The logarithmic average of Sinai's walk in random environment

by

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Dedicated to Professor Endre CSÁKI on the occasion of his 65-th birthday

Summary. We study the logarithmic average of Sinai's one-dimensional random walk in random environment.

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1. Introduction.

Let $\Xi = \{\xi_i, i \in \mathbb{Z}\}$ be a sequence of iid variables taking values in the interval $(0,1)$. We consider the random walk in random environment (RWRE) (S_n): Conditioned on each realization of Ξ, $(S_n, n \geq 0)$ is a Markov chain on $\mathbb{Z}$ such that $S_0 = 0$ and

$$\mathbb{P}\Big(S_{n+1} = \begin{cases} i+1 \\ i-1 \end{cases} \Big| \ S_n = i, \ \Xi\Big) = \begin{cases} \xi_i \\ 1-\xi_i \end{cases}, \tag{1.1}$$

where here and in the sequel, $\mathbb{P}$ denotes the total probability and $\mathbb{E}$ the associated expectation. To simplify the discussion, we also assume that there exists some constant $c \in (0, 1/2)$ such that $c \leq \xi_i \leq 1-c$.

Solomon [25] obtained the recurrence/transience criteria:

$$\mathbb{P}\Big((S_n) \text{ is recurrent }\Big) = 1 \Longleftrightarrow \mathbb{E}\log\frac{1-\xi_i}{\xi_i} = 0, \tag{1.2}$$

We shall only deal with the recurrent case in this note. The rate of convergence was characterized by

Theorem A (Sinai [24]). *Suppose that*

$$\mathbb{E}\log\frac{1-\xi_i}{\xi_i} = 0, \qquad \sigma^2 \stackrel{\text{def}}{=} \mathbb{E}\log^2\frac{1-\xi_i}{\xi_i} > 0, \tag{1.3}$$

we have

$$\frac{\sigma^2\, S_n}{\log^2 n} \xrightarrow{(d)} \mathcal{L}, \tag{1.4}$$

where the limit law $\mathcal{L}$ was given explicitly by Kesten [17] and Golosov [10] as follows:

$$\mathbb{P}\Big(\mathcal{L} \in dx\Big)/dx = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\Big(-\frac{(2k+1)^2\pi^2}{8}\,|x|\Big). \tag{1.5}$$

See Révész [19, Part III] and Hughes [13] for references and studies on RWRE. There are also many recent works, see e.g. the references in Hu and Shi [12] and Shi [23].

Here, we are interested in the logarithmic average in Sinai's renormalization. The main result is

Theorem 1.1. *Fix a constant $0 < c < \pi^2\sigma^2/8$. Under (1.3), $\mathbb{P}$-almost surely for every a.e. continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\sup_{x\in\mathbb{R}} |f(x)|e^{-c|x|} < \infty$, we have*

$$\frac{1}{\log\log n} \sum_{k=3}^{n} \frac{1}{k\log k} f\Big(\frac{S_k}{\log^2 k}\Big) \xrightarrow{\text{a.s.}} \mathbb{E}\Big(f\big(\frac{\mathcal{L}}{\sigma^2}\big)\Big), \tag{1.6}$$

where the law of $\mathcal{L}$ is given by (1.5).

The condition on f can be weakened, see Berkes, Csáki and Horváth [2] for the minimal conditions on the logarithmic average of the (usual) random walk, see also Ibragimov and Lifshits [14]. There are a huge related references on the studies of the almost sure (or pointwise) central limit theorem for random walk and Brownian motion, see e.g. Brosamler [3], Schatte [21], Fisher [9], Lacey and Philipp [15], and Csáki, Földes and Révész [7] (random walk), Csáki and Földes [5] (local time and additive functional), Csörgő and Horváth [8] (rate of convergence and invariance principle). See Berkes [1] for a survey on the almost sure central limit results.

