\|u_j\|_{X^1_T} \lesssim p_k(\|u_0\|_{H^s})(\|Q_j u_0\|_{H^{s'}} + \|f_j\|_{X^1_T}).$$

Summing this inequality over $j$ finishes the proof of Proposition 3.2.  

We also need $L^k_T L^\infty_x$-norm estimates.

**Proposition 3.3.** Let $u \in S_T$ be a solution of (3.12) with initial data $u_0 \in \dot{H}^s$. Then there exist $T > 0$ and a positive nondecreasing polynomial function $p_k$ such that

$$\|u\|_{L^k T \dot{H}^s_x} \lesssim p_k(\|u_0\|_{H^s})(\|u_0\|_{H^{s'}} + \|f\|_{X^1_T}).$$

Moreover, if $u_0 \in \dot{H}^s \cap \dot{H}^k, s \in \mathbb{R}$, then

$$\|u\|_{L^k T \dot{H}^s_x} \lesssim p_k(\|u_0\|_{H^s})(\|u_0\|_{H^{s'}} + \|f\|_{X^1_T}).$$  

**Proof.** Since $f = \pi(u_0, u) - u^k \partial_x u$ and $u \in S(\mathbb{R}^2)$, we can rewrite our equation as

$$u = u_L - \int_0^t V(t - t')(f - \pi(u_0, u)) dt'.$$
By virtue of Lemmas 3.2 and 3.3, we deduce
\[ \|u\|_{L^k_t L^\infty_x} \lesssim \|u_0\|_{\dot{H}^s} + \|f\|_{\dot{N}^s_x} + \|\pi(u_0, u)\|_{\dot{N}^s_x}, \]
and also
\[ \|u\|_{L^\infty_t \dot{H}^s_x} \lesssim \|u_0\|_{\dot{H}^s} + \|f\|_{\dot{N}^s_x} + \|\pi(u_0, u)\|_{\dot{N}^s_x}. \]
Next, we get
\[ \|\pi(u_0, u)\|_{\dot{N}^s_x} \lesssim \left( \sum_j \left[ 2^{j(s+1/2)} \|u_{0, <j}\|^k \|u_{<j}\|_{L^1_t L^\infty_x}^2 \right] \right)^{1/2} \]
\[ \lesssim \|u_0\|_{L^k_t L^\infty_x} \left( \sum_j \left[ 2^{j(s+1/2)} \|u_{<j}\|_{L^\infty_t L^2_x}^2 \right] \right)^{1/2} \]
\[ \lesssim \|u_0\|_{L^k_t L^\infty_x} \|u\|_{\dot{X}^s_t} \]
\[ \lesssim p_k(\|u_0\|_{\dot{H}^s})(\|u_0\|_{\dot{H}^s} + \|f\|_{\dot{N}^s_x}) \]
by Proposition 3.2. ■

4 Well-Posedness for \( k \geq 4 \)

4.1 Nonlinear estimates

Now we estimate the right-hand side of (3.12) in \( \dot{N}^s_x \)-norm.

**Proposition 4.1.** For any \( u \in \dot{X}^s_t \cap L^k_t L^\infty_x \), we have
\[ \|\pi(u_0, u) - \pi(u, u)\|_{\dot{N}^s_x} \lesssim \|u_0 - u\|_{L^k_t L^\infty_x} \left( \|u_0\|_{L^k_t L^\infty_x}^{k-1} + \|u\|_{L^k_t L^\infty_x}^{k-1} \right) \|u\|_{\dot{X}^s_t} \]
and
\[ \|g\|_{\dot{N}^s_x} \lesssim \|u\|_{L^k_t L^\infty_x}^{k-1} \|u\|_{\dot{X}^s_t}^2. \]

