Penalization of the Wiener Measure and Principal Values

by

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Summary. Let $\mathbb{Q}_\epsilon^{(\mu)}$ be absolutely continuous with respect to the Wiener measure with density $\mathbb{D}_t = \exp\left(\int_0^t h(B_s)dB_s - \frac{1}{2}\int_0^t h^2(B_s)ds\right)$, where $h = \frac{\mu}{x}\mathbb{1}_{(|x|\geq\epsilon)}$ and $\epsilon > 0$. When $\mu \geq 1/2$, we show that as $\epsilon \to 0_+$, there exists a "penalization effect" of the Wiener measure through the densities $\mathbb{D}_t$ such that the limit law does not charge the paths hitting 0. The remaining case $\mu < 1/2$ together with a more general form of h are also studied.

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1. Introduction.

Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. On the canonical space $\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, we consider the probabilities :

- $\mathbb{W}$ the Wiener measure.
- $\mathbb{Q}^{(h)}$ the law of the process $X^{(h)}$, the unique strong solution of the equation (cf. Zvonkin [21]):

$$X_t = B_t + \int_0^t h(X_s)ds, \qquad t > 0, \tag{1.1}$$

  where B is a one-dimensional Brownian motion starting from 0.
- $\widehat{\mathbb{Q}}^{(h)}$ the law of the process:

$$\widehat{B}_t^{(h)} = B_t - \int_0^t h(B_s)ds, \qquad t > 0. \tag{1.2}$$

According to Girsanov's theorem:

- The probability law $\mathbb{Q}^{(h)}_{|\mathcal{F}_t}$ is absolutely continuous with respect to $\mathbb{W}_{|\mathcal{F}_t}$ with density ($(\mathcal{F}_t)$ being the natural filtration of the canonical process):

$$\mathbb{D}_t^{(h)} = \exp\left(\int_0^t h(B_s)dB_s - \frac{1}{2}\int_0^t h^2(B_s)ds\right), \qquad t > 0. \tag{1.3}$$

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- Under $\mathbb{Q}^{(h)}$, $\widehat{B}^{(h)}$ is a Brownian motion. Moreover,

$$\text{(1.4)} \qquad \text{for every } t, \qquad \mathcal{F}_t^{\widehat{B}^{(h)}} = \mathcal{F}_t^B,$$

where $\mathcal{F}_t^X$ denotes the natural filtration of a process X. (see e.g. Revuz and Yor [13, pp. 367])

When h is not bounded, these results are no more true (and even meaningless). Nevertheless, for some functions h, the process $\hat{B}^{(h)}$ makes sense. For example, if h has a singularity at 0, we can define $\int_0^t h(B_s)ds \overset{\text{def}}{=} \lim_{\epsilon\to 0+} \int_0^t h(B_s)\mathbb{1}_{(|B_s|\geq\epsilon)}ds$ when the limit exists. For instance, this is the case for $h(x) = \text{sgn}(x)|x|^{-\alpha}$ for $\alpha < 3/2$, due to the Hölder continuity of Brownian local times. In that case, we denote the limit by p.v. $\int_0^t h(B_s)ds$ as Cauchy's principal values. We refer to Biane and Yor [2], Yor [20] and Yor [19, Chap. 10] for studies and references on principal values.

In this paper, we shall concentrate on the case $h(x) = \frac{\mu}{x}$, where $\mu \in \mathbb{R}\backslash\{0\}$ denotes some fixed parameter. We firstly consider $\mu = 1$. Let

$$\hat{B}_t = B_t - \text{p.v.} \int_0^t \frac{ds}{B_s} = \lim_{\epsilon\to 0} \hat{B}_t^{(h_\epsilon)},$$

where $h_\epsilon(x) = \frac{1}{x}\mathbb{1}_{(|x|\geq\epsilon)}$. Contrarily to the bounded case (see (1.4)), the filtration of $\hat{B}$ is strictly included in the filtration of B (see [2,15]). An open problem is to characterize this loss of information (see Yor [19, Chap. 17]). Instead of discussing this difficult problem here, we consider the following question:

For $h(x) = \frac{1}{x}$, what can we say about (1.1) and (1.3)?

It is well known that (1.1) does not admit uniqueness in law, since the law $\mathbb{P}_0^{(3)}$ of the 3-dimensional Bessel process (BES(3)) is a solution as well as the law $\hat{\mathbb{P}}_0^{(3)}$ of the opposite of a BES(3). Nevertheless, we can think that there exists a probability $\mathbb{Q}^{(h)}$ (singular with respect to $\mathbb{W}$) which morally satisfies:

$$\mathbb{Q}^{(h)}_{|\mathcal{F}_t} = \exp\Big(\int_0^t h(B_s)dB_s - \frac{1}{2}\int_0^t h^2(B_s)ds\Big)\,\mathbb{W}_{|\mathcal{F}_t}, \qquad t > 0,$$

This density is meaningless, and we shall study the law $\mathbb{Q}_\epsilon^{(\mu)}$ of the solution of (1.1) with $h(x) = \frac{\mu}{x}\mathbb{1}_{(|x|\geq\epsilon)}$, with $\mu \in \mathbb{R}\backslash\{0\}$ and $\epsilon > 0$. We obtain (as expected!) the following

Theorem 1.1 **($\mu = 1$).** *When $\epsilon \to 0$, $\mathbb{Q}_\epsilon^{(1)}$ converges to $\frac{1}{2}\mathbb{P}_0^{(3)} + \frac{1}{2}\widehat{\mathbb{P}}_0^{(3)}$, where $\mathbb{P}_0^{(3)}$ denotes the law of a 3-dimensional Bessel process $(R_3(t), t \geq 0)$ starting from 0, and $\widehat{\mathbb{P}}_0^{(3)}$ denotes the distribution of $(-R_3(t), t \geq 0)$.*

Remark (M. Yor). Let $h(x) = \frac{1}{x}$. The family of the laws of all solutions of (1.1) is exactly $\left\{a\mathbb{P}_0^{(3)} + (1-a)\widehat{\mathbb{P}}_0^{(3)},\ 0 \leq a \leq 1\right\}$.

We give a first proof of Theorem 1.1 in Section 2 by using the absolute continuity (1.3): $d\mathbb{Q}_\epsilon^{(1)}|_{\mathcal{F}_t} = \mathbb{D}_t^{(h_\epsilon)}\, d\mathbb{P}|_{\mathcal{F}_t}$, and by studying carefully the density term $\mathbb{D}_t^{(h_\epsilon)}$. The proof shows that the penalization effect of the density $\mathbb{D}_t^{(h_\epsilon)}$ is to prevent the paths from hitting 0 when $\epsilon \to 0$. This should be compared with the polymer measure which is defined as the limit of the Wiener measure in $\mathbb{R}^d$ with densities preventing the paths from self intersecting. See [3] and [5] for some related references.

Another proof of Theorem 1.1 will be given in Section 3 based on the one-dimensional diffusion theory, and we shall remark that the penalization parameter $\mu \in \mathbb{R}$ plays an important role. More precisely, we summarize the different cases as follows:

- if $\mu \geq \frac{1}{2}$, the above phenomenon of penalization of 0 holds, see Proposition 3.1.
- if $-\frac{1}{2} < \mu < \frac{1}{2}$, $\mathbb{Q}_\epsilon^{(\mu)}$ converges to that of a symmetric Bessel process of dimension $\delta = 2(1+\mu)$ in the terminology of Watanabe [17, 18], see Proposition 3.2.
- if $\mu = -\frac{1}{2}$, this case is studied in Section 4 by using Krein's spectral theory. See Theorem 4.1 for the behaviour of the solution of (1.1).
- if $\mu < -\frac{1}{2}$, there exists a stationary measure of the corresponding diffusion, see Proposition 3.3.

Finally, we consider some general form of h in Section 5 by using Girsanov's transform.

Before closing this introduction, we would like to say that the proof of Theorem 1.1 presented in Section 2 is much more complicated than that in Section 3, but this proof, based on the decomposition of Brownian path, clearly shows where the "penalization effect" comes from, and also brings some by-product such as Lemma 2.3 and Corollary 2.1.