The proof of Theorem 1.1 relies on an analysis of Brownian valley. We shall introduce in the next section the associated diffusion in random environment, and prove the corresponding result for the diffusion process. By using the Skorokhod embedding in random environment, we give the proof of Theorem 1.1 in Section 3.

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2. Diffusion with random potential.

Consider a process $\{V(x);\, x \in \mathbb{R}\}$ with locally bounded trajectories which are continuous from right and have limit from left. The process V plays the role of random potential. We can formally define a process $\{X(t);\, t \geq 0\}$ by the equation

$$\begin{cases} dX(t) = d\beta(t) - \frac{1}{2}V'(X(t))dt \\ X(0) = 0 \end{cases}, \tag{2.1}$$

where $\{\beta(t);\, t \geq 0\}$ is an independent one–dimensional Brownian motion independent of V. Rigorously speaking, instead of writing the formal derivative of V in (2.1), we should consider X as a diffusion process (conditioning on each realization of V) with generator

$$\frac{1}{2} e^{V(x)} \frac{d}{dx}\Big(e^{-V(x)} \frac{d}{dx} \Big). \tag{2.2}$$

See Brox [4], Schumacher [22], Kawazu et al. [16], Tanaka [26] together with their references.

Here, we assume the following hypothesis: On a possibly enlarged probability space, there exists a coupling of V and a standard two–sided Brownian motion $\{W(y);\, y \in \mathbb{R}\}$, and a constant $\sigma > 0$ such that for all $n \geq 1$,

$$\mathbb{P}\Big(\sup_{|x|\leq n} \big| V(x) - \sigma W(x) \big| \geq C_1 \log n\Big) \leq \frac{C_2}{n^{C_3}}, \tag{2.4}$$

with $C_i > 0$ $(1 \leq i \leq 3)$. For instance, the well–known Komlós–Major–Tusnády [18] strong approximation theorem tells us that (2.4) will be satisfied for V a step function on $\mathbb{R}$ defined as the partial sum of iid (bounded) variables, cf. (3.1) below.

The goal of this section is to prove the following almost sure central limit theorem for $X(t)$:

Theorem 2.1. *Fix a constant $0 < c < \pi^2\sigma^2/8$. Under (2.4), $\mathbb{P}$-almost surely for every a.e. continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\sup_{x\in\mathbb{R}} |f(x)|e^{-c|x|} < \infty$, we have*

$$\frac{1}{\log\log T} \int_2^T \frac{dt}{t\log t}\, f\Big(\frac{X(t)}{\log^2 t}\Big) \xrightarrow{\text{a.s.}} \mathbb{E}\Big(f\Big(\frac{\mathcal{L}}{\sigma^2}\Big)\Big), \qquad T \to \infty, \tag{2.5}$$

where $\mathcal{L}$ is defined in (1.5).

The proof of Theorem 2.1 is based on a localization argument. Firstly, we introduce the Brownian valley. Write $W_+(x) \stackrel{\text{def}}{=} W(x)$ and $W_-(x) \stackrel{\text{def}}{=} W(-x)$ for $x \geq 0$ for the two independent Brownian motions. For the sake of notational convenience, we shall write, for any continuous process Y and all $t > 0$

$$\overline{Y}(t) \stackrel{\text{def}}{=} \sup_{0\leq s\leq t} Y(s), \tag{2.6}$$

$$\underline{Y}(t) \stackrel{\text{def}}{=} \inf_{0\leq s\leq t} Y(s), \tag{2.7}$$

$$Y^{\#}(t) \stackrel{\text{def}}{=} \sup_{0\leq s\leq t} (Y(s) - \underline{Y}(s)). \tag{2.8}$$

For the Brownian motion W_+, define

$$d_+(r) \stackrel{\text{def}}{=} \inf\{t > 0 : W_+^{\#}(t) > r\}, \qquad r > 0. \tag{2.9}$$