**Remark 4.1.** Thanks to the first estimate, we will be able to get the required contraction factor by taking advantage of the difference \( u_0 - u \) combined with Lemma 3.4. Concerning the second estimate, we will exploit the square on a norm which does not contain any \( L^\infty_x \) component (the \( \dot{X}^s_t \)-norm).
Proof. Set $u_j = Q_j u$, $u_{<j} = Q_{<j} u$, etc. Then,

$$
\| \pi(u_0, u) - \pi(u, u) \|_{\mathcal{X}_T^s} \lesssim \left( \sum_j 2^{j(s-1/2)} \| \partial_x ((u_{0, <j})^k - (u_{<j})^k) u_{<j} \|_{L^1_T L^2_x}^2 \right)^{1/2}
$$

$$
\lesssim \left( \sum_j 2^{j(s+1/2)} \| (u_{0, <j})^k - (u_{<j})^k \|_{L^1_T L^2_x} \| u_{<j} \|_{L^\infty_T L^2_x}^2 \right)^{1/2}
$$

$$
\lesssim \| u_0 - u \|_{L^1_T L^2_x} \left( \| u_0 \|_{L^2_x}^{k-1} + \| u \|_{L^1_T L^2_x} \| u \|_{\mathcal{X}_T^s} \right).
$$

Recalling that $g = \sum_j g_j$ with

$$
g_j = -\partial_x Q_j \left( \sum_{r < j} (Q_{<r} u)^2 (Q_{<r} u)^{k-1} \right),
$$

we bound the second term by

$$
\| g \|_{\mathcal{X}_T^s} \lesssim \left( \sum_j \left[ 2^{j(s+1/2)} \sum_{r < j} \| (u_{<r})^k \|_{L^1_T L^2_x} \right] \right)^{1/2}
$$

$$
\lesssim \left( \sum_j \left[ \sum_{r < j} 2^{j(s+1/2)} \| u_{<r} \|_{L^1_T L^2_x} \| u_{<r} \|_{L^2_x}^{1/2} \| u_{<r} \|_{L^\infty_T L^2_x} \| u_{<r} \|_{L^2_x}^{k-1} \right] \right)^{1/2}
$$

$$
\lesssim \| u \|_{L^1_T L^2_x}^{k-1} \sup_r 2^{2r/4} \| u_{<r} \|_{L^2_x}^{1/2} \| u_{<r} \|_{L^\infty_T L^2_x} \left( \sum_j \sum_{r < j} 2^{j(s+1/2)} \right) \left( \sum_j \left[ 2^{j(s+1/2)} \| u_{<r} \|_{L^2_x}^{1/2} \right] \right)^{1/2}
$$

$$
\lesssim \| u \|_{L^1_T L^2_x}^{k-1} \| u \|_{\mathcal{X}_T^s} \left( \sum_j 2^{j(s+1/2)} \right)^{1/2}
$$

where we used discrete Young inequality.

### 4.2 Existence in $\dot{H}^s(\mathbb{R})$

Consider the map $F$ defined as

$$
F(u) = U(t)u_0 - \int_0^t U(t - t') f(t') dt',
$$
where
\[ f = \pi(u_0, u) - \pi(u, u) + g. \]

We shall contract \( F \) in the intersection of two balls:

\[ B_M(u_0, T) = \{ u \in \dot{X}_T^{s_k} \cap L_T^k L_x^\infty : \| u - u_0 \|_{L_T^k L_x^\infty} \leq \delta \}, \]

and

\[ B_S(T) = \{ u \in \dot{X}_T^{s_k} \cap L_T^k L_x^\infty : \| u \|_{X^{s_k}} \leq \delta \}, \]

endowed with the norm

\[ \| u \|_{\dot{Y}_T} = \| u \|_{\dot{X}_T^{s_k}} + \| u \|_{L_T^k L_x^\infty}. \]

The space \( S_T \) is dense in \( \dot{Y}_T \) and thus all the estimates derived in Section 3.2 hold true for \( u \in B_M \cap B_S \).

