On the joint asymptotic behaviours of ranked heights of Brownian excursions

by

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Dedicated to Professor Pál Révész on the occasion of his 65-th birthday

Summary. In this paper, we present some joint asymptotic behaviours of the heights of Brownian excursions.

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1. Introduction

Let $\{B(t), t \geq 0\}$ be a one-dimensional standard Brownian motion starting from 0 and consider the sequence

$$M_1(t) \geq M_2(t) \geq ... \geq M_n(t) \geq ... \tag{1.1}$$

the ranked heights of all the excursions of the reflected Brownian motion $|B|$ up to time t (including the meander height $\sup_{g_t \leq u \leq t} |B(u)|$, where g_t is the last zero of B before t). This gives a natural way to order the countable many Brownian excursions. We refer to Csáki, Erdős and Révész [3] and Révész [17, Chap. XIII] for asymptotic studies on excursion lengths of random walk and Brownian motion, and to Pitman and Yor [12, 13] for the laws of excursion lengths. The ordered excursion heights were studied first by Pitman and Yor [14, 15], who also considered a general class of recurrent self-similar Markov processes.

Here, we are interested in the joint asymptotic behaviours of $\{M_n(t), n \geq 1\}$ as $t \to \infty$.

Theorem 1.1. *Let $f(t) > 0$ be a nondecreasing function. For fixed $0 \leq r \leq 1$,*

$$\mathbb{P}\Big(M_1(t) > \sqrt{t}\, f(t) \text{ and } M_2(t) > r\sqrt{t}\, f(t), \text{ i.o.}\Big) = \begin{cases} 0 \\ 1 \end{cases},$$

according as

$$\int^{\infty} \frac{dt}{t} f(t) \exp\left(-\frac{(1+2r)^2 f^2(t)}{2}\right) \begin{cases} < \infty \\ = \infty \end{cases},$$

where, here and in the sequel, "i.o." *means "infinitely often" as the relevant index goes to infinity.*

Theorem 1.2. *The set of vectors*

$$\left(\frac{M_1(t)}{\sqrt{2t\log\log t}}, \frac{M_2(t)}{\sqrt{2t\log\log t}}\right), \qquad t > 3,$$

is relatively compact in $\mathbb{R}^2$ and its set of limit points is given by

$$\{(x,y) \in R^2 : 0 \leq y \leq x,\ x + 2y \leq 1\}.$$

It is an interesting open problem to give a Strassen's [18] functional law for the vector of processes $(2t\log\log t)^{-1/2}\big(M_1(ut), M_2(ut)\big)_{0\leq u\leq 1}$.

The lower functions of $M_1(t)$ and $M_n(t)$ for $n \geq 2$, are considerably different (cf. [4]), which leads us to consider the joint lower functions of $(M_1(t), M_2(t))$:

Theorem 1.3. *Let $f(t) > 0$ be a nondecreasing function. For each $1 \le \sigma \le \infty$,*

$$\mathbb{P}\left(M_1(t) < \sigma \frac{\sqrt{t}}{f(t)} \text{ and } M_2(t) < \frac{\sqrt{t}}{f(t)}, \;\; \text{i.o.} \right) = \begin{cases} 0 \\ 1 \end{cases},$$

according as

$$\int^{\infty} \frac{dt}{t} f^2(t) \exp\Big(- \frac{\pi^2 f^2(t)}{8} \Big) \begin{cases} < \infty \\ = \infty \end{cases}, \qquad \text{if } \; \sigma = 1, \tag{1.2}$$

$$\int^{\infty} \frac{dt}{t} f^4(t) \exp\Big(- \frac{\pi^2 f^2(t)}{8} \Big) \begin{cases} < \infty \\ = \infty \end{cases}, \qquad \text{if } \; 1 < \sigma < 2, \tag{1.3}$$

$$\int^{\infty} \frac{dt}{t} f^6(t) \exp\Big(- \frac{\pi^2 f^2(t)}{8} \Big) \begin{cases} < \infty \\ = \infty \end{cases}, \qquad \text{if } \; \sigma = 2, \tag{1.4}$$

$$\int^{\infty} \frac{dt}{t} f^2(t) \exp\Big(- \frac{\pi^2 f^2(t)}{2\sigma^2} \Big) \begin{cases} < \infty \\ = \infty \end{cases}, \qquad \text{if } \; 2 < \sigma < \infty, \tag{1.5}$$

$$\int^{\infty} \frac{dt}{t f(t)} \begin{cases} < \infty \\ = \infty \end{cases}, \qquad \text{if } \; \sigma = \infty. \tag{1.6}$$

Notice that (1.2) is the well-known Chung's test, whereas (1.6) is taken from [4]. See [2] for the joint lower functions of $(\sup_{0 \le s \le t} B(s), - \inf_{0 \le s \le t} B(s))$ (and random walk case), where Chung-type and Hirsch-type tests are unified. In particular, the above result merely says that if $M_1(t), M_2(t)$ are small, the critical ratio of $M_2(t)/M_1(t)$ is $1/2$. We shall study the ratio $M_{n+1}(t)/M_n(t)$ for $n \ge 1$, as $t \to \infty$, and the result reads as follows:

Theorem 1.4. *Let $f > 0$ be a nondecreasing function. For $n \ge 1$, we have*

$$\mathbb{P}\left(\frac{M_{n+1}(t)}{M_n(t)} < \frac{1}{f(t)}, \text{ i.o.} \right) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t f^n(t)} \begin{cases} < \infty \\ = \infty \end{cases}. \tag{1.7}$$

The rest of this paper is organized as follows: In Section 2, we compute some Laplace transforms needed to obtain the joint small and large deviations for the heights. The proofs of Theorems 1.1 and 1.2 are given in Section 3, whereas Section 4 is devoted to prove Theorem 1.3, and Theorem 1.4 is proved in Section 5.

We write $f(x) \sim g(x)$ (resp: $f(x) \asymp g(x)$ for $x \in I$, with I being some interval of $\mathbb{R}$) as $x \to x_0$ if $\lim_{x \to x_0} f(x)/g(x) = 1$ (resp: $0 < C_1 \le \inf_{x \in I} f(x)/g(x) \le \sup_{x \in I} f(x)/g(x) \le C_2 < \infty$). Unless stated otherwise, $(C_i, 1 \le i \le 15)$ denote some (unimportant) positive constants.

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2. Preliminaries

Define

$$H_n(r) \stackrel{\text{def}}{=} \inf\{t > 0 : M_n(t) > r\}, \qquad r > 0. \tag{2.1}$$

Recall the following characterization of the law of the sequence $\{M_i(H_n(1)), i \geq 1; H_n(1)\}$:

Proposition 2.1 ([4]). *The two sequences $\{M_i(H_n(1)), i \leq n\}$ and $\{M_j(H_n(1)), j \geq n+1\}$ are independent. Furthermore, we have*

(2.2) *For $n \geq 2$, $\left\{0 \leq \frac{1}{M_1(H_n(1))} \leq \ldots \leq \frac{1}{M_{n-1}(H_n(1))} \leq 1\right\}$ has the same law as the rearranged nondecreasing sequence of $n-1$ i.i.d. uniform variables in $[0,1]$;*

(2.3) *The law of $\{M_j(H_n(1)), j \geq n+1\}$ is characterized as follows: for every measurable function $f \geq 0$, we have*

$$\mathbb{E} \exp\left(- \sum_{j \geq n+1} f(M_j(H_n(1))) \right) = \left(1 + \int_0^1 \frac{dx}{x^2} \, (1 - e^{-f(x)}) \right)^{-n};$$

(2.4) *The law of $H_n(1)$ conditioning on $\{M_i(H_n(1)), i \geq 1\}$ is determined as follows: for every $\lambda > 0$, we have*

$$\mathbb{E}\Big(e^{-\frac{\lambda^2}{2} H_n(1)} \,\Big|\, \big\{ M_i(H_n(1)) = x_i > 0, i \neq n \big\} \Big) = \frac{\lambda}{\sinh(\lambda)} \prod_{i \neq n} \left(\frac{\lambda x_i}{\sinh(\lambda x_i)} \right)^2.$$

Lemma 2.2. *Fix $n \geq 1$ and $0 < r \leq 1$. We have for $\lambda > 0$ that*

$$\mathbb{E} e^{-\frac{\lambda^2}{2} H_n(1)} \mathbb{1}_{(H_n(1) < H_{n+1}(r))} = e^{-\lambda(n-1)} \left(\frac{\sinh(\lambda r)}{\sinh(\lambda)\cosh(\lambda r)} \right)^n, \tag{2.5}$$

$$\mathbb{E} e^{-\frac{\lambda^2}{2} H_{n+1}(r)} \mathbb{1}_{(H_n(1) > H_{n+1}(r))} = \cosh^{-(n+1)}(\lambda r) \left(e^{-\lambda r n} - e^{-\lambda n} \left[\frac{\sinh(\lambda r)}{\sinh(\lambda)} \right]^n \right). \tag{2.6}$$

Proof. By conditioning on $\{M_i(H_n(1), i \neq n\}$, it follows from (2.4) that the LHS of (2.5) equals

$$\begin{aligned}
&= \frac{\lambda}{\sinh(\lambda)} \mathbb{E} \prod_{i \neq n} \left[\frac{\lambda M_i(H_n(1))}{\sinh(\lambda M_i(H_n(1)))} \right]^2 \mathbb{1}_{(M_{n+1}(H_n(1)) < r)} \\
&= \frac{\lambda}{\sinh(\lambda)} \mathbb{E} \prod_{i \leq n-1} \left[\frac{\lambda M_i(H_n(1))}{\sinh(\lambda M_i(H_n(1)))} \right]^2 \\
&\qquad \times \mathbb{E} \prod_{i \geq n+1} \left[\frac{\lambda M_i(H_n(1))}{\sinh(\lambda M_i(H_n(1)))} \right]^2 \mathbb{1}_{(M_{n+1}(H_n(1)) < r)},
\end{aligned} \tag{2.7}$$

by using the fact that $\{M_i(H_n(1), i < n\}$ and $\{M_i(H_n(1), i > n\}$ are independent. It follows from (2.2) that

