Existence and stability of a solution with a new prescribed behavior for a heat equation with or without a critical nonlinear gradient term

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Introduction: The equation

We consider the following PDE:

$$\begin{cases} \partial_t u &= \Delta u + \mu|\nabla u|^q + |u|^{p-1}u, \\ u(\cdot,0) &= u_0 \end{cases}$$

where:

- $p > 3$, $\mu \in \mathbb{R}$, $0 \le q \le q_c = \frac{2p}{p+1}$,
- $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$,
- $u_0 \in W^{1,\infty}(\mathbb{R}^N)$.

History of the equation $\mu = 0$

This is the simplest case of a heat equation which may exhibit blow-up.

Indeed, if $u_0 \equiv \alpha_0 > 0$, then

$$u(x,t) = u(t) = \kappa (T-t)^{-\frac{1}{p-1}} \to \infty \text{ as } t \to T,$$

with

$$\kappa = (p-1)^{-\frac{1}{p-1}} \text{ and } T = \frac{1}{(p-1)\alpha_0^{p-1}}$$

is called *the blow-up time*.

This is a *lab model* to develop new techniques for the study of blow-up in more physical situations *(Ginzburg-Landau, Navier-Stokes, etc...)*,

History of the equation $\mu \neq 0$

- *Introduction*: Chipot-Weissler (1989), mathematical motivation ($\mu < 0$).
- *Population dynamics interpretation*: Souplet (1996).
- *Mathematical analysis*: Chipot, Weissler, Peletier, Kawohl, Fila, Quittner, Deng, Alfonsi, Tayachi, Souplet, Snoussi, Galaktionov, Vázquez, Ebde, Z., Nguyen, ...
- *Elliptic version* : Chipot, Weissler, Serrin, Zou, Peletier, Voirol, Fila, Quittner, Bandle ...

Two limiting cases

- When $\mu = 0$, this is the well-known *semilinear heat equation*:

$$\partial_t u = \Delta u + |u|^{p-1} u.$$

- When $\mu = +\infty$, we recover (after rescaling) the *Diffusive Hamilton-Jacobi equation*:

$$\partial_t u = \Delta u + |\nabla u|^q.$$

A critical case: $q = q_c$

When $\mu \in \mathbb{R}$ and $w(y,s)$ is the similarity variables version of $u(x,t)$:

$$w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = \frac{x}{\sqrt{T-t}} \text{ and } s = -\log(T-t),$$

we have for all $s \geq -\log T$ and $y \in \mathbb{R}^N$:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w + \mu e^{\alpha s} |\nabla w|^q.$$

with

$$\alpha = \frac{q(p+1)}{2(p-1)} - \frac{p}{p-1} = \frac{(p+1)}{2(p-1)}(q - q_c) \text{ and } q_c = \frac{2p}{p+1}.$$

Therefore, we have 3 cases:

- subcritical when $q < q_c$: we have a "perturbation" of the semilinear heat equation;
- supercritical when $q > q_c$: we are in the Hamilton-Jacobi limit;
- critical when $q = q_c$: this is the aim of the talk.

Other indications for the criticality of $q_c = 2p/(p+1)$

- *Scaling:* Only when $q = q_c$, we have "u solution $\Rightarrow u_\lambda$ solution", where
$$u_\lambda(x,t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t),\ \forall \lambda > 0,\ \forall t > 0,\ x \in \mathbb{R}^N,$$
as for the equation without gradient term ($\mu = 0$).

- *Large time behavior:* it depends on whether $q < q_c$, $q = q_c$, $q > q_c$; see Snoussi-Tayachi-Weissler (1999) and Snoussi-Tayachi (2007).

- *Blow-up behavior for $\mu < 0$:* it depends also on whether $q < q_c$, $q = q_c$, $q > q_c$; see Souplet (2001, 2005), Chlebik, Fila and Quittner (2003) (Bounded Domain)

- Also for the elliptic version.

Cauchy problem and blow-up solutions

- *Cauchy problem*:
  - for $\mu = 0$, wellposed in $L^\infty(\mathbb{R}^N)$,
  - for $\mu \neq 0$, wellposed in $W^{1,\infty}(\mathbb{R}^N)$ (fixed point argument, see Alfonsi-Weissler (1993), Souplet-Weissler (1999)).
- *Blow-up solutions*: If $T < \infty$, then $\lim_{t\to T} \|u(t)\|_{W^{1,\infty}(\mathbb{R}^N)} = \infty$.

  Definition: x_0 is a blow-up point if $\exists (t_n, x_n) \to (T, x_0)$ s.t. $|u(x,t)| \to \infty$ as $n \to \infty$.

Aim of the talk

Take

$$q = q_c \text{ and either } \mu = 0 \text{ or } \mu \neq 0.$$

We have 3 goals:

- construct a blow-up solution,
- determine its blow-up profile,
- prove its stability (with respect to perturbations in initial data).

Remark. The case $q < q_c$ is treated as a perturbation of the case $\mu = 0$, but we don't mention that here (see Ebde and Zaag (2011)).

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Case $\mu = 0$: the standard semilinear heat equation

- The (generic) profile is given by

$$(T-t)^{1/(p-1)}u(z\sqrt{(T-t)|\log(T-t)|},t) \sim f_0(z) \text{ as } t \to T,$$

where

$$f_0(x) = \left(p-1+b_0|x|^2\right)^{-1/(p-1)} \text{ and } b_0 = (p-1)^2/(4p).$$

See Galaktionov-Posashkov (1985), Berger-Kohn (1988), Herrero-Velázquez (1993).
The constructive existence proof by Bricmont-Kupiainen (1994), Merle-Z. (1997) is based on:
  - The reduction of the problem to a finite-dimensional one;
  - The solution of the finite-dimensional problem thanks to the degree theory.
- Other profiles are possible.

