All about blow-up for a semilinear wave equation in one space dimension

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The equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $p > 1$,
$u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$,
$u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R})$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$
and

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})} = \sup_{a\in\mathbb{R}} \left(\int_{a-1}^{a+1} |v(x)|^2 dx \right)^{1/2}.$$

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THE CAUCHY PROBLEM IN $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$

It is a consequence of:

- the Cauchy problem in $H^1 \times L^2(\mathbb{R})$,
- the finite speed of propagation.

Maximal solution in $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$

- either it exists for all $t \in [0, \infty)$ (**global solution**),
- or it exists for all $t \in [0, \bar{T})$ (**singular solution**).

Existence of singular solutions

It's a consequence of ODE techniques and the finite speed of propagation; see also the energy argument by Levine 1974:

if $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$ *and* $\int_{\mathbb{R}} \left(\frac{1}{2}(u_1)^2 + \frac{1}{2}(\partial_x u_0)^2 - \frac{1}{p+1}|u_0|^{p+1}\right) dx < 0$, *then* u *is not global*.

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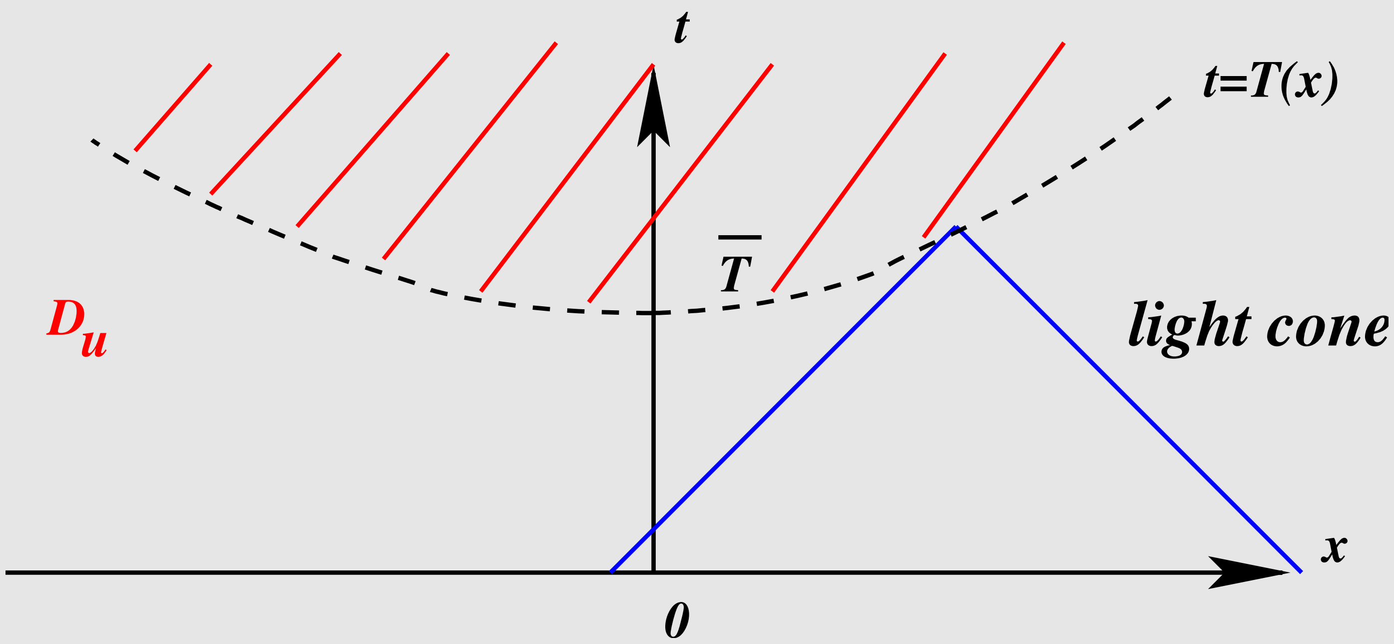
Singular solutions: the maximal influence domain



