Tightness of localization and return time in random environment

by

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Summary. Consider a class of diffusions with random potentials which behave asymptotically as Brownian motion. We study the tightness of localization around the bottom of some Brownian valley, and determine the limit distribution of the return time to the origin after a typical time. Via the Skorokhod embedding in random environment, we also solve the return time problem for Sinai's walk.

Keywords. Tightness of localization, return time, valley, diffusion with random potential, Sinai's walk in random environment.

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1. Introduction

Let $\Xi = \{\xi_i, i \in \mathbb{Z}\}$ be a sequence of iid variables taking values in $(0,1)$. A random walk (S_n) in the random environment Ξ (RWRE) can be understood as follows: for each realization of Ξ, $(S_n, n \geq 0)$ is a Markov chain on $\mathbb{Z}$ such that $S_0 = 0$ and

$$\mathbb{P}\Big(S_{n+1} = \begin{cases} i+1 \\ i-1 \end{cases} \Big| S_n = i,\ \Xi\Big) = \begin{cases} \xi_i \\ 1-\xi_i \end{cases}, \tag{1.1}$$

where here and in the sequel, $\mathbb{P}$ denotes the total probability and $\mathbb{E}$ the associated expectation. We always suppose that ξ_i is not a constant random variable to avoid the usual Bernoulli random walk case. The above setup can be realized via product spaces, see Solomon [30], who also obtained the recurrence/transience criteria. Among the random environment problems considered in mathematical physics, the RWRE (S_n) may be the most elementary model and has drawn much attention both from mathematics and from physics, see e.g. Hughes [15].

Kesten et al. [19] characterized the rate of convergence of S_n in the transient case, whereas the recurrent case was solved by Sinai [29] who showed that $S_n / \log^2 n$ converges in law under $\mathbb{P}$ (cf. Section 6 below). We refer to Révész (1990, Part III) for the literature about the studies of (S_n) prior to 1990. For more recent studies, see e.g. [4], [5], [10], [12, 13], [28], [35] and the references therein.

The continuous analogue of RWRE was firstly introduced in Schumacher [27] and Brox [2] as follows: Consider a process $\{V(x);\, x \in \mathbb{R}\}$ with "càdlàg" (i.e. continuous from the right and having limits from the left) and locally bounded trajectories, which plays the role of random potential. We can formally define a process $\{X(t);\, t \geq 0\}$ from the equation

$$\begin{cases} dX(t) = d\beta(t) - \frac{1}{2}V'(X(t))dt \\ X(0) = 0 \end{cases}, \tag{1.2}$$

where $\{\beta(t);\, t \geq 0\}$ is a one–dimensional Brownian motion independent of V. For some physicists points of view about this model (1.2), see Bouchaud et al. [1], and some recent papers [6], [9], [21]. Rigorously speaking, instead of writing the formal derivative of V in (1.2), we should consider X as a diffusion process (conditioning on each realization of V) with generator

$$\frac{1}{2} e^{V(x)} \frac{d}{dx}\Big(e^{-V(x)} \frac{d}{dx}\Big).$$

Taking a Brownian potential $V = W$ in (1.2), where $W = \{W(x), x \in \mathbb{R}\}$ denotes a (two-sided) one–dimensional Brownian motion starting from 0 (independent of β), we denote by $\{X_W(t);\, t \geq 0\}$ the associated diffusion. Brox [2] showed the analogue of Sinai's renormalization for X_W. Furthermore, Tanaka [31, 32] obtained a deeper localization result: if we denote by $b(\log t)$ the bottom of a certain valley of W, whose mathematical definition will be given in (2.6) below, we have

$$X_W(t) - b(\log t) \qquad \text{converges in law as } t \to \infty. \tag{1.3}$$

Furthermore, the convergence in terms of process $\{X_W(t+\cdot) - b(\log t)\}$ was obtained in Tanaka [31,32]. Therefore Sinai's renormalization follows immediately from the self-similarity $b(\log t) \overset{\text{law}}{=} (\log^2 t)\; b(1)$. We refer to Tanaka [33] for the references prior to 1994 about the studies of (1.2) in the one-dimensional and higher dimensional cases. See also some recent papers [3], [14], [16], [22, 23], [34] together with their references.

Here, we generalize the Brownian potential to some potential V which asymptotically behaves as a Brownian motion, and we want to see how this perturbation influences the properties of $X(t)$ such as the localization (1.3). Remark that this perturbation may bring on big fluctuations (see e.g. Shi [28] for the fluctuations of local times).

We assume the following setting: On a possibly enlarged probability space, there exists a coupling of V and a standard two–sided Brownian motion $\{W(y);\, y \in \mathbb{R}\}$, and a constant $\sigma > 0$ such that for all $n \geq 1$,

$$\mathbb{P}\Big(\sup_{|x|\leq n} \big|\, V(x) - \sigma W(x) \,\big| \geq C_1 \,\log n\Big) \leq \frac{C_2}{n^{C_3}}, \tag{1.4}$$

with $C_i > 0$ $(1 \leq i \leq 3)$. For instance, the well–known Komlós–Major–Tusnády [20] strong approximation theorem tells us that (1.4) will be satisfied for V a step function on $\mathbb{R}$ defined as the partial sum of iid (bounded) variables, cf. (6.9).

Firstly, we estimate the deviation of $X(t)$ from $b(\frac{\log t}{\sigma})$:

Theorem 1.1. *Assuming (1.4). For each small $\delta > 0$, there exist a constant $C_4 = C_4(\sigma, C_1, C_2, C_3) > 0$ and $t_0 = t_0(\delta)$ sufficiently large such that for all $t \geq t_0$ and $\lambda \geq 1$, we have*

$$\mathbb{P}\Big(\big|X(t) - b(\frac{\log t}{\sigma})\big| > \lambda\Big) \leq C_4 \frac{\log\log t}{\sqrt{\lambda}} + C_4 (\log t)^{-1+\delta}, \tag{1.5}$$

with $b(\frac{\log t}{\sigma})$ being defined in (2.6) below. Consequently, the family of laws of $\big\{\big(X(t) - b(\frac{\log t}{\sigma})\big) / \log^2 \log t, t \geq 10\big\}$ is tight. In particular,

$$\frac{\sigma^2 X(t)}{\log^2 t} \xrightarrow{(d)} b(1), \qquad t \to \infty. \tag{1.6}$$

Remark 1.1. The uniform estimate (1.5) is useful in further studies of random environment, such applications will be explored elsewhere. In contrast with (1.3), we believe that $X(t) - b(\frac{\log t}{\sigma})$, as $t \to \infty$, does *not* converge in law in general for any potential satisfying (1.4), due to the possible big fluctuations of the potential V (which also partially explain the term of $\log\log t$ in (1.5)). We refer to Kawazu et al. [17] for the tightness of $\{X(t)/\log^\alpha t, t \geq 2\}$ (where $\frac{1}{\alpha} > 0$ is the self-similarity order of the environment), and the corresponding convergence in law in more general settings.

Following previous works [29, 2, 31] on the localization problem, the proof of Theorem 1.1 can be outlined in two steps: Outside an event of small probability, the diffusion hits the bottom $b(\frac{\log t}{\sigma})$ quickly; after this hitting time, the diffusion starting from $b(\frac{\log t}{\sigma})$ behaves as a diffusion with stationary distribution. Instead of using the coupling technique as in Brox [2] and in Tanaka [31], we shall study in the second step the speed of convergence of a diffusion towards its stationary distribution, and the first step can be achieved by using some estimates of hitting times which are presented in Section 3.

Now, we consider the problem of return time at 0 for X. Define

$$D_X(t) \stackrel{\text{def}}{=} \inf\{s > t : X(s) = 0\}, \qquad t \geq 0. \tag{1.7}$$

Notice that under (1.4), $D_X(t)$ is almost surely finite. We intend to characterize the limit distribution of $D_X(t)$ as $t \to \infty$. The localization arguments say that $X(t)$ lives near to the bottom $b(\frac{\log t}{\sigma})$ of a certain Brownian valley; hence the problem of return time consists in studying how long the random walk takes to exit this valley and to hit the origin, which can be viewed as looking for a better understanding of what happens after the localization of $X(t)$. See Fisher et al. [9] and Le Doussal and Monthus [21] for several interesting exponents related to return time problem under physical consideration.

Our result reads as follows:

Theorem 1.2. *Consider a diffusion X with potential V satisfying (1.4). As $t \to \infty$,*

$$\frac{\log\Big(D_X(t) - t\Big)}{\log t} \xrightarrow{(d)} \Lambda(1), \qquad \textit{under } \mathbb{P}, \tag{1.8}$$

where $\Lambda(1)$ is defined in (2.10) below, and admits the following density:

$$\mathbb{P}\Big(\Lambda(1) \in dx\Big) = \Big(\frac{2x}{3}\mathbb{1}_{(0 \leq x < 1)} + \frac{2}{3x^2}\mathbb{1}_{(x \geq 1)}\Big)\, dx. \tag{1.9}$$

Remark 1.2. In the Brownian potential case, the fact that σ does not appear in (1.8) can be understood as follows: Let us denote by $X_{\sigma W}$ the diffusion given in (1.2) with $V = \sigma W$, where W is a Brownian motion independent of β. Then, by the self-similarity (cf. Brox [2]): $\{X_{\sigma W}(t), t \geq 0\} \stackrel{\text{law}}{=} \{\sigma^{-2} X_W(\sigma^4 t), t \geq 0\}$, showing that under logarithm scale, the limit of $\log \big(D_{X_{\sigma W}}(t) - t\big) / \log t$ (if it exists) does not depend on σ.

The proof of Theorem 1.2 relies on some quenched estimates of hitting times, which roughly assert that $D_X(t) - t$, the first hitting time at 0 of a diffusion starting from $X(t)$, is proportional in the logarithm scale to the height of some new Brownian valley, and the density of the limit law $\Lambda(1)$ follows from some explicit computations of the law of this height given in Section 2.

The rest of this paper is organized as follows: In Section 6, we study the return time problem for Sinai's walk by using the Skorokhod embedding in random environment. The estimates of hitting times are stated in Section 3. Theorem 1.1 is proven in Section 4, whereas Section 5 is devoted to proving Theorem 1.2.

Throughout this paper, unless stated otherwise, we always assume (1.4), and the constants $(C_j, 5 \leq j \leq 13)$ may depend on (σ, C_1, C_2, C_3).