Remark. We will give the proof in this case.

Subcritical case: $\mu \neq 0$ and $q < q_c = \frac{2p}{p+1}$

Ebde and Z. (2011) could adapt the previous existence strategy and *find the same behavior as for $\mu = 0$*, since the gradient term is subcritical in size in similarity variables:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w + \mu e^{\alpha s} |\nabla w|^q.$$

with

$$\alpha = \frac{(p+1)}{2(p-1)}(q - q_c) < 0.$$

Remark. We don't mention the proof in this case.

Critical case: $q = q_c$, with $-2 < \mu < 0$ and $p - 1 > 0$ small

Exact self-similar blow-up solution by Souplet, Tayachi and Weissler (1996):

$$u(x,t) = (T-t)^{-1/(p-1)} W\left(\frac{|x|}{\sqrt{T-t}}\right)$$

where W satisfies the following elliptic equation:

$$W'' + \frac{N-1}{r}W' - \frac{1}{2}rW' - \frac{W}{p-1} + W^p + \mu|W'|^{q_c} = 0.$$

Critical case: $\mu \neq 0$ and $q = q_c$; A numerical result

A similar profile to the case $\mu = 0$ was discovered *numerically* by Van Tien Nguyen (2014):

$$(T-t)^{1/(p-1)} u(z\sqrt{(T-t)|\log(T-t)|}, t) \sim f_0(z) \text{ as } t \to T,$$

where

$$f_\mu(x) = \left(p - 1 + b_\mu |x|^2\right)^{-1/(p-1)}$$

with

$$b_\mu > 0 \text{ and } b_0 = (p-1)^2/(4p),$$

the same as for $\mu = 0$.

Remark: We initially wanted to confirm this result, and ended by finding a *new* type of behavior.

Critical case: $\mu > 0$ and $q = q_c$; Our new profile

Theorem (Tayachi and Z.) *There exists a solution $u(x,t)$ s.t.:*

- *Simultaneous Blow-up: Both u and ∇u blow up as $t \to T > 0$ only at the origin;*
- *Blow-up Profile:*

$$(T-t)^{\frac{1}{p-1}} u(z\sqrt{T-t}|\log(T-t)|^{\frac{p+1}{2(p-1)}}, t) \sim \bar{f}_\mu(z) \text{ as } t \to T$$

with

$$\bar{f}_\mu(z) = \left(p - 1 + \bar{b}_\mu |z|^2\right)^{-\frac{1}{p-1}} \text{ with } \bar{b}_\mu = \frac{1}{2}(p-1)^{\frac{p-2}{p-1}} \left(\frac{(4\pi)^{\frac{N}{2}}(p+1)^2}{p\int_{\mathbb{R}^N}|y|^q e^{-|y|^2/4}dy}\right)^{\frac{p+1}{p-1}} \mu^{-\frac{p+1}{p-1}} > 0.$$

- *Final profile When $x \neq 0$, $u(x,t) \to u(x,T)$ as $t \to T$ with*

$$u(x,T) \sim \left(\frac{\bar{b}_\mu}{2} \frac{|x|^2}{|\log|x||^{\frac{p+1}{p-1}}}\right)^{-\frac{1}{p-1}} \text{ as } x \to 0.$$

Comments

The exhibited behavior is new in two respects:

- *The scaling law*: $\sqrt{T-t}|\log(T-t)|^{\frac{p+1}{2(p-1)}}$ instead of the laws of the case $\mu = 0$, $\sqrt{(T-t)|\log(T-t)|}$ or $(T-t)^{\frac{1}{2m}}$ where $m \geq 2$ is an integer;
- *The profile function*: $\bar{f}_\mu(z) = \left(p-1+\bar{b}_\mu|z|^2\right)^{-\frac{1}{p-1}}$ is different from the profile of the case $\mu = 0$, namely $f_0(z) = \left(p-1+b_0|z|^2\right)^{-\frac{1}{p-1}}$, in the sense that $\bar{b}_\mu \neq b_0$.

Note in particular, that

$$\bar{b}_\mu \to \infty \text{ as } \mu \to 0.$$

Remark: Our solution is different already in the scaling from the numerical solution of Van Tien Nguyen, which is in the $\mu = 0$ style.

Idea of the proof

We follow *the constructive existence proof* used by Bricmont-Kupiainen (1994), Merle-Z. (1997) for the *standard semilinear heat equation*.

That method is based on:

- The reduction of the problem to a finite-dimensional one ($N+1$ parameters);
- The solution of the finite-dimensional problem thanks to the degree theory.

Stability of the constructed solution for $\mu \geq 0$

Thanks to the interpretation of the $(N+1)$ parameters of the finite-dimensional problem in terms of the blow-up time (in $\mathbb{R}$) and the blow-up point (in $\mathbb{R}^N$), the existence proof yields the following:

Theorem (Merle and Z. ($\mu = 0$), Tayachi and Z. ($\mu > 0$): Stability)

The constructed solution is stable with respect to perturbations in inital data in $W^{1,\infty}(\mathbb{R}^N)$.

Applications: Perturbed Hamilton-Jacobi Equation ($\mu > 0$)

Corollary (Tayachi and Z.)

After an appropriate scaling, our results yield stable blow-up solutions for the following Viscous Hamilton-Jacobi equation:

$$\partial_t v = \Delta v + |\nabla v|^q + \nu |v|^{p-1} v;$$

with

$$\nu > 0, \; 3/2 < q < 2, \; p = \frac{q}{2-q}.$$

The solution and its gradient blow up simultaneously, only at one point.