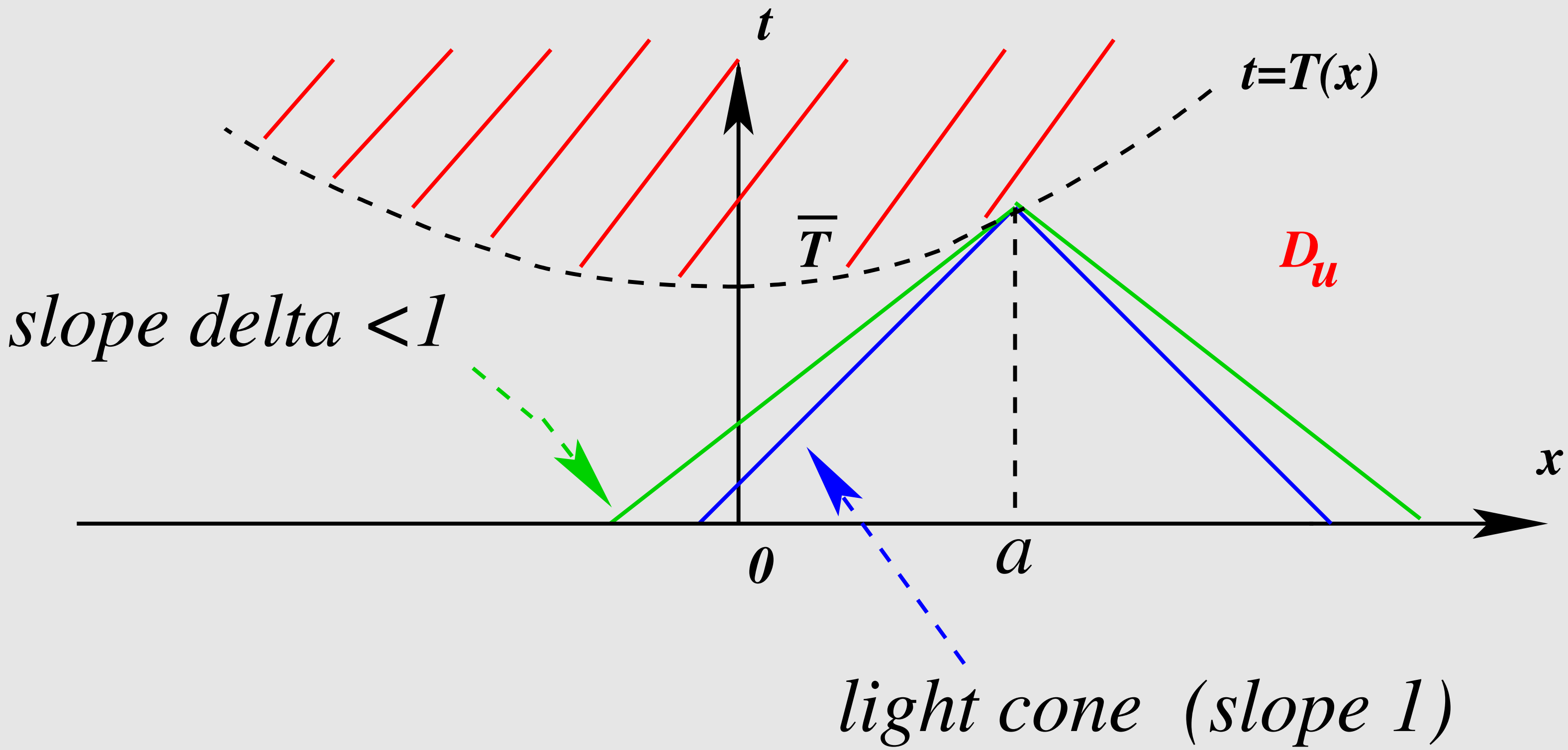
The blow-up set $x \mapsto T(x)$ is 1-Lipschitz (**finite speed of propagation**).

Remark: $\bar{T} = \inf T(x)$ is the **blow-up time**. For all $x \in \mathbb{R}^N$, there exists a "local" blow-up time $T(x)$.

The aim of this talk: To describe precisely the blow-up set, and the solution near the blow-up set, *for an arbitrary blow-up solution*.

Definition: Non characteristic points and characteristic points

A point a is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



The point is said *characteristic* if not.

- Notation: $\mathcal{R} \subset \mathbb{R}$ is the set of all *non* characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}$).

Known results, for an arbitrary solution

- The blow-up set $\Gamma = \{(x, T(x))\} \subset \mathbb{R}^2$.

- By definition, Γ is 1-Lipschitz.

- $\mathcal{R} \neq \emptyset$ (Indeed, $\bar{x}$ such that $T(\bar{x}) = \min_{x\in\mathbb{R}} T(x)$ is non characteristic).

- Caffarelli and Friedman (1985 and 1986) had two criteria to have $\mathcal{R} = \mathbb{R}$ and $x \mapsto T(x)$ of class C^1 (using the **positivity of the fundamental solution**):

  ▷ either when $p \geq 3$, with $u_0 \geq 0$, $u_1 \geq 0$ and $(u_0, u_1) \in C^4 \times C^3(\mathbb{R})$,

  ▷ or under conditions on initial data that ensure that

$$u \geq 0 \text{ and } \partial_t u \geq (1 + \delta_0)|\partial_x u|$$

  for some $\delta_0 > 0$.

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Questions and new results

- ▷ **Existence**
  - Are there characteristic points? *yes,* $\mathcal{S} \neq \emptyset$.
- ▷ **Regularity**
  - Is $\mathcal{R}$ open? *yes*
  - Is Γ (or $\Gamma_{\mathcal{R}}$) of class C^1 ? *yes*
  - "How is" $\mathcal{S}$? *isolated points*
  - How does Γ look like near $\mathcal{S}$? *corner shaped*
- ▷ **Asymptotic behavior (profile)**
  - How does the solution behave near a non characteristic point? *we have the profile*
  - and near a characteristic point? *we have a precise decomposition into solitons*

Rk. Regularity and asymptotic behavior are linked.

Extension to the radial case outside the origin

If $u = u(r,t)$ satisfies for all $r > 0$,

$$\partial_t^2 u = \partial_r^2 u + \frac{(N-1)}{r}\partial_r u + |u|^{p-1}u$$

then all our results in one dimension extend to this case, as long as we consider the behavior outside the origin.