(2.8)

$$\mathbb{E} \prod_{i \le n-1} \Big[\frac{\lambda M_i(H_n(1))}{\sinh(\lambda M_i(H_n(1)))}\Big]^2 = \left(\int_0^1 du \left(\frac{\lambda/u}{\sinh(\lambda/u)}\right)^2\right)^{n-1} = \left(\frac{\lambda}{\sinh(\lambda)}\, e^{-\lambda}\right)^{n-1},$$

whereas by means of (2.3), the second expectation term in (2.7) is equal to

$$\begin{aligned}\mathbb{E} \prod_{i \ge n+1} \Big[\frac{\lambda M_i(H_n(1))}{\sinh(\lambda M_i(H_n(1)))} \mathbb{1}_{(M_i(H_n(1))<r)}\Big]^2 &= \left(1 + \int_0^1 \frac{dx}{x^2}\Big[1 - \left(\frac{\lambda x}{\sinh(\lambda x)} \mathbb{1}_{(x<r)}\right)^2\Big]\right)^{-n} \\ &= (\lambda \coth(\lambda r))^{-n},\end{aligned}$$

which in view of (2.7) and (2.8), implies (2.5). The proof of (2.6) is quite similar: by using Brownian scaling and applying Proposition 2.1 to $H_{n+1}(1)$ instead of $H_n(1)$, we obtain the desired result. □

Corollary 2.3. *For $0 < r \le 1$, we have*

$$\mathbb{E} e^{-\frac{\lambda^2}{2} H_1(1) \wedge H_2(r)} = \frac{\sinh(\lambda(1-r))}{\cosh^2(\lambda r)\, \sinh \lambda} + \frac{\tanh(\lambda r)}{\sinh \lambda}, \tag{2.9}$$

$$\mathbb{E} e^{-\frac{\lambda^2}{2} H_1(1) \vee H_2(r)} = \frac{2 \sinh(\lambda(1-r))}{\sinh(2\lambda)\, \cosh(\lambda r)} + \frac{\sinh(\lambda r)\, e^{-\lambda}}{\sinh \lambda\, \cosh^2(\lambda r)}, \tag{2.10}$$

$$\mathbb{P}\Big(M_{n+1}(H_n(1)) < r\Big) = r^n, \qquad n \ge 1. \tag{2.11}$$

Proof. By letting $r = 1$ in (2.5), we recover Pitman and Yor's formula [14]: for $\lambda > 0$,

$$\mathbb{E} \exp\left(-\frac{\lambda^2}{2} H_n(1)\right) = e^{-\lambda(n-1)} \cosh^{-n}(\lambda). \tag{2.12}$$

By taking $\lambda = 0$ in (2.5), we get (2.11). Adding (2.5) and (2.6) gives (2.9), and (2.10) follows from (2.9) and (2.12). □

We end this section by two elementary results:

Lemma 2.4. *Let X_1, X_2 be two independent nonnegative random variables such that*

$$\mathbb{P}(X_i > t) \asymp t^{\gamma_i} \exp(-c_i\, t), \qquad \forall t \ge 1,\ i = 1, 2, \tag{2.13}$$

for some constants $c_1 \ge c_2 > 0$ and $\gamma_1, \gamma_2 > -1$. Then

$$\mathbb{P}(X_1 + X_2 > t) \asymp \begin{cases} t^{\gamma_2} \exp(-c_2\, t), & \text{if } 0 < c_2 < c_1, \\ t^{\gamma_1+\gamma_2+1} \exp(-c_2\, t), & \text{if } c_1 = c_2, \end{cases} \qquad t \ge 1. \tag{2.14}$$

Lemma 2.5. *Suppose Y_1 and Y_2 are two independent positive random variables such that*

$$\mathbb{P}(Y_i < \epsilon) \asymp \epsilon^{\alpha_i} \exp\left(-\frac{\beta_i}{\epsilon}\right), \qquad 0 < \epsilon < 1, \tag{2.15}$$

for some constants $\beta_i > 0$ and $\alpha_i \in \mathbb{R}$. Then

$$\mathbb{P}(Y_1 + Y_2 < \epsilon) \asymp \epsilon^{\alpha_1+\alpha_2-1/2} \exp\left(-\frac{(\sqrt{\beta_1}+\sqrt{\beta_2})^2}{\epsilon}\right), \qquad 0 < \epsilon < 1. \tag{2.16}$$

3. Joint upper functions

Lemma 3.1. *Fix $0 \le r \le 1$. Uniformly for $\lambda \ge 1$,*

$$\mathbb{P}\left(M_1(1) > \lambda \text{ and } M_2(1) > r\,\lambda\right) \asymp \frac{1}{\lambda} \exp\left(-\frac{(1+2r)^2\lambda^2}{2}\right). \tag{3.1}$$