Throughout this paper, we denote by R_δ a Bessel process starting from 0, of dimension $\delta > 0$, and by $\mathbb{P}_0^{(\delta)}$ its law. For a process X, we denote $l_t^a(X)$ its local time process (when it exists) defined by:

$$l_t^a(X) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{(|X_s - a| \leq \epsilon)} ds.$$

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2. Proof of Theorem 1.1: Penalization through the densities.

We shall prove that for any continuous bounded functional F on $\mathcal{C}([0,t],\mathbb{R})$,

$$\mathbb{Q}_\epsilon^{(1)}\Big(F\Big) \to \frac{1}{2}\mathbb{P}_0^{(3)}\Big(F\Big) + \frac{1}{2}\widehat{\mathbb{P}}_0^{(3)}\Big(F\Big), \qquad \epsilon \to 0.$$

We prove this only for $t = 1$, the general $t > 0$ follows by using the same method or from the scaling property. The proof consists of four steps:

Step 1: On $\mathcal{F}_1$, $\mathbb{Q}_\epsilon^{(1)}$ is absolutely continuous with respect to $\mathbb{W}$ with density:

$$\begin{aligned} \mathbb{D}_1^{(\epsilon)} &= \exp\Big(\int_0^1 \frac{dB_s}{B_s} \mathbb{1}_{(|B_s| \geq \epsilon)} \; - \; \frac{1}{2} \int_0^1 \frac{ds}{B_s^2} \mathbb{1}_{(|B_s| \geq \epsilon)} \Big) \\ &= \Big(1 \vee \frac{|B_1|}{\epsilon} \Big) \exp\Big(- \frac{1}{2\epsilon} l_1^\epsilon(|B|) \Big). \end{aligned} \tag{2.1}$$

To obtain (2.1), we have from the Itô-Tanaka formula that

$$|B_t| \vee \epsilon = \epsilon + \int_0^t d(|B_s|) \mathbb{1}_{(|B_s| \geq \epsilon)} + \frac{1}{2} l_t^\epsilon(|B|),$$

$$\log(|B_t| \vee \epsilon) = \log(\epsilon) + \int_0^t \frac{dB_s}{B_s} \mathbb{1}_{(|B_s| \geq \epsilon)} - \frac{1}{2} \int_0^t \frac{ds}{B_s^2} \mathbb{1}_{(|B_s| \geq \epsilon)} + \frac{1}{2\epsilon} l_t^\epsilon(|B|).$$

And (2.1) follows. □

Step 2: Decomposition of the Brownian path (see [13, Exercise (XII.3.8)]): Let $g \overset{\text{def}}{=} \sup\{t \leq 1 : B_t = 0\}$. Then,

- $(b(s) \overset{\text{def}}{=} \frac{1}{\sqrt{g}} B_{gs}, s \leq 1)$ is a standard Brownian bridge,
- $(m(s) \overset{\text{def}}{=} \frac{1}{\sqrt{1-g}} |B_{g+(1-g)s}|, s \leq 1)$ is a standard Brownian meander,
- b, m, g and $\operatorname{sgn}(B_1)$ are independent.

We shall now express (2.1) in terms of b, m, g, $\operatorname{sgn}(B_1)$. On one hand, we need to express $F(B)$ in terms of these processes. This is the content of the following lemma:

Lemma 2.1. *Let $\mathcal{C}_0([0,1],\mathbb{R}) \overset{\text{def}}{=} \{\omega$ continuous on $[0,1] : \omega(0) = \omega(1) = 0\}$. To any measurable functional F on $\mathcal{C}([0,1],\mathbb{R})$, we can associate a measurable functional $\tilde{F}$ on $\mathcal{C}_0([0,1],\mathbb{R}) \times [0,1] \times \mathcal{C}([0,1],\mathbb{R})$ such that:*

i) $\tilde{F}(\omega; 0; \omega') = F(\omega')$

ii) The following equality holds :

$$F((B_s)_{s \leq 1}) = \tilde{F}((b_s)_{s \leq 1};\, g\,;\, \operatorname{sgn}(B_1)(m_s)_{s \leq 1}). \tag{2.2}$$

Moreover, if F is continuous, then:

$$t \to \tilde{F}(\omega; t; \omega') \text{ is continuous at } t = 0.$$

Proof of Lemma 2.1. Pick up $(\omega, t, \omega') \in \mathcal{C}_0([0,1],\mathbb{R}) \times [0,1] \times \mathcal{C}([0,1],\mathbb{R})$, we define

$$\tilde{F}(\omega, t, \omega') \overset{\text{def}}{=} F(\omega\,|t\,|\omega'),$$

where $\omega\,|t\,|\omega'$ is the continuous path defined by:

$$(\omega\,|t\,|\omega')(s) \overset{\text{def}}{=} \begin{cases} \sqrt{t}\,\omega(\frac{s}{t}), & \text{if } s \leq t \\ \sqrt{1-t}\,\omega'(\frac{s-t}{1-t}), & \text{if } t < s \leq 1. \end{cases}$$

Then (i) and (ii) follow, and it is not difficult to prove that for fixed ω, ω', $t \to (\omega \,|\, t \,|\, \omega')$ is continuous at $t = 0$. $\square$

On the other hand, we have:

$$l_1^{\epsilon}(|B|) = \sqrt{1-g}\, l_1^{\frac{\epsilon}{\sqrt{1-g}}}(m) + \sqrt{g}\, l_1^{\frac{\epsilon}{\sqrt{g}}}(|b|).$$

Thus,

$$\begin{aligned}\mathbb{Q}_{\epsilon}^{(1)}\Big(F\Big) &= \mathbb{E}\Big[\tilde{F}((b_s); g; \mathrm{sgn}(B_1)(m_s))\,\Big(1 \vee \frac{\sqrt{1-g}\, m_1}{\epsilon}\Big) \\ &\qquad \exp(-\frac{\sqrt{g}}{2\epsilon} l_1^{\frac{\epsilon}{\sqrt{g}}}(|b|)) \exp(-\frac{\sqrt{1-g}}{2\epsilon} l_1^{\frac{\epsilon}{\sqrt{1-g}}}(m))\Big] \\ &= \frac{1}{2}\mathbb{Q}_{\epsilon,+}^{(1)}\Big(F\Big) + \frac{1}{2}\mathbb{Q}_{\epsilon,-}^{(1)}\Big(F\Big),\end{aligned}$$

where $\mathbb{Q}_{\epsilon,\pm}^{(1)}\Big(F\Big) \stackrel{\text{def}}{=} \mathbb{Q}_{\epsilon}^{(1)}\Big(F \,\Big|\, \mathrm{sgn}(B_1) = \pm 1\Big)$. We shall now study the term $\mathbb{Q}_{\epsilon,+}^{(1)}$, the other term $\mathbb{Q}_{\epsilon,-}^{(1)}$ follows from symmetry.

Step 3: Conditioning by g: We recall that g is arcsine distributed. Then,

$$\begin{aligned}\mathbb{Q}_{\epsilon,+}^{(1)}\Big(F\Big) &= \frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{x(1-x)}}\mathbb{E}\Big[\tilde{F}(b; x; m)\,\Big(1 \vee \frac{\sqrt{1-x}\, m_1}{\epsilon}\Big) \\ &\qquad \exp(-\frac{\sqrt{x}}{2\epsilon} l_1^{\frac{\epsilon}{\sqrt{x}}}(|b|)) \exp(-\frac{\sqrt{1-x}}{2\epsilon} l_1^{\frac{\epsilon}{\sqrt{1-x}}}(m))\Big] \\ &= \frac{1}{\pi}\int_0^{\frac{1}{\epsilon^2}} \frac{dy}{\sqrt{y(1-\epsilon^2 y)}}\mathbb{E}\Big[\exp\big(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|)\big)\, A_F^{\epsilon}(y, b)\Big],\end{aligned} \tag{2.3}$$

where for any path $\omega \in \mathcal{C}([0,1] \to \mathbb{R})$,

$$A_F^{\epsilon}(y, \omega) \stackrel{\text{def}}{=} \mathbb{E}\Big[\tilde{F}(\omega; \epsilon^2 y; m)(\epsilon \vee \sqrt{1-\epsilon^2 y}\, m_1) \exp(-\frac{\sqrt{1-\epsilon^2 y}}{2\epsilon} l_1^{\frac{\epsilon}{\sqrt{1-\epsilon^2 y}}}(m))\Big], \tag{2.4}$$

where the expectation is taken with respect to m. The asymptotic behaviour of A_F^{ϵ} is given by:

Lemma 2.2. *Let $y > 0$ and $\omega \in C([0,1], \mathbb{R})$. Then as $\epsilon \to 0$,*

$$\begin{aligned}A_F^{\epsilon}(y, \omega) &\to \sqrt{\frac{\pi}{2}}\,\mathbb{E}\Big[\tilde{F}(\omega; 0; R_3)\Big]\,\mathbb{E}\Big[\exp(-\frac{1}{2} l_{\infty}^1(R_3))\Big] \\ &= \sqrt{\frac{\pi}{8}}\mathbb{E}\Big[F(R_3)\Big].\end{aligned} \tag{2.5}$$

Proof of Lemma 2.2: From Imhof [6]'s absolute continuity between the law of m and R_3,

$$A_F^{\epsilon}(y, \omega) = \sqrt{\frac{\pi}{2}}\,\mathbb{E}\Big[\tilde{F}(\omega; \epsilon^2 y; R_3)\,(\frac{\epsilon}{R_3(1)} \vee \sqrt{1-\epsilon^2 y})\, \exp(-\frac{\sqrt{1-\epsilon^2 y}}{2\epsilon} l_1^{\frac{\epsilon}{\sqrt{1-\epsilon^2 y}}}(R_3))\Big].$$

Let us denote

$$B_F^\epsilon(y,\omega) = \mathbb{E}\Big[\tilde F(\omega;0;R_3)\ \exp(-\frac{\sqrt{1-\epsilon^2 y}}{2\epsilon} l_1^{\frac{\epsilon}{\sqrt{1-\epsilon^2 y}}}(R_3))\Big].$$

Then,

$$A_F^\epsilon(y,\omega) - \sqrt{\frac{\pi}{2}}\, B_F^\epsilon(y,\omega) \to 0, \qquad \epsilon \to 0.$$

This follows from the a.s. convergence of

$$\tilde F(\omega;\epsilon^2 y;R_3)(\frac{\epsilon}{R_3(1)} \vee \sqrt{1-\epsilon^2 y}) \ \to\ \tilde F(\omega;0;R_3), \qquad \epsilon \to 0,$$

(see Lemma 2.1) and the integrability of $1/R_3(1)$ ($\tilde F$ is bounded). Now,

$$B_F^\epsilon(y,\omega) = \mathbb{E}[\tilde F(\omega;0;R_3)\exp(-\frac{1}{2} l^1_{\frac{1-\epsilon^2 y}{\epsilon^2}}(R_3^{(\epsilon)}))]$$

where $R_3^{(\epsilon)}(t) = \sqrt{\frac{1-\epsilon^2 y}{\epsilon^2}} R_3\Big(\frac{\epsilon^2}{1-\epsilon^2 y} t\Big)$ is a 3-dimensional Bessel process. Since $\frac{\epsilon^2}{1-\epsilon^2 y}$ goes to 0 as $\epsilon \to 0$, it follows from the ergodicity of the scaling transformation that R_3 and $R_3^{(\epsilon)}$ are asymptotically independent, i.e.:

$$(R_3, R_3^{(\epsilon)}) \xrightarrow{(d)} (R_3, \tilde R_3), \qquad \epsilon \to 0,$$

where $\tilde R_3$ is an independent copy of R_3. See e.g. [13, Exercise (XIII.1.17)].

It follows that

$$B_F^\epsilon(y,\omega) \to \mathbb{E}\Big[\tilde F(\omega;0;R_3)\Big]\,\mathbb{E}\Big[\exp(-\frac{1}{2} l^1_\infty(\tilde R_3))\Big], \qquad \epsilon \to 0,$$

and (2.5) follows from $\tilde F(\omega;0;R_3) = F(R_3)$ and $\mathbb{E}\Big[\exp(-\frac{1}{2} l^1_\infty(R_3))\Big] = 1/2$. □

Step 4: convergence of $Q_+^{(\epsilon)}$:

Lemma 2.3. *We have*

$$\int_0^\infty \frac{dy}{\sqrt{y}}\,\mathbb{E}\Big[\exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|))\Big] < \infty.$$

Proof: We write (2.3) for $F \equiv 1$. Then,

$$1 = \frac{1}{\pi}\int_0^\infty \int_{\mathcal{C}([0,1]\to\mathbb{R})} 1_{(y \le \frac{1}{\epsilon^2})} \frac{dy}{\sqrt{y(1-\epsilon^2 y)}}\, P_{0,0}(db)\, \exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|))\, A_F^\epsilon(y,b),$$

where $P_{0,0}$ denotes the law of the Brownian bridge. The function in the integral converges $dydP_{0,0}(db)$ a.s. to $\sqrt{\frac{\pi}{8y}}\exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|))$.

By Fatou's lemma,

$$\frac{1}{\sqrt{8\pi}}\int_0^\infty \int_C \frac{dy}{\sqrt{y}} P_{0,0}(db)\exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|)) \leq \liminf_{\epsilon\to 0} Q^{(\epsilon)}(1) = 1,$$

as desired. □

In order to pass to the limit in the integral (2.3), we shall decompose the integral in $\int_0^{\frac{\eta}{\epsilon^2}} \ldots + \int_{\frac{\eta}{\epsilon^2}}^{\frac{1}{\epsilon^2}} \ldots$, where $\eta \in]0,1[$ is a fixed number whose value will be determined later. Let us denote

$$\Phi_1(\epsilon,\eta) = \frac{1}{\pi}\int_0^{\frac{\eta}{\epsilon^2}} \frac{dy}{\sqrt{y(1-\epsilon^2 y)}} \mathbb{E}\Big[\exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|)) A_F^\epsilon(y,b)\Big] \tag{2.6}$$

where A_F^ϵ is defined by (2.4) and for all y, b, $|A_F^\epsilon(y,b)| \leq K$ for some constant K (since $\mathbb{E}[m_1] < \infty$). Using Lemmas 2.2 and 2.3, we deduce from the dominated convergence theorem that (η being fixed)

$$\Phi_1(\epsilon,\eta) \to C\,\mathbb{E}\Big[F(R_3)\Big], \qquad \epsilon \to 0,$$

where

$$C = \frac{1}{\sqrt{8\pi}}\int_0^\infty \frac{dy}{\sqrt{y}}\,\mathbb{E}\Big[\exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|))\Big]. \tag{2.7}$$

It remains to choose η such that the remaining term is arbitrarily small when $\epsilon \to 0$. Let

$$\Phi_2(\epsilon,\eta) = \frac{1}{\pi}\int_{\frac{\eta}{\epsilon^2}}^{\frac{1}{\epsilon^2}} \frac{dy}{\sqrt{y(1-\epsilon^2 y)}}\,\mathbb{E}\Big[\exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|))\,A_F^\epsilon(y,b)\Big].$$

Since $\epsilon \vee \sqrt{1-\epsilon^2 y}\, m_1 \leq \epsilon + \sqrt{1-\epsilon^2 y}\, m_1$ and $\tilde{F} \leq K'$, we deduce from (2.4) that

$$\Phi_2(\epsilon,\eta) \leq K'\left\{\int_{\frac{\eta}{\epsilon^2}}^{\frac{1}{\epsilon^2}} \frac{dy}{\sqrt{y}}\,\mathbb{E}\Big[\exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|))\Big]\,\mathbb{E}\Big[m_1\Big] + \epsilon\int_{\frac{\eta}{\epsilon^2}}^{\frac{1}{\epsilon^2}} \frac{dy}{\sqrt{y(1-\epsilon^2 y)}}\right\}.$$

Now,

$$\epsilon\int_{\frac{\eta}{\epsilon^2}}^{\frac{1}{\epsilon^2}} \frac{dy}{\sqrt{y(1-\epsilon^2 y)}} = 2\int_0^{\sqrt{1-\eta}} \frac{dz}{\sqrt{1-z^2}} \to 0, \qquad \eta \to 1.$$