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2. Brownian valley

For the sake of notational convenience, we shall constantly write, for any continuous process Y and all $t > 0$

$$\overline{Y}(t) \stackrel{\text{def}}{=} \sup_{0 \leq s \leq t} Y(s), \tag{2.1}$$

$$\underline{Y}(t) \stackrel{\text{def}}{=} \inf_{0 \leq s \leq t} Y(s), \tag{2.2}$$

$$Y^{\#}(t) \stackrel{\text{def}}{=} \sup_{0 \leq s \leq t} (Y(s) - \underline{Y}(s)). \tag{2.3}$$

Consider $\{W(x), x \in \mathbb{R}\}$ a two-sided Brownian motion, starting from 0; i.e. $W_+ \equiv \{W(t), t \geq 0\}$ and $W_- \equiv \{W(-t), t \geq 0\}$ are two independent Brownian motions, both starting from 0. For the Brownian motion W_+, define

$$d_+(r) \stackrel{\text{def}}{=} \inf\{t > 0 : W_+^{\#}(t) > r\}, \qquad r > 0. \tag{2.4}$$

Let $b_+(r)$ be the localization of the minimum of W_+ over $[0, d_+(r)]$:

$$b_+(r) \stackrel{\text{def}}{=} \inf\{0 \leq u \leq d_+(r) : W_+(s) = \underline{W_+}(d_+(r))\}, \qquad r > 0. \tag{2.5}$$

Let us define similarly $d_-(r) \geq 0$ and $b_-(r) \geq 0$, by replacing (W_+, d_+) by (W_-, d_-) in (2.4) and (2.5). In the literature, the triplet $(-d_-(r), 0, d_+(r))$ is called a Brownian valley containing 0 of depth r, its bottom $b(r)$ is defined as follows:

$$b(r) \stackrel{\text{def}}{=} \begin{cases} b_+(r), & \text{if} \quad \overline{W_+}(d_+(r)) < \overline{W_-}(d_-(r)) \\ -b_-(r), & \text{otherwise} \end{cases}, \qquad r > 0. \tag{2.6}$$

The law of $b(r)$ was independently computed by Kesten [18] and Golosov [11]. We define a new valley around $b_+(r)$. Let

$$\theta_+(r) \stackrel{\text{def}}{=} \overline{W_+}(b_+(r)) - W_+(b_+(r)), \qquad r > 0, \tag{2.7}$$

$$\zeta_+(r) \stackrel{\text{def}}{=} \inf\{t > b_+(r) : W_+(t) > \overline{W_+}(b_+(r))\}, \qquad r > 0, \tag{2.8}$$

$$\Lambda_+(r) \stackrel{\text{def}}{=} W_+^{\#}(\zeta_+(r)), \qquad r > 0. \tag{2.9}$$

Remark that $\Lambda_+(r) \geq \theta_+(r)$, and both the strict inequality and the equality can be realized with positive probability. We can think $\Lambda_+(r)$ as the (absolute) height of a valley of W_+ around $b_+(r)$. Let us define similarly $\Lambda_-(r)$ for the negative part W_-. Finally, let

$$\Lambda(r) \stackrel{\text{def}}{=} \begin{cases} \Lambda_+(r), & \text{if} \quad b(r) > 0 \\ \Lambda_-(r), & \text{otherwise} \end{cases}, \qquad r > 0. \tag{2.10}$$

Observe $\Lambda(r) \stackrel{\text{law}}{=} r\,\Lambda(1)$ for a fixed $r > 0$. The main purpose of this section is to identify the law of $\Lambda(1)$. Remark from (2.5) and (2.7), $\theta_+(1) + W_+(d_+(1))) \geq W_+(d_+(1))) - W_+(b_+(1))) = 1$.

Lemma 2.1. *The variable $\big(1 - W_+(d_+(1)))\big)$ is exponentially distributed with parameter 1, and for $v > 0$,*

$$\begin{aligned} &\mathbb{P}\Big(\theta_+(1) \in dx \,\Big|\, W_+(d_+(1))) = 1 - v\Big) \\ &= \Big(\mathbb{1}_{(v \leq x < 1)} \frac{v}{x^2}\, e^v + \mathbb{1}_{(1 \vee v \leq x < 1+v)} (1 + v - x) e^{1+v-x}\Big) dx. \end{aligned} \tag{2.11}$$

Moreover, $\overline{W_+}(d_+(1))$ *and* $W_+^{\#}(b_+(1))$ *are uniformly distributed on* $[0,1]$.

Proof of Lemma 2.1. Let us identify $W(t) \equiv W_+(t)$ for $t \geq 0$. According to Lévy's identity (cf. [25, Theorem (VI.2.3)]), we can write $W(t) - \underline{W}(t) = |B(t)|, t \geq 0$, for some Brownian motion B, and $-\underline{W}(t) = L(t), t \geq 0$, the process of local time at 0 of B. It turns out that

$$d_+(1) = \inf\{t > 0 : |B(t)| > 1\} \stackrel{\text{def}}{=} \sigma(1), \tag{2.12}$$

$$b_+(1) = \sup\{t \leq \sigma(1) : B(t) = 0\} \stackrel{\text{def}}{=} g(\sigma(1)). \tag{2.13}$$

Hence $W_+^{\#}(b_+(1)) = \sup_{0 \leq s \leq g(\sigma(1))} |B(s)|$ is uniformly distributed in $[0,1]$ according to Williams' path-decomposition (cf. [25, Theorem (VII.4.9)]). Moreover, $\overline{W}(b_+(1)) = \sup_{0 \leq s \leq g(\sigma(1))} \big(|B(s)| - L(s)\big)$, and we have

$$W(d_+(1))) = 1 + W(b_+(1)) = 1 - L(\sigma(1)), \tag{2.14}$$

$$\theta_+(1) = \sup_{0 \leq s \leq g(\sigma(1))} \big(|B(s)| - L(s)\big) + L(\sigma(1)), \tag{2.15}$$

$$\begin{aligned} \overline{W}(d_+(1)) &= \max\big(\overline{W}(b_+(1)), W(d_+(1))\big) \\ &= \max\big(\theta_+(1) - 1, 0\big) + W(d_+(1)). \end{aligned} \tag{2.16}$$

Denote by $(\mathbf{e}(t), t \geq 0)$ the excursion process related to B, with Itô measure $\mathbf{n}$ (cf. [25, Chap. XII]). Let us write $\mathbf{m}(\epsilon) \stackrel{\text{def}}{=} \sup\{|\epsilon(s)|, s \geq 0\}$ for a generic excursion ϵ. It is well known that $L(\sigma(1))$ is the first entrance time of $\mathbf{e}(t)$ into $\mathcal{A} \stackrel{\text{def}}{=} \{\epsilon : \mathbf{m}(\epsilon) > 1\}$, hence is exponentially distributed with parameter $\mathbf{n}(\mathbf{m}(\epsilon) > 1) = 1$. Consider the PPP $\widehat{\mathbf{e}}$ the restriction of $\mathbf{e}$ on $\mathcal{A}^c$, the complement of $\mathcal{A}$. The characteristic measure $\widehat{\mathbf{n}}$ of $\widehat{\mathbf{e}}$ is: $\widehat{\mathbf{n}} = \mathbb{1}_{\mathcal{A}^c}\, \mathbf{n}$, and $\widehat{\mathbf{e}}$ is independent of $L(\sigma(1))$. It turns out that for $0 < y < 1$ and $v > 0$,

$$\begin{aligned} &\mathbb{P}\Big(\sup_{0 \leq s \leq g(\sigma(1))} \big(|B(s)| - L(s)\big) < y \,\big|\, L(\sigma(1)) = v \Big) \\ &= \mathbb{P}\Big(\sup_{0 \leq s < v} \big(\mathbf{m}(\widehat{\mathbf{e}}(s)) - s\big) < y \,\big|\, L(\sigma(1)) = v \Big) \\ &= \mathbb{P}\Big(\sup_{0 \leq s < v} \big(\mathbf{m}(\widehat{\mathbf{e}}(s)) - s\big) < y \Big) \\ &= \exp\Big(- \int_0^v ds \int_{\mathbf{m}(\epsilon) \leq 1} \mathbf{n}(d\epsilon) \Big(1 - \mathbb{1}_{(\mathbf{m}(\epsilon) - s \leq y)}\Big)\Big) \\ &= \exp\Big(- \int_0^v ds \int_0^1 \frac{dz}{z^2} \mathbb{1}_{(z \geq s+y)}\Big) \\ &= \begin{cases} \frac{y}{v+y}\, e^v, & \text{if } \ v + y < 1, \\ y e^{1-y}, & \text{if } \ v + y \geq 1, \end{cases} \end{aligned} \tag{2.17}$$

where the third equality follows from the exponential formula for the multiplicative function $v \to \mathbb{1}_{(\sup_{0\le s<v}(\mathbf{m}(\widehat{\mathbf{e}}(s))-s)<y)}$. In view of (2.14)–(2.16), some lines of elementary computations based on (2.17) show (2.11) and prove that $\overline{W_+}(d_+(1))$ is uniformly distributed. $\square$

The following description of the law of the process $\{W_+(x+b_+(r)), x \ge -b_+(r)\}$ is due to Tanaka [31] ((i) is immediate from the strong Markov property together with the fact that $W_+(d_+(r)) - W_+(b_+(r)) = r$, and we have rewritten Tanaka [31]'s result as (iii) by using Pitman's identity for Bessel process of dimension 3 (cf. [25, Theorem (VI.3.5)])):

Fact 2.1 (Tanaka [31]). Recall (2.4) and (2.5). Fix $r > 0$ and denote by $R_3 = \{R_3(t), t \ge 0\}$ a Bessel process of dimension 3, starting from 0. The following three ingredients are independent, and

(i) The process $\{W_+(x+d_+(r)) - W_+(b_+(r)), x \ge 0\}$ behaves as a Brownian motion starting from r;

(ii) The law of $\{W_+(x+b_+(r)) - W_+(b_+(r)), 0 \le x \le d_+(r) - b_+(r)\}$ is that of R_3 till its first hitting time at r;

(iii) The law of $\{W_+(b_+(r)-x) - W_+(b_+(r)), 0 \le x \le d_+(r) - b_+(r)\}$ is that of R_3 till $\wp(r)$, where $\wp(r) \stackrel{\text{def}}{=} \sup\{0 < t < \varsigma(r) : R_3(t) = J_{R_3}(t)\}$, with $\varsigma(r) \stackrel{\text{def}}{=} \inf\{t > 0 : R_3(t) - J_{R_3}(t) = r\}$ and $J_{R_3}(t) \stackrel{\text{def}}{=} \inf\{R_3(s) : s \ge t\}$.

