# Energy methods and blow-up rate for semilinear wave equations

Hatem ZAAG

CNRS and LAGA Université Paris 13

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Joint work with:
M.A. Hamza (Faculté des Sciences de Tunis).

# Introduction: The equation

$$\begin{cases} \partial_t^2 u = \Delta u + |u|^{p-1}u + f(u) + g(x,t,\nabla u,\partial_t u), \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

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where:

$1 < p < p_S \equiv 1 + \frac{4}{N-2}$,

$u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$,

$u_0 \in H^1(\mathbb{R}^N)$, $u_1 \in L^2(\mathbb{R}^N)$,

and

$$\begin{array}{rcl} |f(u)| & \leq & M(1+|u|^q), \text{ with } (q < p, \ \ M > 0), \\ |g(x,t,\nabla u,\partial_t u)| & \leq & M(1+|\nabla u| + |\partial_t u|), \end{array}$$

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with $f$ Lipschitz-continuous, and $g$ locally Lipschitz-continuous.

**Rk.** The Klein-Gordon equation $\partial_t^2 u = \Delta u + |u|^{p-1}u + \alpha u$ is included in our framework.

# Another critical exponent below Sobolev

Introducing the *conformal* exponent:

$$p_c = 1 + \frac{4}{N-1} < p_S = 1 + \frac{4}{N-2},$$

3 cases will be considered:

- subconformal exponent $p < p_c$
- conformal exponent $p = p_c$
- superconformal exponent $p_c < p < p_S$.

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**In the literature**, the exponent $p_c$ is called *conformal* since only for $p = p_c$, there is some $\alpha \in \mathbb{R}$ such that the unperturbed equation

$$\partial_t^2 u = \Delta u + |u|^{p-1} u$$

is invariant under the *conformal transformation* $u(x,t) \mapsto U(\xi,\tau)$ defined by

$$U(\xi,,\tau) = (t^2 - |x|^2)^\alpha u(x,t) \text{ with } \xi = \frac{x}{t^2 - |x|^2} \text{ and } \tau = \frac{t}{t^2 - |x|^2}.$$

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**Rk.** When $p = p_c$, we take $\alpha = \frac{N-1}{2}$. Surprisingly, in our analysis, we never use this invariance....

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- We may solve the solution for $t \le 0$, since the equation is invariant by time-symmetry. Here, we only consider $t \ge 0$;
- We may solve the Cauchy problem in $H^s \times H^{s-1}$ for $s < 1$ (see Lindblad and Sogge 1995), but then, we are out of the energy space.

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$$\mathcal{E}(u) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) dx.$$

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  If it is negative at $t = 0$, then the solution blows up in finite time.

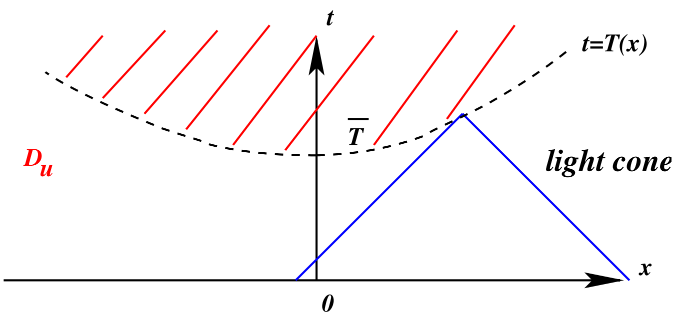
- *ODE techniques*: truncate at $t = 0$ any blow-up solution of the ODE, and you get a (non-trivial) blow-up solution of the PDE, thanks to the finite speed of propagation. This works also for the equation with perturbations.

## Definition: the maximal influence domain

We consider an arbitrary blow-up solution $u(x,t)$.
From the finite speed of propagation, its domain of definition is

$$D_u = \{(x,t) \mid 0 \le t < T(x)\}$$

where $x \mapsto T(x)$ is 1-Lipschitz.