Of course, the blow-up profile is given after an appropriate scaling.

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A formal approach to find the ansatz ($N = 1$)

Following the standard semilinear heat equation case, we work in similarity variables:

$$w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = \frac{x}{\sqrt{T-t}} \text{ and } s = -\log(T-t).$$

We need to *find a solution* for the following equation defined for all $s \geq s_0$ and $y \in \mathbb{R}$:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{w}{p-1} + |w|^{p-1} w + \mu |\partial_y w|^q,$$

such that

$$0 < \epsilon_0 \leq \|w(s)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\epsilon_0} \text{ (type 1 blow-up)}.$$

Simple ideas

Idea 1: Look for a (non trivial) stationary solution:

- for $\mu = 0$ and $p < p_S \equiv \frac{N+2}{N-2}$, we know from Giga and Kohn that we have only 3 solutions: κ, $-\kappa$, 0, where

$$\kappa \equiv (p-1)^{-\frac{1}{p-1}}.$$

  Problem: These are "trivial" solutions.

- $\mu = 0$ and $p \geq p_S \equiv \frac{N+2}{N-2}$: there are some non trivial solutions.
- for $\mu \neq 0$, there is a non trivial solution, by Souplet, Tayachi and Weissler (1996), for p close to 1 (self-similar solution in the $u(x,t)$ setting).

Idea 2: Since $w \equiv \kappa \equiv (p-1)^{-\frac{1}{p-1}}$ is a trivial solution, let us look for a solution w such that

$$w \to \kappa, \quad \text{as} \quad s \to \infty.$$

Inner expansion

We write

$$w = \kappa + \overline{w},$$

and *look for* $\overline{w}$ such that

$$\overline{w} \to 0 \text{ as } s \to \infty.$$

The equation to be satisfied by $\overline{w}$ is the following:

$$\partial_s \overline{w} = \mathcal{L}\overline{w} + \overline{B}(\overline{w}) + \mu|\nabla \overline{w}|^{q_c},$$

where $\mu \in \mathbb{R}$ and $q_c = \frac{2p}{p+1}$,

$$\mathcal{L}v = \partial_y^2 v - \frac{1}{2} y \partial_y v + v,$$

and

$$\overline{B}(\overline{w}) = |\overline{w} + \kappa|^{p-1}(\overline{w} + \kappa) - \kappa^p - p\kappa^{p-1}\overline{w}.$$

Note that $\overline{B}$ is quadratic:

$$\left|\overline{B}(\overline{w}) - \frac{p}{2\kappa}\overline{w}^2\right| \leq C|\overline{w}^3|.$$

The linear operator

Note that $\mathcal{L}$ is self-adjoint in $D(\mathcal{L}) \subset L^2_\rho(\mathbb{R})$ where

$$L^2_\rho(\mathbb{R}) = \left\{ f \in L^2_{loc}(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} (f(y))^2 \rho(y) dy < \infty \right\}$$

and

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{\sqrt{4\pi}}.$$

The spectrum of $\mathcal{L}$ is explicitly given by

$$spec(\mathcal{L}) = \left\{ 1 - \frac{m}{2} \;\middle|\; m \in \mathbb{N} \right\}.$$

All the eigenvalues are simple, and the eigenfunctions h_m are (rescaled) Hermite polynomials, with

$$\mathcal{L} h_m = \left(1 - \frac{m}{2}\right) h_m.$$

In particular, for $\lambda = 1,\ \frac{1}{2},\ 0$, the eigenfunctions are $h_0(y) = 1,\ h_1(y) = y$ and $h_2(y) = y^2 - 2$.

Naturally, we expand $\overline{w}(y,s)$ according to the eigenfunctions of $\mathcal{L}$:

$$\overline{w}(y,s) = \sum_{m=0}^{\infty} \overline{w}_m(s) h_m(y).$$

Since h_m for $m \geq 3$ correspond to negative eigenvalues of $\mathcal{L}$, assuming $\overline{w}$ even in y, we may consider that

$$\overline{w}(y,s) = \overline{w}_0(s)h_0(y) + \overline{w}_2(s)h_2(y),$$

with

$$\overline{w}_0,\ \overline{w}_2\ \to 0 \text{ as } \ s \to \infty.$$

Plugging this in the equation to be satisfied by $\overline{w}$:

$$\partial_s \overline{w} = \mathcal{L}\overline{w} + \overline{B}(\overline{w}) + \mu|\partial_y \overline{w}|^{q_c},$$

we first see that

$$\mu|\partial_y \overline{w}|^{q_c} = \mu 2^{q_c}|y|^{q_c}|\overline{w}_2|^{q_c},$$

then, projecting on h_0 and h_2, we get the following ODE system:

$$\overline{w}_0' = \overline{w}_0 + \frac{p}{2\kappa}\left(\overline{w}_0^2 + 8\overline{w}_2^2\right) + \tilde{c}_0|\overline{w}_2|^{q_c} + O\left(|\overline{w}_0|^3 + |\overline{w}_2|^3\right),$$

$$\overline{w}_2' = 0 + \frac{p}{\kappa}\left(\overline{w}_0\overline{w}_2 + 4\overline{w}_2^2\right) + \tilde{c}_2|\overline{w}_2|^{q_c} + O\left(|\overline{w}_0|^3 + |\overline{w}_2|^3\right),$$

where

$$1 < q_c = \frac{2p}{p+1} < 2, \quad \tilde{c}_0 = \mu 2^{q_c}\left(\int_{\mathbb{R}} |y|^{q_c}\rho\right), \quad \tilde{c}_2 = \mu q_c 2^{q_c-2}\left(\int_{\mathbb{R}} |y|^{q_c}\rho\right).$$

Note that the sign of $\tilde{c}_0$ and $\tilde{c}_2$ is the same as for μ.
Remark. The heart of the argument is to find a solution for this system.
The solution will depend on the value of μ, singularly when $\mu \to 0$.