The plan

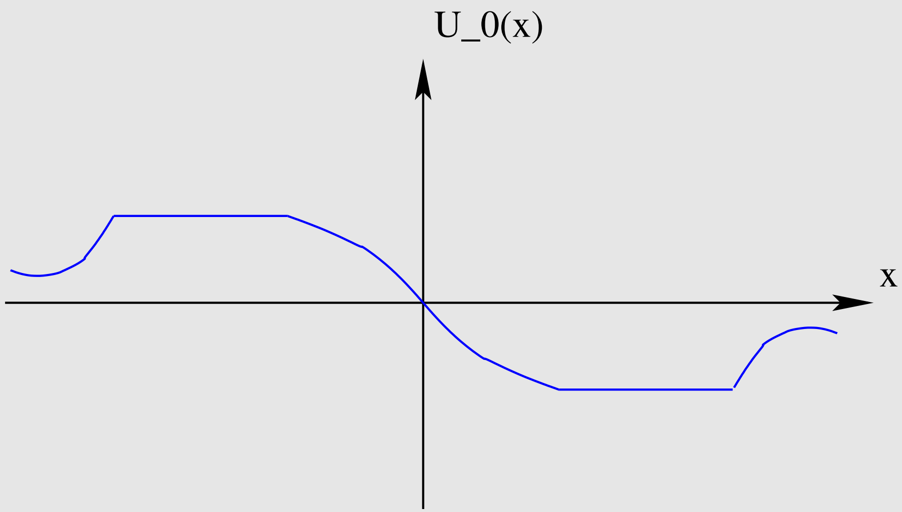
- Part 1: Existence of characteristic points.
- Part 2: A Liouville theorem and regularity of the blow-up set.
- Part 3: A Lyapunov functional and the blow-up rate.
- Part 4: Asymptotic behavior near *non characteristic* points (the blow-up profile).
- Part 5: Asymptotic behavior near *characteristic* points (decomposition into solitons).

Part 1: Existence of characteristic points

We recall: Any *solution to the Cauchy problem has (at least) a non characteristic point (the minimum of the blow-up set).*

Th. There exist *initial data which give solutions with a characteristic point.*

Example: We take odd initial data, with two large plateaus of different signs. Then, the solution blows up, and the origin is a characteristic point with $\forall t < T(0),\ u(0,t) = 0$.



Th. If we perturb the constructed initial data, then the new solution blows up and has a characteristic point.

Part 2: Regularity of the blow-up set

- Near a non characteristic point:

  Th. *The set of non characteristic points* $\mathcal{R}$ **is open** *and* $T(x)$ **is of class** C^1 *on this set* ($C^{1,\alpha}$ *by N. Nouaili CPDE 2008*).

- Near a characteristic point:

  Th. *The set of characteristic points* $\mathcal{S}$ *is made of* **isolated points.**
  If $a \in \mathcal{S}$*, then* $T_l'(a) = 1$ *and* $T_r'(a) = -1$.

  Cor. *There is no solution with* $a \in \mathcal{S}$ *and* $T'(a) = 1$.

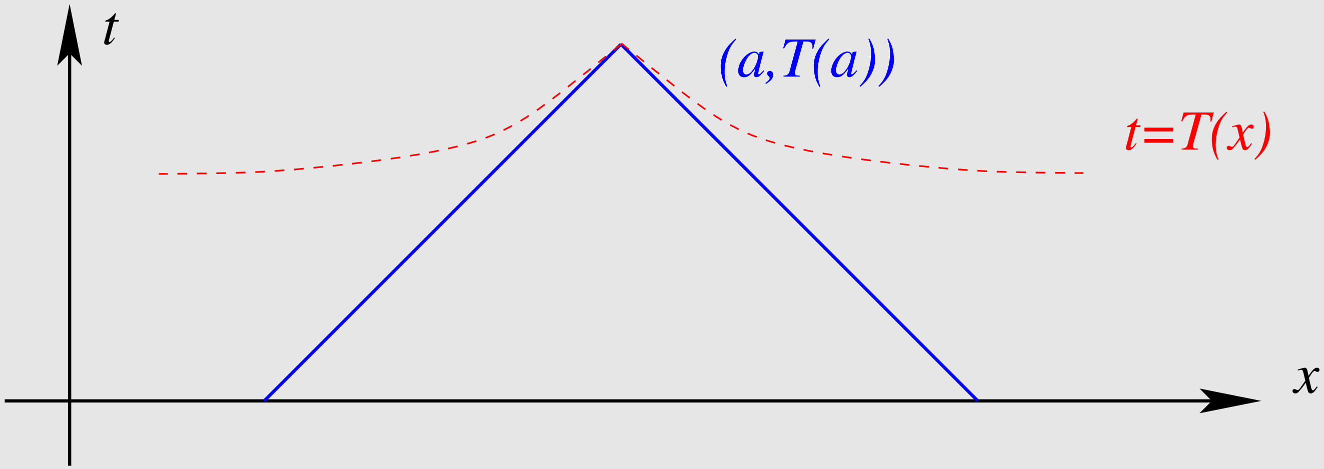
Part 2: The corner property near a *characteristic* point

Th. (the corner property) *If $a \in \mathcal{S}$, then for all x near a,*

$$\frac{1}{C}|x-a||\log|x-a||^{-\gamma(a)} \le T(x) - T(a) + |x-a| \le C|x-a||\log|x-a||^{-\gamma(a)} \tag{1}$$

where

$$\gamma(a) = \frac{(k(a)-1)(p-1)}{2} \text{ with } k(a) \in \mathbb{N}, k(a) \ge 2.$$