We end this section by proving (1.9), which we state as a lemma:

Lemma 2.2. *Recall (2.10). We have*

$$\mathbb{P}\Big(\Lambda(1) \in dx\Big) = \Big(\frac{2x}{3}\mathbb{1}_{(0\le x<1)} + \frac{2}{3x^2}\mathbb{1}_{(x\ge 1)}\Big)\,dx. \tag{2.18}$$

Proof of Lemma 2.2. From (2.6) and (2.10), we have by symmetry that for all $z > 0$,

$$\begin{aligned}\mathbb{P}\Big(\Lambda(1) > z\Big) &= 2\,\mathbb{P}\Big(\Lambda_+(1) > z; \overline{W_+}(d_+(1)) < \overline{W_-}(d_-(1))\Big)\\ &= 2\,\mathbb{E}\Big(\big(1 - \overline{W_+}(d_+(1))\big)\,\mathbb{1}_{(\Lambda_+(1)>z)}\Big),\end{aligned} \tag{2.19}$$

using the fact that $\overline{W_-}(d_-(1)) \stackrel{\text{law}}{=} \overline{W_+}(d_+(1))$ is uniformly distributed (cf. Lemma 2.1). Write $\widehat{W}(t) \stackrel{\text{def}}{=} W_+(b_+(1)+t) - W_+(b_+(1)), t \ge 0$. Recall (2.7)–(2.9). We have

$$\zeta_+(1) - b_+(1) = \inf\{t > 0 : \widehat{W}(t) > \theta_+(1)\} \stackrel{\text{def}}{=} \widehat{\sigma}(\theta_+(1)), \tag{2.20}$$

$$\Lambda_+(1) = \theta_+(1) + \sup\{-\widehat{W}(s) : 0 \le s \le \widehat{\sigma}(\theta_+(1))\} \ge \theta_+(1), \tag{2.21}$$

where $\widehat{\sigma}(r) \stackrel{\text{def}}{=} \inf\{t > 0 : \widehat{W}(t) > r\}$ denotes the first hitting time at r of $\widehat{W}$. Remark that $d_+(1) - b_+(1) = \widehat{\sigma}(1)$. According to Fact 2.1, $\{\widehat{W}(t), 0 \le t \le \widehat{\sigma}(1)\}$ is a three-dimensional Bessel process starting from 0, and independent of $\theta_+(1)$; The process $\widehat{W}$ after $\widehat{\sigma}(1)$, $\{\widehat{W}(t+\widehat{\sigma}(1)), t \ge 0\}$, is a Brownian motion starting from 1, and is independent of $\sigma\{\widehat{W}(t), 0 \le t \le \widehat{\sigma}(1)\,;\theta_+(1)\}$. Combined this with (2.21), we have

$$\begin{aligned}
&\mathbb{P}\Big(\Lambda_+(1) > y \,\big|\, \theta_+(1) = x > 1\Big) \\
&\quad = \mathbb{P}\Big(\text{a Brownian motion starting from 1 hits } -(y-x) \text{ before } x\Big) \\
&\quad = \frac{x-1}{y}, \qquad y \ge x, \qquad (2.22)
\end{aligned}$$

and $\Lambda_+(1) = \theta_+(1)$ on $\{\theta_+(1) \le 1\}$ (remarking from (2.22) that $\mathbb{P}\big(\Lambda_+(1) = \theta_+(1) \,|\theta_+(1) = x > 1\big) = 1/x$). Let $0 < z < 1$. Using Lemma 2.1, since $\overline{W_+}(d_+(1)) = W_+(d_+(1))$ on the event $\{\Lambda_+(1) \le z\} = \{\theta_+(1) \le z \le 1\}$, we deduce from (2.19) and (2.11) that

$$\begin{aligned}
\mathbb{P}\Big(\Lambda(1) > z\Big) &= 1 - 2\mathbb{E}\big(1 - \overline{W_+}(d_+(1))\big)\, \mathbb{1}_{(\Lambda_+(1) \le z)} \\
&= 1 - 2\mathbb{E}\big(1 - W_+(d_+(1))\big)\, \mathbb{1}_{(\theta_+(1) \le z)} \\
&= 1 - 2\int_0^z dv v e^{-v} \int_v^z dx \frac{v}{x^2} e^v \\
&= 1 - \frac{1}{3} z^2, \qquad\qquad 0 < z \le 1. \qquad (2.23)
\end{aligned}$$

For $z > 1$, it follows from (2.16), (2.19) and (2.22) that

$$\begin{aligned}
\mathbb{P}\Big(\Lambda(1) > z\Big) &= 2\mathbb{E}\Big(\big[2 - W_+(d_+(1)) - \theta_+(1)\big]\Big(\mathbb{1}_{(\theta_+(1) > z)} + \mathbb{1}_{(1 < \theta_+(1) \le z)} \frac{\theta_+(1) - 1}{z}\Big)\Big) \\
&= \frac{2}{3z}, \qquad (2.24)
\end{aligned}$$

after some lines of computations based on (2.11); hence Lemma 2.2 follows from (2.23) and (2.24). □

3. Hitting times

Throughout this section, we assume (1.4). Consider a finite stopping time $\Upsilon_0 \ge 0$ (with respect to $\{\sigma(X(s), 0 \le s \le t;\, V, W), t \ge 0\}$). Define

$$\mathbb{X}(t) \stackrel{\text{def}}{=} X(t + \Upsilon_0) - X(\Upsilon_0), \qquad t \ge 0, \qquad (3.1)$$

$$\mathbb{V}(x) \stackrel{\text{def}}{=} V(x + X(\Upsilon_0)) - V(X(\Upsilon_0)), \qquad x \in \mathbb{R}, \qquad (3.2)$$

$$\mathbb{W}(x) \stackrel{\text{def}}{=} W(x + X(\Upsilon_0)) - W(X(\Upsilon_0)), \qquad x \in \mathbb{R}, \qquad (3.3)$$

$$\begin{cases} \mathbb{V}_+(t) \stackrel{\text{def}}{=} \mathbb{V}(t), & \mathbb{V}_-(t) \stackrel{\text{def}}{=} \mathbb{V}(-t), \\ \mathbb{W}_+(t) \stackrel{\text{def}}{=} \mathbb{W}(t), & \mathbb{W}_-(t) \stackrel{\text{def}}{=} \mathbb{W}(-t), \end{cases} \qquad t \ge 0. \qquad (3.4)$$

When $\Upsilon_0 \equiv 0$, we are simply considering the process X. We shall also simply write $\mathbb{W}(t), \mathbb{V}(t)$ instead of $\mathbb{W}_+(t), \mathbb{V}_+(t)$ if there is no risk of confusion. Notice that the process $\mathbb{X}$ is related to its potential $\mathbb{V}$ the same way X is to V. For any càdlàg process $(Y(t), t \geq 0)$, define

$$\Theta_Y(r) \stackrel{\text{def}}{=} \begin{cases} \inf\{t > 0 : Y(t) > r\}, & \text{if} \quad r \geq 0, \\ \inf\{t > 0 : Y(t) < r\}, & \text{if} \quad r < 0. \end{cases} \tag{3.5}$$

The following estimate of $\Theta_{\mathbb{X}}(r)$ is in fact a quenched-type result, with the error probability estimated under the total probability $\mathbb{P}$, and may be of independent interest (for instance, in the study of the increments of (X_t) and (S_n)):

Lemma 3.1. *Recall (2.1), (2.3), and (3.1)–(3.5). Under (1.4), there exists a constant $C_5(\sigma, C_1, C_2, C_3) > 0$ such that for every $r \geq C_5$, there exists a measurable event $E_1 = E_1(r)$ such that*

$$\mathbb{P}\Big(E_1^c\Big) \leq \exp\Big(-\log^2 r\Big), \tag{3.6}$$

and on $E_1 \cap \{|X(\Upsilon_0)| \leq \exp\big(\log^2 r\big)\}$, we have

$$\log \Theta_{\mathbb{X}}(r) \leq \sigma \max\Big(\mathbb{W}_+^{\#}(r),\, \mathbb{U}_-\big(\overline{\mathbb{W}_+}(r) + \log^5 r\big)\Big) + 2\log^5 r, \tag{3.7}$$

whereas on $E_1 \cap \{|X(\Upsilon_0)| \leq \exp\big(\log^2 r\big)\} \cap \{\overline{\mathbb{W}_+}(r) \geq 2\log^5 r\}$, we have

$$\log \Theta_{\mathbb{X}}(r) \geq \sigma \max\Big(\mathbb{W}_+^{\#}(r) - \log^5 r,\, \mathbb{U}_-\big(\overline{\mathbb{W}_+}(r) - \log^5 r\big)\Big), \tag{3.8}$$

where $\mathbb{U}_-(x) \stackrel{\text{def}}{=} x + \sup\{-\mathbb{W}_-(s), 0 \leq s \leq \Theta_{\mathbb{W}_-}(x)\}$, for all $x > 0$. The same estimates hold for $\Theta_{\mathbb{X}}(-r)$ by interchanging in all the above the roles of $\mathbb{W}_+$ and of $\mathbb{W}_-$.