Let $b_+(r)$ be the localization of the minimum of W_+ over $[0, d_+(r)]$:

$$b_+(r) \stackrel{\text{def}}{=} \inf\{0 \leq u \leq d_+(r) : W_+(s) = \underline{W_+}(d_+(r))\}, \qquad r > 0. \tag{2.10}$$

Define similarly $d_-(r) \ge 0$ and $b_-(r) \ge 0$, by replacing (W_+, d_+) by (W_-, d_-) in (2.9) and (2.10). In the literature, the triplet $(-d_-(r), 0, d_+(r))$ is called a Brownian valley containing 0 of depth r, its bottom $b(r)$ is defined as follows:

$$
(2.11) \qquad b(r) \stackrel{\text{def}}{=} \begin{cases} b_+(r), & \text{if} \quad \overline{W_+}(d_+(r)) < \overline{W_-}(d_-(r)) \\ -b_-(r), & \text{otherwise} \end{cases}, \qquad r > 0.
$$

The identity $b(1) \stackrel{\text{law}}{=} \mathcal{L}$ (in (1.5)) was obtained by Kesten [17] and by Golosov [10]. Recall the following result whose proof can be found in [11]:

Fact 2.2. *Assuming (2.4). There exist a constant $C_4 = C_4(\sigma, C_1, C_2, C_3) > 0$ such that for all $t \ge 3$, we have*

$$
(2.12) \qquad \mathbb{P}\Big(\big| X(t) - b(\frac{\log t}{\sigma}) \big| > \log t \Big) \le C_4 \Big(\log t \Big)^{-1/3}.
$$

Let us recall the following fact stated in Berkes et al. [2, Lemma 1], which can be proven by using the arguments of Schatte [21, section 2.3]:

Fact 2.3. *Assume that the convergence like (2.5) holds for f being any indicator function of intervals and for $f = f_0 : \mathbb{R} \to \mathbb{R}_+$ a fixed a.e. continuous function, then (2.5) holds for all a.e. continuous function f such that $|f(x)| \le f_0(x)$. The null probability set can be chosen universally for all such functions f.*

Lemma 2.4. *Recall (2.11). Almost surely for every f satisfying the condition of Theorem 2.1, we have*

$$
(2.13) \qquad \frac{1}{\log T} \int_1^T \frac{dt}{t} f(\frac{b(t)}{t^2}) \xrightarrow{\text{a.s.}} \mathbb{E}\Big(f(b(1)) \Big), \qquad T \to \infty.
$$

Proof of Lemma 2.4. According to Fact 2.3, it suffices to prove (2.13) for a fixed a.e. continuous function f. The proof is based on the fact that the Brownian scaling transform is ergodic (cf. Csáki and Földes [6] for related discussions). By using the change of variable $t = e^s$, the LHS of (2.13) equals (write $\log T = N$)

$$
\frac{1}{N} \int_0^N ds\, f(e^{-2s}\, b(e^s)) \xrightarrow{\text{a.s.}} \Theta, \qquad N \to \infty,
$$

where this almost sure convergence follows from the ergodic theorem and by using the fact that the process $\widehat{b} \equiv (e^{-2s} b(e^s), s \ge 0)$ is (strictly) stationary. The limit variable Θ

is measurable with respect to the σ-fields which is invariant by translation on the process $\widehat{b}$. If we have proven that this invariant σ-fields is trivial, necessarily we have $\Theta = \mathbb{E}\Theta = \mathbb{E}\Big(f(b(1))\Big)$, as desired.

