Blow-up behavior for subconformal semilinear wave equations

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Joint work with

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Introduction : The equation

$$\begin{cases} \partial_t^2 u = \Delta u + |u|^{p-1} u, \\ \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $1 < p < p_c = 1 + \frac{4}{N-1}$, $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$, $u_0 \in H^1(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N)$.
Rk.: $p_c \equiv 1 + \frac{4}{N-1} < 1 + \frac{4}{N-2}$, the Sobolev critical exponent.

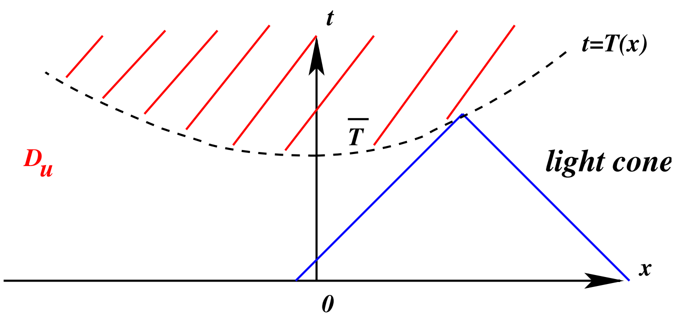
Earlier work: Levine 1974, Caffarelli and Friedman 1985, Ginibre, Soffer and Velo 1992, Kichenassamy and Littman 1993, Alinhac 1995, Lindblad and Sogge 1995, Shatah and Struwe 1998, Killip, Stroval and Vişan 2012, Donninger and Shorkhüber 2012, Schlag, Krieger, Nakanishi, etc...

Singular solutions: the maximal influence domain

We consider an arbitrary blow-up solution $u(x,t)$.
From the finite speed of propagation, its domain of definition is

$$D_u = \{(x,t) \mid 0 \le t < T(x)\}$$

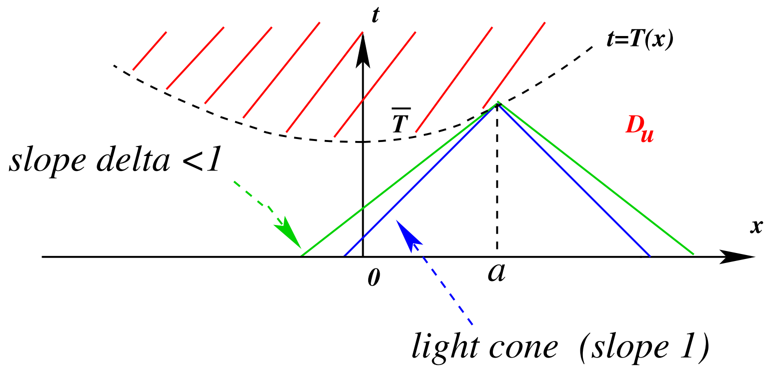
where $x \mapsto T(x)$ is 1-Lipschitz.



Remark: For all $x \in \mathbb{R}^N$, there exists a "local" blow-up time $T(x)$.

Definition: Non characteristic points and characteristic points

A point a is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



The point is said *characteristic* if not.

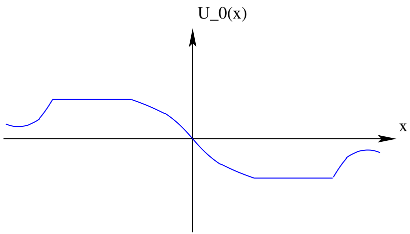
- Notation: $\mathcal{R} \subset \mathbb{R}^N$ is the set of all *non* characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}^N$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}^N$).

Case $N = 1$ (and $p > 1$): Existence results

Rk. All blow-up solutions have non-characteristic points ($x_0 = \arg\min T(x)$);

Th (Merle, Z.): There *exist* solutions with characteristic points.

Example: We take odd initial data, with two large plateaus of different signs. Then, the solution blows up, and the origin is a characteristic point with $\forall t < T(0)$, $u(0,t) = 0$.