Proof of Lemma 3.1. Consider $r \gg 1$ in this proof. The case of $\Upsilon_0 = 0$ has been proven in [13], but the arguments presented there will also apply to the general case. In fact, although the process $\mathbb{W}$ is not a Brownian motion, but we deduce from (1.4) that for $C_6 \stackrel{\text{def}}{=} 1 + 3/C_3$

$$\begin{aligned} &\mathbb{P}\Big(\sup_{|x| \leq \exp\left(C_6 \log^2 r\right)} \big|\mathbb{V}(x) - \sigma\mathbb{W}(x)\big| \geq \log^3 r;\, |X(\Upsilon_0)| \leq \exp\big(\log^2 r\big)\Big) \\ &\leq \mathbb{P}\Big(\sup_{|x| \leq 2\exp\left(C_6 \log^2 r\right)} \big|V(x) - \sigma W(x)\big| \geq \log^3 r\Big) \\ &\leq \exp\Big(-3\log^2 r\Big). \end{aligned} \tag{3.9}$$

An advantage of working with $\mathbb{W}$ rather than with $\mathbb{V}$ is that the modulus of continuity of $\mathbb{W}$ is easier to control, since

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{0\le u\le v\le r; v-u\le \log r} \big|\mathbb{W}(u)-\mathbb{W}(v)\big| > \log^3 r;\ |X(\Upsilon_0)|\le \exp\big(\log^2 r\big)\Big)\\
&\le \mathbb{P}\Big(\sup_{|u|,|v|\le 2\exp\left(\log^2 r\right); |v-u|\le \log r} \big|W(u)-W(v)\big| > \log^3 r\Big)\\
&\le \exp\Big(-3\log^2 r\Big), \qquad (3.10)
\end{aligned}
$$

by using the estimate for Brownian oscillations (cf. Csörgő and Révész [7, pp. 24]). Also the range of $\mathbb{W}$ is not too big; indeed, we have

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{0\le u\le r} \big|\mathbb{W}(u)\big| \ge \exp\big(\log^2 r\big);\ |X(\Upsilon_0)|\le \exp\big(\log^2 r\big)\Big)\\
&\le \mathbb{P}\Big(\sup_{|u|\le 2\exp\left(\log^2 r\right)} \big|W(u)\big| \ge \frac{1}{2}\exp\Big(\log^2 r\Big)\Big)\\
&\le \exp\Big(-3\log^2 r\Big), \qquad (3.11)
\end{aligned}
$$

by using the Gaussian tail estimate.

From (3.9)–(3.11), the arguments used in [13] to treat the case of $\Upsilon_0 = 0$ work well here after some slight modifications. For the sake of completeness and to avoid tedious details, we shall only briefly point out the arguments, and the explicit estimates will be omitted. Recall (3.1)–(3.3), using the Feller space-change representation of a real-valued diffusion, we obtain the following version of $\mathbb{X}$: there exists a one-dimensional Brownian motion γ starting from 0 such that

$$
\mathbb{X}(t) = \mathbb{A}^{-1}(\gamma(\mathbb{T}^{-1}(t))), \qquad t\ge 0, \qquad (3.12)
$$

where, $\mathbb{A}^{-1}$, $\mathbb{T}^{-1}$ denote the inverse of the increasing processes $\mathbb{A}$ and $\mathbb{T}$ defined by

$$
\mathbb{A}(x) \stackrel{\text{def}}{=} \int_0^x dy \exp\Big(\mathbb{V}(y)\Big), \qquad x\in\mathbb{R}, \qquad (3.13)
$$

$$
\mathbb{T}(t) \stackrel{\text{def}}{=} \int_0^t ds \exp\Big(-2\mathbb{V}(\mathbb{A}^{-1}(\gamma(s)))\Big), \qquad t\ge 0. \qquad (3.14)
$$

The inverses $\mathbb{A}^{-1}$, $\mathbb{T}^{-1}$ are well-defined since a.s., $\mathbb{A}(\infty) = \infty$ and $\mathbb{A}(-\infty) = -\infty$ from (1.4) and (3.2)–(3.3). Observe that by using the Markov property of X conditioning on the environment (V, W), the Brownian motion γ is independent of $X(\Upsilon_0)$. Write

$\rho(x) \stackrel{\text{def}}{=} \inf\{t > 0 : \gamma(t) > x\}$, the first hitting time at $x > 0$ of γ, and $\{L(t,x), t \geq 0, x \in \mathbb{R}\}$ the family of local times of γ. Using the occupation time formula,

$$\Theta_{\mathbb{X}}(r) = \Big(\int_0^r + \int_{-\infty}^0 \Big) ds\, e^{-\mathbb{V}(s)}\, L\big(\rho(\mathbb{A}(r)), \mathbb{A}(s)\big) \stackrel{\text{def}}{=} I_1(r) + I_2(r), \tag{3.15}$$

with obvious definitions. Let us first study $I_1(r)$. Using the Ray-Knight theorem (cf. [25, pp. 433]), we have

$$I_1(r) = \int_0^r ds\, e^{-\mathbb{V}(s)}\, \mathbb{A}(r)\, R^2(1 - \mathbb{A}(s)/\mathbb{A}(r)), \tag{3.16}$$

for a two-dimensional Bessel process R starting from 0, and independent of $(\mathbb{V}, \mathbb{W})$. From (3.9)–(3.11), we can deduce from (3.16) the existence of E_2 satisfying $\mathbb{P}\Big(E_2^c\Big) \leq \exp\Big(- 2\log^2 r\Big)$, and on $E_2 \cap \{|X(\Upsilon_0)| \leq \exp\big(\log^2 r\big)\}$,

$$\Big| \log I_1(r) - \sigma \mathbb{W}^{\#}(r) \Big| \leq \log^5 r. \tag{3.17}$$

We use the same idea to study $I_2(r)$. From the second Ray-Knight theorem [25, pp. 439], there exists a squared 0-dimensional Bessel process Z independent of $(\mathbb{V}, \mathbb{W})$, with $Z(0)$ exponentially distributed with parameter $1/2$, such that

$$I_2(r) = \mathbb{A}(r) \int_0^{\kappa(r)} ds e^{-\mathbb{V}_-(s)}\, Z(\mathbb{A}_-(s)/\mathbb{A}(r)), \tag{3.18}$$

where $\mathbb{A}_-(t) \stackrel{\text{def}}{=} \int_0^t du \exp\big(\mathbb{V}_-(u)\big)$ (recalling (3.4)), for $t \geq 0$, and

$$\kappa(r) \stackrel{\text{def}}{=} \inf\{s > 0 : Z(\mathbb{A}_-(s)/\mathbb{A}(r)) = 0\}. \tag{3.19}$$

By using (3.9) and (3.10), there exists an event E_3 such that $\mathbb{P}\Big(E_3^c\Big) \leq \exp\big(- 2\log^2 r\big)$, and on $E_3 \cap \{|X(\Upsilon_0)| \leq \exp\big(\log^2 r\big)\}$,

$$\big| \log \mathbb{A}(r) - \overline{\mathbb{W}}(r) \big| \leq 3 \log^3 r. \tag{3.20}$$

Similarly, we can bound $\log \mathbb{A}_-(r)$ by $\overline{\mathbb{W}_-}(r)$. Going back to (3.19), we can show that there exists an event E_4 such that $\mathbb{P}\Big(E_4^c\Big) \leq \exp\big(- 2\log^2 r\big)$, and on $E_4 \cap \{|X(\Upsilon_0)| \leq \exp\big(\log^2 r\big)\}$,

$$\Theta_{\mathbb{W}_-}((\overline{\mathbb{W}}(r) - \log^5 r)^+) \leq \kappa(r) \leq \Theta_{\mathbb{W}_-}(\overline{\mathbb{W}}(r) + \log^5 r), \tag{3.21}$$

where $\Theta_{\mathbb{W}_-}(r)$ denotes the hitting time of r by $\mathbb{W}_-$ (recalling (3.5)). Since in the logarithm scale, the term $Z(\mathbb{A}_-(s)/\mathbb{A}(r))$ does not contribute much to $I_2(r)$, applying (3.20) and (3.21) to (3.18), we get the existence of E_5 such that $\mathbb{P}\Big(E_5^c\Big) \le \exp\big(-2\log^2 r\big)$, and on $E_5 \cap \{|X(\Upsilon_0)| \le \exp\big(\log^2 r\big)\} \cap \{\overline{\mathbb{W}}(r) > 2\log^5 r\}$,

$$\sigma \mathbb{U}_-\big(\overline{\mathbb{W}}(r) - \log^5 r\big) \le \log I_2(r) \le \sigma \mathbb{U}_-\big(\overline{\mathbb{W}}(r) + \log^5 r\big), \tag{3.22}$$

which in view of (3.17) implies Lemma 3.1. □

Using the monotonicity and (3.10), we can obtain the following strengthened form of Lemma 3.1, which will be applied in the sequel to $\Upsilon_0 \equiv 0$ in the proof of Theorem 1.1, and to $\Upsilon_0 \equiv t$ (some constant) in the proof of Theorem 1.2:

Proposition 3.2. *Recall (2.1)–(2.3) and (3.1)–(3.5). Assuming (1.4), for every $r_1 \ge C_7(\sigma, C_1, C_2, C_3)$ (C_7 being some large constant), there exists an event $E_6 = E_6(r_1)$ such that for all $r \ge r_1$, on $E_6 \cap \{|X(\Upsilon_0)| \le \exp\big(\log^2 r\big)\} \cap \{\overline{\mathbb{W}_+}(r) \ge 3\log^5 r\}$, we have*

$$\log \Theta_{\mathbb{X}}(r) \le \sigma \max\Big(\mathbb{W}_+^{\#}(r),\ \mathbb{U}_-\big(\overline{\mathbb{W}_+}(r) + 2\log^5 r\big)\Big) + 3\log^5 r, \tag{3.23}$$

$$\log \Theta_{\mathbb{X}}(r) \ge \sigma \max\Big(\mathbb{W}_+^{\#}(r) - 2\log^5 r,\ \mathbb{U}_-\big(\overline{\mathbb{W}_+}(r) - 2\log^5 r\big)\Big), \tag{3.24}$$

where $\mathbb{U}_-(x)$ has been defined in Lemma 3.1, and

$$\mathbb{P}\Big(E_6^c\Big) \le \exp\Big(-\frac{1}{2}\log^2 r_1\Big). \tag{3.25}$$

Similar estimates (3.23) and (3.24) hold for $\Theta_{\mathbb{X}}(-r)$ by interchanging $\mathbb{W}_+$ and $\mathbb{W}_-$.