To complete the proof, we remark the following ergodic scaling property: For $\lambda > 0$, define $W_\lambda(x) \stackrel{\text{def}}{=} \lambda^{-1/2} W(\lambda x)$ for $x \in \mathbb{R}$. We have

$$(2.14) \qquad \Big(W(x), W_\lambda(x),\, x\in\mathbb{R}\Big) \xrightarrow{(d)} \Big(W(x), \widetilde{W}(x),\, x\in\mathbb{R}\Big), \qquad \lambda\to\infty,$$

where $\widetilde{W}$ is an independent copy of W. See e.g. Revuz and Yor [20, Exercise (XIII.1.17)] for (2.14). Define b_λ related to W_λ the same way b to W. Observe that for any $s_0 \in \mathbb{R}$, $\widehat{b}(s_0+s) = b_\lambda(s)$, $s\ge 0$, for $\lambda = e^{2s_0}$. We apply (2.14) and obtain that the σ-fields invariant with respect to translation on $\widehat{b}$ is trivial, completing the proof. $\square$

Proof of Theorem 2.1. According to Fact 2.3, it suffices to prove (2.5) for each indicator function of intervals and for $f(x) = e^{c|x|}$. Let us show (2.5) for $f(x) = e^{c|x|}$ with $c < \pi^2\sigma^2/8$, the indicator functions can be similarily treated by using the monotonicity.

For notational convenience, we assume that $\sigma \equiv 1$. In view of Lemma 2.4, it suffices to prove that

$$(2.15) \qquad \frac{1}{\log\log T}\int_2^T \frac{dt}{t\log t}\Big|f\big(\frac{X(t)}{\log^2 t}\big) - f\big(\frac{b(\log t)}{\log^2 t}\big)\Big| \xrightarrow{\text{a.s.}} 0, \qquad T\to\infty.$$

Write for simplification $\Delta_t \stackrel{\text{def}}{=} \big|X(t) - b(\log t)\big|$. Since $|f(x+v)-f(x)| \le c|v| f(|x|+|v|)$, the LHS of (2.15) is less than

$$\begin{aligned} &\le I_1(T) + \frac{1}{\log\log T}\int_2^T \frac{dt}{t\log t}\, 1\!\!1_{(\Delta_t\le \log t)}\, \frac{c}{\log t}\, f\big(\frac{b(\log t)}{\log^2 t} + \frac{1}{\log t}\big) \\ (2.16) \qquad &= I_1(T) + o(1), \qquad T\to\infty, \qquad \text{a.s.} \end{aligned}$$

by using Lemma 2.4, and where

$$(2.17) \qquad I_1(T) \stackrel{\text{def}}{=} \frac{1}{\log\log T}\int_2^T \frac{dt}{t\log t}\, 1\!\!1_{(\Delta_t\ge\log t)}\Big|f\big(\frac{X(t)}{\log^2 t}\big) - f\big(\frac{b(\log t)}{\log^2 t}\big)\Big|.$$

Let us show $I_1(T)$ converges to 0 almost surely. Recall the following LIL for $X(t)$ ($\sigma = 1$)

$$(2.18) \qquad \limsup_{t\to\infty} \frac{\sup_{0\le s\le t}|X(s)|}{\log^2 t\, \log\log\log t} = \frac{8}{\pi^2} \qquad \text{a.s.},$$

$$(2.19) \qquad \limsup_{t\to\infty} \frac{\sup_{1\le s\le t}|b(\log s)|}{\log^2 t\, \log\log\log t} \le \frac{8}{\pi^2} \qquad \text{a.s.},$$

see [12, Theorem 1.6] for the proof of (2.18), and since $-d_-(r) \le b(r) \le d_+(r)$, the estimate (2.19) in fact has been obtained in the proof of (2.18). Applying (2.18)–(2.19) to (2.17), we obtain a finite variable $0 < K(\omega) < \infty$ such that

$$I_1(T) \le \frac{K(\omega)}{\log\log T} \int_3^T \frac{dt}{t\log t}\, \mathbb{1}_{(\Delta_t \ge \log t)} \Big(\log\log t\Big)^2 \stackrel{\text{def}}{=} \frac{K(\omega)}{\log\log T} I_2(T). \tag{2.20}$$