Th. (Merle-Z.) If we perturb the constructed initial data, then the new solution blows up and has a characteristic point.

Case $N = 1$ (and $p > 1$): Asymptotic behavior

Introducing similarity variables

$$w_{x_0}(y,s) = (T(x_0) - t)^{\frac{2}{p-1}} u(x,t) \text{ with } y = \frac{x - x_0}{T(x_0) - t} \text{ and } s = -\log(T(x_0) - t),$$

and the soliton

$$\kappa(d,y) = \kappa_0(p) \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}},$$

we have as $s \to \infty$:
- if $x_0 \in \mathcal{R}$, then $w_{x_0}(y,s) \to \pm\kappa(d(x_0), y)$;
- if $x_0 \in \mathcal{S}$, then $w_{x_0}(y,s) \sim \pm \sum_{i=1}^{k} (-1)^i \kappa(d_i(s), y)$ **(multi-solitons)**

with
$k \geq 2$ and $d_i(s) = \tanh(C_0(i - \frac{k+1}{2}) \log s + C_1)$.
Th. (Côte, Z.) : Every multi-soliton modality does occur.

Illustration with hyperbolic coordinates when $x_0 \in \mathcal{S}$

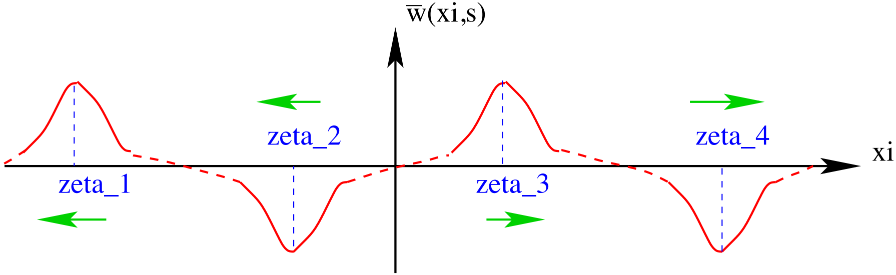
Introducing for $\xi \in \mathbb{R}$,

$$\bar{w}_{x_0}(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_{x_0}(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(s) = C_0(i - \frac{k+1}{2}) \log s + C_1),$$

we get

$$\|\bar{w}_{x_0}(\xi, s) - \epsilon(x_0)\kappa_0 \sum_{i=1}^{k(x_0)} (-1)^i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

and with $k(x_0) = 4$ and $\epsilon(x_0) = -1$:



Behavior of the solitons' centers

$(\zeta_i)_{i=1,\dots,k}$ is a solution to the system

$$\dot{\zeta}_i = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)}, \quad i = 1, \dots, k,$$

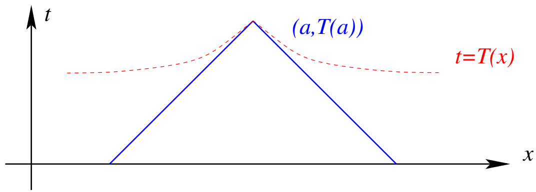
with the convention $\zeta_0(s) \equiv -\infty$, $\zeta_{k+1}(s) \equiv +\infty$. Note that the barycenter is *conserved* $\frac{1}{k}(\zeta_1(s) + \cdots + \zeta_k(s)) \equiv \bar{\zeta}(x_0)$. One can compute explicitely:

$$\zeta_i(s) = \left(i - \frac{k+1}{2}\right)\frac{(p-1)}{2}\ln s + \alpha_i + \bar{\zeta}(x_0),$$

where $\alpha_i = \alpha_i(p,k)$ are chosen adequately.

Regularity of the blow-up curve

- $\mathcal{R}$ is open and $T_{|\mathcal{R}}$ is C^1; more precisley, if $d(x_0)$ is such that $w_{x_0}(y,s) \sim \pm\kappa(d(x_0), y)$, then, $T'(x_0) = d(x_0)$;
- $\mathcal{S}$ is finite on compact sets, and T is corner shaped near $a \in \mathcal{S}$.