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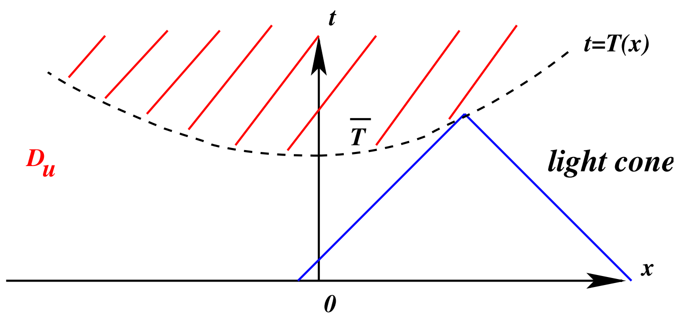
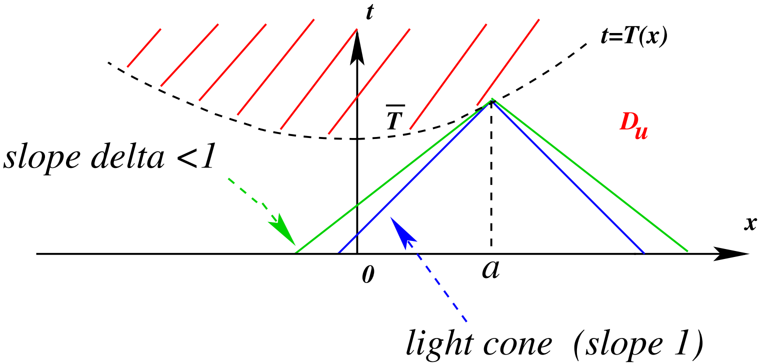



**Remark**: For all $x \in \mathbb{R}^N$, there exists a "local" blow-up time $T(x)$.

## Definition: Non characteristic points and characteristic points

A point $a$ is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



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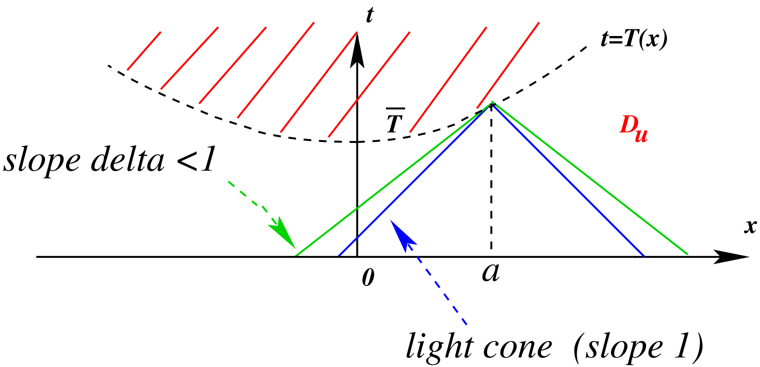


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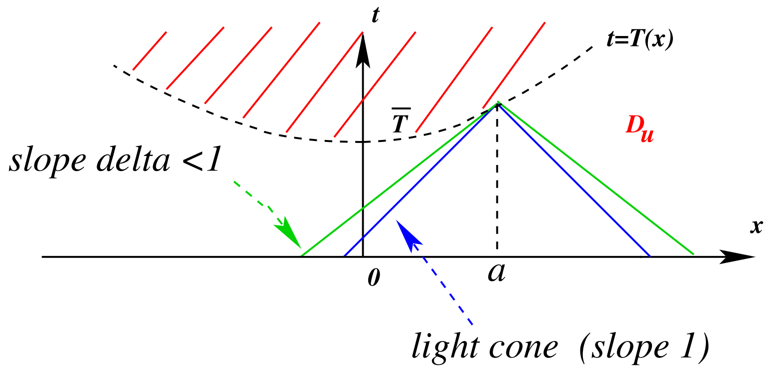


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- Notation: $\mathcal{R} \subset \mathbb{R}^N$ is the set of all *non* characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}^N$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}^N$).

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We consider $u(x,t)$ a blow-up solution, with blow-up graph $\{t = T(x)\}$.

**Our goal**: obtain the blow-up rate, i.e. an estimate of the norm of the solution at blow-up.

**Rk.** We will in fact bound $L^2$ averages of $u$, $\partial_t u$ and $\nabla u$ in sections of the backward light cone with vertex $(a, T(a))$.

# Strategy of the energy method of Merle-Z., 2003-2005

It was first developed for the *unperturbed* equation

$$\partial_t^2 u = \Delta u + |u|^{p-1} u$$

with *subconformal* exponent $p < p_c$.

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2. Multiplying the equation by $\partial_s w_{x_0}$ then integrating, we get a *first identity*, which provides a *Lyapunov functional* $E_p(w_{x_0}(s)) \le C$.

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3. The derivation of a blow-up criterion which implies that $E_p(w_{x_0}(s)) \geq 0$.
4. Multiplying the equation, this time by $w_{x_0}$ then integrating, we get a *second identity*.
5. Starting from these bounds on $E_p(w_{x_0}(s))$, we use interpolation, Gagliardo-Nirenberg estimates, coercivity and covering to bound $w_{x_0}(s)$ itself.