Let $\alpha > 0$ be arbitrarily small, we can choose η such that:

$$K'\epsilon\int_{\frac{\eta}{\epsilon^2}}^{\frac{1}{\epsilon^2}} \frac{dy}{\sqrt{y(1-\epsilon^2 y)}} \leq \frac{\alpha}{2}.$$

Thanks to Lemma 2.3, there exists ϵ_0 such that for $\epsilon \leq \epsilon_0$,

$$K'\int_{\frac{\eta}{\epsilon^2}}^{\frac{1}{\epsilon^2}} \frac{dy}{\sqrt{y}}\,\mathbb{E}\Big[\exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|))\Big]\,\mathbb{E}\Big[m_1\Big] \leq \frac{\alpha}{2}.$$

Thus, we have obtained:

$$\mathbb{Q}^{(1)}_{\epsilon,+}\Big(F\Big) \to C\,\mathbb{E}\Big[F(R_3)\Big], \qquad \epsilon \to 0,$$

where C is given by (2.7) and necessarily $C = 1$.
In the same way,

$$\mathbb{Q}^{(1)}_{\epsilon,-}\Big(F\Big) \to \mathbb{E}[F(-R_3)], \qquad \epsilon \to 0,$$

proving the theorem. □

The following consequence may have an independent interest:

Corollary 2.1. *We have*

$$\int_0^\infty \frac{dy}{\sqrt{y}}\mathbb{E}[\exp(-\frac{\sqrt{y}}{2} l_1^{\frac{1}{\sqrt{y}}}(|b|))] = \sqrt{8\pi}.$$

3. Convergences in law.

In this section and the forthcoming one, we shall consider a family of processes $(X_\epsilon(t), t \geq 0)$, which are the unique (strong) solutions of the following equations:

$$X_\epsilon(t) = B(t) + \mu \int_0^t \frac{ds}{X_\epsilon(s)} \mathbb{1}_{(|X_\epsilon(s)|>\epsilon)}. \tag{3.1}$$

We shall simply write $(X(t) \equiv X_1(t), t \geq 0)$ for the solution of (3.1) corresponding to $\epsilon = 1$. By scaling, we have

$$(X_\epsilon(t), t \geq 0) \overset{\text{law}}{=} (\epsilon X(t/\epsilon^2), t \geq 0) \tag{3.2}$$

We are interested in the convergence in law of X_ϵ as $\epsilon \to 0$. When $\mu = 1$, we have shown in Theorem 2.1 how the penalization prevents X_ϵ from hitting the origin, this fact holds for all $\mu \geq 1/2$. We shall also show that when the penalization parameter μ is small, the limit process can hit the origin.

Lemma 3.1. *Assume $\mu > -1/2$. As $\epsilon \to 0$, we have*

$$\Big|X_\epsilon(\cdot)\Big| \xrightarrow{(d)} R_\delta(\cdot),$$

where R_δ denotes a Bessel process of dimension $\delta \overset{\text{def}}{=} 1 + 2\mu > 0$ starting from 0, and $\xrightarrow{(d)}$ means the convergence in law on $\mathcal{C}(\mathbb{R}_+ \to \mathbb{R})$ endowed with the topology of uniform convergence on each compact interval.

Proof. Define $Y_\epsilon(t) \overset{\text{def}}{=} X_\epsilon^2(t)$, for $t \geq 0$. Using Itô's formula, the process Y_ϵ satisfies

$$\begin{cases} Y_\epsilon(t) = 2\int_0^t \sqrt{Y_\epsilon(s)}\, dW(s) + \int_0^t ds\big(1 + 2\mu \mathbb{1}_{(Y_\epsilon(s)>\epsilon^2)}\big), \\ Y_\epsilon(t) \geq 0, \end{cases} \qquad t \geq 0, \tag{3.3}$$

with a Brownian motion W, which *a priori* depends on ϵ. It is elementary to verify that (3.3) admits the uniqueness in law (e.g. by using Zvonkin's method [21] and then using Engelbert and Schmidt's criteria, see Karatzas and Shreve [7, Chap. 5.5]). Hence, it suffices to show that the solutions of (3.3) converge in law to $R^2_\delta(\cdot)$. But this can be done in a standard way, see e.g. the proof of Stroock and Varadhan [16, Theorem 11.3.3]. The only remaining fact to verify is that

$$\lim_{\epsilon\to 0} \mathbb{E} \int_0^t ds\, \mathbb{1}_{(Y_\epsilon(s)\le \epsilon^2)} = 0.$$

To this end, we use the comparison theorem (cf. [13]) to the two diffusions Y_ϵ and $R^2_{\delta'}$ which satisfies the following equation:

$$\begin{cases} R^2_{\delta'}(t) = 2\int_0^t R_{\delta'}(s)\, dW(s) + \delta' t, \\ R_{\delta'}(t) \ge 0, \end{cases} \qquad t \ge 0,$$

with the dimension $0 < \delta' < \min(1, 1+2\mu)$. It follows that almost surely for all $t \ge 0$, $Y_\epsilon(t) \ge R^2_{\delta'}(t)$. It follows that

$$\mathbb{E} \int_0^t ds\, \mathbb{1}_{(Y_\epsilon(s)\le \epsilon^2)} \le \mathbb{E} \int_0^t ds\, \mathbb{1}_{(R_{\delta'}(s)\le \epsilon)} \to 0, \qquad \epsilon \to 0,$$

completing the proof. □

Remark 3.1. To our best knowledge, the comparison theorems for two diffusions often require some regularities of drift terms (at least one of the two drift terms), this is why we are not able to deduce the convergence of Y_ϵ directly from (3.3).

Proposition 3.1. *Assume that $\mu \ge 1/2$. The law of $X_\epsilon(\cdot)$ converges to $\frac{1}{2}\mathbb{Q}_\delta + \frac{1}{2}\widehat{\mathbb{Q}}_\delta$, where $\mathbb{Q}_\delta$ denotes the law of R_δ a Bessel process starting from 0, of dimension $\delta \stackrel{\text{def}}{=} 1 + 2\mu \ge 2$, and $\widehat{\mathbb{Q}}_\delta$ denotes the law of $-R_\delta(\cdot)$.*

Proof. Fix a small $\eta > 0$. Since the Bessel process R_δ does not visit the origin after time 0 ($\delta \ge 2$), we deduce from Lemma 3.1 that $(X_\epsilon(s), \eta \le s \le t)$ will keep the same sign as $X_\epsilon(\eta)$, with probability approaching 1 when $\epsilon \to 0$. This fact together with Lemma 3.1 imply that the process $(X_\epsilon(s), \eta \le s \le t)$ will converge in law to $(UR_\delta(s), \eta \le s \le t)$, as $\epsilon \to 0$, where U is independent of R_δ and $U \stackrel{\text{law}}{=} \text{sgn}(X_\epsilon(\eta))$. The symmetry implies that U is a (symmetric) Bernoulli variable and the rest of the proof is completed by letting $\eta \to 0$. □

Remark 3.2. Proposition 3.1 admits a natural generalization to the multidimensional case: By considering a Lipschitz drift term which coincides with $\frac{\mu x}{|x|^2}$ when $|x| \ge \epsilon$, we can prove in the same way that the associated diffusion converges in law to $\mathbf{U_d}(du) \times \mathbb{Q}_\delta(u)$, where $\mathbf{U_d}$ denotes the uniform probability measure on the sphere $\{u \in \mathbb{R}^d : |u| = 1\}$, and $\mathbb{Q}_\delta(u)$ the law of the process uR_δ.