Furthermore, for some $\gamma = \gamma(p) > 0$,

$$T(x) - T(x_0) + |x - x_0| \sim \frac{\gamma e^{2\zeta_0 \operatorname{sgn}(x_0 - x)} |x - x_0|}{|\ln |x - x_0||^{\frac{(k(x_0)-1)(p-1)}{2}}} \text{ as } x \to x_0,$$

where $k(x_0)$ is the solitons' number and $\zeta_0(x_0)$ is their barycenter. Note that

- The number of solitons $k(x_0)$ can be "seen" on the blow-up curve.
- The blow-up curve is never symmetric with respect to x_0, unless maybe if the barycenter of the solitons $\zeta_0(x_0) = 0$.

(relatively) "Easy" Generalizations

- **When $N = 1$ with lower order perturbations (M.A. Hamza and Z. 2013)**:

$$\partial_t^2 u = \partial_x^2 u + |u|^{p-1}u + f(u) + +g(\partial_t u, \partial_x u, x, t)$$

with

$$|f(u)| \le C(1 + |u|^q), \quad |g(\partial_t u, \partial_x u, x, t)| \le M(1 + |\partial_t u| + |\partial_x u| + |u|^q) \text{ and } q < p.$$

- **When $N \ge 2$, $p < p_c$, with radial symmetry, outside the origin**:

$$\partial_t^2 u = \partial_r^2 u + (N-1)\frac{\partial_r u}{r} + |u|^{p-1}u$$

(this is because the term $\frac{\partial_r u}{r}$ appears as a lower order perturbation).

- **A mixture of both cases (radial + perturbations)**,
- **When $u \in \mathbb{C}$ (by A. Azaiez 2013)**.

And what about $N \geq 2$ with u not necessarily radial?

We know the blow-up rate:

- If $x_0 \in \mathcal{R}$, then

$$0 < \epsilon_0(N,p) \leq \|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(|y|<1)} \leq K(u_0, u_1);$$

- If $x_0 \in \mathcal{S}$, then

$$\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(|y|<\frac{1}{2})} \leq K(u_0, u_1).$$

Question: Can we derive the limit of $w_{x_0}(y,s)$ as $s \to \infty$?

Candidates for the limit as $s \to \infty$?

Note that $w_{x_0}(y,s)$ satisfies for all $|y| < 1$ and $s \geq -\log T(x_0)$:

$$\partial_s^2 w = \frac{1}{\rho(y)}\mathrm{div}[\rho(\nabla w - (y\cdot\nabla w)) - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w$$
$$-\frac{p+3}{p-1}\partial_s w - 2y\cdot\nabla\partial_s w \qquad (1)$$

where $\rho(y) = (1-|y|^2)^\alpha$ and $\alpha = \dfrac{2}{p-1} - \dfrac{N-1}{2} > 0$ since $p < p_c$.

Candidates for the limit as $s \to \infty$?

Thanks to dissipation, there is a Lyapunov functional, and we can prove that when $x_0 \in \mathcal{R}$,

$$\inf_{v \in S, v \neq 0} \|w_{x_0}(s) - v\|_{H^1 \times L^2(|y|<1)} \to 0 \text{ as } s \to \infty,$$

where S is the set of all finite-energy stationary solutions of equation (1).
The problem: unlike when $N = 1$, we don't have $S = S_0$ where

$$S_0 \equiv \{0, \pm\kappa(d, y) \mid d \in \mathbb{R}^N,\ |d| < 1\} \text{ and } \kappa(d, y) = \kappa_0(p) \frac{(1 - |d|^2)^{\frac{1}{p-1}}}{(1 + d \cdot y)^{\frac{2}{p-1}}}.$$

We only have $S_0 \subset\neq S$.
Consequence: it is certainly not true that for any solution and any $x_0 \in \mathcal{R}$,

$$w_{x_0}(y, s) \to \pm\kappa(d(x_0), y) \text{ as } s \to \infty.$$

Consequence for our framework

We restrict ourselves to the subset $\mathcal{R}_0 \subset \mathcal{R}$ where

$$\mathcal{R}_0 = \{x_0 \in \mathcal{R} \mid w_{x_0} \to \pm\kappa(d, y) \text{ for some } |d| < 1\}.$$

Rk.