Proof. Recall (2.1):

$$\mathbb{P}\left(M_1(1) > \lambda \text{ and } M_2(1) > r\,\lambda\right) = \mathbb{P}\left(H_1(1) \vee H_2(r) < x\right), \tag{3.2}$$

with $x \stackrel{\text{def}}{=} \lambda^{-2} \le 1$. For $0 < a < b$, denote by $T^{(3)}_{a\to b}$ a positive r.v. such that

$$\mathbb{E}\exp\left(-\frac{\lambda^2}{2}T^{(3)}_{a\to b}\right) = \frac{b\sinh(\lambda a)}{a\sinh(\lambda b)}, \qquad \lambda > 0. \tag{3.3}$$

We simply write $T^{(3)}_b$ for $T^{(3)}_{0\to b}$ (whose Laplace transform can be obtained from (3.3) by letting $a \to 0$). In fact, $T^{(3)}_{a\to b}$ is distributed as the first hitting of b by a three-dimensional Bessel process starting from a (cf. [10] and [16]). It follows from (2.10) that

$$\begin{aligned}\mathbb{P}\left(H_1(1) \vee H_2(r) < x\right) = (1-r)\mathbb{P}\left(T^{(3)}_{1-r\to 2} + r^2 T^{(1)}_1 < x\right) + \\ + r\,\mathbb{P}\left(T^{(3)}_{r\to 1} + r^2\left(T^{(1)}_1 + \widehat{T}^{(1)}_1\right) + T_1(B) < x\right),\end{aligned} \tag{3.4}$$

where, $T^{(1)}_1$ (resp: $T_1(B)$) is distributed as the first hitting time of 1 by $|B|$ (resp: B), and $\widehat{T}^{(1)}_1 \stackrel{\text{law}}{=} T^{(1)}_1$, and all the variables in (3.4) are mutually independent.

For $0 < a < b$, the following formula (3.5) is taken from Borodin and Salminen [1, pp. 339]):

$$\mathbb{P}\Big(T^{(3)}_{a\to b} \in dt\Big)\Big/dt = \frac{b}{a}\sum_{k=-\infty}^{\infty} \frac{b-a+2kb}{\sqrt{2\pi t^3}} \exp\left(-\frac{(b-a+2kb)^2}{2t}\right) \tag{3.5}$$

$$= \frac{4\pi\, b}{a}\sum_{k=1}^{\infty} (-1)^{k-1}\, k \sin\left(\frac{ka\pi}{b}\right) \exp\left(-\frac{k^2\pi^2\, t}{2b^2}\right), \tag{3.6}$$

where (3.6) follows from Poisson's summation formula (cf. Feller [8, pp.592]). Recall that

$$\mathbb{P}\Big(T^{(1)}_1 < x\Big) \asymp \mathbb{P}\Big(T_1(B) < x\Big) \asymp x^{1/2} \exp\Big(-\frac{1}{2x}\Big), \qquad 0 < x < 1, \tag{3.7}$$

see also Gruet and Shi [9]. Applying Lemma 2.5 to (3.4) and using (3.7), (3.5) for the small deviation of $T^{(3)}_{a\to b}$, (3.1) follows. $\square$

Proof of Theorem 1.1. From (3.1), scaling and monotonicity, it is routine to prove the convergence part of Theorem 1.1 (cf. the pioneer paper of Erdős [7]), the details are omitted.

To show the divergent part, we only need to consider the "critical" case: there exists some $t_0 > 0$ such that

$$\frac{1}{3}\sqrt{\log\log t} \le f(t) \le 3\sqrt{\log\log t}, \qquad t \ge t_0, \tag{3.8}$$

see e.g. [7] for a rigorous justification. We only consider the case $0 < r \le 1$, since the case of $r = 0$ is described by the classical Erdős-Feller-Kolmogorov-Petrowsky test. Recall (2.1) and define

$$\begin{aligned} t_i &\overset{\text{def}}{=} \exp\Big(\theta\frac{i}{\log i}\Big),\\ \lambda_i &\overset{\text{def}}{=} \sqrt{t_i} f(t_i),\\ A_i &\overset{\text{def}}{=} \Big\{t_{i-1} < H_1(\lambda_i) \vee H_2(r\lambda_i) < t_i\Big\}, \end{aligned}$$

where $\theta > 0$ is a constant whose value will be determined later. It suffices to prove that

$$\mathbb{P}(A_i;\ \text{i.o.}) = 1. \tag{3.9}$$

Observe that from (3.1) and using scaling, we can find some sufficiently large $\theta > 0$ such that

$$\begin{aligned} \mathbb{P}(A_i) &= \mathbb{P}(H_1(\lambda_i) \vee H_2(r\lambda_i) < t_i) - \mathbb{P}(H_1(\lambda_i) \vee H_2(r\lambda_i) < t_{i-1})\\ &\asymp \frac{1}{f(t_i)} \exp\left(-\frac{(1+2r)^2 f^2(t_i)}{2}\right), \end{aligned} \tag{3.10}$$