# Related approaches

Our approach is related to that developed by Giga by Kohn in the 80' for the semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1}u,$$

then completed by Giga, Matsui, Sasayama in 2004, proving that when $1 < p < p_S$,

$$\kappa(p) \le (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty} \le C(u_0).$$

# The blow-up rate for the subconformal case without perturbations

**Theorem (Merle-Z. 2005)**

*Consider* $u(x,t)$ *a blow-up solution of* $\partial_t^2 u = \Delta u + |u|^{p-1}u$ *with* $p < p_c$.

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*If* $x_0 \in \mathcal{R}$ *and* $\frac{T(x_0)}{2} \le t < T(x_0)$, *then we have*

$$\begin{aligned} 0 < \varepsilon_0 \le \; & (T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{\frac{N}{2}}} \\ + \; & (T(x_0)-t)^{\frac{2}{p-1}+1} \left( \frac{\|\partial_t u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{\frac{N}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{\frac{N}{2}}} \right) \le K, \end{aligned}$$

*where* $\varepsilon_0 = \varepsilon_0(N,p)$ *and* $K = K(u_0,u_1,x_0)$.

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**Rk.** The blow-up rate is given by the solution of the associated ODE $u'' = u^p$.

# The framework of the proof: the similarity variables

Introducing the similarity variables' transformation:

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}}u(x,t) \text{ with } y = \frac{x-x_0}{T(x_0)-t} \text{ and } s = -\log(T(x_0)-t),$$

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we see that for all $(y,s) \in B(0,1) \times [-\log T(x_0), +\infty)$, we have (with $w = w_{x_0}$)

$$\partial_s^2 w = \frac{1}{\rho_p}\mathrm{div}[\rho_p(\nabla w - (y\cdot\nabla w)y)] - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w$$

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where $\rho_p(y) = (1-|y|^2)^{\alpha(p)}$ and $\alpha(p) = \dfrac{2}{p-1} - \dfrac{N-1}{2} > 0$ if $p < p_c$.

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Carrying on *the strategy* (blow-up criterion, interpolation, Gagliardo-Nirenberg, Coercivity and Covering), we bound the norm of $w(s)$.

# The subconformal case with perturbations

Let us recall the equation:

$$\partial_t^2 u = \Delta u + |u|^{p-1}u + f(u) + g(x,t,\nabla u,\partial_t u),$$

where:

$$\begin{aligned} |f(u)| &\leq M(1+|u|^q), \text{ with } (q<p, \ \ M>0), \\ |g(x,t,\nabla u,\partial_t u)| &\leq M(1+|\nabla u|+|\partial_t u|), \end{aligned}$$

with $f$ Lipschitz-continuous, and $g$ locally Lipschitz-continuous.

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Let us focus on the *existence of a Lyapunov functional*?

# Similarity variables in the perturbed subconformal case

Introducing the same *similarity variables'* transformation:

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(x,t) \text{ with } y = \frac{x-x_0}{T(x_0)-t} \text{ and } s = -\log(T(x_0)-t),$$

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*What about the size of the perturbative terms?*

# Size of the perturbative terms

Recalling that

$$\begin{aligned} |f(u)| &\leq M(1+|u|^q), \text{ with } (q<p, \quad M>0), \\ |g(x,t,\nabla u,\partial_t u)| &\leq M(1+|\nabla u|+|\partial_t u|), \end{aligned}$$

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*Good news !!! All the perturbative terms come with a negative exponential !!!*

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where $\theta >> 1$, $\gamma = \min(\frac{1}{2}, \frac{p-q}{p-1}) > 0$, and

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Note that

$$H_p(w) \sim E_p(w) \text{ as } s \to \infty.$$

# The dissipation of the new Lyapunov functional

**Theorem (Hamza-Z., 2012)**

*For $s$ large enough, we have*

$$\frac{d}{ds}H_p(w(s)) \le -\alpha \int_B (\partial_s w)^2 \frac{\rho_p}{1-|y|^2} dy,$$

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We recall the equation

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In similarity variables, we have the same energy (here the weight $\rho_{p_c} \equiv 1$):

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Indeed, in the subconformal case, the dissipation was

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**Rk.** Because of the degeneracy, we need a covering method to overcome this degeneracy.

# The perturbed conformal case ($p = p_c$)

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We need a new idea....

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Here, the weight is $\rho_{p_c} \equiv 1$, and the integration by parts gives a boundary term.