Now, applying Fact 2.2 yields that

$$\mathbb{E}\Big(I_2(T)\Big) \le \int_2^T \frac{dt}{t\log t} \frac{C_4\Big(\log\log t\Big)^2}{\log^{1/3} t} = O(1), \qquad T \to \infty,$$

showing that $I_2(\infty) < \infty$, a.s.. This combining with (2.20) and (2.16) prove (2.15), as desired. □

3. Proof of Theorem 1.1.

We firstly give the Skorokhod embedding in random environment. Consider the step potential V such that $V(0) = 0$,$V(x)$ is constant on $[n-1, n)$, and

$$V(n) - V(n-) = \log \frac{1-\xi_n}{\xi_n}, \qquad n \in \mathbb{Z}. \tag{3.1}$$

Let $\{X_V(t), t \ge 0\}$ be the diffusion associated with the potential V via the equation (2.1). The following Skorokhod embedding was stated in Schumacher [22], Kawazu-Tamura-Tanaka [16], see [12] for the proof (there also exists a strong approximation of local times).

Fact 3.1 (Skorokhod embedding). *Assuming (1.3). Define*

$$\mu_k \stackrel{\text{def}}{=} \inf\{t > \mu_{k-1} : |X_V(t) - X_V(\mu_{k-1})| = 1\}, \tag{3.2}$$

with $\mu_0 \stackrel{\text{def}}{=} 0$. *Therefore*

- $\{X_V(\mu_n), n \ge 0\} \stackrel{\text{law}}{=} \{S_n, n \ge 0\}$;
- $\{\mu_n - \mu_{n-1}, n \ge 1\}$ *are iid, and* $\mu_1 \stackrel{\text{law}}{=} \inf\{t > 0 : |W(t)| = 1\}$.

To deduce Theorem 1.1 from Fact 3.1 and Theorem 2.1, we have to bound some increments of $(X_V(t))$. The following result is elementary:

Lemma 3.2. *We have*

$$\mu_n = n + \mathcal{O}(n^{2/3}), \qquad n \to \infty, \qquad \text{a.s.} \tag{3.3}$$

$$\lim_{n\to\infty} \Big(\log n\Big) \min_{1\le k\le n} \Big(\mu_k - \mu_{k-1}\Big) = \frac{1}{2}, \qquad \text{a.s..} \tag{3.4}$$

Proof of Lemma 3.2. Using Fact 3.1, we have $\mathbb{E}\big(\mu_1\big) = 1$, and (3.3) follows from the usual LIL for the partial sum μ_n. The well-known estimate for the Brownian hitting time yields that

$$\lim_{x\to\infty} \frac{1}{x} \log \mathbb{P}\Big(\frac{1}{\mu_1} > x\Big) = -\frac{1}{2}.$$

This in view of the standard theory for the extreme values of the iid variables imply that

$$\lim_{n\to\infty} \frac{1}{\log n} \max_{1\le k\le n} \frac{1}{\mu_k - \mu_{k-1}} = 2, \qquad \text{a.s.,}$$

proving (3.4). □

Lemma 3.3. *Recall (3.1) and assume (1.3). We have*

$$\limsup_{t_2>t_1\to\infty} \frac{\big|X_V(t_2) - X_V(t_1)\big|}{1 + (t_2 - t_1)\, \log t_2} \le 3, \qquad \mathbb{P}\text{–a.s.} \tag{3.5}$$

Proof of Lemma 3.3. This is an argument ω-by-ω. Consider large t_1, t_2. Let $\mu_k \le t_1 < \mu_{k+1}$ and $\mu_l \le t_2 < \mu_{l+1}$. Then $l \ge k$ and $k \sim \log t_1$ and $l \sim \log t_2$ in view of (3.3). By using (3.2),

$$|X_V(t_2) - X_V(t_1)| \le l - k + 1.$$

Suppose $l \ge k+1$. From (3.4), we deduce that

$$l - k - 1 \le (\mu_l - \mu_{k+1})\, \frac{5}{2} \log l \le (t_2 - t_1) \frac{5}{2} \log l \le 3(t_2 - t_1) \log t_2,$$

implying (3.5). □

We need the following estimate concerning the Brownian valley. Recall (2.9)–(2.11).

