Construction d'une solution explosive stable pour l'équation Complexe de Ginzburg-Landau dans un cas critique avec détermination du profil

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The complex Ginzburg Landau (CGL) equation

We consider the following equation

$$\begin{array}{rcl} \partial_t u & = & (1+i\beta)\Delta u + (1+i\delta)|u|^{p-1}u - \gamma u \\ u(x,0) & = & u_0(x) \text{ for } x \in \mathbb{R}^N, \end{array} \qquad (CGL)$$

where

- $p > 1$, β, δ and γ are real.
- $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{C}$.
- $u_0 \in L^\infty(\mathbb{R}^N, \mathbb{C})$.

Content

Physical motivation and Mathematical relevance for CGL

- Physical motivation: When $p = 3$, CGL appears:
  - in the description of plane Poiseuille flow, see Stewartson and Stuart (1971) and Hocking, Stuart and Stewartson (1971);
  - in the context of the binary mixture, see Kolodner, Bensimon and Surko (1988).

- Mathematical relevance: Classical tools break down:
  - Maximum principle;
  - Variational formulation;
  - Energy methods.

History of blow-up in CGL equation

- $p = 3$, Formal approach by Hocking and Stewartson (1972), Popp, Stiller, Kuznetsov and Kramer (1998), under some condition on β and δ:
  - Existence of blow-up solutions;
  - Determination of the blow-up profile.
- Rigorous approach for $p > 1$: Construction, profile and stability, under some condition on β and δ,
  - when $\beta = 0$, see Zaag (1998);
  - when $\beta \neq 0$, see Masmoudi and Zaag (2008).
- Case $\beta = \delta$: This is variational. Results by Cazenave, Dickstein and Weissler.

Cauchy problem and blow-up

- Cauchy problem: Welpossedness in $L^\infty(\mathbb{R}^N, \mathbb{C})$.
  For other spaces, see Ginibre and Velo (1996-1997), Cazenave (2003), Ogawa and Yokota (2004).
- Blow-up solutions: If $T < \infty$, then $\lim_{t\to T} \|u(t)\|_{L^\infty} = +\infty$.
- Blow-up point: The point a is a blow-up point if and only if there exists $(a_n, t_n) \to (a, T)$ as $n \to +\infty$ such that $|u(a_n, t_n)| \to +\infty$.

Content

Case $\beta = \delta = 0$, the heat equation

- The generic profile is given by

$$(T-t)^{\frac{1}{p-1}} u(z\sqrt{(T-t)|\log(T-t)|}, t) \sim f_0(z),$$

where $f_0(z) = (p-1+b_0|z|^2)^{-\frac{1}{p-1}}$ and $b_0 = \frac{(p-1)^2}{4p}$
See Galaktionov-Posashkov (1985), Berger-Kohn (1988), Herrero-Velázquez (1993).
The constructive existence proof by Bricmont-Kupiainen (1994), Merle-Zaag (1997) is based on:
  - The reduction of the problem to a finite-dimensional one.
  - The solution of the finite-dimensional problem thanks to the degree theory.
- Other profiles are possible.

Case $(\beta, \delta) \neq (0, 0)$

If

$$p - \delta^2 - \beta\delta(p+1) > 0, \qquad\qquad (subcritical)$$

then, Masmoudi and Zaag (2008):

- constructed a solution such that

$$(T-t)^{\frac{1+i\delta}{p-1}} |\log(T-t)|^{-i\mu} u(z\sqrt{(T-t)|\log(T-t)|}, t) \sim f(z),$$

where $f(z) = \kappa^{-i\delta}(p-1+b|z|^2)^{-\frac{1+i\delta}{p-1}}$, $\kappa = (p-1)^{-\frac{1}{p-1}}$

$$b = \frac{(p-1)^2}{4(p-\delta^2-\beta\delta(p+1))} \text{ and } \mu = -\frac{2b\beta}{(p-1)^2}(1+\delta^2);$$

- proved the stability with respect to initial data.

Question: What happens in the *critical case*?

Theorem (Nouaili and Zaag; Existence of a blow-up solution with determination of its profile)

If

$$\beta = 0 \text{ and } p = \delta^2,$$

then, there exists a solution $u(x,t)$ blowing up at time $T > 0$ only at the origin, s.t.

- *Blow-up profile*

$$(T-t)^{\frac{1+i\delta}{p-1}} |\log(T-t)|^{-i\mu} u(z\sqrt{(T-t)}|\log(T-t)|^{\frac{1}{4}}, t) \sim f_c(z)$$

as $t \to T$, where

$$f_c(z) = (p - 1 + b_c|z|^2)^{-\frac{1+i\delta}{p-1}},$$

$$b_c = \frac{(p-1)^2}{8\sqrt{p(p+1)}} \text{ and } \mu = \frac{8\delta b^2}{(p-1)^4}(1+p).$$

Comments

The blow-up behavior is *new* in two aspects:

- The scaling law: $\sqrt{(T-t)|\log(T-t)|^{\frac{1}{4}}}$ instead of the law of the subcritical case, $\sqrt{(T-t)|\log(T-t)|}$.
- The profile function: $f_c(z) = (p - 1 + b_c|z|^2)^{-\frac{1+i\delta}{p-1}}$ depends on a constant b_c different from the subcritical case.