(Case $\mu = 0$) Looking at the equation to be satisfied by $\overline{w}_2$

Let us write it as follows:

$$\overline{w}_2' = \frac{4p}{\kappa}\overline{w}_2^2(1 + O(\overline{w}_2)) + \frac{p}{\kappa}\overline{w}_0\overline{w}_2 + O\left(|\overline{w}_0|^3\right).$$

Assuming that

$$|\overline{w}_0| \ll |\overline{w}_2|, \text{ (H1)}$$

we end-up with

$$\overline{w}_2' \sim \frac{4p}{\kappa}\overline{w}_2^2,$$

which yields

$$\overline{w}_2 \sim -\frac{\kappa}{4ps}.$$

(Case $\mu = 0$) Looking at the equation to be satisfied by $\overline{w}_0$

Let us recall it

$$\overline{w}_0' = \overline{w}_0 + \frac{p}{2\kappa}\left(\overline{w}_0^2 + 8\overline{w}_2^2\right) + O\left(|\overline{w}_0|^3 + |\overline{w}_2|^3\right),$$

then write it as follows:

$$\overline{w}_0' = \overline{w}_0\left(1 + O\left(\overline{w}_0\right)\right) + \frac{4p}{\kappa}\overline{w}_2^2\left(1 + O\left(\overline{w}_2\right)\right).$$

Assuming that

$$|\overline{w}_0'| \ll |\overline{w}_0|,\ |\overline{w}_0'| \ll |\overline{w}_2|^2,\ \text{(H2)}$$

we end-up with

$$\overline{w}_0 \sim -\frac{4p}{\kappa}\overline{w}_2^2 \sim -\frac{\kappa}{4ps^2} \ll \overline{w}_2 \sim -\frac{\kappa}{4ps}.$$

Note that both hypotheses (H1) and (H2) are satisfied by the solution we have just found.

Case $\mu \neq 0$

Let us recall the system to be satisfied by $\overline{w}_0$ and $\overline{w}_2$:

$$\overline{w}_0' = \overline{w}_0 + \frac{p}{2\kappa}\left(\overline{w}_0^2 + 8\overline{w}_2^2\right) + \tilde{c}_0|\overline{w}_2|^{q_c} + O\left(|\overline{w}_0|^3 + |\overline{w}_2|^3\right),$$

$$\overline{w}_2' = 0 + \frac{p}{\kappa}\left(\overline{w}_0\overline{w}_2 + 4\overline{w}_2^2\right) + \tilde{c}_2|\overline{w}_2|^{q_c} + O\left(|\overline{w}_0|^3 + |\overline{w}_2|^3\right),$$

where

$$1 < q_c = \frac{2p}{p+1} < 2, \quad \tilde{c}_0 = \mu 2^{q_c}\left(\int_{\mathbb{R}} |y|^{q_c}\rho\right), \quad \tilde{c}_2 = \mu q_c 2^{q_c-2}\left(\int_{\mathbb{R}} |y|^{q_c}\rho\right).$$

Note that the sign of $\tilde{c}_0$ and $\tilde{c}_2$ is the same as for μ.

(Case $\mu \neq 0$) Looking at the equation to be satisfied by $\overline{w}_2$

Let us write it as follows:

$$\overline{w}_2' = \tilde{c}_2|\overline{w}_2|^{q_c}\left(1 + O\left(|\overline{w}_2|^{2-q_c}\right)\right) + \frac{p}{\kappa}\left(\overline{w}_0\overline{w}_2\right) + O\left(|\overline{w}_0|^3\right).$$

Assuming that

$$|\overline{w}_0\overline{w}_2| \ll |\overline{w}_2|^{q_c},\ |\overline{w}_0|^3 \ll |\overline{w}_2|^{q_c},\ \text{(H1)}$$

we end-up with

$$\overline{w}_2' \sim sign(\mu)|\tilde{c}_2||\overline{w}_2|^{q_c},$$

which yields

$$\overline{w}_2 = -sign(\mu)\frac{B}{s^{\frac{1}{q_c-1}}},\ \text{for some } B > 0.$$

(remember that $q_c = \frac{2p}{p+1} \in (1,2)$).

(Case $\mu \neq 0$) Looking at the equation to be satisfied by $\overline{w}_0$

Let us write it as follows:

$$\overline{w}_0' = \overline{w}_0 \left(1 + O\left(\overline{w}_0\right)\right) + \tilde{c}_0 |\overline{w}_2|^{q_c} \left(1 + O\left(|\overline{w}_2|^{2-q_c}\right)\right).$$

Assuming that

$$|\overline{w}_0'| \ll \overline{w}_0, \ |\overline{w}_0'| \ll |\overline{w}_2|^{q_c}, \ \text{(H2)}$$

we end-up with

$$\overline{w}_0 \sim -\tilde{c}_0 |\overline{w}_2|^{q_c} \sim -\frac{\tilde{c}_0 B^{q_c}}{s^{\frac{q_c}{q_c-1}}} \ll \overline{w}_2 = -sign(\mu) \frac{B}{s^{\frac{1}{q_c-1}}}.$$

Note that both hypotheses (H1) and (H2) are satisfied by the found solution.