Proof of Proposition 3.2. Applying Lemma 3.1 to $r = k \ge r_1 \ge C_7 \ge C_5$ gives the existence of $E_7(k)$ such that

$$\mathbb{P}\Big(E_7^c(k)\Big) \le \exp\Big(-\log^2 k\Big),$$

and (3.7)–(3.8) hold on $E_7(k) \cap \{|\mathbb{X}(\Upsilon_0)| \le \exp\big(\log^2 k\big)\}$. It follows from (3.10) that $\mathbb{P}\Big(E_8^c(k)\Big) \le \exp\Big(-3\log^2 k\Big)$, where

$$E_8^c(k) \stackrel{\text{def}}{=} \Big\{\sup_{0 \le u \le v \le k+1;\, v-u \le 1} \big|\mathbb{W}_+(u) - \mathbb{W}_+(v)\big| > \log^3(k+1)\Big\} \cap \Big\{|X(\Upsilon_0)| \le e^{\log^2(k+1)}\Big\},$$

Define

$$E_6(r_1) \stackrel{\text{def}}{=} \bigcap_{k \ge [r_1]} \Big(E_7(k) \cap E_8(k)\Big),$$

and (3.25) follows for sufficiently large C_7. For all $r \geq r_1$, $k \leq r < k+1$ for some k, we have from the monotonicity that on $E_6(r_1) \cap \{|X(\Upsilon_0)| \leq \exp\left(\log^2 r\right)\}$,

$$\begin{aligned}
\log \Theta_{\mathbb{X}}(r) &\leq \log \Theta_{\mathbb{X}}(k+1) \\
&\leq \sigma \max\left(\mathbb{W}_+^{\#}(k+1), \mathbb{U}_-\left(\overline{\mathbb{W}_+}(k+1) + \log^5(k+1)\right)\right) + 2\log^5(k+1) \\
&\leq \sigma \max\left(\mathbb{W}_+^{\#}(k), \mathbb{U}_-\left(\overline{\mathbb{W}_+}(k) + 2\log^5 k\right)\right) + 3\log^5 k,
\end{aligned}$$

provided that C_7 is sufficiently large, and (3.23) follows. (3.24) can be proven in the same way by using (3.8) instead of (3.7). □

4. Proof of Theorem 1.1

Firstly, we state a preliminary lemma to estimate the speed of convergence in law of a diffusion towards its stationary distribution. Without any loss of generality, we suppose that Y is of natural scale.

Lemma 4.1. *Let Y be a (regular) diffusion on $\mathbb{R}$, on its natural scale and starting from $y \in \mathbb{R}$; Its Feller space-change representation is:*

$$Y(t) \stackrel{\text{def}}{=} B(a^{-1}(t)), \qquad t \geq 0, \tag{4.1}$$

$$a(t) \stackrel{\text{def}}{=} \int_0^t h(B(u))\, du, \qquad h > 0, \;\; m \stackrel{\text{def}}{=} \int_{\mathbb{R}} h(x) dx < \infty, \tag{4.2}$$

where B is a one-dimensional Brownian motion starting from y, and a^{-1} denotes the inverse of the increasing function a. If we denote by $\nu(dx) \stackrel{\text{def}}{=} \frac{h(x)}{m} dx$ the stationary distribution of Y, we have for all $t > 0$

$$\|\mathcal{L}(Y(t)) - \nu\| \leq \nu\{x : |x-y| \geq t^{1/4}\} + \left(1 + \frac{C_8 m}{t^{1/4}}\right) t^{-1/4}, \tag{4.3}$$

where $\|\cdot\|$ denotes the total variation of the measure and $\mathcal{L}(Y(t))$ the law of $Y(t)$, and $C_8 > 0$ is some universal constant.

Proof of Lemma 4.1. The proof uses the coupling method, which we learnt from Rogers and Williams [26, pp.303]. It is well-known that ν defined above is the stationary distribution of Y (see e.g. [26, pp. 277, 303]). Let $\widetilde{Y}$ be an independent diffusion with the same semigroup as Y, and with the initial distribution ν. Let

$$T_{co} \stackrel{\text{def}}{=} \inf\{t > 0 : Y(t) = \widetilde{Y}(t)\},$$

be the coupling time of Y and $\widetilde{Y}$. The coupling inequality says

$$\|\mathcal{L}(Y(t)) - \nu\| \le \mathbb{P}\Big(T_{co} > t\Big), \tag{4.4}$$

since $\widetilde{Y}(t) \overset{\text{law}}{=} \nu$. Let us bound the coupling time T_{co}. Applying the Dubins-Schwarz' representation to the two independent continuous martingales $Y(t)$ and $\widetilde{Y}(t)$ gives

$$M(t) \overset{\text{def}}{=} Y(t) - \widetilde{Y}(t) = \gamma(\langle M\rangle_t) - \widetilde{Y}(0) + y,$$

where γ is a one-dimensional Brownian motion starting from 0, and $\langle M\rangle_t = \langle Y\rangle_t + \langle\widetilde{Y}\rangle_t \ge \langle Y\rangle_t = a^{-1}(t)$. Denote by $\Theta_\gamma(x)$ the hitting time of $x \in \mathbb{R}$ by γ. It follows that

$$\begin{aligned}
\mathbb{P}\Big(T_{co} > t\Big) &= \mathbb{P}\Big(\Theta_\gamma(-(\widetilde{Y}(0) - y)) > \langle M\rangle_t\Big) \\
&\le \mathbb{P}\Big(\Theta_\gamma(-(\widetilde{Y}(0) - y)) > t\Big) + \mathbb{P}\Big(\langle M\rangle_t < t\Big) \\
&\le \mathbb{P}\Big(|\widetilde{Y}(0) - y| > t^{1/4}\Big) + \mathbb{P}\Big(\Theta_\gamma(t^{1/4}) > t\Big) + \mathbb{P}\Big(a^{-1}(t) < t\Big).
\end{aligned} \tag{4.5}$$

Using (4.2), the third probability term in (4.5) is

$$\mathbb{P}\Big(a(t) > t\Big) = \mathbb{P}\Big(\int_{\mathbb{R}} dx h(x) L_\gamma(t,x) > t\Big) \le \mathbb{P}\Big(m L^*_\gamma(t) > t\Big) \le \frac{m\mathbb{E}L^*_\gamma(1)}{\sqrt{t}}, \tag{4.6}$$

where $L_\gamma(t,x)$ denotes the local time of γ at level x up to time t, and its supremum $L^*_\gamma(t) \overset{\text{def}}{=} \sup_{x\in\mathbb{R}} L_\gamma(t,x) \overset{\text{law}}{=} \sqrt{t}L^*_\gamma(1)$. Take $C_8 \overset{\text{def}}{=} \mathbb{E}L^*_\gamma(1) < \infty$. Since $\Theta_\gamma(x) \overset{\text{law}}{=} x^2/\mathcal{N}^2$, for a standard centered Gaussian variable $\mathcal{N}$ of variance 1, we deduce from the boundedness of the Gaussian density that the second probability term in (4.5) is bounded by $t^{-1/4}$, which in view of (4.6) implies (4.3), as desired. □

Proof of Theorem 1.1. Fix a small $\delta > 0$. We shall prove the existence of a constant $C_9 > 0$ and $t_0 = t_0(\delta) > 0$ such that for all $\lambda \ge 1$ and $t \ge t_0$,

$$\mathbb{P}\Big(\{|X(t) - b(\frac{\log t}{\sigma})| > \lambda\} \cap \{b(\frac{\log t}{\sigma}) > 0\}\Big) \le C_9 \frac{\log\log t}{\sqrt{\lambda}} + C_9\big(\log t\big)^{-1+\delta}. \tag{4.7}$$

The proof of (4.7) is divided into two steps. Consider large $t \ge t_0 \gg 1$. Recall the notation introduced in Section 2 and (3.1)–(3.5). For simplification, we write in this proof:

$$F \overset{\text{def}}{=} \Big\{b(\ell) > 0\Big\} = \Big\{\overline{W_+}(d_+(\ell)) < \overline{W_-}(d_-(\ell))\Big\}, \qquad \text{with } \ell \overset{\text{def}}{=} \frac{\log t}{\sigma}. \tag{4.8}$$

First step: Show that under F, the process X will hit $b_+(\ell)$ quickly before t with large probability (i.e. (4.13)–(4.14) below). The idea is to work with some "favorite" events. Define

$$\begin{aligned}
\epsilon &\stackrel{\text{def}}{=} \frac{\log^6 \ell}{\ell}, \\
E_9 &\stackrel{\text{def}}{=} \Big\{\ell^\delta \le \overline{W_+}(d_+(\ell)) < \overline{W_-}(d_-(\ell)) - \epsilon\ell\Big\}, \\
E_{10} &\stackrel{\text{def}}{=} \Big\{W_+^\#(b_+(\ell)) < (1-\epsilon)\ell\Big\} \cap \Big\{W_-^\#(b_-(\ell)) < (1-\epsilon)\ell\Big\}, \\
E_{11} &\stackrel{\text{def}}{=} \Big\{\ell^\delta \le b_+(\ell) \le d_+((1+\epsilon)\ell) \le \ell^3\Big\} \cap \Big\{d_-(\ell) \le \ell^3\Big\}, \\
E_{12} &\stackrel{\text{def}}{=} \cap_{j=9}^{11} E_j,
\end{aligned}$$

where here and in the sequel,

$$U_-(x) \stackrel{\text{def}}{=} x + \sup\{-W_-(s) : 0 \le s \le \Theta_{W_-}(x)\}, \qquad x > 0. \tag{4.9}$$

with $\Theta_{W_-}(x) \stackrel{\text{def}}{=} \inf\{u > 0 : W_-(u) > x\}$ being defined in (3.5). Let us show

$$E_{12} \subset \Big\{U_-\big(\overline{W_+}(b_+(\ell)) + 2\log^5 b_+(\ell)\big) < (1 - \frac{\epsilon}{2})\ell\Big\}. \tag{4.10}$$

To see (4.10), since $\overline{W_-}(d_-(\ell)) = \max\big(\overline{W_-}(b_-(\ell)), W_-(d_-(\ell))\big)$, on E_{12}, there are only two possibilities:

Either $\overline{W_-}(b_-(\ell)) > \overline{W_+}(d_+(\ell)) + \epsilon\ell$, implying that $U_-\big(\overline{W_+}(b_+(\ell)) + 2\log^5 b_+(\ell)\big) \le U_-\big(\overline{W_-}(b_-(\ell)) - \epsilon\ell/2\big) \le W_-^\#(b_-(\ell)) \le (1-\epsilon)\ell$;

Or $W_-(d_-(\ell)) > \overline{W_+}(d_+(\ell)) + \epsilon\ell$, implying that $U_-\big(\overline{W_+}(b_+(\ell)) + 2\log^5 b_+(\ell)\big) \le U_-\big(W_-(d_-(\ell)) - \epsilon\ell/2\big) \le U_-\big(W_-(d_-(\ell))\big) - \epsilon\ell/2 = (1-\epsilon/2)\ell$, by using the fact that $U_-(x) \le U_-(x+y) - y$ for $x, y \ge 0$, showing (4.10), as desired.