Gathering Propositions 3.2, 3.3, and 4.1 (with \( s = s_k \)), we find that there exists \( C = C(u_0) > 1 \) such that

\[ \| F(u) \|_{\dot{X}_T^{s_k}} \leq C \| (U(t)u_0) \|_{\dot{X}_T^{s_k}} + C \left( 1 + \| u - u_0 \|_{L_T^k L_x^\infty}^{k-1} \right) \| u \|_{\dot{X}_T^{s_k}}^2 \]
\[ + C \| u - u_0 \|_{L_T^k L_x^\infty} \left( 1 + \| u - u_0 \|_{L_T^k L_x^\infty}^{k-1} \right) \| u \|_{\dot{X}_T^{s_k}}. \]

and in the same way

\[ \| F(u) - u_0 \|_{L_T^k L_x^\infty} \leq \| U(t)u_0 - u_0 \|_{L_T^k L_x^\infty} + C \left( 1 + \| u - u_0 \|_{L_T^k L_x^\infty}^{k-1} \right) \| u \|_{\dot{X}_T^{s_k}}^2 \]
\[ + C \| u - u_0 \|_{L_T^k L_x^\infty} \left( 1 + \| u - u_0 \|_{L_T^k L_x^\infty}^{k-1} \right) \| u \|_{\dot{X}_T^{s_k}}. \]

From Definition 3.2 and Lemma 3.4 (adapted to the linear group \( U(t) \)), we see that we can choose \( T = T(u_0) \) small enough so that the quantities \( \| U(t)u_0 \|_{\dot{X}_T^{s_k}} \) and \( \| U(t)u_0 - u_0 \|_{L_T^k L_x^\infty} \) are smaller than \( \varepsilon = \frac{1}{128C} \). Thus, if \( u \in B_M \cap B_S \), then

\[ \| F(u) \|_{\dot{X}_T^{s_k}} \leq 4C \varepsilon + 4C \delta^2 \]

and

\[ \| F(u) - u_0 \|_{L_T^k L_x^\infty} \leq 4C \varepsilon + 4C \delta^2. \]
Now we take $\delta = \frac{1}{8C}$ so that $F(u)$ belongs to $B_M \cap B_S$. Similarly, for any $u_1$ and $u_2$ in $B_M \cap B_S$, one has
\[
\|F(u_1) - F(u_2)\|_{\dot{Y}_T} \lesssim C \|f(u_1) - f(u_2)\|_{\dot{X}^k_T}
\]
\[
\lesssim C \|u_0 - u_1\|_{L^\infty_T L^\infty_x} \left( 1 + \|u_1\|_{L^2_T L^2_x}^{k-1} + \|u_2\|_{L^2_T L^2_x}^{k-1} \right) \|u_1 - u_2\|_{\dot{X}^k_T}
\]
\[
+ C \|u_1\|_{\dot{X}^k_T} \left( \|u_1\|_{L^2_T L^2_x}^{k-2} + \|u_2\|_{L^2_T L^2_x}^{k-2} \right) \|u_1 - u_2\|_{L^2_T L^2_x}
\]
\[
+ C \|u_2\|_{L^2_T L^2_x}^{k-1} \left( \|u_1\|_{\dot{X}^k_T} + \|u_2\|_{\dot{X}^k_T} \right) \|u_1 - u_2\|_{\dot{X}^k_T}.
\]
(4.1)

Therefore,
\[
\|F(u_1) - F(u_2)\|_{\dot{Y}_T} \lesssim (\epsilon + \delta) \|u_1 - u_2\|_{\dot{Y}_T}
\]
and for $\epsilon, \delta$ small enough, $F : B_M \cap B_S \to B_M \cap B_S$ is contractive. There exists a solution $u$ in $B_M \cap B_S$ to $F(u) = u$.