When $\mu < 1/2$, the limit process can hit 0. Firstly, we have

Proposition 3.2 ($-\frac{1}{2} < \mu < \frac{1}{2}$)**.** *Let $\delta \equiv 1 + 2\mu \in (0,2)$. As $\epsilon \to 0$, the law $\mathbb{Q}_\epsilon^{(\mu)}$ converges to that of the continuous process $(\mathbb{U}_\delta(t), t \geq 0)$, which is determined by the following time-change:*

$$\mathbb{U}_\delta(t) = 2\alpha\, sgn(B(\rho^{-1}(t))\, \big|B(\rho^{-1}(t))\big|^{1/(2\alpha)}, \tag{3.4}$$

with $\alpha \overset{\text{def}}{=} \frac{1}{2} - \mu \in (0,1)$ and ρ^{-1} the inverse of ρ:

$$\rho(t) \overset{\text{def}}{=} \int_0^t |B(s)|^{\frac{1}{\alpha}-2} ds, \qquad t \geq 0. \tag{3.5}$$

Remark 3.3. According to the terminology of Watanabe [17, 18], $\mathbb{U}_\delta$ is a symmetric Bessel process of dimension δ. More precisely, $\mathbb{U}_\delta$ is the unique diffusion on $\mathbb{R}$ such that $\mathbb{U}_\delta(0) = 0$ and

(i) For every $f \in \mathcal{C}^2(\mathbb{R})$ with compact support and vanishing on a neighborhood of 0, the process $f(\mathbb{U}_\delta(t)) - \int_0^t ds\big(\frac{1}{2} f''(\mathbb{U}_\delta(s)) + \frac{(\delta-1)}{2}\frac{f'(\mathbb{U}_\delta(s))}{\mathbb{U}_\delta(s)}\big)$ is a martingale.
(ii) Almost surely for $t > 0$, $\int_0^t \mathbb{1}_{\{0\}}(\mathbb{U}_\delta(s))\, ds = 0$.
(iii) The function $\int_0^{\cdot} |y|^{1-\delta} dy$ is a scale function of $\mathbb{U}_\delta$.

We can verify that the process $\mathbb{U}_\delta$ defined via (3.4) and (3.5) satisfies the properties (i)–(iii). For more details on the skew and bilateral Bessel processes, we refer to Watanabe [17,18].

Proof. The proof follows from Feller's time change for one-dimensional diffusion. Recall (3.2). Denote by S the scale function of the diffusion X:

$$S(x) \overset{\text{def}}{=} \int_0^x S'(y)dy \overset{\text{def}}{=} \int_0^x \big(\mathbb{1}_{(|y|\leq 1)} + \mathbb{1}_{(|y|>1)}|y|^{-2\mu}\big)dy, \qquad x \in \mathbb{R}. \tag{3.6}$$

We have

$$Z(t) \overset{\text{def}}{=} S(X(t)) = \int_0^t S'(X(s))dB(s) = \int_0^t \sigma(Z(s))dB(s), \tag{3.7}$$

where $\sigma(x) \overset{\text{def}}{=} S'(S^{-1}(x))$, and $S^{-1}(\cdot)$ denotes the inverse function of $S(\cdot)$. Recall that $\alpha = \frac{1}{2} - \mu$. It is elementary to obtain that

$$\epsilon S^{-1}(z\epsilon^{-2\alpha}) \to \text{sgn}(z)\big(2\alpha\,|z|\big)^{1/(2\alpha)}, \qquad \epsilon \to 0,\ z \in \mathbb{R}, \tag{3.8}$$

$$\sigma(x) \sim \big(2\alpha\,|x|\big)^{1-1/(2\alpha)}, \qquad |x| \to \infty. \tag{3.9}$$

In view of scaling, we write

$$\Big(X_\epsilon(t), t \geq 0\Big) \overset{\text{law}}{=} \Big(\epsilon S^{-1}\big(\epsilon^{-2\alpha}\,\epsilon^{2\alpha} Z(t/\epsilon^2)\big), t \geq 0\Big),$$

therefore it suffices to show that

$$\Big(\epsilon^{2\alpha} Z(t/\epsilon^2), t \geq 0\Big) \xrightarrow{(d)} \Big((2\alpha)^{2\alpha-1}\, B(\rho^{-1}(t)), t \geq 0\Big), \qquad \epsilon \to 0. \tag{3.10}$$

To this end, applying Dubins-Schwarz' representation theorem to the continuous martingale $Z(t)$ shows that for some Brownian motion W, we have

$$Z(t) = W(\psi^{-1}(t)),$$

where ψ^{-1} denotes the inverse of ψ, and

$$\psi(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} L(t,x)\sigma^{-2}(x)dx,$$

with $L(t,x)$ the local times of W. Using the scaling property, we obtain:

$$\Big(\epsilon^{2\alpha} Z(t/\epsilon^2), t \geq 0\Big) \stackrel{\text{law}}{=} \Big(W(\psi_\epsilon^{-1}(t)), t \geq 0\Big), \tag{3.11}$$

where ψ_ϵ^{-1} denotes the inverse of ψ_ϵ, and

$$\psi_\epsilon(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} L(t,x)\epsilon^{2-4\alpha}\sigma^{-2}(x/\epsilon^{2\alpha})\,dx.$$

Using (3.9), we obtain that uniformly on each interval $[0,T]$ for $T>0$,

$$\psi_\epsilon(t) \to (2\alpha)^{\frac{1}{\alpha}-2}\int_{\mathbb{R}} L(t,x)\,|x|^{\frac{1}{\alpha}-2}\,dx, \qquad 0 \leq t \leq T.$$

Applying this observation to (3.11), we obtain:

$$\Big(\epsilon^{2\alpha} Z(t/\epsilon^2), t \geq 0\Big) \xrightarrow{(d)} \Big(B(\rho^{-1}(t(2\alpha)^{2-1/\alpha})), t \geq 0\Big), \qquad \epsilon \to 0,$$

showing (3.10) by using the scaling property. □

Proposition 3.3. *Assume that* $\mu < -1/2$. *For* $t > 0$,

$$\frac{X_\epsilon(t)}{\epsilon} \xrightarrow{(d)} \frac{|1+2\mu|}{4|\mu|}\left(\mathbb{1}_{(|x|\leq 1)} + |x|^{2\mu}\,\mathbb{1}_{(|x|>1)}\right)dx, \qquad \epsilon \to 0.$$

Proof. We use again (3.6)–(3.9). Observe that the diffusion Z is on its natural scale, and its speed measure has a finite mass on $\mathbb{R}$. It turns out (see e.g. Rogers and Williams [14, Theorem V.54.5]) that

$$Z(u) \xrightarrow{(d)} \pi(dx), \qquad u \to \infty,$$

where $\pi(dx) \stackrel{\text{def}}{=} \sigma^{-2}(x)dx/\int_{\mathbb{R}}\sigma^{-2}(y)dy$ is the stationary measure of Z. Now, observe that

$$\frac{X_\epsilon(t)}{\epsilon} \stackrel{\text{law}}{=} X_1(t/\epsilon^2) = S^{-1}(Z(t/\epsilon^2)),$$

yielding the desired result after elementary calculations. □

The case $\mu = -1/2$ is discussed in the next section.

4. Krein's spectral theory: Application to the case $\mu = -1/2$.

The main result of this section is the following:

Theorem 4.1. *Let $\mu = -1/2$ and fix $v > 0$. Recall that X_ϵ denotes the solution of (3.1). As $\epsilon \to 0$, we have*

$$\sqrt{\log(1/\epsilon)} \sup_{0 \le s \le v} \left|X_\epsilon(s)\right| \xrightarrow{(d)} \sqrt{\frac{v}{\mathbf{e}}},$$
$$\frac{\log|X_\epsilon(v)|}{\log(1/\epsilon)} \xrightarrow{(d)} -\mathbf{U},$$

where $\mathbf{e}$ *and* $\mathbf{U}$ *are respectively exponential distributed with parameter* 1 *and uniform on* $[0,1]$.