Conclusion for the inner expansion

Recalling the ansatz

$$w(y,s) = \kappa + \overline{w}(y,s) = \kappa + \overline{w}_0(s)h_0(y) + \overline{w}_2(s)h_2(y) \text{with } h_2(y) = y^2 - 2,$$

we end-up with:

$$w(y,s) = \kappa - \tilde{b}\frac{y^2}{s^{2\beta}} + \frac{2\tilde{b}}{s^{2\beta}} + o\left(\frac{1}{s^{2\beta}}\right),$$

with

- for $\mu = 0$: $\tilde{b} = \frac{\kappa}{4p}$ and $\beta = \frac{1}{2}$;
- for $\mu \neq 0$: $\tilde{b} = sign(\mu)B$ and $\beta = \frac{1}{2(q_c-1)} = \frac{p+1}{2(p-1)} > \frac{1}{2}$, with $B = \left[2^{q_c-2}q_c(q_c-1)\int_{\mathbb{R}}|y|^{q_c}\rho\right]^{-\frac{1}{q_c-1}}\mu^{-\frac{p+1}{p-1}}$.

Remark: This expansion is valid in L^2_ρ and uniformly on compact sets by parabolic regularity. However, for y bounded, we see no shape: the expansion is asymptotically a constant.

Idea: *What if $z = \frac{y}{s^\beta}$ is the relevant space variable for the solution shape?*

Outer expansion

To have a *shape*, following the inner expansion, (*valid for* $|y|$ *bounded*),

$$w(y,s) = \kappa - \tilde{b}z^2 + \frac{2\tilde{b}}{s^{2\beta}} + o\left(\frac{1}{s^{2\beta}}\right) \text{ with } z = \frac{y}{s^\beta},$$

let us look for a solution of the following form (*valid for* $|z|$ *bounded*):

$$w(y,s) = f(z) + \frac{a}{s^{2\beta}} + O\left(\frac{1}{s^\nu}\right),\ \nu > 2\beta,$$

with $z = \frac{y}{s^\beta}, f(0) = \kappa$ and f bounded.
Plugging this ansatz in the equation,

$$\partial_s w = \partial_y^2 w - \frac{1}{2}y\partial_y w - \frac{w}{p-1} + |w|^{p-1}w + \mu|\partial_y w|^{q_c},$$

then, keeping only the main order, we get

$$-\frac{1}{2}zf'(z) - \frac{1}{p-1}f(z) + (f(z))^p = 0,$$

hence, $f(z) = \left(p - 1 + \bar{b}|z|^2\right)^{-\frac{1}{p-1}}$, for some constant $\bar{b} > 0$.

Matching asymptotics

For y bounded, both the inner expansion (*valid for* $|y|$ *bounded*)

$$w(y,s) = \kappa - \tilde{b}\frac{y^2}{s^{2\beta}} + \frac{2\tilde{b}}{s^{2\beta}} + o\left(\frac{1}{s^{2\beta}}\right),$$

and the outer expansion (*valid for* $|z|$ *bounded*)

$$w(y,s) = f(z) + \frac{a}{s^{2\beta}} + O(\frac{1}{s^{\nu}}) \text{ where } \nu > 2\beta, \quad z = \frac{y}{s^{\beta}}$$

and

$$f(z) = \left(p - 1 + \bar{b}|z|^2\right)^{-\frac{1}{p-1}} = \kappa - \frac{\bar{b}\kappa}{(p-1)^2}z^2 + O(z^4),$$

have to agree. Therefore,

$$\tilde{b} = \frac{\bar{b}\kappa}{(p-1)^2} \text{ and } a = 2\tilde{b}.$$

Matching asymptotics (continued)

In particular:

- For $\mu = 0$, $\tilde{b} = \frac{\kappa}{4p}$, hence $\bar{b} = \frac{(p-1)^2}{4p}$ and $a = \frac{\kappa}{2p}$;
- For $\mu \neq 0$, $\tilde{b} = sign(\mu)B$ with $B = \left[2^{q_c-2}q_c(q_c-1)\int_{\mathbb{R}}|y|^{q_c}\rho\right]^{-\frac{1}{q_c-1}}\mu^{-\frac{p+1}{p-1}}$.

  Since $B > 0$ and $\bar{b} > 0$ from the inner and the outer expansions, it follows that

$$\mu > 0, \quad \bar{b} = \frac{(p-1)^2}{\kappa}B \text{ and } a = 2B, \text{ with } B = \left[2^{q_c-2}q_c(q_c-1)\int_{\mathbb{R}}|y|^{q_c}\rho\right]^{-\frac{1}{q_c-1}}\mu^{-\frac{p+1}{p-1}}.$$

Conclusion of the formal approach

We have just derived the blow-up profile for $|y| \leq Ks^\beta$:

$$\varphi(y,s) = \bar{f}\left(\frac{y}{s^\beta}\right) + \frac{a}{s^{2\beta}} = \left(p-1+\bar{b}\frac{|y|^2}{s^{2\beta}}\right)^{-\frac{1}{p-1}} + \frac{a}{s^{2\beta}},$$

where:

- For $\mu = 0$,

$$\beta = \frac{1}{2}, \quad \bar{b} = \frac{(p-1)^2}{4p} \text{ and } a = \frac{\kappa}{2p};$$

- Form $\mu \neq 0$,

$$\beta = \frac{p+1}{2(p-1)}, \quad \bar{b} = \frac{(p-1)^2}{\kappa} B \text{ and } a = 2B,$$

  with

$$B = \left[2^{q_c-2} q_c(q_c-1)\int_{\mathbb{R}} |y|^{q_c}\rho\right]^{-\frac{1}{q_c-1}} \mu^{-\frac{p+1}{p-1}}.$$

Strategy of the proof

We follow the strategy used by Bressan (1992), Bricmont and Kupiainen (1994) then Merle and Z. (1997) for the standard semilinear heat equation, based on:

- The reduction of the problem to a finite-dimensional one;
- The solution of the finite-dimensional problem thanks to the degree theory.