which in view of the divergence of the integral test yields that

$$\sum_i \mathbb{P}(A_i) = \infty. \tag{3.11}$$

To apply the Borel-Cantelli lemma, we shall estimate the second moment $\mathbb{P}(A_i \cap A_j)$ for $j > i \geq i_0$, where i_0 denotes some large constant. We first estimate

$$\mathbb{P}\Big(M_1(t_i) > r\lambda_j\Big) = \mathbb{P}\Big(M_1(1) > r\lambda_j/\sqrt{t_i}\Big) \leq \frac{C_3\sqrt{t_i}}{rf(t_j)\sqrt{t_j}} \exp\Big(-\frac{f^2(t_j)t_j}{2t_i}\Big), \tag{3.12}$$

where the inequality follows from (3.1) by taking $r = 0$ there. Define

$$\beta(t) \stackrel{\text{def}}{=} |B\big(H_1(\lambda_i) \vee H_2(r\lambda_i) + t\big)|, \qquad t \geq 0.$$

Using strong Markov property, the process β is a reflected Brownian motion, starting from $|B(H_1(\lambda_i) \vee H_2(r\lambda_i))| = \lambda_i$ or $r\lambda_i$, and independent of $\mathcal{F}_{H_1(\lambda_i)\vee H_2(r\lambda_i)}$ (where, here and in the sequel, $\big(\mathcal{F}_t, t \geq 0\big)$ denotes the natural filtration generated by B).

Observe that on the event $A_i \cap A_j \cap \big\{M_1(t_i) \leq r\lambda_j\big\}$, there are at least two excursions of β before time $t_j - t_{i-1}$, whose heights are larger than $r\lambda_j$ and λ_j. Write $a \stackrel{\text{def}}{=} |B(H_1(\lambda_i) \vee H_2(r\lambda_i))| = \lambda_i$ or $r\lambda_i$, we introduce a Brownian motion $\widehat{B}$ by $\beta(t) = |a + \widehat{B}(t)|$ for $t \geq 0$, $\widehat{B}$ is independent of $\mathcal{F}_{H_1(\lambda_i)\vee H_2(r\lambda_i)}$. Therefore,

$$A_i \cap A_j \cap \big\{M_1(t_i) \leq r\lambda_j\big\} \subset A_i \cap \big\{\widehat{M_1}(t_j - t_{i-1}) \geq \lambda_j - a\big\} \cap \big\{\widehat{M_2}(t_j - t_{i-1}) \geq (r\lambda_j - a)^+\big\},$$

with $\widehat{M_1}(\cdot)$, $\widehat{M_2}(\cdot)$ related to $\widehat{B}$ the same way as $M_1(\cdot), M_2(\cdot)$ are related to B. It follows that

$$\begin{aligned}
&\mathbb{P}\Big(A_i \cap A_j \cap \big\{M_1(t_i) \leq r\lambda_j\big\}\Big) \\
&\leq \mathbb{P}(A_i)\mathbb{P}\left(\left\{\widehat{M_1}(1) \geq \frac{\lambda_j - \lambda_i}{\sqrt{t_j - t_{i-1}}}\right\} \cap \left\{\widehat{M_2}(1) \geq \frac{(r\lambda_j - \lambda_i)^+}{\sqrt{t_j - t_{i-1}}}\right\}\right) &(3.13)\\
&\leq \mathbb{P}(A_i)\frac{C_4}{(\lambda_j - \lambda_i/r)^+ + 1} \exp\left(-\frac{(1+2r)^2((\lambda_j - \lambda_i/r)^+)^2}{2(t_j - t_{i-1})}\right), &(3.14)
\end{aligned}$$

by applying (3.1) to $\lambda = (\lambda_j - \lambda_i/r)^+$. Using (3.1) with $r = 0$, it follows from (3.13) that

$$\begin{aligned}
\mathbb{P}\Big(A_i \cap A_j \cap \big\{M_1(t_i) \leq r\lambda_j\big\}\Big) &\leq \mathbb{P}(A_i)\,\mathbb{P}\left(M_1(1) \geq \frac{\lambda_j - \lambda_i}{\sqrt{t_j - t_{i-1}}}\right) \\
&\leq C_5\mathbb{P}(A_i)\,\frac{\sqrt{t_j - t_{i-1}}}{\lambda_j - \lambda_i} \exp\left(-\frac{(\lambda_j - \lambda_{i-1})^2}{2(t_j - t_{i-1})}\right). &(3.15)
\end{aligned}$$

Using (3.12), (3.14) and (3.15), it is elementary to obtain that

$$\mathbb{P}(A_i \cap A_j) \le \begin{cases} C_6 \mathbb{P}(A_i)\, \mathbb{P}(A_j), & \text{if} \quad j-i \ge \log^2 i, \\ C_7 \mathbb{P}(A_i) j^{-C_8}, & \text{if} \quad \log i \le j-i < \log^2 i, \\ C_9 \mathbb{P}(A_i)\, e^{-C_{10}(j-i)}, & \text{if} \quad 2 \le j-i < \log i. \end{cases} \tag{3.16}$$