Before we present the proof, we recall some facts on Krein's correspondence, see Dym and McKean [4], Kotani and Watanabe [10] and Kasahara [8] together with their references for details. Let $\mathcal{M}$ be the class of functions $m : [0,\infty] \to [0,\infty]$ which are nondecreasing, right-continuous and $m(\infty) = \infty$. Put $m(0-) = 0$ and let $\ell = \sup\{x : m(x) < \infty\}$ ($\ell = 0$ if $m(x) \equiv \infty$). When $\ell > 0$, consider the following integral equations:

$$\phi(x,\lambda) = 1 + \lambda \int_0^x dy \int_{0-}^{y+} \phi(z,\lambda) dm(z), \qquad 0 \le x < \ell, \tag{4.1}$$

$$\psi(x,\lambda) = x + \lambda \int_0^x dy \int_{0-}^{y+} \psi(z,\lambda) dm(z), \qquad 0 \le x < \ell, \tag{4.2}$$

$$h(\lambda) \overset{\text{def}}{=} \int_0^\ell \frac{dx}{\phi^2(x,\lambda)} = \lim_{x\to\infty} \frac{\psi(x,\lambda)}{\phi(x,\lambda)}. \tag{4.3}$$

When $m(x) \equiv \infty$, we define $h(\lambda) \equiv 0$. The correspondence $m(x) \leftrightarrow h(\lambda)$ is called Krein's correspondence and h is called the characteristic function of m. When $m(x) \not\equiv 0$, $h \not\equiv \infty$ and h can be represented as follows:

$$h(\lambda) = c + \int_{0-}^\infty \frac{d\nu(t)}{\lambda + t}, \tag{4.4}$$

where $c \ge 0$ denotes the left endpoint of the support of m, and $\nu(dt)$ is a nonnegative measure on $[0,\infty)$ such that $\int_{0-}^\infty \frac{d\nu(t)}{1+t} < \infty$. We call $d\nu(t)$ the spectral measure. Let $\mathcal{H} \overset{\text{def}}{=} \{h : \text{ of form (4.4)}\} \cup \{h \equiv 0\} \cup \{h \equiv \infty\}$.

Theorem (Krein [11], Kasahara [8]). *Krein's correspondence*

$$m \in \mathcal{M} \longleftrightarrow h \in \mathcal{H}$$

is one to one and onto. Let $m_n \in \mathcal{M} \longleftrightarrow h_n \in \mathcal{H}$. We denote by ϕ_n and ψ_n the solutions of (4.1)-(4.2) related to m_n. The following three assertions are equivalent:
(i) As $n \to \infty$, $m_n(x) \to m_0(x)$ at each continuity point $x \in [0, \infty)$ of $m_0(x)$.
(ii) As $n \to \infty$, $(\phi_n(x, \lambda), \psi_n(x, \lambda)) \to (\phi_0(x, \lambda), \psi_0(x, \lambda))$ for $x \in [0, \infty)$ and $\lambda > 0$.
(iii) As $n \to \infty$, $h_n(\lambda) \to h_0(\lambda)$ for $\lambda > 0$.

We shall make use of the following scaling property:

Lemma 4.1. *Let $m(x) \longleftrightarrow h(\lambda)$ be in Krein's correspondence. For $a, b > 0$, we define $\tilde{m}(x) = \frac{b}{a} m(\frac{x}{a})$, and denote by $\tilde{h}$, $\tilde{\nu}(t)$, $\tilde{\phi}, \tilde{\psi}$ the associated characteristic function, the spectral measure and the fundamental solutions. We have for all $x, t, \lambda \geq 0$,*

$$\tilde{h}(\lambda) = a\, h(b\lambda),\ \tilde{\nu}(t) = \frac{a}{b}\nu(bt),\ \tilde{\phi}(x, \lambda) = \phi(\frac{x}{a}, b\lambda),\ \tilde{\psi}(x, \lambda) = a\ \psi(\frac{x}{a}, b\lambda). \tag{4.5}$$

When $m(x) < \infty$, the functions $\phi(x, \xi), \psi(x, \xi)$ are analytic on ξ, and the scaling property can be extended to all $\xi \in \mathcal{C}$.

Krein's theory together with Kotani and Watanabe's extension are very powerful to treat the real-valued diffusion of generator $\frac{d}{dM}\frac{d}{dz}$ (here, $dM(z)$ is a Radon measure on $\mathbb{R}$). See Bertoin [1] for a nice application to Brownian principal additive functionals. See also Kasahara et al. [9]. Here, we study a particular example of diffusion, and the method can be applied to a more general class of diffusions.

Proposition 4.1. *Let Z be a diffusion on $\mathbb{R}$ of generator $\frac{1}{2}\sigma^2(z)\frac{d^2}{dz^2}$ with $\sigma(z) = 1 \vee \sqrt{(2|z| - 1)^+}$ for $z \in \mathbb{R}$ (cf. (3.12)–(3.13) with $\mu = -1/2$). As $t \to \infty$, we have*

$$\frac{\log t}{t} \sup_{0 \leq s \leq t} \big|Z(s)\big| \xrightarrow{(d)} \frac{1}{\mathbf{e}}, \tag{4.6}$$

$$\frac{\log |Z(t)|}{\log t} \xrightarrow{(d)} \mathbf{U}, \tag{4.7}$$

where $\mathbf{e}$ and $\mathbf{U}$ denote respectively the exponential distribution with parameter 1 and the uniform distribution on $[0, 1]$.

Proof. We consider the two measures $m_1, m_2 \in \mathcal{M}$ determined by $m_1(dx) = 2\sigma^{-2}(x)dx$ and $m_2(dx) = 2\sigma^{-2}(-x)dx$ ($x \geq 0$). Since $m_1 = m_2$ by the symmetry of σ, we can only consider $m \equiv m_1$. Consider Krein's correspondence $h(\lambda) \longleftrightarrow m(x)$ and let $(\phi(x, \lambda), \psi(x, \lambda))$ and $d\nu(t)$ be the fundamental solutions and the spectral measure respectively. We define the extensions of $\phi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ on $\mathbb{R}$ by: $\phi(z, \lambda) \stackrel{\text{def}}{=} \phi(|z|, \lambda)$ and $\psi(z, \lambda) = \operatorname{sgn}(z)\, \psi(|z|, \lambda)$ for $z \in \mathbb{R}$. Put

$$u_+(z, \lambda) = \phi(z, \lambda) - \frac{1}{h(\lambda)}\psi(z, \lambda), \qquad z \in \mathbb{R},\ \lambda > 0,$$

$$u_-(z, \lambda) = \phi(z, \lambda) + \frac{1}{h(\lambda)}\psi(z, \lambda),$$

$$g_\lambda(z, y) \stackrel{\text{def}}{=} g_\lambda(y, z) \stackrel{\text{def}}{=} \frac{1}{2}\, h(\lambda) u_+(z, \lambda) u_-(y, \lambda), \qquad z \geq y \in \mathbb{R}.$$

We refer to Kotani and Watanabe [10] for details. Therefore g_λ is the density of the resolvent operator of Z with respect to its speed measure. We denote by $p(t,y)$ the density of the law of $Z(t)$ with respect to the speed measure: $\mathbb{P}\big(Z(t)\in dz\big) = p(t,z)2\sigma^{-2}(z)dz$. We have $p(t,z)=p(t,|z|)$ and for $x\ge 0$,

$$\int_0^\infty e^{-\lambda t}2p(t,x)dt = 2g_\lambda(0,x) = h(\lambda)u_+(x,\lambda) = \int_{0-}^\infty \frac{\phi(x,-\xi)}{\lambda+\xi}d\nu(\xi), \tag{4.8}$$

where the last equality follows from the following spectral representation:

$$h(\lambda)u_+(x\vee y,\lambda)\phi(x\wedge y,\lambda) = \int_{0-}^\infty \frac{\phi(x,-\xi)\phi(y,-\xi)}{\lambda+\xi}d\nu(\xi), \qquad x,y\ge 0, \tag{4.9}$$

see Dym and McKean [4, pp. 176]. Inverting the Laplace transform of (4.8), we get

$$p(t,x) = \frac{1}{2}\int_{0-}^\infty e^{-t\,\xi}\,\phi(x,-\xi)\,d\nu(\xi), \qquad t>0, x\ge 0. \tag{4.10}$$

The above integral is absolutely convergent, since we have e.g. $\int_{0-}^\infty \frac{\phi^2(x,-\xi)}{\lambda+\xi}d\nu(\xi) < \infty$ by taking $x=y$ in (4.9).