Now, we show that conditioning on F, E_{12} has large probability in the sense that

$$\mathbb{P}\Big(F \cap E_{12}^c\Big) \le 3\ell^{1-\delta}, \qquad \ell \ge \ell_0, \tag{4.11}$$

where here and in the sequel, ℓ_0 denotes some large constant depending on $(\sigma, C_1, C_2, C_3, \delta)$ whose value can change from one line to the next. In fact, Lemma 2.1 tells us that $\overline{W_+}(d_+(1))$, $W_+^\#(b_+(1))$ are uniformly distributed in $[0,1]$, and since W_- and W_+ are independent, we have by scaling that $\mathbb{P}\Big(F \cap E_9^c\Big) \le \mathbb{P}\Big(\overline{W_+}(d_+(1)) < \ell^{-(1-\delta)}\Big) + \mathbb{P}\Big(\overline{W_-}(d_-(1)) - \epsilon \le \overline{W_+}(d_+(1)) < \overline{W_-}(d_-(1))\Big) \le \ell^{-(1-\delta)} + \epsilon \le 2\ell^{-(1-\delta)}$, and $\mathbb{P}\Big(E_{10}^c\Big) \le 2\epsilon$. Finally, since $b_+(\ell), d_+(\ell)$ are self-similar of order 2, we obtain:

$$\mathbb{P}\Big(E_{11}^c\Big) \le \mathbb{P}\Big(b_+(1) \le \ell^{\delta-2}\Big) + 2\mathbb{P}\Big(d_+(1+\epsilon) \ge \ell\Big), \tag{4.12}$$

and it is easy to estimate the two probabilities in RHS of (4.12). Firstly, using (2.12) and scaling, we have $\mathbb{E}d_+(1+\epsilon) = (1+\epsilon)^2$, which implies that $\mathbb{P}\Big(d_+(1+\epsilon) \ge \ell\Big) \le (1+\epsilon)^2/\ell$; Using (2.13) and Williams' path-decomposition for the Brownian motion B at $g(\sigma(1))$ (cf. [25, Theorem (VII.4.9)]), we get $\mathbb{E}\exp\big(-\frac{\lambda^2}{2}b_+(1)\big) = \tanh(\lambda)/\lambda$ for $\lambda > 0$. Therefore Chebychev's inequality yields:

$$\mathbb{P}\Big(b_+(1) \le \ell^{\delta-2}\Big) \le e^{\lambda^2 \ell^{\delta-2}/2} \tanh(\lambda)/\lambda \le e^{1/2} \ell^{-(1-\delta/2)},$$

by choosing $\lambda = \ell^{(1-\delta/2)}$. Assembling all above estimates gives (4.11).

Recalling (3.1)–(3.5) and (4.10). Applying (3.23) to $\Upsilon_0 = 0$ (so $\mathbb{X}, \mathbb{W}$ are just X, W), and $r_1 = \ell^\delta$, we find an event $E_{13} = E_{13}(\ell)$ such that $\mathbb{P}\Big(E_{13}^c\Big) \le \exp\big(-\frac{1}{2}\log^2(\delta\ell)\big) \le \ell^{-2}$ and

$$\Theta_X(b_+(\ell)) \le \exp\Big(\sigma\ell(1-\epsilon/2) + 3\log^5(\ell^3)\Big) \le \epsilon\, t \le t/2, \qquad \text{on } E_{12} \cap E_{13}, \tag{4.13}$$

and by using (4.11), we have

$$\mathbb{P}\Big(F \cap \big(E_{12} \cap E_{13}\big)^c\Big) \le 4\ell^{-(1-\delta)}, \qquad \ell \ge \ell_0. \tag{4.14}$$

Observe that

$$c_-(\ell) \stackrel{\text{def}}{=} \inf\{s > 0 : W_-(s) > \frac{\epsilon\ell}{2} + \overline{W_+}(d_+(\ell))\} < d_-(\ell), \qquad \text{on } E_9. \tag{4.15}$$

We show that under E_{12}, $\{X(s), 0 \le s \le t\}$ exits from the interval $[-c_-(\ell), d_+((1+\epsilon)\ell]$ only with small probability. In fact, applying (3.24) to $\Upsilon_0 \equiv 0$, $r_1 = \ell^\delta$ gives the existence of E_{14} such that $\mathbb{P}\Big(E_{14}^c\Big) \le \ell^{-2}$ and on $E_{12} \cap E_{14}$,

$$\log\Theta_X(d_+((1+\epsilon)\ell)) \ge \sigma\big(W_+^{\#}(d_+((1+\epsilon)\ell)) - 2\log^5(\ell^3)\big) \ge \sigma(1+\epsilon/2)\ell > \log t, \tag{4.16}$$

and by interchanging W_+ and W_-, on $E_{12} \cap E_{14} \cap \big\{c_-(\ell) \ge \ell^\delta\big\}$, we have

$$\log\Theta_X(-c_-(\ell)) \ge \sigma U_+(\overline{W_-}(c_-(\ell)) - 2\log^5(\ell^3)) \ge \sigma U_+(\overline{W_+}(\ell) + \epsilon\ell/3) > \log t, \tag{4.17}$$

where $U_+(x)$ relates to W_+ the same way U_- relates to W_- (cf. (4.9)). Notice that on E_{12}, $\overline{W_+}(d_+(\ell)) \ge \ell^\delta$, it follows from (4.15) that

$$\mathbb{P}\Big(E_{12} \cap \{c_-(\ell) \le \ell^\delta\}\Big) \le \mathbb{P}\Big(\Theta_{W_-}(\ell^\delta) < \ell^\delta\Big) = \mathbb{P}\Big(|\mathcal{N}| \ge \ell^{\delta/2}\Big) \le 2\exp\big(-\ell^\delta/2\big), \tag{4.18}$$

since $\Theta_{W_-}(x) \stackrel{\text{law}}{=} x^2\mathcal{N}^{-2}$ for a standard Gaussian variable $\mathcal{N}$. Define

$$E_{15} \stackrel{\text{def}}{=} E_{12} \cap E_{13} \cap E_{14} \cap \big\{c_-(\ell) \ge \ell^\delta\big\} \subset F.$$

In view of (4.16)–(4.18), we have shown that for $\ell \geq \ell_0$,

$$(4.19)\quad -c_-(\ell) \leq X(s) \leq d_+((1+\epsilon)\ell),\ \forall\, 0 \leq s \leq t, \text{ on } E_{15}; \text{ and } \mathbb{P}\Big(F \cap E_{15}^c\Big) \leq 5\ell^{-(1-\delta)}.$$

Second step: We study the tightness of the laws of the post-process $X(s+\Theta_X(b_+(\ell)))$ for $0 \leq s \leq \big(t - \Theta(b_+(\ell))\big)^+$. Taking $\Upsilon_0 = \Theta(b_+(\ell))$ in (3.1)–(3.3), so that $\mathbb{X}(s) \stackrel{\text{def}}{=} X(s + b_+(\ell)) - b_+(\ell)$, for $s \geq 0$, and $\mathbb{V}(x) \stackrel{\text{def}}{=} V(x+b_+(\ell)) - V(b_+(\ell))$ and $\mathbb{W}(x) \stackrel{\text{def}}{=} W(x+b_+(\ell)) - W(b_+(\ell))$, for $x \in \mathbb{R}$. Recall Fact 2.1 for the law of $\mathbb{W}$ (it is obvious that $\mathbb{W}(x), x \leq -b_+(\ell)$ is an independent Brownian motion starting from $-W(b_+(\ell))$).

Denote by J_ℓ the interval $[-c_-(\ell) - b_+(\ell), d_+((1+\delta)\ell) - b_+(\ell)]$. Recall (4.19), under $E_{15} \subset F$, the process $(\mathbb{X}(s), 0 \leq s \leq t)$ stays in J_ℓ.

Let us consider a random potential U which coincides with $\mathbb{V}$ in J_ℓ, i.e.: $U(x) \stackrel{\text{def}}{=} \mathbb{V}(x)$ for $x \in J_\ell$, and $U(x) \stackrel{\text{def}}{=} |x|$ otherwise. Recall (3.12)–(3.14). We can define a diffusion Z starting from 0 with potential U, exactly the same way $\mathbb{X}$ relates to $\mathbb{V}$, and with the same driving Brownian motion γ. Therefore, before its exit time of J_ℓ, the path of $\mathbb{X}$ coincides with that of Z, which implies in view of (4.19) and (4.13) that for any $\lambda > 0$,

$$\begin{aligned}
\mathbb{P}\Big(\{|X(t) - b(\ell)| \geq \lambda\} \cap F\Big) &\leq \mathbb{P}\Big(F \cap E_{15}^c\Big) + \mathbb{P}\Big(\{|X(t) - b(\ell)| \geq \lambda\} \cap E_{15}\Big) \\
&\leq 5\ell^{-1+\delta} + \mathbb{P}\Big(\{|Z\big(t - \Theta(b_+(\ell))\big)| \geq \lambda\} \cap E_{15}\Big), \\
&\leq 5\ell^{-1+\delta} + \mathbb{E}\Big[\mathbb{1}_{E_{12}} \sup_{s \geq t/2} \mathbb{P}\Big(\ |Z(s)| \geq \lambda \,\big|\, (V, W)\Big)\Big], \qquad (4.20)
\end{aligned}$$

since $E_{15} \subset E_{12} \cap E_{13} \subset F$. Remark that conditioning on the potential U, $\int_{\mathbb{R}} dx e^{-U(x)} < \infty$, a.s., therefore Z converges towards its stationary distribution, and we shall estimate the speed of this convergence with the help of Lemma 4.1.