Idea of the proof

We follow the the constructive existence proof introduced by Bricmont-Kupiainen (1994), Merle-Zaag (1997) for the standard semilinear heat equation, and used by Masmoudi and Zaag (2008) for the CGL equation in the subcritical case.

The method is based on:

- the reduction of the problem to a finite-dimensional one $(N+1)$ parameters;
- the solution of the finite-dimensional problem thanks to the degree theory.

Stability of the constructed solution

Thanks to the interpretation of the $(N+1)$ parameters of the finite-dimensional problem in terms of the blow-up time (in $\mathbb{R}$) and the blow-up point (in $\mathbb{R}^N$), the existence proof yields the following:

Theorem (Nouaili and Zaag: Stability)

The constructed solution is stable with respect to perturbations in initial data:
Consider initial data $\hat{u}_0$ of the solution of (CGL) with blow-up time $\hat{T}$, blow-up point $\hat{a}$ and profile f_c centred at $(\hat{T}, \hat{a})$.
Then, $\exists \mathcal{V}$ neighborhood of $\hat{u}_0$ s.t. $\forall u_0 \in \mathcal{V}$, $u(x,t)$ the solution of (CGL) blows up at time T, at a point a, with the profile f_c centred at (T, a).

Content

A formal approach to find the ansatz ($N = 1$)

- Herrero, Galaktionov and Velázquez (1991), Tayachi and Zaag (2015).
- Following the standard semilinear heat equation case, we work in *similarity variables*:

$$w(y,s) = (T-t)^{\frac{1+i\delta}{p-1}} u(x,t), \quad y = \frac{x}{\sqrt{T-t}} \text{ and } s = -\log(T-t).$$

We need to find a solution for the following equation defined for all $s \geq s_0$ and $y \in \mathbb{R}$:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{1+i\delta}{p-1} w + (1+i\delta)|w|^{p-1} w,$$

such that

$$0 < \varepsilon_0 \leq \|w(s)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\varepsilon_0}.$$

Inner expansion

We write

$$w = e^{i\mu logs}(v(y,s)+\kappa)$$

and we look for v such that

$$v \to 0 \text{ as } s \to \infty.$$

The equation satisfied by $v = \Re v + i\Im v$ is the following

$$\partial_s v = \tilde{\mathcal{L}} v + f(v) - i\frac{\mu}{s}(v+\kappa),$$

where

$$\tilde{\mathcal{L}} v = \mathcal{L}_0 v + (1+i\delta)\Re v \text{ with } \mathcal{L}_0 = \partial_y^2 v - \frac{1}{2} y \partial_y v,$$

$$f(v) = (1+i\delta)\left(|v+\kappa|^{p-1}(v+\kappa) - \kappa^p - \frac{v}{p-1} - v_1\right)$$

Note that f is quadratic.

The linear operator $\tilde{\mathcal{L}}$

Note that $\mathcal{L}_0$ is self-adjoint in in L^2_ρ, *but not $\tilde{\mathcal{L}}$!!!!* where

$\mathrm{L}^2_\rho = \{g \in \mathrm{L}^2_{loc}(\mathbb{R}) | \int_{\mathbb{R}} (g(y))^2 \rho(y) dy < \infty\}$ and $\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{\sqrt{4\pi}}$.The spectrum of $\tilde{\mathcal{L}}$ is given by

$$spec(\tilde{\mathcal{L}}) = \{1 - \frac{m}{2} | m \in \mathbb{N}\}.$$

The eigenfuntions are given by $(1 + i\delta)h_m$ and ih_m, where h_m are rescaled Hermite polynomials

$$\tilde{\mathcal{L}}((1 + i\delta)h_m) = (1 - \tfrac{m}{2})(1 + i\delta)h_m,$$
$$\tilde{\mathcal{L}}(ih_m) = -\tfrac{m}{2} ih_m.$$

In particular, for $\lambda = 1, \frac{1}{2}$, the eigenfunctions are $(1 + i\delta)h_0(y)$ and $(1 + i\delta)h_1(y)$.
for $\lambda = 0$, the eigenfunctions are $(1 + i\delta)h_2(y)$ and $ih_0(y)$.