The next step is to show that $u \in C([-T, +T], \dot{H}^k_x(\mathbb{R}))$. Using (3.19) and Proposition 4.1, we obtain first that $u \in L^\infty_T \dot{H}^k_x$. Now for any $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, writing $u(t)$ as
\[
u(t) = U(t - t_1)u(t_1) - \int_{t_1}^t U(t - t')f(u(t'))dt',
\]
we get
\[
\|u(t_1) - u(t_2)\|_{\dot{H}^k} \lesssim \sup_{t \in [t_1, t_2]} \|u(t) - u(t_1)\|_{\dot{H}^k}
\]
\[
\lesssim \sup_{t \in [t_1, t_2]} \|u(t_1) - U(t - t_1)u(t_1)\|_{\dot{H}^k} + \left\| \int_{t_1}^t U(t - t')f(u(t'))dt' \right\|_{L^\infty(t_1, t_2; \dot{H}^k)}.
\]
On the other hand, the previous estimates show that $U(t - t')f(u(t')) \in \dot{H}^k_x L^1_{loc}(0, T)$. It follows from the dominated convergence theorem that $\|u(t_1) - u(t_2)\|_{\dot{H}^k} \to 0$ as $t_1 \to t_2$.

4.3 Uniqueness in $\dot{Z}_T$

Consider $u_{0,1}, u_{0,2} \in \dot{H}^k_x$ two initial data and $u_1, u_2$ belonging to our resolution space $\dot{Z}_T$ (see Theorem 2.1) satisfying
\[
u_1(t) = U_1(t)u_{0,1} - \int_0^t U_1(t - t')f_1(u_1)(t')dt',
\]
\[
u_2(t) = U_2(t)u_{0,2} - \int_0^t U_2(t - t')f_2(u_2)(t')dt',
\]
where $U_j(t)\phi$ is solution to

$$\partial_t u + \mathcal{H} \partial_x^2 u + \pi(u_{0,j}, u) = 0, \quad u(0) = \phi$$

and $f_j$ is defined by

$$f_j(u) = \pi(u_{0,j}, u) - \pi(u, u) + g(u).$$

We intend to show that there exists a nondecreasing polynomial function $P \geq 1$ such that

$$\|u_1 - u_2\|_{\dot{Z}_T} \lesssim P\left(\|u_1\|_{\dot{Z}_T} + \|u_2\|_{\dot{Z}_T} + \|u_{0,1}\|_{\dot{H}^k} + \|u_{0,2}\|_{\dot{H}^k}\right)[\|u_{0,1} - u_{0,2}\|_{\dot{H}^k}]
+ \left(\|u_1\|_{\dot{X}^k_T} + \|u_2\|_{\dot{X}^k_T}\right)\|u_1 - u_2\|_{\dot{Z}_T}. \quad (4.2)$$

Taking $T > 0$ so that $\|u_j\|_{\dot{X}^k_T}$ is small enough, we see that the uniqueness of the solution to (gBO) in the whole space $\dot{Z}_T$ and the fact that the flow map is locally Lipschitz from $\dot{H}^k(\mathbb{R})$ to $\dot{Z}_T$ follow directly from (4.2).

One has

$$\|U_1(t)u_{0,1} - U_2(t)u_{0,2}\|_{\dot{Z}_T} \lesssim \|U_1(t)(u_{0,1} - u_{0,2})\|_{\dot{Z}_T} + \|(U_1(t) - U_2(t))u_{0,2}\|_{\dot{Z}_T}.$$

The first term in the right-hand side is bounded by $p_k(\|u_{0,1}\|_{\dot{H}^k})\|u_{0,1} - u_{0,2}\|_{\dot{H}^k}$. To treat the second one, we note that $(U_1(t) - U_2(t))u_{0,2}$ is solution to

$$\partial_t u + \mathcal{H} \partial_x^2 u + \pi(u_{0,1}, U_2(t)u_{0,2}) - \pi(u_{0,2}, U_2(t)u_{0,2})$$

with zero initial data. Hence, by Propositions 3.2 and 3.3,

$$\|(U_1(t) - U_2(t))u_{0,2}\|_{\dot{Z}_T} \lesssim p_k(\|u_{0,1}\|_{\dot{H}^k})\|\pi(u_{0,1}, U_2(t)u_{0,2}) - \pi(u_{0,2}, U_2(t)u_{0,2})\|_{\dot{X}^k_T}
\lesssim p_k(\|u_{0,1}\|_{\dot{H}^k})\|u_{0,1} - u_{0,2}\|_{\dot{H}^k}.$$