- If $N = 1$, we already proved that $\mathcal{R} = \mathcal{R}_0$;
- If $N \geq 2$, we have $\mathcal{R}_0 \neq \emptyset$, thanks to explicit blow-up solutions related to $\kappa(d, y)$.

Stability results related to $\mathcal{R}_0$

- **Result 1: w.r.t to the blow-up point**. $\mathcal{R}_0$ is open and $T_{|\mathcal{R}_0}$ is C^1. In fact, $\nabla T(x_0) = d(x_0)$ where $d(x_0)$ is s.t.

$$w_{x_0}(y,s) \to \pm\kappa(d(x_0),y), \text{ hence } u(x,t) \sim \pm \frac{\kappa_0(p)(1-|d(x_0)|^2)^{\frac{1}{p-1}}}{(T(x_0)-t+d(x_0)\cdot(x-x_0))^{\frac{2}{p-1}}}.$$

- **Result 2: w.r.t. initial data**. Take $\hat{u}$ a blow-up solution and $\hat{a} \in \mathcal{R}_0(\hat{u})$. Then, there exists $\epsilon_0 > 0$ such that $B(\hat{a},\epsilon_0) \subset \mathcal{R}_0(u)$ for any solution satisfying

$$\|(u_0,\partial_t u_0) - (\hat{u}_0,\partial_t \hat{u}_0)\| \leq \epsilon_0.$$

- **Corollary: Persistence of a local minimum**. Assume in addition that $x \mapsto \hat{T}(x)$ achieves a local minimum at $\hat{a}$. Then, $x \mapsto T(x)$ achieves also a local minimum in $B(\hat{a},\epsilon_0)$.
- We have the same statement with a local maximum.

Ingredients of the proof

As for $N = 1$, the proof relies on two ingredients:

- A very good understanding of the dynamics of the equation in similarity variables near $\pm\kappa(d, y)$;
- A rigidity theorem for ancient solutions of $\partial_t^2 u = \Delta u + |u|^{p-1}u$ defined in an infinite backward non-characteristic cone.

However, **the proof is far from being a simple adaptation of the case $N = 1$.**

In the following, we will present these two ingredients and insist on the difference with the case $N = 1$.

Ingredient 1: the dynamics near $\kappa(d,y)$

I recall the equation (1)

$$\partial_s^2 w = \frac{1}{\rho(y)}\mathrm{div}[\rho(\nabla w - (y\cdot\nabla w)) - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w$$

$$-\frac{p+3}{p-1}\partial_s w - 2y\cdot\nabla\partial_s w$$

where $\rho(y) = (1-|y|^2)^\alpha$ and $\alpha = \dfrac{2}{p-1} - \dfrac{N-1}{2} > 0$ since $p < p_c$.

I recall **the soliton**:

$$\kappa(d,y) = \kappa_0(p)\frac{(1-|d|^2)^{\frac{1}{p-1}}}{(1+d\cdot y)^{\frac{2}{p-1}}}.$$

Ingredient 1: the dynamics near $\kappa(d, y)$ (cont.)