This strategy was later adapted for:

- the present equation with *subcritical* gradient exponent $q < q_c$ in Ebde and Z. (2011);
- the Ginzburg-Landau equation:

$$\partial_t u = (1 + i\beta)\Delta u + (1 + i\delta)|u|^{p-1}u - \gamma u$$

  in Z. (1998) and Masmoudi and Z. (2008);
- the supercritical gKdV and NLS in Côte, Martel and Merle (2011);
- the semilinear wave equation

$$\partial_t^2 u = \partial_x^2 u + |u|^{p-1}u$$

  in Côte and Z. (2013), for the construction of a blow-up solution showing multi-solitons.

Construction of solutions of PDEs with prescribed behavior

More generally, we are in the framework of constructing a solution to some PDE with some *prescribed behavior*:

- NLS: Merle (1990), Martel and Merle (2006);
- KdV (and gKdV): Martel (2005), Côte (2006, 2007),
- water waves: Ming-Rousset-Tzvetkov (2013),
- Schrödinger maps: Merle-Raphaël-Rodniansky (2013),
- etc....

The strategy of the proof ($N = 1$)

We recall our aim: to construct a solution $w(y, s)$ of the equation in similarity variables:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{w}{p-1} + |w|^{p-1} w + \mu |\partial_y w|^{q_c},$$

such that

$$w(y, s) \sim \varphi(y, s) \text{ where } \varphi(y, s) = \left(p - 1 + \bar{b} \frac{|y|^2}{s^{2\beta}}\right)^{-\frac{1}{p-1}} + \frac{a}{s^{2\beta}}.$$

Idea: We linearize around $\varphi(y, s)$ by introducing

$$v(y, s) = w(y, s) - \varphi(y, s).$$

In that case, our aim becomes to construct $v(y,s)$ such that

$$\|v(s)\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

and $v(y,s)$ satisfies for all $s \geq s_0$ and $y \in \mathbb{R}$,

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y,s),$$

where

$$\mathcal{L}v = \partial_y^2 v - \frac{1}{2} y \partial_y v + v,$$

$$V(y,s) = p\,\varphi(y,s)^{p-1} - \frac{p}{p-1},$$

$$B(v) = |\varphi + v|^{p-1}(\varphi + v) - \varphi^p - p\varphi^{p-1} v,$$

$$G(\partial_y v) = \mu|\partial_y \varphi + \partial_y v|^{q_c} - \mu|\partial_y \varphi|^{q_c},$$

$$R(y,s) = -\partial_s \varphi + \partial_y^2 \varphi - \frac{1}{2} y \partial_y \varphi - \frac{\varphi}{p-1} + \varphi^p + \mu|\partial_y \varphi|^{q_c}.$$

Effect of the different terms

- The linear term: Its spectrum is given by $\{1 - \frac{m}{2}, \mid m \in \mathbb{N}\}$ and its eigenfunctions are Hermite polynomials with $\mathcal{L}h_m = (1 - \frac{m}{2})h_m$.
  Note that we have two positive directions $\lambda = 1, \ \frac{1}{2}$ and a null direction $\lambda = 0$.
- The potential term V: it has two fundamental properties:
  (i) $V(.,s) \to 0$ in $L^2_\rho(\mathbb{R})$ as $s \to \infty$. In practice, the effect of V in the blow-up area $(|y| \leq Ks^\beta)$ is regarded as a perturbation of the effect of $\mathcal{L}$ (except on the null mode).
  (ii) $V(.,s) \to -\frac{p}{p-1}$ as $s \to \infty$. and $\frac{|y|}{s^\beta} \to \infty$. Since $-\frac{p}{p-1} < -1$ and 1 is the largest eigenvalue of the operator $\mathcal{L}$, outside the blow-up area (i.e. for $|y| \geq Ks^\beta$), we may consider that the operator $\mathcal{L} + V$ has negative spectrum, hence, easily controlled.
- The nonlinear term in v: It is quadratic: $|B(v)| \leq C|v|^2$,
- The nonlinear term in $\partial_y v$: It is sublinear: $\|G(\partial_y v)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{s}}\|\partial_y v\|_{L^\infty(\mathbb{R})}$.
- The rest term: It is small: $\|R(.,s)\|_{L^\infty} \leq \frac{C}{s}$.

Space localization

- From the properties of the profile and the potential, the variable

$$z = \frac{y}{s^\beta}$$

plays a fundamental role, and our analysis will be different in the regions

$$|z| > K \text{ and } |z| < 2K.$$

Decomposition of $v(y,s)$ into “”inner” and “outer” parts

Consider a cut-off function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K\,s^\beta}\right),$$

where $\chi_0 \in C^\infty([0,\infty),[0,1])$, s.t. $\text{supp}(\chi_0) \subset [0,2]$ and $\chi_0 \equiv 1$ in $[0,1]$. Then, introduce

$$v(y,s) = v_{inner}(y,s) + v_{outer}(y,s),$$

with

$$v_{inner}(y,s) = v(y,s)\chi(y,s) \text{ and } v_{outer}(y,s) = v(y,s)\big(1-\chi(y,s)\big).$$

Note that

$$\text{supp } v_{inner}(s) \subset [-2Ks^\beta, 2Ks^\beta],\ \text{supp } v_{outer}(s) \subset (-\infty, -Ks^\beta] \cup [Ks^\beta, \infty).$$

Remark: $v_{outer}(y,s)$ is easily controlled, because $\mathcal{L} + V$ has a negative spectrum (less than $1 - \frac{p}{p-1} + \epsilon < 0$).