Comments

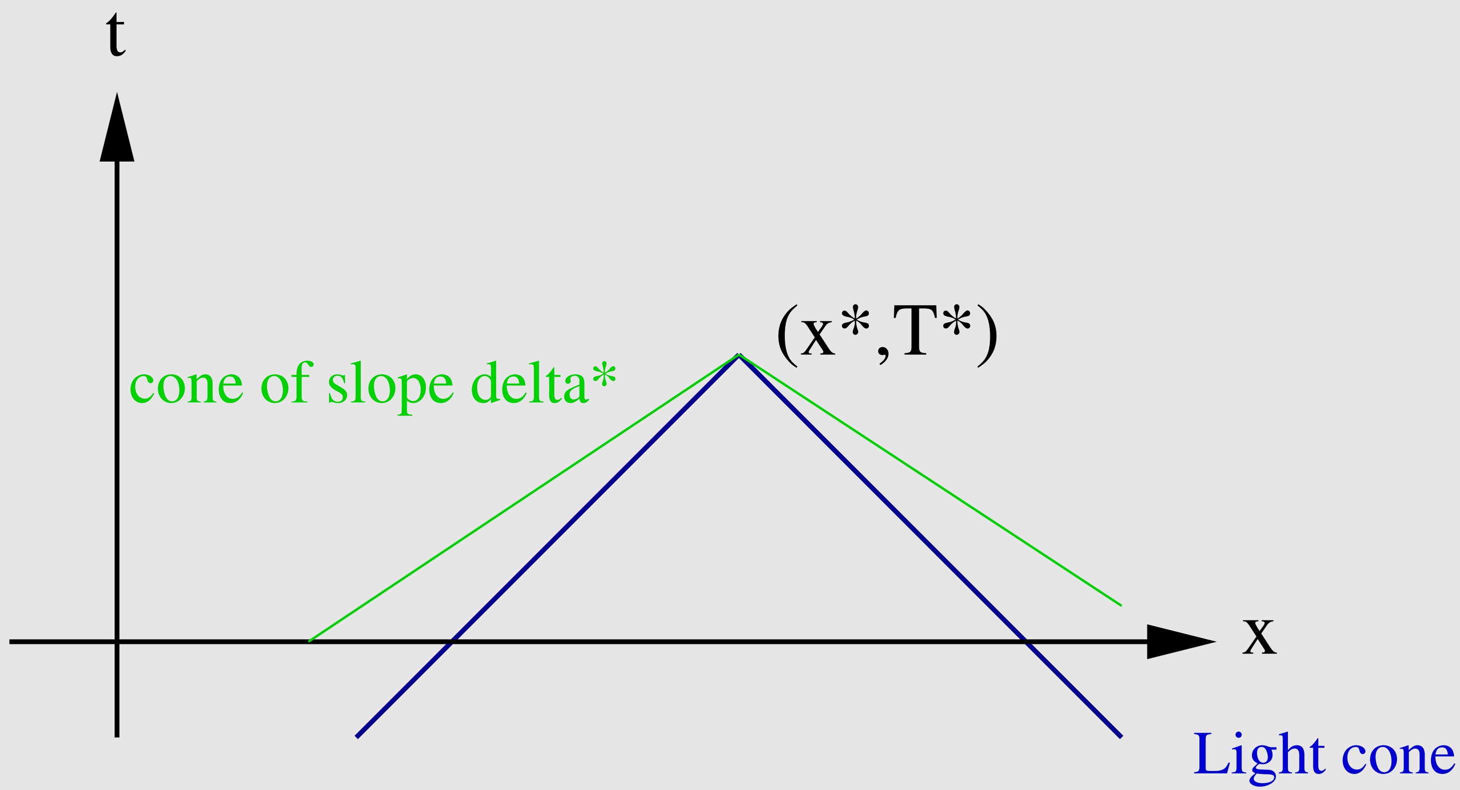
Idea of the proof:
The techniques are based on

- ▷ - a very good understanding of the **behavior of the solution in selfsimilar variables in the energy space** related to the selfsimilar variable (see Part 3 of this talk).
- ▷ - a **Liouville Theorem** (see next slide).

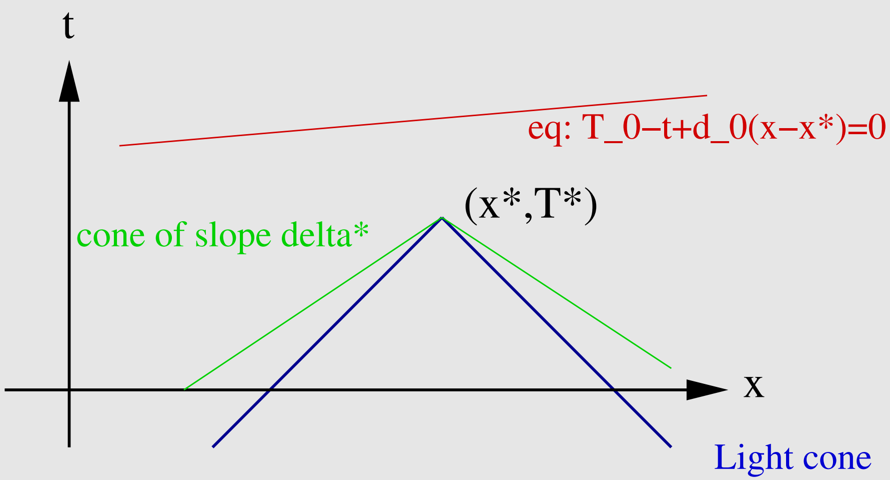
A Liouville Theorem

Th. Consider $u(x,t)$ a solution of $u_{tt} = u_{xx} + |u|^{p-1}u$ such that:

- u is defined in the *infinite* green cone,
- u is less than $(T^* - t)^{-\frac{2}{p-1}}$ (in L^2 average).



A Liouville Theorem



Then,
- either $u \equiv 0$,
- or there exists $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ and $\theta_0 = \pm 1$ such that u is actually defined below the red line by

$$u(x,t) = \theta_0 \kappa_0(p) \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T_0 - t + d_0(x - x^*))^{\frac{2}{p-1}}}.$$

Remark: u blows up on the red line.

Comments

- ▷ The limiting case $\delta^* = 1$ is still open.