It follows that

$$\liminf_{n\to\infty} \frac{\sum_{i_0 \le i,j \le n} \mathbb{P}(A_i \cap A_j)}{\Big(\sum_{i_0 \le i \le n} \mathbb{P}(A_i)\Big)^2} \le C_6,$$

which in view of (3.11), according to Kochen and Stone's version of the Borel-Cantelli lemma (cf. [11]) implies $\mathbb{P}(A_i; \text{i.o.}) \ge 1/C_6 > 0$. This probability in fact equals 1 from Kolmogorov's 0-1 law. □

We present a lemma before proving Theorem 1.2:

Lemma 3.2. *For each* $0 < \delta < 1/9$, *there exists a constant* $C_{11} = C_{11}(\delta) > 0$ *such that for all* $x > 0$,

$$\mathbb{P}\Big(M_1(1) + 2M_2(1) > x\Big) \le C_{11} \exp\Big(-\frac{(1-\delta)^3\, x^2}{2}\Big). \tag{3.17}$$

Proof. The probability term in (3.17) is bounded by

$$\mathbb{P}\Big(M_1(1) > (1-\delta)x\Big) + \mathbb{P}\Big(M_2(1) > \frac{1-\delta}{2}\, x\Big) + \sum_{1 \le k \le 1/\delta} I_k, \tag{3.18}$$

with $I_k \overset{\text{def}}{=} \mathbb{P}\Big(k\delta x < M_1(1) \le (k+1)\delta x\, ;\, M_2(1) > \frac{1-(k+1)\delta}{2}\, x\Big)$. Using Lemma 3.1, the first two probability terms in (3.18) are bounded by $C_{12} \exp\Big(-\frac{(1-\delta)^3\, x^2}{2}\Big)$, and

$$I_k \le \mathbb{P}\left(M_1(1) > k\delta x\, ;\, M_2(1) > \frac{1-(k+1)\delta}{2}\, x\right) \le C_{13} \exp\Big(-\frac{(1-\delta)^3\, x^2}{2}\Big),$$

which in view of (3.18), yields the desired estimate. □

Proof of Theorem 1.2. Denoting by K the limit set stated in Theorem 1.2, we first show that with probability one, for every point $(a,b) \in K$, there exists some sequence $\Big(\frac{M_1(t_n)}{\sqrt{2t_n \log\log t_n}}, \frac{M_2(t_n)}{\sqrt{2t_n \log\log t_n}}\Big)_{n\ge 1}$ which converges to (a,b) as $t_n \to \infty$. This is in fact a direct consequence of Strassen's functional LIL for Brownian motion: Consider a continuous function $f : [0,1] \to \mathbb{R}$ defined by

$$f(t) \overset{\text{def}}{=} \begin{cases} -t, & \text{if} \;\; 0 \le t \le b, \\ \\ -b\frac{1+a}{1-b} + \frac{a+b}{1-b}t, & \text{if} \;\; b \le t \le 1. \end{cases} \tag{3.19}$$

Therefore $\int_0^1 f'^2(t)dt = b + (a+b)^2/(1-b) \le 1$ since $(a,b) \in K$. According to Strassen's LIL, there exists a sequence $(t_k = t_k(\omega))_{k\ge 1}$ such that

$$\sup_{0\le s\le 1}\left|\frac{B(st_k)}{\sqrt{2t_k \log\log t_k}} - f(s)\right| \to 0, \qquad k\to\infty,$$

yielding that

$$\frac{M_1(t_k)}{\sqrt{2t_k \log\log t_k}} \to a, \qquad \frac{M_2(t_k)}{\sqrt{2t_k \log\log t_k}} \to b, \qquad k\to\infty,$$

as desired. To end this proof, we fix a small $\epsilon > 0$ and write $d(z,K) \stackrel{\text{def}}{=} \inf_{z'\in K} |z-z'|$ for $z \in \mathbb{R}^2$. It suffices to show that with probability one,

$$d\left(\left(\frac{M_1(t)}{\sqrt{2t\log\log t}}, \frac{M_2(t)}{\sqrt{2t\log\log t}}\right), K\right) \to 0, \qquad t\to\infty. \tag{3.20}$$