Let $A_U(x) \stackrel{\text{def}}{=} \int_0^x dy \exp\big(U(z)\big), x \in \mathbb{R}$, be the scale function of Z. Applying Lemma 4.1 to $Y = A_U(Z)$, $y = 0$ and $h(x) \stackrel{\text{def}}{=} \exp\big(-2U(A_U^{-1}(x))\big), x \in \mathbb{R}$ gives that for all $\lambda > 0$ and $s \geq t/2$

$$\begin{aligned}
\mathbb{P}\Big(|Z(s)| > \lambda \,\big|\, (V, W)\Big) &\leq \nu_U[A_U(\lambda), \infty) + \nu_U(-\infty, A_U(-\lambda)] + \nu_U[A_U((t/2)^{1/4}), \infty) + \\
&\qquad \nu_U(-\infty, A_U(-(t/2)^{1/4})] + \Big(1 + \frac{C_8 m_U}{(t/2)^{1/4}}\Big)(t/2)^{-1/4} \\
&\leq \frac{1}{m_U}\Big(\int_{|x| \geq \lambda} + \int_{|x| \geq (t/2)^{1/4}}\Big) e^{-U(x)} dx + 2\Big(1 + \frac{C_8 m_U}{t^{1/4}}\Big) t^{-1/4}, \qquad (4.21)
\end{aligned}$$

with $\nu_U(dx) \stackrel{\text{def}}{=} \frac{1}{m_U} \exp\big(-2U(A_U^{-1}(x))\big)dx$ and $m_U = \int_{\mathbb{R}} dx \exp\big(-U(x)\big)$. To bound the integrals in (4.21), we define $C_{10} = 3 + 3/C_3$, $C_{11} = C_1 C_{10}$, $C_{12} = 2C_{11} + 4$ and

$$E_{16} \stackrel{\text{def}}{=} \Big\{ \sup_{|x| \leq \ell^{C_{10}}} \big|V(x) - W(x)\big| \leq C_{11} \log \ell \Big\},$$

$$E_{17} \stackrel{\text{def}}{=} \Big\{ \sup_{-\ell^3 \le x \le y \le x+1 \le \ell^3} \big|W(x) - W(y)\big| \le \log \ell \Big\},$$

$$E_{18} \stackrel{\text{def}}{=} \Big\{ \inf_{x \in J_\ell; |x| \ge \lambda} \mathbb{W}(x) \ge C_{12} \log \ell \Big\}, \qquad (\inf_{\emptyset} \equiv +\infty),$$

$$E_{19} \stackrel{\text{def}}{=} E_{12} \cap E_{16} \cap E_{17} \cap E_{18}.$$

To estimate $\mathbb{P}\Big(F \cap E_{19}^c\Big)$, we use (1.4), which gives:

$$\mathbb{P}\Big(E_{16}^c\Big) \le \ell^{-2}, \qquad \ell \ge \ell_0. \tag{4.22}$$

The Brownian oscillation estimate (cf. [7, pp.24]) yields:

$$\mathbb{P}\Big(E_{17}^c\Big) \le \ell^{-2} \qquad \ell \ge \ell_0. \tag{4.23}$$

Recall Fact 2.1. $\{\mathbb{W}(x), -b_+(\ell) \le x \le d_+(\ell) - b_+(\ell)\}$ coincides with the path of a two-sided Bessel process R of dimension 3 starting from 0: $R(x) \equiv R_+(x)\mathbb{1}_{(x \ge 0)} + R_-(-x)\mathbb{1}_{(x<0)}$ for $-\wp_-(\ell) \le x \le \Theta_{R_+}(\ell)$, where $\wp_-(\ell)$ is defined from R_- the same way $\wp(r)$ is from R_3 in Fact 2.1 (iii). For $R = R_+$ or R_-, denote by $\mathcal{L}_R(x) = \sup\{t > 0 : R(t) \le x\}$, the last exit time of x by the three-dimensional Bessel process R, Williams' time-reversal for R at $\mathcal{L}_R(x)$ (cf. [25, Corollary(VII.4.6)]) implies that $\mathcal{L}_R(x) \stackrel{\text{law}}{=} x^2 \mathcal{N}^{-2}$ for a standard Gaussian variable $\mathcal{N}$. It turns out that for $\lambda > 0$,

$$\mathbb{P}\Big(\mathcal{L}_R(C_{12} \log \ell) \ge \lambda\Big) \le \frac{C_{12} \log \ell}{\sqrt{\lambda}}. \tag{4.24}$$

Remark from (4.15), $W_-(c_-(\ell)) - \inf_{0 \le x \le c_-(\ell)} W_-(x) < \ell$ implies that

$$\begin{aligned} \inf_{-c_-(\ell) \le x \le 0} W(x) = \inf_{0 \le x \le c_-(\ell)} W(x) &\ge W_-(c_-(\ell)) - \ell \\ &= \overline{W_+}(d_+(\ell)) - \ell + \frac{\epsilon \ell}{2} \\ &\ge W_+(b_+(\ell)) + C_{12} \log \ell, \qquad \ell \ge \ell_0, \quad \text{on } E_9, \end{aligned} \tag{4.25}$$

since $W_+(d_+(\ell)) = W_+(b_+(\ell)) + \ell$. Using Fact 2.1 (i), we have

$$\begin{aligned} &\mathbb{P}\Big(\inf_{d_+(\ell) \le x \le d_+((1+\epsilon)\ell)} \big(W_+(x) - W_+(b_+(\ell))\big) \ge C_{12} \log \ell \Big) \\ &\ge \mathbb{P}\Big(\text{a Brownian motion starting from } \ell \text{ hits } (1+\epsilon)\ell \text{ before } C_{12} \log \ell\Big) \\ &\ge 1 - 2\epsilon = 1 - \frac{2 \log^6 \ell}{\ell}. \end{aligned} \tag{4.26}$$

In view of (4.24)–(4.26), we arrive at

$$\mathbb{P}\Big(E_{18}^c \cap E_{15}\Big) \le \frac{C_{13}\log \ell}{\sqrt{\lambda}} + 4\ell^{-(1-\delta)},$$

which in view of (4.11), (4.22) and (4.23) implies that

$$\mathbb{P}\Big(E_{19}^c \cap F\Big) \le \frac{C_{13}\log \ell}{\lambda} + 5\ell^{-(1-\delta)}. \tag{4.27}$$

On E_{19}, for $x \in J_\ell$ and $|x| \ge \lambda$, we have $U(x) = \mathbb{V}(x) \ge \mathbb{W}(x) - C_{11}\log \ell \ge (C_{12}-C_{11})\log \ell$, which implies that

$$\begin{aligned}
\int_{|x|\ge\lambda} e^{-U(x)}dx &\le \Big(\int_{x\le -c_-(\ell)-b_+(\ell)} + \int_{x\ge d_+((1+\epsilon)\ell)}\Big) e^{-|x|}dx + \int_{x\in J_\ell, |x|\ge\lambda} e^{-(C_{12}-C_{11})\log \ell}dx \\
&\le 2\int_{|x|\ge \ell^\delta} e^{-|x|}dx + 2\ell^3\,\ell^{-(C_{12}-C_{11})} \\
&\le 3\ell^{-(C_{12}-C_{11}-3)}, \qquad (4.28)
\end{aligned}$$

and for all $x \in J_\ell$, $U(x) = \mathbb{V}(x) \ge \mathbb{W}(x) - C_{11}\log \ell \ge -C_{11}\log \ell$,

$$m_U = \Big(\int_{x\notin J_\ell} + \int_{x\in J_\ell}\Big) e^{-U(x)}dx \le 2\int_{|x|\ge \ell^\delta} e^{-|x|}dx + 2\ell^3 \ell^{C_{11}} \le 3\ell^{3+C_{11}}. \tag{4.29}$$

We can bound m_U below as follows: for $|x| \le 1$, $U(x) \le \mathbb{W}(x) + C_{11}\log \ell \le (1+C_{11})\log \ell$, hence

$$m_U \ge \inf_{|x|\le 1} e^{-U(x)} \ge \ell^{-1-C_{11}}. \tag{4.30}$$

Recall that $t = e^\ell$. From (4.28)–(4.30), we have shown that on E_{19},

$$\frac{1}{m_U}\int_{|x|\ge\lambda} e^{-U(x)}dx \le 3\ell^{-(C_{12}-2C_{11}-4)} = 3\ell^{-2}, \tag{4.31}$$

$$\frac{1}{m_U}\int_{|x|\ge (t/2)^{1/4}} e^{-U(x)}dx = \frac{1}{m_U}\int_{|x|\ge (t/2)^{1/4}} e^{-|x|}dx \le \ell^{-2}. \tag{4.32}$$

Observe that E_{19} is $\sigma(V,W)$-measurable. Applying (4.30)–(4.32) to (4.21) gives that for all $\lambda \ge 1$ and $s \ge t/2$, we have

$$\mathbb{P}\Big(|Z(s)| > \lambda \,\Big|\, (V,W)\Big) \le 5\ell^{-2}, \qquad \text{on } E_{19} \subset E_{12},$$

which together with (4.20), imply that

$$\mathbb{P}\Big(\{|X(t) - b(\ell)| \ge \lambda\} \cap F\Big) \le 10\ell^{-1+\delta} + \mathbb{P}\Big(F \cap E_{19}^c\Big),$$

yielding (4.7) by means of (4.27). The case $\{b(\frac{\log t}{\sigma})<0\}$ follows from (4.7) by interchanging (V_+,W_+) with (V_-,W_-), hence yielding Theorem 1.1 . $\square$

5. Proof of Theorem 1.2

Fix $c>0$. We first prove

$$\mathbb{P}\Big(D_X(t)-t>t^c; b(\frac{\log t_0}{\sigma})>0\Big)=\mathbb{P}\Big(\Lambda(1)>c; b(1)>0\Big)+o(1), \qquad t\to\infty. \tag{5.1}$$

Fix a small $\epsilon>0$. Consider a small $\delta>0$ and define

$$E_{20}\stackrel{\text{def}}{=}\Big\{\big|X(t)-b(\frac{\log t}{\sigma})\big|\le \delta\log^2 t\Big\},$$
$$E_{21}\stackrel{\text{def}}{=}\Big\{\epsilon\log^2 t\le \big|b(\frac{\log t}{\sigma})\big|\le \frac{\log^2 t}{\epsilon}\Big\},$$
$$E_{22}\stackrel{\text{def}}{=}\Big\{\sup_{|x|,|y|\le\frac{2}{\epsilon}\log^2 t;|x-y|\le 3\delta\log^2 t}\big|W(x)-W(y)\big|\le\epsilon\log t\Big\}$$
$$E_{23}\stackrel{\text{def}}{=}\Big\{\sup_{0\le s\le b_+(\frac{\log t}{\sigma})}W(s)-W(b_+(\frac{\log t}{\sigma}))\ge 3\epsilon\log t\Big\}.$$

Notice that $\mathbb{P}\Big(E_{22}^c\Big)$ does not depend on t. Choose a sufficiently small but fixed $0<\delta=\delta(\epsilon)\le\epsilon/2$ such that $\mathbb{P}\Big(E_{22}^c\Big)\le\epsilon$. It follows from Theorem 1.1 that for all large $t\ge t_0(\delta,\epsilon)$, we have $\mathbb{P}\Big(E_{20}^c\Big)\le\epsilon$, hence we deduce from Brownian scaling that

$$\sup_{t\ge t_0}\mathbb{P}\Big(\big(\cap_{j=20}^{23}E_j\big)^c\Big)=o(1), \qquad \epsilon\to 0. \tag{5.2}$$