The proof:

- ▷ The proof has a completely different structure from the proof for the heat equation.
- ▷ The proof is based on various energy arguments and on a dynamical result.

Part 3: A Lyapunov functional and the blow-up rate

Selfsimilar transformation for all $x_0 \in \mathbb{R}$

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x-x_0}{T(x_0)-t},\ s = -\log(T(x_0)-t).$$

(x,t) in the light cone of vertex $(x_0, T(x_0)) \iff (y,s) \in (-1,1)\times[-\log T(x_0), \infty)$.

Equation on $w = w_{x_0}$: For all $(y,s) \in (-1,1)\times[-\log T(x_0),\infty)$:

$$\partial^2_{ss} w - \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) + \frac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y\partial^2_{sy} w$$

where $\rho(y) = (1-|y|^2)^{\frac{2}{p-1}}$

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The case of radial solutions

If $u = u(r,t)$ satisfies for all $r > 0$,

$$\partial_t^2 u = \partial_r^2 u + \frac{(N-1)}{r}\partial_r u + |u|^{p-1}u$$

and for $r_0 > 0$, $w_{r_0}(y,s)$ is defined by

$$w_{r_0}(y,s) = (T(r_0)-t)^{\frac{2}{p-1}} u(r,t),\ y = \frac{r-r_0}{T(r_0)-t},\ s = -\log(T(r_0)-t),$$

then $w_{r_0}(y,s)$ satisfies the following:

$$\partial_{ss}^2 w - \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) + \frac{2(p+1)}{(p-1)^2}w - |w|^{p-1}w + \frac{(N-1)e^{-s}}{r_0+ye^{-s}}\partial_y w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y\partial_{sy}^2 w.$$

In particular, the new term $\frac{(N-1)e^{-s}}{r_0+ye^{-s}}\partial_y w$ is negligeable (unlike when $r_0 = 0$).

Back to 1 d: A Lyapunov functional (Antonini-Merle)

$$E(w) = \int_{-1}^{1} \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy,$$

Thanks to a Hardy-Sobolev inequality, $E = E(w, \partial_s w)$ is well defined in the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(B) \mid \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (\partial_y q_1)^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

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Properties of the Lyapunov functional E

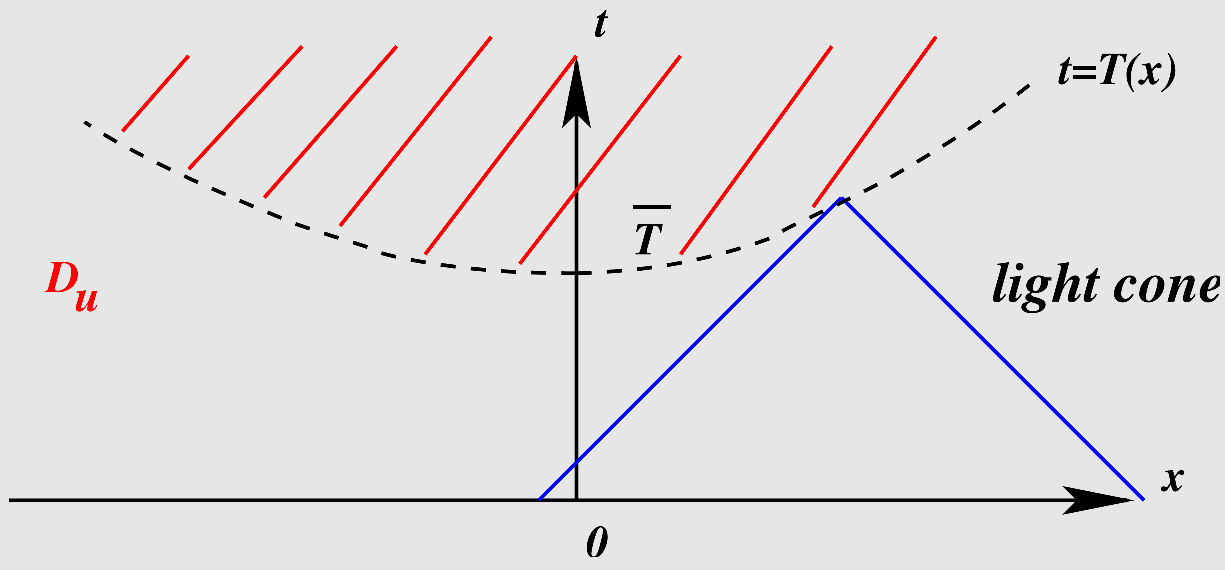
Lemma 1 (Monotonicity (Antonini-Merle)) *For all* s_1 *and* s_2*:*

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1}\int_{s_1}^{s_2}\int_{-1}^{1}(\partial_s w)^2(1-|y|^2)^{\frac{2}{p-1}-1}dyds.$$

Lemma 2 (A blow-up criterion) *Consider a solution* W *such that* $E(W(s_0)) < 0$ *for some* $s_0 \in \mathbb{R}$*, then* W *blows up in finite time* $S > s_0$*.*

The blow-up rate

We look for a *local* **blow-up rate** near the singular surface (i.e. near every local blow-up time, $t \to T(x_0)$), in $H^1 \times L^2$ **of the section of the light cone.**



Hint: Is the rate given by the associated ODE $v'' = v^p$?

An upper bound on the blow-up rate in selfsimilar variables

Th. For all $x_0 \in \mathbb{R}$ and $s \geq -\log T(x_0) + 1$,

$$\int_{-1}^{1} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2(1-|y|^2) + \frac{(p+1)}{(p-1)^2} w^2 + \frac{1}{p+1}|w|^{p+1} \right) \rho dy \leq K$$

where the constant K depends only on p and an upper bound on $T(x_0)$, $1/T(x_0)$ and $\|(u_0, u_1)\|$.