Define $r_k \stackrel{\text{def}}{=} \exp\left(k^{1-\epsilon/6}\right)$ for $k \ge 1$. Using Lemma 3.2,

$$\begin{aligned}
&\mathbb{P}\left(d\left(\left(\frac{M_1(r_k)}{\sqrt{2r_k\log\log r_k}}, \frac{M_2(r_k)}{\sqrt{2r_k\log\log r_k}}\right), K\right) > \epsilon\right)\\
&\le \mathbb{P}\Big(M_1(r_k) + 2M_2(r_k) > \frac{\epsilon}{2}\sqrt{2r_k\log\log r_k}\Big)\\
&\le C_{14}(\epsilon)\exp\Big(-(1+\epsilon/3)\log\log r_k\Big)\\
&\le C_{15}k^{-(1+\epsilon)/4},
\end{aligned}$$

which is summable in k. The Borel-Cantelli lemma implies that with probability one, for all large k, we have $d\left(\left(\frac{M_1(r_k)}{\sqrt{2r_k\log\log r_k}}, \frac{M_2(r_k)}{\sqrt{2r_k\log\log r_k}}\right), K\right) \le \epsilon$. For $r_k \le t < r_{k+1}$, $(1-\epsilon)\frac{M_i(r_k)}{\sqrt{2r_k\log\log r_k}} \le \frac{M_i(t)}{\sqrt{2t\log\log t}} \le (1+\epsilon)\frac{M_i(r_{k+1})}{\sqrt{2r_{k+1}\log\log r_{k+1}}}$ with $i = 1$ or 2, and (3.20) follows. □

4. Joint lower functions

Lemma 4.1. *Fix $\sigma \in [1, \infty]$. Uniformly for $0 < \epsilon < 1$,*

$$\mathbb{P}\Big(M_1(1) < \sigma\epsilon \text{ and } M_2(1) < \epsilon\Big) \asymp \begin{cases} \exp\Big(-\frac{\pi^2}{8\epsilon^2}\Big), & \text{if } \sigma = 1, \\ \epsilon^{-2}\exp\Big(-\frac{\pi^2}{2\epsilon^2}\Big), & \text{if } 1 < \sigma < 2, \\ \epsilon^{-4}\exp\Big(-\frac{\pi^2}{2\epsilon^2}\Big), & \text{if } \sigma = 2, \\ \exp\Big(-\frac{\pi^2}{2\epsilon^2\sigma^2}\Big), & \text{if } 2 < \sigma < \infty, \\ \epsilon, & \text{if } \sigma = \infty. \end{cases} \tag{4.1}$$

Proof. The case $\sigma = \infty$ was obtained in [4], whereas the case $\sigma = 1$ is the classical Chung's small deviation (cf. [5]). We only consider $1 < \sigma < \infty$. Recall (2.1), we deduce from the self-similarity that

$$\mathbb{P}\Big(M_1(1) < \sigma\epsilon \text{ and } M_2(1) < \epsilon\Big) = \mathbb{P}\Big(H_1(1) \wedge H_2(r) > t\Big), \tag{4.2}$$

with $t \stackrel{\text{def}}{=} \epsilon^{-2}\sigma^{-2}$ and $r = 1/\sigma < 1$. Recall (3.3). It follows from (2.9) that for all $t > 0$,

$$\mathbb{P}\Big(H_1(1) \wedge H_2(r) > t\Big) = (1-r)\mathbb{P}\Big(T^{(3)}_{1-r\to 1} + r^2(T^{(1)}_1 + \widehat{T}^{(1)}_1) > t\Big) + r\mathbb{P}\Big(r^2\Lambda + T^{(3)}_{0\to 1} > t\Big), \tag{4.3}$$

where all the r.v. in RHS of (3.6) are independent, and $\widehat{T}^{(1)}_1 \stackrel{\text{law}}{=} T^{(1)}_1$ is distributed as the first hitting time of 1 by $|B|$; the Laplace transform at $\lambda^2/2$ of Λ is $\frac{\tanh\lambda}{\lambda}$. The density function of Λ can be obtained by inverting the Laplace transform (cf. [6, pp. 257]), which implies that

$$\mathbb{P}\Big(\Lambda > x\Big) \asymp \exp\Big(-\frac{\pi^2}{8}\,x\Big) \asymp \mathbb{P}\Big(T^{(1)}_1 > x\Big), \qquad x > 0. \tag{4.4}$$

where the last asymptotic equivalence is well-known. Applying Lemma 2.4 to (4.3) with (4.4), and (3.6) for the tail probability of $T^{(3)}_{1-r\to 1}$, the desired estimates in (4.1) follow. □

Proof of Theorem 1.3. Applying Lemma 4.1, the proof is similar to that of Theorem 1.1, we omit the details. □

5. Ratio.

Lemma 5.1. *Fix $n \ge 1$. Uniformly for $0 < \epsilon < 1$,*

$$\mathbb{P}\Big(\inf_{H_n(1) \le t \le H_n(2)} \frac{M_{n+1}(t)}{M_n(t)} < \epsilon \Big) \asymp \epsilon^n. \tag{5.1}$$

Proof. By monotonicity,

$$\frac{M_{n+1}(H_n(1))}{2} \le \inf_{H_n(1) \le t \le H_n(2)} \frac{M_{n+1}(t)}{M_n(t)} \le M_{n+1}(H_n(2)) \stackrel{\text{law}}{=} 2M_{n+1}(H_n(1)),$$

yielding (5.1) in view of (2.11). $\square$

Proof of Theorem 1.4. Observe that with probability one, for some large $r_0 = r_0(\omega)$,

$$r \le H_n(r) \le r^3, \qquad r \ge r_0, \tag{5.2}$$

which can be easily obtained from the upper and lower functions for $M_n(\cdot)$ (cf. [4]). We omit the details of the proof of the convergent part of Theorem 1.4 which can be easily obtained by using (5.1) and (5.2).