And now, **the dynamics near the soliton**:

There exists $\epsilon_0 > 0$ *s.t. if* w *is a solution of* (1) *with*

$$\|w(0) - \kappa(d^*)\|_{\mathcal{H}} \leq \epsilon_0 \text{ for some } |d^*| < 1,$$

then:

- *either* $w(s) \to 0$ *as* $s \to \infty$;
- *or* $w(s) \to \kappa(d_\infty)$ *for some* d_∞ *close to* d^*;
- *or* $w(s)$ *blows up in finite time.*

Idea of the proof

Linearizing equation (1) around $\kappa(d,\cdot)$, we find the following eigenvalues:

- $\lambda = 1$, with multiplicity 1 and the same eigenfunction as for $N = 1$ (depending only on $|d|$);
- $\lambda = 0$, with multiplicity N, with one eigenfunction which depends only on $|d|$ and is the same as for $N = 1$, and the $N - 1$ other eigenfunctions depend also on the $N - 1$ angular directions of d (*new feature*)
- An infinite-dimensional negative subspace.

Control of the nonnegative directions

For the $N+1$ nonnegative directions, we use a modulation technique, choosing parameters $d(s) \in \mathbb{R}^N$ and $\nu(s) \in \mathbb{R}$ such that we kill the $N+1$ nonnegative directions of $q(y,s)$, where

$$q(y,s) = w(y,s) - \kappa^*(d,\nu,y).$$

and

$$\kappa^*(d,\nu,y) = \kappa_0(p)\frac{(1-|d|^2)^{\frac{1}{p-1}}}{(1+\nu+d\cdot y)^{\frac{2}{p-1}}}.$$

Rk. Note that:

- When $\nu = 0$, $\kappa^*(d,0,y) = \kappa(d,y)$;
- For any $\mu \in \mathbb{R}$, the function $(y,s) \mapsto \kappa^*(d,\mu e^s,y)$ is a particular solution of equation (1).

Control of the negative directions (of infinite dimension)

We do it thanks to the Lyapunov functional of the linearized equation (obtained by multiplication of the equation by the time derivative, then, integrating).

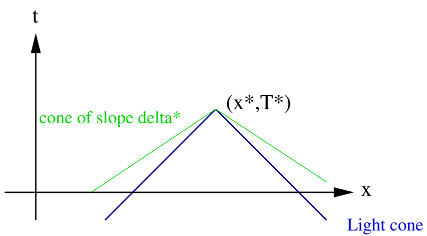
The difficulties with respect to $N = 1$:

- Handling the angular derivatives (a new feature);
- Deriving spectral properties of the operator $v \mapsto \frac{1}{\rho(y)}\mathrm{div}[\rho(\nabla v - (y \cdot \nabla v))$ (related to the Laplace-Beltrami operator);
- Deriving some sharp Hardy-Sobolev inequality;
- The modulation technique is more intricate.

Ingredient 2: A rigidity (or Liouville) Theorem

The result: Consider $u(x,t)$ a solution of $\partial_t^2 u = \Delta u + |u|^{p-1}u$ such that:
- u is defined in the *infinite* green cone with $\delta^* < 1$,
- u is less than $(T^* - t)^{-\frac{2}{p-1}}$ (in L^2 average),
- In the blue cone (a light cone),

$$u(x,t) = \kappa_0(p)\frac{(1-|d^*|^2)^{\frac{1}{p-1}}}{(T^* - t + d^* \cdot (x - x^*))^{\frac{2}{p-1}}} \text{ for some } |d^*| < 1. \qquad (2)$$



Then, (2) holds in the green cone.

Comments on the Liouville Theorem

When $N = 1$, we didn't have the third hypothesis. Thus, we had more possibilities in the conclusion.

In fact, both for $N = 1$ and $N \geq 2$, we apply the rigidity theorem to data satisfying this third hypothesis.

Proving a rigidity theorem without the third hypothesis is blocked by the non-availability of the classification of the finite-energy self-similar solutions.

The proof of the Liouville Theorem

Thanks to the third hypothesis (not used when $N = 1$), we know that for any $a \in \mathbb{R}$, $w_a(s) \to \kappa(d)$ as $s \to -\infty$:

$\Longrightarrow$ We can use our sharp knowledge of the dynamics of the equation near $\kappa(d, y)$ given in the first ingredient.

Thank you for your attention.