Decomposition of the "inner" part

We decompose v_{inner}, according to the sign of the eigenvalues of $\mathcal{L}$:

$$v_{inner}(y,s) = \sum_{m=0}^{2} v_m(s)h_m(y) + v_-(y,s),$$

where v_m is the projection of v_{inner} (and not v on h_m, and $v_-(y,s) = P_-(v_{inner})$ with P_- being the projection on the negative subspace $E_- \equiv Span\{h_m \mid m \geq 3\}$ of the operator $\mathcal{L}$.

Remark: $v_-(y,s)$ is easily controlled because the spectrum of $\mathcal{L}$ restricted to E_- is less than $-\frac{1}{2}$.

It remains then to control v_0, v_1 and v_2.

Control of v_2

This is delicate, because it corresponds to the direction $h_2(y)$, the null mode of the linear operator $\mathcal{L}$.
Projecting the equation

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y,s)$$

on $h_2(y)$, and recalling that $\mathcal{L}h_2 = 0$, we need to refine the contributions of Vv and $G(\partial_y v)$ to the linear term (this is delicate), and write:

$$v_2'(s) = -\frac{2}{s}v_2(s) + O\left(\frac{1}{s^{4\beta}}\right) + O\left(\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right).$$

Working in the slow variable $\tau = \log s = \log|\log(T-t)|$,

we see that

$$\frac{d}{d\tau}v_2 = -2v_2 + O\left(\frac{1}{s^{4\beta-1}}\right) + O\left(s\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right),$$

which shows a negative eigenvalue.
Conclusion: v_2 can be controlled as well.

We are left only with two components v_0 and v_1: A finite dimensional problem.

Dealing with v_0 and v_1

These remaining components correspond repectively to the projections along $h_0(y) = 1$ and $h_1(y) = y$, the *positive* directions of $\mathcal{L}$. Projecting the equation

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y,s)$$

on $h_m(y)$ with $m = 0,\ 1$, we write

$$v_0'(s) = v_0(s) + O\left(\frac{1}{s^{2\beta+1}}\right) + O\left(\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right),$$

$$v_1'(s) = \frac{1}{2}v_1(s) + O\left(\frac{1}{s^{2\beta+1}}\right) + O\left(\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right).$$

Since all the other components are easy to control, we may assume that

$$v(y,s) = v_0(s)h_0(y) + v_1(s)h_1(s) = v_0(s) + v_1(s)y,$$

ending with a "baby" problem, *which is two-dimensional*, with initial data at $s = s_0$ given by

$$v_0(s_0) = d_0, \quad v_1(s_0) = d_1,$$

Solution of the baby problem

Recall the baby problem:

$$v_0'(s) = v_0(s) + O\left(\frac{1}{s^{2\beta+1}}\right) + O\left(v_0(s)^2\right) + O\left(v_1(s)^2\right),$$
$$v_1'(s) = \frac{1}{2}v_1(s) + O\left(\frac{1}{s^{2\beta+1}}\right) + O\left(v_0(s)^2\right) + O\left(v_1(s)^2\right),$$

with initial data at $s = s_0$ given by

$$v_0(s_0) = d_0, \quad v_1(s_0) = d_1.$$

This problem can be easily solved by contradiction, based on *Index Theory*:

There exist a particular value $(d_0, d_1) \in \mathbb{R}^2$ *such that the "baby" problem has a solution* $(v_0(s), v_1(s))$ *which converges to* $(0,0)$ *as* $s \to \infty$.

Conclusion for the full problem

For the full problem (which is *infinite-dimensional*), recalling that

$$v(y,s) = v_{inner}(y,s) + v_{outer}(y,s) = \sum_{m=0}^{2} v_m(s)h_m(y) + v_-(y,s) + v_{outer}(y,s),$$

and that all the three other components correspond to negative eigenvalues, hence easily converging to zero, we have the following statement:

Consider the equation

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y,s)$$

equipped with initial data at $s = s_0$:

$$\psi_{s_0,d_0,d_1}(y) = \Big(d_0h_0(y) + d_1h_1(y)\Big)\chi(2y,s_0).$$

Then, there exists a particular value (d_0,d_1) such that the corresponding solution $v(y,s)$ exists for all $s \geq s_0$ and $y \in \mathbb{R}$, and satisfies

$$\|v(y,s)\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty.$$

End of the proof of the existence proof

Introducing

$$T = e^{-s_0},$$

and recalling that

$$v(y,s) = w(y,s) - \varphi(y,s) \text{ and } u(x,t) = (T-t)^{-\frac{1}{p-1}} w\left(\frac{x}{\sqrt{T-t}}, -\log(T-t)\right),$$

and

$$\varphi(y,s) = \bar{f}_\mu\left(\frac{y}{s^\beta}\right) + \frac{a}{s^{2\beta}},$$

we derive the existence of $u(x,t)$, a solution to the equation

$$\partial_t u = \Delta u + \mu|\nabla u|^{q_c} + |u|^{p-1}u,$$

such that

$$(T-t)^{\frac{1}{p-1}} u(z\sqrt{T-t}|\log(T-t)|^{\frac{p+1}{2(p-1)}}, t) \sim \bar{f}_\mu(z) \text{ as } t \to T.$$

Using refined parabolic regularity estimates, we derive that:

- $u(x,t)$ blows up only at the origin;
- the final profile satisfies $u(x,T) \sim \left(\frac{2}{\bar{b}_\mu}|x|^{-2}\,|\log|x||^{\frac{p+1}{p-1}}\right)^{\frac{1}{p-1}}$ as $x \to 0$.

Proof of the stability result ($N = 1$)

Let us recall the statement:

Theorem (Tayachi and Z.: Stability)

The constructed solution is stable with respect to perturbations in inital data in $W^{1,\infty}(\mathbb{R}^N)$.