We also need to bound

$$\left\| \int_0^t (U_1(t - t') f_1(u_1) - U_2(t - t') f_2(u_2)) dt' \right\|_{\dot{Z}_T}
\lesssim \left\| \int_0^t U_1(t - t')(f_1(u_1) - f_1(u_2)) dt' \right\|_{\dot{Z}_T} \quad (4.3)
+ \left\| \int_0^t U_1(t - t')(f_1(u_2) - f_2(u_2)) dt' \right\|_{\dot{Z}_T} \quad (4.4)
+ \left\| \int_0^t (U_1(t - t') - U_2(t - t')) f_2(u_2) dt' \right\|_{\dot{Z}_T} \quad (4.5)$$
Here, (4.3) is bounded by

\[ (4.3) \lesssim p_k(\|u_{0,1}\|_{H^s}) \|f_1(u_1) - f_1(u_2)\|_{N \beta T} \]

and we can use (4.1) to get the desired estimate. Term (4.4) is bounded by

\[ (4.4) \lesssim p_k(\|u_{0,1}\|_{H^s}) \|\pi(u_{0,1}, u_2) - \pi(u_{0,2}, u_2)\|_{N \beta T} \]

\[ \lesssim P(\|u_2\|_{Z_T} + \|u_{0,1}\|_{H^s} + \|u_{0,2}\|_{H^s}) \|\pi(u_{0,1}, \psi) - \pi(u_{0,2}, \psi)\|_{N \beta T}. \]

Finally, note that \( f_0^t(U_1(t - t') - U_2(t - t')) f_2(u_2) dt' \) is solution to

\[ \partial_t u + H\partial_x^2 u + \pi(u_{0,1}, u) = \pi(u_{0,2}, \psi) - \pi(u_{0,1}, \psi), \]

with zero initial data, and where \( \psi = f_0^t U_2(t - t') f_2(u_2) dt' \). It follows that

\[ (4.5) \lesssim p_k(\|u_{0,1}\|_{H^s}) \|\pi(u_{0,2}, \psi) - \pi(u_{0,1}, \psi)\|_{N \beta T} \]

\[ \lesssim P(\|u_{0,1}\|_{H^s} + \|u_{0,2}\|_{H^s}) \|\pi(u_{0,1}, \psi) - \pi(u_{0,2}, \psi)\|_{N \beta T}. \]

Gathering all these estimates, we obtain (4.2).

### 4.4 Existence and uniqueness in \( H^s(\mathbb{R}) \), \( s \geq \beta \)

Now we consider the problem in nonhomogeneous spaces \( H^s(\mathbb{R}) \). The main task is to obtain a suitable control for the \( L^2 \)-norm of the solution. Define the spaces \( X^s_T = X^0_T \cap X^s_T \) and \( N^{s,\theta} = N^{0,\theta}_T \cap N^{s,\theta}_T \).

We closely follow the proof of Theorem 2.1 and show that \( F \) is a contraction in the intersection of

\[ B_M(u_0, T) = \{ u \in X^s_T \cap L^k \cap L^\infty_T : \|u - u_0\|_{L^k L^\infty_T} \leq \delta \} \]

and

\[ B_S(T) = \{ u \in X^s_T \cap L^k \cap L^\infty_T : \|u\|_{X^s_T} \leq \delta \}, \]

endowed with the norm

\[ \|u\|_{Y_s} = \|u\|_{X^s_T} + \|u\|_{L^k L^\infty_T}. \]
Using Propositions 3.2, 3.3, and 4.1 (applied with $s \geq s_k$ and $s = 0$) and the embedding $\mathcal{N}_T^{s_k,1} \hookrightarrow \mathcal{N}_T^{s_k,1}$ for $s \geq s_k$, we find

$$\|F(u)\|_{X_T^s} \leq C \|U(t)u_0\|_{X_T^s} + C \left(1 + \|u - u_0\|_{L_T^k L_x^\infty}^{k-1}\right) \|u\|_{X_T^s}^2$$

and

$$\|F(u) - u_0\|_{L_T^k L_x^\infty} \leq \|U(t)u_0 - u_0\|_{L_T^k L_x^\infty} + C \left(1 + \|u - u_0\|_{L_T^k L_x^\infty}^{k-1}\right) \|u\|_{X_T^s}^2$$