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**Idea**: Let us rewrite the linear term in a divergence form, involving $\rho_{\bar{p}}$, for some $\bar{p} < p = p_c$. More precisely, we write

$$\frac{1}{\rho_{p_c}} \text{div}[\rho_{p_c}(\nabla w - (y \cdot \nabla w) y)] = \frac{1}{\rho_{\bar{p}}} \text{div}[\rho_{\bar{p}}(\nabla w - (y \cdot \nabla w) y)] + 2\alpha(\bar{p}) y \cdot \nabla w$$

where

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**Conclusion**: With a Lyapunov functional involving the weight $\rho_{\bar p}$, we will have a dissipation supported by the unit ball, though with a perturbation term, namely $2\alpha(\bar p)y\cdot\nabla w$, fortunately satisfying $\alpha(\bar p) \to 0$ as $\bar p \to p_c$.

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- for the full system, including the term $2\alpha(\bar{p})y\cdot\nabla w$, we can find a new functional $\bar{H}_{\bar{p},p}$, which satisfies

$$\bar{H}_{\bar{p},p}(w) \sim \bar{E}_{\bar{p},p}(w) \text{as } s\to\infty$$

  and

$$\frac{d}{ds}\bar{H}_{\bar{p},p}(w(s)) \le \frac{\alpha(\bar{p})(p_c+3)}{2}\bar{H}_{\bar{p},p}(w(s)) - C(N,p)\alpha(\bar{p})\int_{|y|<1}(\partial_s w)^2\frac{\rho_{\bar{p}}}{1-|y|^2}dy + Ce^{-\gamma s}$$

  for some $\gamma > 0$.

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With *the strategy* (interpolation, Gagliardo-Nirenberg, Coecrcivity and covering technique), we end-up with a *rough estimate* on $w(s)$, namely

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**Rk.** This is exponential growth, but note that

$$\alpha(\bar{p}) \to \alpha(p_c) \text{ as } \bar{p} \searrow p_c,$$

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This time we will see that the perturbation terms are already small.

# Smallness of the perturbation terms

Indeed, recalling that

$$e^{-\frac{2ps}{p-1}}\left|f\left(e^{\frac{2s}{p-1}}w\right)\right| \le Me^{-\frac{2ps}{p-1}} + Me^{-\frac{2(p-q)s}{p-1}}|w|^q$$

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then choosing $\bar p$ close enough to $p_c$, so that $\alpha(\bar p)$ is small,we see that we can make the **perturbation terms arbitrarily exponentially small**. Hence, we are left with the unperturbed equation in the conformal case, with exponentially small terms, and *the strategy* (Interpolation, Gagliardo-Nirenberg, Coercivity and covering technique) works and yields the boundedness of $w(s)$.

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The equation in similarity variables has the same form as for $p \le p_c$, namely:

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This doesn't work, simply because some integration by parts formula fails.

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**Idea**: As for the perturbed conformal case, we rewrite the superconformal case as a perturbation of the conformal case, with the same large perturbation, namely $-2\alpha(p)y\cdot\nabla w$, where $\alpha(p)$ cannot be considered small this time....

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More precisely, we write our equation as follows (remember that $\rho_{p_c}\equiv 1$)

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Remember that $\alpha(p)<0$.

Applying the strategy of the conformal case, we find a Lyapunov functional (with the weight $\rho_{p_c}\equiv 1$) which is bounded by $e^{-\frac{(p+3)\alpha(p)s}{2}}$. Applying *the strategy* (Interpolation, Gagliardo-Nirenberg, Coercivity and Covering), we bound $w(s)$ by $e^{-\frac{(p+3)\alpha(p)s}{4}}$.

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- This rate doesn't seem to be optimal. Anyway, there is no example satisfying this bound.

# An upper bound on the blow-up rate in the superconformal case

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**Theorem (Hamza-Z. 2014)**

*If* $p_c < p < p_S$ *and* $x_0 \in \mathbb{R}^N$*, then we have the following:*

$$(T(x_0) - t)^{-\frac{(p-1)N}{p+3}} \int_{|x-x_0|<T(x_0)-t} u(x,t)^2 dx \to 0 \text{ as } t \to T(x_0),$$

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- These results hold also for the perturbed case, including the Klein-Gordon equation.
- Killip, Stoval and Vişan proved a slightly weaker version thanks to a different approach (energy method in the backward light cone in the $u(x,t)$ setting).

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**Conclusion**: The same blow-up rate as in the unperturbed case (Giga-Kohn, Giga-Matsui-Sasayama).

Thank you for your attention.