Proof: It follows from the existence proof, through the interpretation of the parameters of *the finite-dimensional problem* in terms of the blow-up time and the blow-up point.

Consider $\hat{u}(x,t)$ the constructed solution, with initial data $\hat{u}_0$, blowing up at time $\hat{T}$ only at one blow-up point $\hat{a}$ (not necessarily 0).
Consider now $u_0 \in W^{1,\infty}(\mathbb{R})$ such that

$$u_0 = \hat{u}_0 + \epsilon_0 \text{ with } \|\epsilon_0\|_{W^{1,\infty}(\mathbb{R})} \text{ small},$$

and $u(x,t)$ the corresponding maximal solution and $T(u_0) \leq +\infty$ its maximal existence time.
We would like to prove that $T(u_0) < +\infty$, and that $u(x,t)$ blows up only at one single point $a(u_0)$ with the same profile as for $\hat{u}(x,t)$, with

$$T(u_0) \to \hat{T} \text{ and } a(u_0) \to \hat{a} \text{ as } u_0 \to \hat{u}_0.$$

Our finite dimensional parameters

At this stage, we don't even know that $T(u_0) < +\infty$, don't mention the blow-up point $a(u_0)$.

Since our goal is to show that $T(u_0)$ and $a(u_0)$ are close to $\hat{T}$ and $\hat{a}$ respectively, let us study ALL the similarity variables versions of $u(x,t)$ considered with *arbitrary* (T,a) close to $(\hat{T},\hat{a})$:

$$w_{u_0,T,a}(y,s) = e^{-\frac{s}{p-1}} u(a + ye^{\frac{s}{2}}, T - e^{-s})$$

and

$$v_{u_0,T,a}(y,s) = w_{u_0,T,a}(y,s) - \varphi(y,s) = e^{-\frac{s}{p-1}} u(a + ye^{\frac{s}{2}}, T - e^{-s}) - \varphi(y,s),$$

where the profile:

$$\varphi(y,s) = \bar{f}_\mu(\frac{y}{s^\beta}) + \frac{a}{s^{2\beta}}.$$

The stability problem as an "existence" problem

Note that *for any* (T, a), $v_{u_0,T,a}(y,s)$ satisfies the *same* equation as for the existence proof:

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y,s),$$

with initial data, say at some time $s = s_0$ large enough, given by

$$v_{u_0,T,a}(y,s_0) = \bar{\psi}_{u_0,T,a}(y) \equiv e^{-\frac{s_0}{p-1}} u(a + ye^{\frac{s_0}{2}}, T - e^{-s_0}) - \varphi(y,s_0).$$

Idea: These initial data depend on two parameters, exactly as in the existence proof, where initial data was

$$\psi_{s_0,d_0,d_1}(y) = \Big(d_0 h_0(y) + d_1 h_1(y)\Big)\chi(2y, s_0),$$

depending also on two parameters.
It happens that the behaviors of

$$(d_0, d_1) \mapsto \psi_{s_0,d_0,d_1} \text{ and } (T,a) \mapsto \bar{\psi}_{u_0,T,a}$$

are similar, so the existence proof starting from $\bar{\psi}_{u_0,T,a}$ works also, in the sense that:

Statement for the "new" existence problem

For $||u_0 - \hat{u}_0||_{W^{1,\infty}}$ *small and* s_0 *large enough, there exists some* $(\bar{T}(u_0), \bar{a}(u_0))$ *such that the solution of equation*

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y, s),$$

with initial data at $s = s_0$ *given by*

$$\bar{\psi}_{u_0,T,a}(y) \equiv e^{-\frac{s_0}{p-1}} u(a + ye^{\frac{s_0}{2}}, T - e^{-s_0}) - \varphi(y, s_0)$$

with $(T, a) = (\bar{T}(u_0), \bar{a}(u_0))$, *converges to* 0 *as* $s \to \infty$.

But remember !

$$\bar{\psi}_{u_0,\bar{T}(u_0),\bar{a}(u_0)}(y) = v_{u_0,\bar{T}(u_0),\bar{a}(u_0)}(y, s_0),$$

so we know that solution: it is simply $v_{u_0,\bar{T}(u_0),\bar{a}(u_0)}(y, s)$!!!!

Therefore, this is in reality the statement we have just proved:

There exists some $(\bar{T}(u_0), \bar{a}(u_0))$ *such that* $\|v_{u_0,\bar{T}(u_0),\bar{a}(u_0)}\|_{W^{1,\infty}} \to 0$ *as* $s \to \infty$.

Going back in the transformations, we see that for all $t \in [0, \bar{T}(u_0))$ and $x \in \mathbb{R}$,

$$u(x,t) = (\bar{T}(u_0) - t)^{-\frac{1}{p-1}} w_{u_0,\bar{T}(u_0),\bar{a}(u_0)}(y,s) = (\bar{T}(u_0) - t)^{-\frac{1}{p-1}} \left[\varphi(y,s) + v_{u_0,\bar{T}(u_0),\bar{a}(u_0)}(y,s)\right]$$

where

$$y = \frac{x - \bar{a}(u_0)}{\sqrt{\bar{T}(u_0) - t}} \text{ and } s = -\log(\bar{T}(u_0) - t).$$

From this identity, we see that:

- The blow-up time of $u(x,t)$ is in fact $\bar{T}(u_0)$;
- $u(x,t)$ blows up at the point $\bar{a}(u_0)$;
- $u(x,t)$ has $\varphi(y,s)$ as blow-up profile, the same as for $\hat{u}(x,t)$,

and this is the desired conclusion for the stability.

Thank you for your attention.