In the same way, one may show that

$$\|F(u_1) - F(u_2)\|_{Y_T} \lesssim C \|f(u_1) - f(u_2)\|_{X_T^{s,1}}$$

$$\lesssim C \|u_0 - u_1\|_{L_T^k L_x^\infty} \left(1 + \|u_1\|_{L_T^k L_x^\infty}^{k-1}\right) \|u_1 - u_2\|_{X_T^s}$$

$$+ C \|u_2\|_{X_T^s} \left(\|u_1\|_{L_T^k L_x^\infty}^{k-1} + \|u_2\|_{L_T^k L_x^\infty}^{k-1}\right) \|u_1 - u_2\|_{L_T^k L_x^\infty}$$

$$+ C \|u_1\|_{X_T^s} \left(\|u_1\|_{L_T^k L_x^\infty}^{k-2} + \|u_2\|_{L_T^k L_x^\infty}^{k-2}\right) \|u_1 - u_2\|_{L_T^k L_x^\infty}$$

$$+ C \|u_2\|_{L_T^k L_x^\infty} \left(\|u_1\|_{X_T^s} + \|u_2\|_{X_T^s}\right) \|u_1 - u_2\|_{X_T^s}.$$

This proves the existence in $H^s(\mathbb{R})$. The end of the proof is identical to that of Theorem 2.1.

5 Well-Posedness for $k = 3$

Let $k = 3$ and $s > 1/3$ be fixed.

The scheme of the proof is the same as for the case $k \geq 4$ with minor modifications. The main new ingredient is the following well-known linear estimate.

**Lemma 5.1 ([10], Proposition 2.5).** Let $0 < T \leq 1$ and $s > 1/3$. Then it holds that

$$\|V(t)\varphi\|_{L_T^k L_x^\infty} \lesssim \varphi\|_{H^s}, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}). \quad (5.1)$$

First, in view of this result, it is clear that Lemma 3.4 holds for $k = 3$ with $u_0 \in H^{s_k}$ replaced by $u_0 \in H^s$. Next, we see that the $\tilde{B}_T^{3/2,2} \left(L_T^2\right)$ -norm which appears in the proof of Proposition 4.1 when estimating the nonlinear term $g$ is not bounded by the $\dot{S}_T^{s,1}$ -norm for $k = 3$. So we modify slightly the space $X_T^{s_k}$ by setting

$$X_T^s = \dot{X}_T^s \cap \dot{X}_T^s \cap \tilde{B}_T^{2,2} \left(L_T^2\right).$$
On one hand, it is clear from Sobolev inequalities that
\[ \| u \|_{\dot{B}_{3}^{\frac{2}{5}, 2}(L^2_T)} \lesssim \| u \|_{\dot{B}_{3}^{\frac{2}{3}, 2}(L^2_T)} \lesssim \| u \|_{X^4_t}. \]

On the other hand, the \( \dot{B}_{3}^{\frac{2}{5}, 2}(L^2_T) \)-norm is acceptable since by (5.1),
\[ \| \dot{V}(t) \psi \|_{\dot{B}_{3}^{\frac{2}{5}, 2}(L^2_T)} \lesssim \left( \sum_j 4^j \| Q_j \dot{V}(t) \psi \|_{L^2_T}^2 \right)^{1/2} \lesssim \left( \sum_j \| Q_j \psi \|_{H^{1/3+2\varepsilon}}^2 \right)^{1/2} \lesssim \| \psi \|_{H^s} \]
for \( \varepsilon \ll 1 \). From this, it is straightforward to check that the subcritical nonhomogeneous versions of Propositions 3.2, 3.3, and 4.1 are valid whenever \( k = 3 \). This essentially proves Theorem 2.3.

Acknowledgments

The author wants to thank Fabrice Planchon for his enthusiastic help and his availability, as well as the referee for helpful comments and suggestions which greatly improved the presentation.

References