Idea of the proof of the upper bound

- Selfsimilar transformation and existence of a Lyapunov functional
- Interpolation to gain regularity
- Gagliardo-Nirenberg estimates.

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Part 4: Asymptotic behavior at a *non characteristic* point

Take $x_0 \in \mathbb{R}$ **non characteristic**. Using a covering argument for x near x_0, we obtain that $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-1,1)}$ is bounded.

Question: Does $w_{x_0}(y,s)$ have a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$).

Remark: In the context of Hamiltonian systems, **this question is delicate**, and there is no natural reason for such a convergence, since **the wave equation is time reversible**.

See for similar difficulty and approach, results for

- the **critical KdV** (Martel and Merle),
- **NLS** (Merle and Raphaël).

Stationary solutions.

We look for solutions of

$$\frac{1}{\rho}\left(\rho(1-y^2)w'\right)' - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w = 0.$$

We work in $\mathcal{H}_0$, the (**stationary energy space**) defined by

$$\mathcal{H}_0 = \{r \in H^1_{loc}(-1,1) \mid \|r\|^2_{\mathcal{H}_0} \equiv \int_{-1}^{1}\left(r'^2(1-y^2)+r^2\right)\rho dy < +\infty\}.$$

Prop. Consider a stationary solution in $\mathcal{H}_0$. Then, either $w \equiv 0$ or there exist $d \in (-1,1)$ and $e = \pm 1$ such that $w(y) = e\kappa(d,y)$ where

$$\forall (d,y) \in (-1,1)^2,\ \ \kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ and } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}.$$

Remark: We have 3 connected components. $E(0) = 0 < E(\pm\kappa(d)) = E(\kappa_0)$.

Blow-up profile near a non characteristic point

Th. *There exist* $C_0 > 0$ *and* $\mu_0 > 0$ *such that if* x_0 *is* **non characteristic**, *then there exist* $d(x_0) \in (-1,1)$, $e(x_0) = \pm 1$ *and* $s^*(x_0) \geq -\log T(x_0)$ *such that:*
(i) For all $s \geq s^*(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \begin{pmatrix} \kappa(d(x_0), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}$$

and $E(w_{x_0}(s) \to E(\kappa_0)$ *where the energy space*

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \mid \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

(ii) $d(x_0) = T'(x_0)$.
Rk. We have exp. fast convergence (hence, C^{1,μ_0} regularity of $\mathcal{R}$, see Nouaili).
Rk. $\|w_{x_0}(y,s) - e(x_0)\kappa(d(x_0), y)\|_{L^\infty(-1,1)} \to 0$.
Rk. The parameter of the profile $d(x_0)$ has a geometrical interpretation $(T'(x_0))$.

Difficulties of the proof of convergence

- ▷ The set of non zero stationary solutions is made up of non isolated solutions (one parameter family):
  $\longrightarrow$ we need **a modulation technique**.
- ▷ The linearized operator around a non zero stationary solution **is non self-adjoint**:
  $\longrightarrow$ we need to use dispersive properties coming from the Lyapunov functional to control the negative part of the spectrum.

Part 5: Asymptotic behavior at a *characteristic* point

Th. If $x_0 \in \mathbb{R}$ is **characteristic**, then, there exist $k(x_0) \geq 2$, $e(x_0) = \pm 1$ and continuous $d_i(s) = -\tanh \zeta_i(s)$ for $i = 1, ..., k$ such that:
(i)

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

(ii) Introducing

$$\bar{w}_{x_0}(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_{x_0}(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(s) = -\tanh^{-1} d_i(s),$$

we get

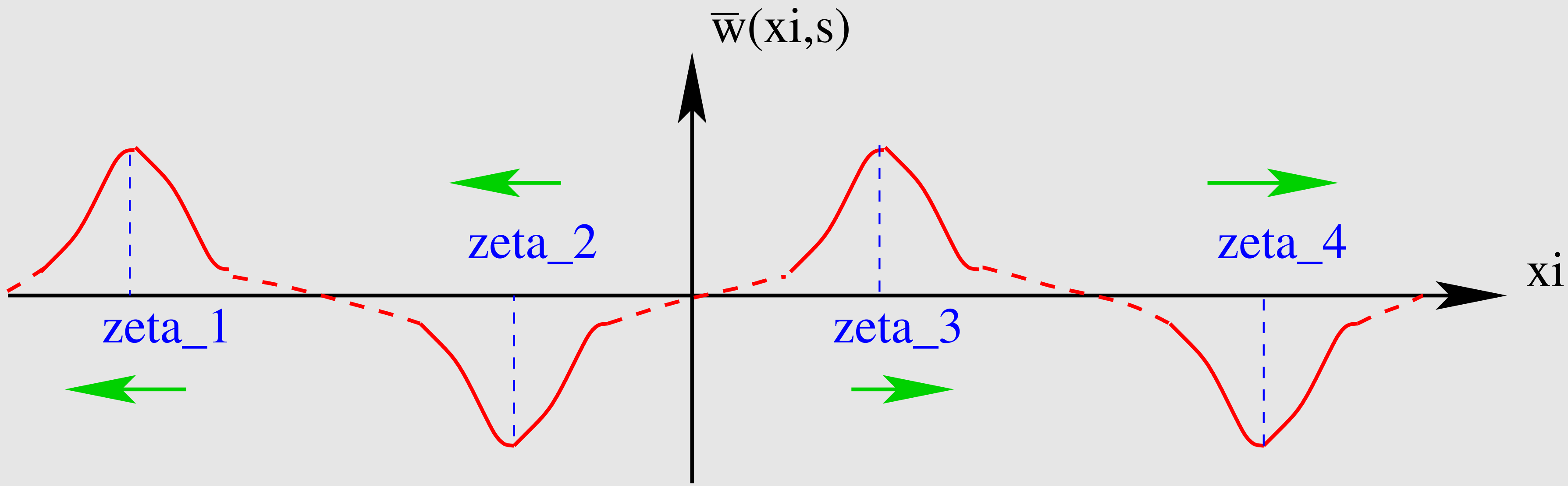
$$\|\bar{w}_{x_0}(\xi, s) - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \cosh^{-\frac{2}{p-1}} (\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