Lemma 3.4. *For all $u_2 > u_1 > 0$,*

$$\mathbb{P}\Big(b(u) \ne b(u_1) \;\; \text{for some } u \in [u_1, u_2]\Big) \le \frac{4(u_2 - u_1)}{u_1}. \tag{3.6}$$

Proof of Lemma 3.4. By scaling, we can take $u_1 = 1$. By symmetry, the probability term in (3.6) is equal to

$$
\begin{aligned}
&= 2\mathbb{P}\Big(\overline{W_+}(d_+(1)) < \overline{W_-}(d_-(1));\ b(u) \neq b_+(1) \ \text{ for some } u \in [1, u_2]\Big) \\
&\leq 2\mathbb{P}\Big(\overline{W_+}(d_+(1)) < \overline{W_-}(d_-(1)) < \overline{W_+}(d_+(u_2))\Big) \\
&\qquad + 2\mathbb{P}\Big(b_+(u) \neq b_+(1) \ \text{ for some } u \in [1, u_2]\Big) \\
&\equiv I_3 + I_4.
\end{aligned}
\tag{3.7}
$$

Observe that $\overline{W_+}(d_+(u_2)) \leq W_+(d_+(1)) + (u_2 - 1)$ and it is not difficult to see that $\overline{W_-}(d_-(1))$ is uniformly distributed on $[0, 1]$ (see e.g. Kesten [17] or Hu [11, Lemma 2.1]). It follows that

$$I_3 \leq 2(u_2 - 1) \equiv 2(u_2 - u_1).$$

Using Lévy's identity (cf. Revuz and Yor [20, Theorem (VI.2.2)]), we deduce that

$$\Big(b_+(r), r \geq 0\Big) \stackrel{\text{law}}{=} \Big(g(T_r(|B|)), r \geq 0\Big),$$

where B denotes a one-dimensional Brownian motion starting from 0, $T_r(|B|) \stackrel{\text{def}}{=} \inf\{t > 0 : |B(t)| > r\}$ and $g(u) \stackrel{\text{def}}{=} \sup\{t \leq u : B(t) = 0\}$ for $u > 0$. By considering the process $(|B(u + T_1(|B|))|, u \geq 0)$, it turns out that

$$
\begin{aligned}
I_4 &\leq 2\,\mathbb{P}\Big(\text{ A reflected Brownian motion starting from 1 hits 0 before hitting } u_2 \Big) \\
&= 2\,\frac{u_2 - 1}{u_2} = 2\,\frac{u_2 - u_1}{u_2},
\end{aligned}
$$

implying (3.6) in view of (3.7). □

Lemma 3.5. *Assume (1.3) and recall Fact 3.1. Write $S_k \stackrel{\text{def}}{=} X_V(\mu_k), k \geq 0$. We have*

$$\sum_{k=3}^{n} \frac{1}{k \ \log^{5/6} k}\, \mathbb{1}_{\left(\left|S_k - b(\frac{\log k}{\sigma})\right| \geq 4 \log k\right)} = \mathcal{O}(1), \qquad n \to \infty, \ \text{a.s.} \tag{3.8}$$

The power 5/6 in the above partial sum of (3.8) can be replaced by any constant larger than 1/2.