Recall (3.1)–(3.5). Let $\Upsilon_0=t\ge t_0$ (t_0 will be sufficiently large), and $\mathbb{X}(s)\stackrel{\text{def}}{=}X(s+t)-X(t)$, $s\ge 0$; $\mathbb{W}(x)\stackrel{\text{def}}{=}W(x+X(t))-W(X(t)), x\in\mathbb{R}$. Applying Proposition 3.2 to $r_1=r_1(\epsilon)$ sufficiently large such that $\exp\Big(-\frac12\log^2 r_1\Big)\le\epsilon$, we obtain an event E_{24} satisfying

$$\mathbb{P}\Big(E_{24}^c\Big)\le\epsilon, \tag{5.3}$$

and for all $r\ge r_1$, on $E_{24}\cap\{|X(t)|\le\exp\big(\log^2 r\big)\}\cap\{\overline{\mathbb{W}_-}(r)\ge 3\log^5 r\}$, we have

$$\log\Theta_{\mathbb{X}}(-r)\le\sigma\max\Big(\mathbb{W}_-^{\#}(r),\mathbb{U}_+(\overline{\mathbb{W}_-}(r)+2\log^5 r)\Big)+3\log^5 r, \tag{5.4}$$
$$\log\Theta_{\mathbb{X}}(-r)\ge\sigma\max\Big(\mathbb{W}_-^{\#}(r)-2\log^5 r,\mathbb{U}_+(\overline{\mathbb{W}_-}(r)-2\log^5 r)\Big), \tag{5.5}$$

where $\mathbb{U}_+(x) \stackrel{\text{def}}{=} x + \sup\{-\mathbb{W}_+(s) : 0 \le s \le \Theta_{\mathbb{W}_+}(x)\}$, and we have interchanged $\mathbb{W}_+$ and $\mathbb{W}_-$ in (3.23) and (3.24). On $\cap_{j=20}^{24} E_j \cap \{b(\frac{\log t}{\sigma}) = b_+(\frac{\log t}{\sigma}) > 0\}$, we apply (5.4) to $r = X(t) \ge (\epsilon - \delta)\log^2 t \ge \frac{1}{2}\epsilon \log^2 t_0 \ge r_1$ gives

$$\begin{aligned}
\log(D_X(t) - t) &= \log \Theta_{\mathbb{X}}(-X(t)) \\
&\le \sigma \max\Big(\mathbb{W}_-^{\#}(b_+(\frac{\log t}{\sigma}) + \delta \log^2 t), \mathbb{U}_+(\overline{\mathbb{W}_-}(b_+(\frac{\log t}{\sigma})) + 2\delta \log^2 t)\Big) + 4\log^5 \log t \\
&\le \sigma \max\Big(\widehat{W}_-^{\#}(b_+(\frac{\log t}{\sigma})), \widehat{U}_+(\overline{\widehat{W}_-}(b_+(\frac{\log t}{\sigma})))\Big) + 5\sigma\epsilon \log t, \qquad t \ge t_0,
\end{aligned} \tag{5.6}$$

provided that t_0 is sufficiently large, and in (5.6), $\widehat{W}_+(s) \stackrel{\text{def}}{=} W(s + b_+(\frac{\log t}{\sigma})) - W(b_+(\frac{\log t}{\sigma}))$, $\widehat{W}_-(x) \stackrel{\text{def}}{=} W(-s + b_+(\frac{\log t}{\sigma})) - W(b_+(\frac{\log t}{\sigma}))$ for $s \ge 0$; and $\widehat{U}_+(x) \stackrel{\text{def}}{=} x + \sup\{-\widehat{W}_+(s) : 0 \le s \le \Theta_{\widehat{W}_+}(x)\}$ for $x \ge 0$. Recall (2.7)–(2.9). Notice that the RHS of (5.6) equals in fact in law $(\Lambda_+(1) + 5\sigma\epsilon)\log t$, which implies in view of (5.2) and (5.3) that

$$\begin{aligned}
&\mathbb{P}\Big(\{\log(D_X(t) - t) \ge c \log t\} \cap \{b(\frac{\log t}{\sigma}) > 0\}\Big) \\
&\le \mathbb{P}\Big(\cup_{j=20}^{24} E_j^c\Big) + \mathbb{P}\Big(\{\Lambda_+(1) + 5\sigma\epsilon \ge c\} \cap \{b(1) > 0\}\Big) \\
&= \mathbb{P}\Big(\{\Lambda(1) \ge c\} \cap \{b(1) > 0\}\Big) + o(1), \qquad \epsilon \to 0,
\end{aligned}$$

yielding the upper bound part of (5.1), the lower bound can be proven in the similar way by using (5.5) instead of (5.4). Obviously the case of $\{b(\frac{\log t}{\sigma}) < 0\}$ can be proven in the same way, and Theorem 1.2 follows. □

6. Sinai's walk in random environment

Recall (1.1). Assume that

$$(\xi_i, i \in \mathbb{Z}) \text{ are i.i.d. and } c \le \xi_i \le 1 - c, \text{ for some constant } c \in (0, 1/2), \tag{6.1}$$

$$\mathbb{E} \log\left(\frac{1 - \xi_i}{\xi_i}\right) = 0, \tag{6.2}$$

$$\mathbb{E} \log\left(\frac{1 - \xi_i}{\xi_i}\right)^2 = \sigma^2 \in (0, \infty), \tag{6.3}$$

Sinai [29] showed that as $n \to \infty$,

$$\frac{\sigma^2 S_n}{\log^2 n} \xrightarrow{(d)} b(1), \qquad \text{under } \mathbb{P}, \tag{6.4}$$

where $b(1)$ is defined by (2.6). We shall call this RWRE (S_n) Sinai's walk. Here, we limit our attention to the first return time $D_S(n)$:

$$D_S(n) \stackrel{\text{def}}{=} \inf\{k > n : S_k = 0\}. \tag{6.5}$$

Similarly to the continuous case, we have

Theorem 6.1. *Assuming (6.1)–(6.3), we have that as $n \to \infty$:*

$$\frac{\log \Big(D_S(n) - n \Big)}{\log n} \xrightarrow{(d)} \Lambda(1), \quad \textit{under } \mathbb{P}, \tag{6.6}$$

with the density of $\Lambda(1)$ given in (1.9).

Define the last zero $G_S(n)$ of S before n by

$$G_S(n) \stackrel{\text{def}}{=} \max\{0 \le k \le n : S_k = 0\}.$$

Theorem 6.1 yields the following convergence in law for the last zero $G_S(n)$ and the age process $n - G_S(n)$:

Corollary 6.1. *Assuming (6.1)–(6.3). Under $\mathbb{P}$, the following two convergences in law hold: as $n \to \infty$*

$$\frac{\log G_S(n)}{\log n} \xrightarrow{(d)} \eta_1, \qquad \mathbb{P}\Big(\eta_1 \in dx\Big) = \frac{2}{3} \mathbb{1}_{(0 \le x < 1)} dx + \frac{1}{3} \delta_1(dx), \tag{6.7}$$

$$\frac{\log \Big(n - G_S(n) \Big)}{\log n} \xrightarrow{(d)} \eta_2, \qquad \mathbb{P}\Big(\eta_2 \in dx\Big) = \frac{2x}{3} \mathbb{1}_{(0 \le x < 1)} dx + \frac{2}{3} \delta_1(dx), \tag{6.8}$$

where δ_1 denotes the Dirac measure at 1.

Notice that (6.7) and (6.8) contrast completely with the classical Lévy's arcsine law for a simple random walk (cf. Feller [8]).

To prove Theorem 6.1, let us consider the step potential V such that $V(0) = 0$ and

$$V(x) \text{ is constant on } [n-1, n) \text{ and } V(n) - V(n-) = \log \frac{1 - \xi_n}{\xi_n}, \qquad n \in \mathbb{Z}. \tag{6.9}$$

Let $\{X(t), t \ge 0\}$ be the diffusion associated with the potential V. The following Skorokhod embedding was stated in Schumacher [27], Kawazu et al. [17], and its proof can be found in [13]:

Fact 6.1. Assuming (6.1)–(6.3). For $k \geq 1$, define $\mu_k \stackrel{\text{def}}{=} \inf\{t > \mu_{k-1} : |X(t) - X(\mu_{k-1})| = 1\}$ with $\mu_0 \stackrel{\text{def}}{=} 0$. Therefore $\{X(\mu_n), n \geq 0\}$ is distributed as $\{S_n, n \geq 0\}$ defined by (1.1). Furthermore $\{\mu_n - \mu_{n-1}, n \geq 1\}$ are iid with common distribution that of the first hitting time at 1 of a reflected Brownian motion starting from 0.

Proof of Theorem 6.1. Using Fact 6.1, we can take $S_n \stackrel{\text{def}}{=} X(\mu_n), n \geq 0$. Remark that $\mu(D_S(n)) = D_X(\mu(n))$ for $n \geq 1$. Fix $c > 0$ and write $m \equiv n + n^c$. Let $E_{25} \stackrel{\text{def}}{=} \Big\{(1-\epsilon)n \leq \mu(n) \leq (1+\epsilon)n\Big\} \cap \Big\{(1-\epsilon)m \leq \mu(m) \leq (1+\epsilon)m\Big\}$. The law of the large numbers shows $\frac{\mu(n)}{n} \to 1$, a.s., which shows $\mathbb{P}\Big(E_{25}^c\Big) = o(1)$ as $n \to \infty$. On E_{25}, we have

$$\Big\{D_S(n) \geq m\Big\} \subset \Big\{D_X(\mu(n)) \geq (1-\epsilon)m\Big\} \subset \Big\{D_X((1+\epsilon)n) \geq (1-\epsilon)m\Big\},$$
$$\Big\{D_X((1-\epsilon)n) \geq (1+\epsilon)m\Big\} \subset \Big\{D_X(\mu(n)) \geq (1+\epsilon)m\Big\} \subset \Big\{D_S(n) \geq m\Big\},$$

implying that

$$\mathbb{P}\Big(D_S(n) \geq m\Big) = \mathbb{P}\Big(D_X(n) \geq m\Big) + o(1), \qquad \epsilon \to 0,$$

which in view of Theorem 1.2, implies the desired result. □

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