Now, we are going to prove (4.6) and (4.7). Let $H(r) \stackrel{\text{def}}{=} \inf\{t>0 : |Z(t)|>r\}$ for $r>0$. Observe that $(e^{-\lambda t}\phi(Z(t),\lambda), 0\le t\le H(r))$ is a bounded continuous martingale. Recall that $\phi(z,\lambda)=\phi(|z|,\lambda)$. The optional stopping time theorem yields

$$\mathbb{E}\exp\Big(-\epsilon\,\lambda\,H(\frac{r}{a})\Big) = \frac{1}{\phi(\frac{r}{a},\epsilon\lambda)}, \tag{4.11}$$

where we take $a\equiv a(\epsilon) \stackrel{\text{def}}{=} \epsilon\,\log(1/\epsilon)$ and $\epsilon>0$ is assumed to be small. It remains to study the behaviour of $\phi(\frac{r}{a},\epsilon\lambda)$ when $\epsilon\to 0$. Define $m_\epsilon(x) = \frac{\epsilon}{a}m(\frac{x}{a})$, and $h_\epsilon, d\nu_\epsilon(t), \phi_\epsilon, \psi_\epsilon$ in the obvious way. Observe that

$$m_\epsilon(x) \;\to\; m_0(x)=1, \qquad x>0,\;\; \epsilon\to 0,$$

where $m_0(x) \stackrel{\text{def}}{=} \mathbb{1}_{(x\ge 0)}$ ($m_0(0-)=0$). It is immediate to obtain that $m_0(x) \longleftrightarrow h_0(\lambda)=1/\lambda$, and $\phi_0(x,\lambda)=1+\lambda x$, $\psi_0(x,\lambda)=x$ for $x,\lambda>0$. Using Lemma 4.1 and Kasahara's continuity theorem, we have that for $x>0$ and $\lambda>0$,

$$\phi(\frac{x}{a},\epsilon\lambda) = \phi_\epsilon(x,\lambda) \to\; 1+\lambda x, \qquad \epsilon\to 0,$$
$$\psi(\frac{x}{a},\epsilon\lambda) = \frac{1}{a}\psi_\epsilon(x,\lambda) \sim\; \frac{x}{\epsilon\log(1/\epsilon)}, \qquad \epsilon\to 0,$$
$$a\,h(\epsilon\lambda) = h_\epsilon(\lambda) \to \frac{1}{\lambda}, \qquad \epsilon\to 0.$$

Finally, we apply (4.4) to h_ϵ and $d\nu_\epsilon$ (with $c=0$ in (4.4)). By considering $h_\epsilon(\lambda)-h_\epsilon(1)$, we deduce from (4.14) that as $\epsilon\to 0$,

$$\frac{1}{h_\epsilon(1)}\frac{\mathbb{1}_{(s\ge 0)}}{1+s}d\nu_\epsilon(s) \text{ weakly converges to the Dirac measure at 0,} \tag{4.12}$$

see e.g. Dym and McKean [4, pp.179].

Going back to (4.11), we obtain that for fixed $r > 0$ and $\lambda > 0$

$$\epsilon\, H(\frac{r}{\epsilon \log(1/\epsilon)}) \xrightarrow{(d)} r\, \mathbf{e}, \qquad \epsilon \to 0.$$

Inverting the above convergence, we obtain (4.6).

To prove (4.7), fix $0 < c < 1$ and $t_0 > 0$. We have from (4.10) that

$$\begin{aligned} \mathbb{P}\Big(|Z(t_0/\epsilon)| < \epsilon^{-c}\Big) &= \int_0^{\epsilon^{-c}} dx 2\sigma^{-2}(x) \int_{0-}^{\infty} e^{-t_0 \xi/\epsilon}\, \phi(x, -\xi)\, d\nu(\xi) \\ &= \frac{2\epsilon}{a} \int_0^{\epsilon^{-c}} dx \sigma^{-2}(x) \int_{0-}^{\infty} d\nu_\epsilon(s) e^{-st_0}\, \phi_\epsilon(ax, -s), \end{aligned} \tag{4.13}$$

by using Lemma 4.1 and the change of variable $\xi = s\epsilon$, with $a \overset{\text{def}}{=} \epsilon \log(1/\epsilon)$. Admitting for the moment that uniformly on $y \in [0, 1]$,

$$\int_{0-}^{\infty} d\nu_\epsilon(s) e^{-st_0}\, \phi_\epsilon(y, -s) \;\to\; 1, \qquad \epsilon \to 0. \tag{4.14}$$

Then, it follows from (4.13) that

$$\mathbb{P}\Big(|Z(t_0/\epsilon)| < \epsilon^{-c}\Big) = \frac{2\epsilon}{a} \int_0^{\epsilon^{-c}} dx \sigma^{-2}(x) (1 + o(1)) = c + o(1), \qquad \epsilon \to 0,$$

showing the desired convergence in law (4.7) since $0 < c < 1$ is arbitrary.

It remains us to prove (4.14). Firstly for some large but fixed $K > 0$, we deduce from Cauchy-Schwarz' inequality that (recalling from (4.4) that $h_\epsilon(1) = \int_{0-}^{\infty} \frac{d\nu_\epsilon(s)}{1+s}$)

$$\begin{aligned} \int_{K+}^{\infty} e^{-st_0}\, \big|\phi_\epsilon(y, -s)\big|\, d\nu_\epsilon(s) &\le \sup_{s \ge K} \Big(e^{-st_0}(1+s)\Big) \Big(\int_{0-}^{\infty} \frac{\phi_\epsilon^2(y, -s)}{1+s} d\nu_\epsilon(s)\Big)^{1/2} \Big(h_\epsilon(1)\Big)^{1/2} \\ &= \sup_{s \ge K} \Big(e^{-st_0}(1+s)\Big)\, h_\epsilon^{3/2}(1)\, u_{+,\epsilon}(y, 1)\, \phi_\epsilon(y, 1), \\ &\le \sup_{s \ge K} \Big(e^{-st_0}(1+s)\Big)\, h_\epsilon^{3/2}(1)\, \phi_\epsilon(1, 1), \end{aligned} \tag{4.15}$$

where the equality is due to the analogue form of (4.9) for $d\nu_\epsilon$, and the last inequality follows from the facts that the function $x \to u_{+,\epsilon}(x, 1) \overset{\text{def}}{=} \phi_\epsilon(x, 1) - \frac{1}{h_\epsilon(1)} \psi_\epsilon(x, 1)$ is positive and nonincreasing hence is bounded by 1 and the function $\phi_\epsilon(\cdot, 1)$ is increasing. In view of (4.15), it suffices to show that for any fixed $K > 0$, when $\epsilon \to 0$, the following convergence holds uniformly for $y \in [0, 1]$,

$$\int_{0-}^{K} d\nu_\epsilon(s) e^{-st_0}\, \phi_\epsilon(y, -s) \;\to\; 1. \tag{4.16}$$

Now consider $0 \le s \le K$. By using e.g. the integral equation for $\phi_\epsilon(\cdot, -s)$ and applying (4.12), it is easy to show that

$$\sup_{0\le s\le K, 0\le y\le 1} \Big|\phi_\epsilon(y,-s) - (1-sy)\Big| \to 0, \qquad \epsilon \to 0.$$

Using the above estimate and applying (4.12), we obtain (4.16) and complete the whole proof. □

Proof of Theorem 4.1. Follows from (3.2), (3.7)–(3.8) with $\alpha = 1$, and Proposition 4.1. □

5. An example of Girsanov's transform.

In the above penalization procedure (Sections 2 and 3), the key property of the function h is that $h(x) \sim \mu/x$ as $x \to 0$. Therefore, it is natural to consider

$$h(x) = f(x) + \frac{\mu}{x}, \qquad x \neq 0,$$

with $f \in \mathcal{C}^1\big(\mathbb{R} \to \mathbb{R}\big)$ a bounded function such that $f(x) \ge f(0) = 0 \ge f(-x)$ for $x \ge 0$ and f' is bounded. Consider the diffusion

$$Y_\epsilon(t) = B(t) + \int_0^t h(Y_\epsilon(s))\, 1\!\!1_{(|Y_\epsilon(s)|\ge\epsilon)}\, ds. \tag{5.1}$$

Using the corresponding result for the case $f \equiv 0$, we obtain:

Theorem 5.1. *Suppose that $\mu \ge 1/2$. When $\epsilon \to 0$, the diffusion Y_ϵ converges in law to*

$$\frac{1}{2}\,\mathbb{Q}^{\mu,f} + \frac{1}{2}\,\widehat{\mathbb{Q}}^{\mu,\widehat{f}},$$

where $\mathbb{Q}^{\mu,f}$ and $\widehat{\mathbb{Q}}^{\mu,\widehat{f}}$ denote respectively the laws of the processes Ξ and $-\widehat{\Xi}$ which are the unique solutions (in law) of:

$$0 \le \Xi_t = B(t) + \int_0^t \big(\frac{\mu}{\Xi_s} + f(\Xi_s)\big) ds,$$
$$0 \le \widehat{\Xi}_t = B(t) + \int_0^t \big(\frac{\mu}{\widehat{\Xi}_s} - f(-\widehat{\Xi}_s)\big) ds.$$

Proof. The proof is based on Girsanov's transform. Recall that the diffusion X_ϵ satisfies

$$X_\epsilon(t) = B(t) + \mu \int_0^t \frac{ds}{X_\epsilon(s)}\, 1\!\!1_{(|X_\epsilon(s)|\ge\epsilon)}.$$

Define $f_\epsilon(x) = f(x)\mathbb{1}_{(|x|\geq\epsilon)}$ and consider the probability Q_ϵ defined by

$$\frac{dQ_\epsilon}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\Big(\int_0^t f_\epsilon(X_\epsilon(s))dB(s) - \frac{1}{2}\int_0^t f_\epsilon^2(X_\epsilon(s))ds\Big). \tag{5.2}$$

It follows from Girsanov's transform that under Q_ϵ, the process X_ϵ has the same law as Y_ϵ under $\mathbb{P}$. We need to treat the density in (5.2). To this end, define $F_\epsilon(x) = \int_0^x f_\epsilon(y)dy$ for $x \in \mathbb{R}$. Using Itô's formula,

$$\begin{aligned}\int_0^t f_\epsilon(X_\epsilon(s))dB(s) = F_\epsilon(X_\epsilon(t)) - \int_0^t \big(\frac{1}{2}f'(X_\epsilon(s)) + \mu\frac{f(X_\epsilon(s))}{X_\epsilon(s)}\big)\mathbb{1}_{(|X_\epsilon(s)|>\epsilon)}\, ds \\ - \frac{1}{2}\Big(f(\epsilon)L_t^\epsilon(X_\epsilon) - f(-\epsilon)L_t^{-\epsilon}(X_\epsilon)\Big).\end{aligned}$$

It turns out that for any bounded continuous functional Φ, we have

$$\mathbb{E}\,\Phi\big(Y_\epsilon(s), 0\leq s\leq t\big) = \mathbb{E}\Big(\Phi(X_\epsilon(s), 0\leq s\leq t)\,\exp\Big(H_t(X_\epsilon) + K_\epsilon(t)\Big)\Big), \tag{5.3}$$

where $H_t(\cdot)$ is a functional defined by:

$$H_t(X) \stackrel{\text{def}}{=} F(X(t)) - \int_0^t \big(\frac{1}{2}f^2(X(s)) + \frac{1}{2}f'(X(s)) + \mu\frac{f(X(s))}{X(s)}\big)\, ds, \tag{5.4}$$

with $F(x) \stackrel{\text{def}}{=} \int_0^x f(y)dy$, and

$$\begin{aligned}K_\epsilon(t) \stackrel{\text{def}}{=} \big(F_\epsilon(X_\epsilon(t)) - F(X_\epsilon(t))\big) - \frac{1}{2}\Big(f(\epsilon)L_t^\epsilon(X_\epsilon) - f(-\epsilon)L_t^{-\epsilon}(X_\epsilon)\Big) \\ + \int_0^t \big(\frac{1}{2}f^2(X_\epsilon(s)) + \frac{1}{2}f'(X_\epsilon(s)) + \mu\frac{f(X_\epsilon(s))}{X_\epsilon(s)}\big)\mathbb{1}_{(|X_\epsilon(s)|\leq\epsilon)}\, ds.\end{aligned}$$

Notice that $|F_\epsilon(x) - F(x)| \leq 2\epsilon\sup_{|y|\leq 1}|f(y)|$ and by assumption, $f(\epsilon) \geq 0 \geq f(-\epsilon)$. It follows that e^{K_ϵ} is uniformly bounded and converges in probability to 1 (recall that $\mathbb{E}\int_0^t ds\mathbb{1}_{(|X_\epsilon(s)|\leq\epsilon)} \to 0$, see Section 3). The functional $H_t(\cdot)$ is continuous by the hypothesis on f, the family $\{\exp\big(H_t(X_\epsilon)\big), \epsilon > 0\}$ is uniformly integrable since for all $p > 1$,

$$\begin{aligned}\mathbb{E}e^{pH_t(X_\epsilon)} &= \mathbb{E}\exp\Big(p\int_0^t f(X_\epsilon(s))dB(s) - \frac{p}{2}\int_0^t f^2(X_\epsilon(s))ds\Big) \\ &\leq \exp\Big(\frac{p(p-1)t}{2}\sup_{x\in\mathbb{R}} f^2(x)\Big) < \infty.\end{aligned}$$

Then we apply the convergence result for X_ϵ to (5.3) and obtain that as $\epsilon \to 0$, the LHS of (5.3) converges to (recalling (5.4) and that R_δ denotes a Bessel process of dimension $\delta = 1 + 2\mu$)

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\Big(\Phi(R_\delta(s), 0\leq s\leq t)\,\exp\Big(H_t(R_\delta)\Big)\Big) + \frac{1}{2}\mathbb{E}\Big(\Phi(-R_s, 0\leq s\leq t)\,\exp\Big(H_t(-R_\delta)\Big)\Big) \\ &\stackrel{\text{def}}{=} \frac{1}{2}\,\mathbb{Q}^{\mu,f}(\Phi) + \frac{1}{2}\,\widehat{\mathbb{Q}}^{\mu,\widehat{f}}(\Phi).\end{aligned}$$

The rest follows from Girsanov's transform in view of the fact that

$$H_t(R_\delta) = F(R_t) - \int_0^t \big(\frac{1}{2}f^2(R_s) + \frac{1}{2}f'(R_s) + \mu\frac{f(R_s)}{R_s}\big)\, ds$$
$$= \int_0^t f(R_s) d\gamma_s - \frac{1}{2}\int_0^t f^2(R_s) ds,$$

if we denote by γ the Brownian driver for the Bessel process R. □

We give an example: Fix $\lambda > 0$ and consider $h(x) = \lambda \coth(\lambda x)$. Let V_λ be the diffusion taking values in $\mathbb{R}_+$ and starting from 0, with infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \lambda\coth(\lambda x)\frac{d}{dx}.$$

In the terminology of Watanabe [18], V_λ is a Bessel diffusion with drift, of dimension 3 and drift parameter $(\frac{\lambda^2}{2}, 0)$. We recall the absolute continuity relation between the law $\mathbb{P}_0^{(3),\lambda}$ of V_λ and $\mathbb{P}_0^{(3)}$ (see Pitman and Yor [12]):

$$d\mathbb{P}_0^{(3),\lambda}\Big|_{\mathcal{F}_t} = \frac{\sinh(\lambda R_t)}{\lambda R_t}\exp(-\frac{\lambda^2 t}{2})\, d\mathbb{P}_0^{(3)}\Big|_{\mathcal{F}_t}, \qquad t > 0. \tag{5.6}$$

In view of Theorems 5.1 and 1.1, we obtain

Corollary 5.1. *Let $\lambda \geq 0$ and denote by $Q_\epsilon^{(\lambda)}$ the law of the solution of (1.1) associated to the function $h = \lambda\coth(\lambda x)\mathbb{1}_{(|x|\geq\epsilon)}$. When $\epsilon \to 0_+$, the sequence $(Q_\epsilon^{(\lambda)})_\epsilon$ converges in distribution to $\frac{1}{2}\mathbb{P}_0^{(3),\lambda} + \frac{1}{2}\widehat{\mathbb{P}}_0^{(3),\lambda}$, where $\widehat{\mathbb{P}}_0^{(3),\lambda}$ is the distribution of $(-V_\lambda(t), t \geq 0)$.*

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