To prove the divergent part, we assume without loss of generality (cf. [7]) that there exists some $t_0 > 0$ such that

$$(\log t)^{1/n} \le f(t) \le (\log t)^{2/n}, \qquad t \ge t_0. \tag{5.3}$$

Define $r_i = 2^i$ and $\widehat{f}(t) \stackrel{\text{def}}{=} f(t^3)$. Observe that in view of (5.2), it suffices to show that with probability one,

$$E_i \stackrel{\text{def}}{=} \Big\{ M_{n+1}(H_n(r_i)) < \frac{r_i}{\widehat{f}(r_i)} \Big\}, \text{ occurs infinitely often.} \tag{5.4}$$

Using (2.11), $\mathbb{P}(E_i) = \Big(\widehat{f}(r_i)\Big)^{-n}$, and therefore

$$\sum_i \mathbb{P}(E_i) = \infty, \tag{5.5}$$

since $\int^\infty dt/(t\widehat{f}^n(t)) = \infty$. For $j > i \ge i_0$, we estimate $\mathbb{P}(E_i \cap E_j)$ to apply Borel-Cantelli lemma. Denote by $d_{H_n(r_i)} \stackrel{\text{def}}{=} \inf\{t > H_n(r_i) : B(t) = 0\}$, the first return time. Define

$$\widetilde{B}(t) \stackrel{\text{def}}{=} B(d_{H_n(r_i)} + t), \qquad t \ge 0,$$

which is a Brownian motion independent of $\mathcal{F}_{d_{H_n(r_i)}}$ (recalling that $(\mathcal{F}_t, t \geq 0)$ is the natural filtration of B). Similarly, define $(\widetilde{M}_n(\cdot), \widetilde{H}_n(\cdot), n \geq 1)$ related to $\widetilde{B}$. For $j > i$, the event $\{M_1(d_{H_n(r_i)}) < r_j\}$ implies $H_n(r_j) = \widetilde{H}_n(r_j) + d_{H_n(r_i)}$, and

$$\begin{aligned} \mathbb{P}\Big(E_i \cap E_j \cap \{M_1(d_{H_n(r_i)}) < r_j\}\Big) &\leq \mathbb{P}\left(E_i \cap \left\{\widetilde{M}_{n+1}(\widetilde{H}_n(r_j)) < \frac{r_j}{\widehat{f}(r_j)}\right\}\right) \\ &= \mathbb{P}(E_i)\mathbb{P}(E_j), \end{aligned} \tag{5.6}$$

by means of the independence of $\widetilde{B}$ and E_i. It remains to consider the event $\{M_1(d_{H_n(r_i)}) \geq r_j\}$. First, we deduce from (2.2) and scaling that

$$\mathbb{P}\Big(M_1(H_n(r)) > xr\Big) = \mathbb{P}\Big(\inf_{1 \leq k \leq n-1} U_k < \frac{1}{x}\Big) \leq \frac{n-1}{x}, \qquad x > 1, r > 0, \tag{5.7}$$

where $(U_k, 1 \leq k \leq n-1)$ denote $n-1$ independent uniform variables. Since $M_1(d_{H_n(r_i)}) = M_1(H_n(r_i)) \vee \sup_{H_n(r_i) \leq t \leq d_{H_n(r_i)}} |B(t)|$, is independent of $M_{n+1}(H_n(1))$, and

$$\begin{aligned} &\mathbb{P}\Big(E_i \cap \{M_1(d_{H_n(r_i)}) \geq r_j\}\Big) \\ &= \mathbb{P}\Big(E_i\Big)\mathbb{P}\Big(M_1(d_{H_n(r_i)}) \geq r_j\Big) \\ &\leq \mathbb{P}(E_i)\Big(\mathbb{P}\Big(M_1(H_n(r_i) > r_j\Big) + \mathbb{P}\Big(\sup_{H_n(r_i) \leq t \leq d_{H_n(r_i)}} |B(t)| > r_j\Big)\Big) \\ &\leq \mathbb{P}(E_i)\Big(\frac{(n-1)r_i}{r_j} + \frac{r_i}{r_j}\Big) \\ &= 2^{-(j-i)}\,\mathbb{P}(E_i). \end{aligned} \tag{5.8}$$

It follows from (5.6) and (5.8) that

$$\liminf_{n \to \infty} \frac{\sum_{i_0 \leq i, j \leq n} \mathbb{P}(E_i \cap E_j)}{\Big(\sum_{i_0 \leq i \leq n} \mathbb{P}(E_i)\Big)^2} \leq 1,$$

which in view of (5.5) and Kochen and Stone's version of Borel-Cantelli lemma, implies (5.4), as desired. $\square$

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