Part 5: Asymptotic behavior at a *characteristic* point (cont.)

(iii) For all $i = 1, ..., k(x_0)$ and s large enough,

$$\left(i - \frac{(k(x_0)+1)}{2}\right)\frac{(p-1)}{2}\log s - C_0 \le \zeta_i(s) \le \left(i - \frac{(k(x_0)+1)}{2}\right)\frac{(p-1)}{2}\log s + C_0.$$

(iv) $E(w_{x_0}(s)) \to k(x_0)E(\kappa_0)$ as $s \to \infty$.



Part 5: Asymptotic behavior at a *characteristic* point (cont.)

Rk.

- As $s \to \infty$, w_{x_0} becomes like a **decoupled** sum of *equidistant* stationary solutions (“solitons”), with *alternate* signs.
- In the ξ variable, half of the solitons go to $-\infty$, and the other half to $+\infty$.
- The main difficulty in the proof is to prove that $k(x_0) \geq 2$ (the case $k(x_0) = 0$ is harder to eliminate).
- The $\zeta_i(s)$ satisfy the following system:

$$\frac{1}{c_1}\zeta_i'(s) = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)} + R_i \text{ with } R_i = o\left(\sum_{j=1}^{k-1} e^{-\frac{2}{p-1}(\zeta_{j+1} - \zeta_j)}\right) \text{ as } s \to \infty.$$

The energy behavior

Defining

$$k(x_0) = 1 \text{ when } x_0 \in \mathcal{R},$$

we get the following:

Cor.
(i) *For all* $x_0 \in \mathbb{R}$ *and* $s \geq -\log T(x_0)$, *we have*

$$E(w_{x_0}(s)) \geq k(x_0) E(\kappa_0).$$

(ii) **(An energy criterion for non characteristic points)** *If for some* $x_0 \in \mathbb{R}$ *and* $s_0 \geq -\log T(x_0)$, *we have*

$$E(w_{x_0}(s_0)) < 2E(\kappa_0),$$

then $x_0 \in \mathcal{R}$.

Blow-up speed or the L^∞ norm behavior

Cor.

(i) **(Case of non-characteristic points)** *If $x_0 \in \mathcal{R}$, then*

$$\frac{(T(x_0)-t)^{-\frac{2}{p-1}}}{C} \le \sup_{|x-x_0|<T(x_0)-t} |u(x,t)| \le C(T(x_0)-t)^{-\frac{2}{p-1}}$$

(i) **(Case of characteristic points)** *If $x_0 \in \mathcal{S}$, then*

$$\frac{|\log(T(x_0)-t)|^{\frac{k(x_0)-1}{2}}}{C(T(x_0)-t)^{\frac{2}{p-1}}} \le \sup_{|x-x_0|<T(x_0)-t} |u(x,t)| \le \frac{C|\log(T(x_0)-t)|^{\frac{k(x_0)-1}{2}}}{(T(x_0)-t)^{\frac{2}{p-1}}}.$$

where $k(x_0) \ge 2$ is the solitons' number in the decomposition of w_{x_0}.

Rk.

When $x_0 \in \mathcal{R}$, the speed is given by the associated ODE $u'' = u^p$.
When $x_0 \in \mathcal{S}$, the speed is higher. It has a $\log$ correction depending on the number of solitons.

Idea of the proof of the results in the *characteristic* case

The results are: the decomposition into solitons, the corner property and the fact that the interior of $\mathcal{S}$ is empty.

6 main steps are needed:

- Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons, with no information on the signs or the distance between the solitons' centers (in the ξ variable).
- Step 2: Characterization of the case $k(x_0) \geq 2$. Proof of *the upper bound* in the corner property if $k(x_0) \geq 2$.
- Step 3: Excluding the case $k(x_0) = 0$ if $x_0 \in \partial\mathcal{S}$ (note that $\partial\mathcal{S} \subset \mathcal{S}$ since $\mathcal{R} = \mathbb{R}\backslash\mathcal{S}$ is open).
- Step 4: Characterization of the case where $x_0 \in \partial\mathcal{S}$ and $k(x_0) = 1$.
- Step 5: We prove that the interior of $\mathcal{S}$ is empty, then that $k(x_0) \geq 2$ for all $x_0 \in \mathcal{S}$ (which gives *the upper bound* in the corner property by Step 2).
- Step 6: We prove that $\mathcal{S}$ is made of isolated points and the *lower bound* in the corner property (omitted here).

Comments

Rk. 1: A good understading of the *non-characteristic* case is *crucial*.

Rk. 2: Excluding the case $k(x_0) = 0$ is more difficult than excluding the case $k(x_0) = 1$.
In particular, we can't exclude directly the case $k(x_0) = 0$ for all $x_0 \in \mathcal{S}$. We do it first when $x_0 \in \partial\mathcal{S}$, then prove that the interior of $\mathcal{S}$ is empty, hence $\partial\mathcal{S} = \mathcal{S}$.

Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons

Take $x_0 \in \mathbb{R}$ a **characteristic** points. We have two estimates:

- $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{\mathcal{H}} \leq C_0$;
- $\int_{-\log T(x_0)}^{\infty} \int_0^1 (\partial_s w_{x_0}(y,s))^2 \frac{\rho}{1-y^2} dyds \leq C_0$.

Rk. Unlike the non characteristic case, we can't have a covering argument, so we can't obtain the $H^1 \times L^2$ norm bounded (in fact, we will show that it is unbounded).

Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons (cont.)

In the $\bar{w}_{x_0}(\xi, s)$ variable, we have

$$\|\bar{w}_{x_0}(\xi, s)\|_{H^1(\mathbb{R})} \leq C_0.$$

For any sequence ξ_n in $\mathbb{R}$, we find a "local" limit in the sense that for some $s_n \to \infty$, we have

$$\bar{w}_{x_0}(\xi + \xi_n, s + s_n) \to \bar{w}^* \text{ as } n \to \infty,$$

uniformly on compact sets for (ξ, s), with w^* a stationary solution, due to the fact that

$$\int_{-\log T(x_0)}^{\infty} \int_0^1 (\partial_s w_{x_0}(y, s))^2 \frac{\rho}{1 - y^2} dyds \leq C_0.$$

Since the energy is bounded, the number of non zero "local limits" is finite, and we end-up with the following result:

Step 1: Decomposition into a decoupled sum of $k(x_0) \geq 0$ solitons (cont.)

Prop. *There exist $k(x_0) \geq 0$ and continuous $d_i(s) \in (-1,1)$ such that*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \sum_{i=1}^{k(x_0)} e_i(x_0) \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

with

$$\zeta_{i+1}(s) - \zeta_i(s) \to \infty \text{ as } s \to \infty \text{ and } d_i(s) = -\tanh \zeta_i(s).$$

Rk.

- If $k(x_0) = 0$, then the above sum is 0.
- At this level, we don't know that $k(x_0) = 0$ and $k(x_0) = 1$ don't occur.
- We have no information on the signs $e_i(x_0)$.
- We have no equivalent for $\zeta_i(s)$ as $s \to \infty$.

— LaTeX prosper —

Step 2: Case $k(x_0) \geq 2$; A differential equation on the solitons' centers

Here, we assume that $k(x_0) \geq 2$ (we don't prove that fact here).
Linearizing the equation in the $w(y,s)$ setting around the sum of the solitons, we get the following system on the solitons' centers in the ξ variable:
for all $i = 1, \dots, k$ and s large enough, we have

$$\frac{1}{c_1}\zeta_i' = -e_{i-1}e_i e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} + e_i e_{i+1} e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)} + R_i$$

where

$$|R_i| \leq C J^{1+\delta_0}, \quad J(s) = \sum_{j=1}^{k-1} e^{-\frac{2}{p-1}(\zeta_{j+1}(s) - \zeta_j(s))},$$

$e_0 = e_{k+1} = 0$, for some $c_1 > 0$ and $\delta_0 > 0$.

Step 2: Case $k(x_0) \geq 2$ (cont.)

Since for all $i = 1, ..., k(x_0) - 1$, we have

$$\zeta_{i+1}(s) - \zeta_i(s) \to \infty \text{ as } s \to \infty,$$

using ODE techniques, we find that

$$e_i e_{i+1} = -1 \text{ and } \zeta_i(s) \sim \left(i - \frac{k(x_0)+1}{2}\right) \frac{(p-1)}{2} \log s.$$

The upper bound on the blow-up rate gives the *upper bound* in the corner property.

Step 3: Excluding the case where $x_0 \in \partial \mathcal{S}$ and $k(x_0) = 0$

By contradiction, if $x_0 \in \partial \mathcal{S}$ and $k(x_0) = 0$, then

$$\|w_{x_0}(s)\|_{\mathcal{H}} \to 0 \text{ and } E(w_{x_0}(s)) \to 0 \text{ as } s \to \infty.$$

Fixing s_0 large enough such that $E(w_{x_0}(s_0)) \leq \frac{1}{4} E(\kappa_0)$, we find x_1 near x_0 such that

$$x_1 \in \mathcal{R} \text{ and } E(w_{x_1}(s_0)) \leq \frac{1}{2} E(\kappa_0).$$

Since $E(w_{x_1}(s)) \to E(\kappa_0)$ as $s \to \infty$ and $E(w_{x_1}(s))$ is decreasing, it follows that

$$E(w_{x_1}(s_0)) \geq E(\kappa_0).$$

Contradiction.

Step 4: Characterization of the case where $x_0 \in \partial \mathcal{S}$ and $k(x_0) = 1$

In this case,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e_1 \begin{pmatrix} \kappa(d_1(s), y) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty \text{ and } E(w_{x_0}(s)) \ge E(\kappa_0).$$

Our "trapping" result implies that for some $d(x_0) \in (-1, 1)$,

$$w_{x_0}(s) \to \kappa(d(x_0)) \text{ as } s \to \infty.$$

Some elementary geometry and the precise knowledge of the case of non characteristic points gives that x_0 is either left-non-characteristic or right-non-characteristic.

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Step 5: Conclusion without Isolatedness

Using the previous steps, we prove in the same time that $k(x_0) \geq 2$ and the interior of $\mathcal{S}$ is empty, together with precise estimate on the location of the solitons' centers.
We also get *the upper bound* in the corner property.