Proof of Lemma 3.5. For notational convenience, we assume $\sigma \equiv 1$. Write $I_5(n)$ the partial sum in (3.8). It follows from (3.3) that for large k,

$$\frac{1}{k \log^{5/6} k} = \frac{1}{[\mu_k] \log^{5/6} [\mu_k]} + \mathcal{O}(k^{-4/3}), \qquad \text{a.s.},$$

where $[x]$ denotes the integer part of $x \geq 0$. This together with (3.5) implies that

$$I_5(n) = \sum_{\mu_k \geq 3, k \leq n} \frac{1}{[\mu_k] \log^{5/6}[\mu_k]} \mathbb{1}_{\left(\left|X_V([\mu_k]) - b(\log k)\right| \geq \log \mu_k\right)} + \mathcal{O}(1)$$

$$= \sum_{j \geq 3} \sum_{k \leq n} \mathbb{1}_{(j \leq \mu_k < j+1)} \frac{1}{j \log^{5/6} j} \mathbb{1}_{(|X_V(j) - b(\log k)| \geq \log j)} + \mathcal{O}(1), \quad \text{a.s.} \tag{3.9}$$

Define for $j \geq 3$

$$E_j \stackrel{\text{def}}{=} \left\{ b(u) = b(\log j) \, , |u - \log j| \leq j^{-1/4} \right\} \cap \left\{ \left| X_V(j) - b(\log j) \right| \leq \log j \right\}. \tag{3.10}$$

In view of (3.3), we have $\log \mu_k = \log k + \mathcal{O}(k^{-1/3}), k \to \infty$, a.s. Therefore, for large j, on E_j, we have $|X_V(j) - b(\log k)| \leq \log j$ if $j \leq \mu_k < j+1$. Hence, the term of partial sum in (3.9) equals

$$= \sum_{3 \leq j \leq 2n} \sum_{k \leq n} \mathbb{1}_{(j \leq \mu_k < j+1)} \frac{1}{j \log^{5/6} j} \mathbb{1}_{E_j^c} + \mathcal{O}(1)$$

$$\leq \sum_{3 \leq j \leq 2n} \frac{1}{j \log^{5/6} j} \Big(N_{j+1} - N_j \Big) \mathbb{1}_{E_j^c} + \mathcal{O}(1)$$

$$\stackrel{\text{def}}{=} I_6(n) + \mathcal{O}(1), \quad \text{a.s.,} \tag{3.11}$$

where N_j is the renewal process defined by

$$N_j \stackrel{\text{def}}{=} \inf\{k \geq 0 : \mu_k \geq j\}.$$

It remains to show that $I_6(n)$ is bounded on n. Using (3.6) and applying (2.12) to X_V $(\sigma \equiv 1)$ give that for large j,

$$\mathbb{P}\Big(E_j^c\Big) \leq 5 j^{-1/4} (\log j)^{-1} + C_4 \Big(\log j \Big)^{-1/3}.$$

It follows from Hölder's inequality that

$$\mathbb{E}\Big(I_6(n)\Big) \leq \sum_{3 \leq j \leq 2n} \frac{1}{j \log^{5/6} j} \left(\mathbb{E}(N_{j+1} - N_j)^3\right)^{1/3} \mathbb{P}\big(E_j^c\big)^{2/3}$$

$$\leq \sum_{3 \leq j \leq 2n} \frac{C_5}{j \log^{19/18} j} = \mathcal{O}(1), \qquad n \to \infty,$$

hence $I_6(\infty) < \infty$, a.s.. This together with (3.9) and (3.11) imply (3.8). □

Proof of Theorem 1.1. In view of Lemma 3.4 and using (2.19), it is immediate to show that

$$\sum_{k=3}^{n} \left| \frac{1}{k \log k} f(b(\frac{\log k}{\sigma})) - \int_{k}^{k+1} \frac{dt}{t \log t} f(b(\frac{\log t}{\sigma})) \right| = \mathcal{O}(1), \qquad n \to \infty, \text{ a.s.,} \tag{3.12}$$

for any function $f : \mathbb{R} \to \mathbb{R}$ in Theorem 1.1. Using Lemma 3.5 and (3.12), the proof of Theorem 1.1 can be achieved by using the same method as that of the proof of Theorem 2.1. The details are omitted. □

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