Naturally, we expand $v(y,s)$ according to the eigenfunctions of $\tilde{\mathcal{L}}$:

$$v(y,s) = (1+i\delta)\sum_{0}^{\infty} \bar{v}_m h_m + i\sum_{0}^{\infty} \hat{v}_m h_m.$$

Since the eigenfunctions $(1+i\delta)h_m$ for $m \geq 3$ and ih_m for $m \geq 1$ correspond to negative eigenvalues of $\tilde{\mathcal{L}}$, assuming v even in y, we may consider that

$$v(y,s) = (1+i\delta)(\bar{v}_0 h_0 + \bar{v}_2 h_2) + i\hat{v}_0 h_0(y)$$

with

$$\bar{v}_0, \bar{v}_2, \hat{v}_0 \to 0 \text{ as } s \to \infty.$$

Plugging this in the equation to be satisfied by v

$$\partial_s v = \tilde{\mathcal{L}} v + f(v) - i\frac{\mu}{s}(v+\kappa),$$

then, projecting on $(1+i\delta)h_0$, $(1+i\delta)h_2$ and ih_0, we get the following ODE system

Our goal is to find a solution for this system

$$
\left\{
\begin{array}{rcl}
\bar v_0' &=& \bar v_0 + \frac{\mu\delta}{s}\bar v_0 + \frac{\mu}{s}\hat v_0 + \frac{1}{2\kappa}\hat v_0^2 - \frac{(p+1)p}{3\kappa^2}\bar v_0^3 - \frac{(p+1)\delta}{\kappa^2}\bar v_0^2\hat v_0 - \frac{(p+1)}{\kappa^2}\bar v_0\hat v_0^2 - 8\frac{(p+1)}{\kappa^2}\bar v_0\bar v_2^2 \\
&& -\frac{\delta}{2\kappa^2}\hat v_0^3 - 8\frac{(p+1)\delta}{\kappa^2}\hat v_0\bar v_2^2 - \frac{64}{3}\frac{(p+1)p}{\kappa^2}\bar v_2^3 + R_1, \\
\bar v_2' &=& \frac{\mu\delta}{s}\bar v_2 - 40\frac{(p+1)p}{\kappa^2}\bar v_2^3 - 8\frac{(p+1)p}{\kappa^2}\bar v_2^2\bar v_0 - 8\frac{(p+1)\delta}{\kappa^2}\bar v_2^2\hat v_0 \\
&& -\frac{(p+1)p}{\kappa^2}\bar v_2\bar v_0^2 - \frac{(p+1)}{\kappa^2}\bar v_2\hat v_0^2 - 2\frac{(p+1)\delta}{\kappa^2}\bar v_2\bar v_0\hat v_0 + R_1 \\
\hat v_0' &=& -\frac{\mu\kappa}{s} - \frac{\mu(1+p)}{s}\bar v_0 - \frac{\mu\delta}{s}\hat v_0 + \frac{1+p}{\kappa}\hat v_0\bar v_0 + \frac{(1+p)\delta}{\kappa}\bar v_0^2 + \frac{8(1+p)\delta}{\kappa}\bar v_2^2 \\
&& -\frac{\delta(p+1)^2}{\kappa^2}\bar v_0^3 + \frac{(2p-1)(p+1)}{\kappa^2}\bar v_0^2\hat v_0 + \frac{3\delta(p+1)}{2\kappa^2}\bar v_0\hat v_0^2 + \frac{(p+1)}{2\kappa^2}\hat v_0^3, \\
&& -24\frac{\delta(p+1)^2}{\kappa^2}\bar v_0\bar v_2^2 + 8\frac{(2p-1)(p+1)}{\kappa^2}\hat v_0\bar v_2^2 - 64\frac{\delta(p+1)^2}{\kappa^2}\bar v_2^3 + R,
\end{array}
\right.
$$

where $R_1 = O(|\bar v_0|^4 + |\hat v_0|^4 + |\bar v_2|^4)$.

Remark:

- Most of the quadratic terms disappear since $p = \delta^2$;
- If $\mu = 0$, then the system has a solution $\sim \log s$, which doesn't go to 0.

Assuming the second equation is driven by the following two terms:

$$\bar{v}_2' \sim -40\frac{p(p+1)}{\kappa^2}\bar{v}_2^3,$$

we make the following ansatz:

$$\bar{v}_2 = \frac{\alpha}{\sqrt{s}}, \quad \hat{v}_0 << \bar{v}_2, \quad \bar{v}_0 << \bar{v}_2 \text{ for some } \alpha \in \mathbb{R}$$

$$\mu\kappa = \frac{8\delta(p+1)}{\kappa}\alpha^2,$$

we obtain,

$$\bar{v}_2 = -\frac{\kappa}{8\sqrt{p(p+1)}}\frac{1}{\sqrt{s}} + O(\frac{1}{s^{7/4}}), \quad \bar{v}_0 = O(\frac{1}{s^{3/2}}), \quad \hat{v}_0 = O(\frac{1}{s^{5/4}})$$

and

$$\mu = \frac{\delta}{8p}.$$

Conclusion for the inner expansion

Recalling the ansatz

$$w(y,s) = e^{i\mu logs}(\kappa + (1+i\delta)\bar{v}_0 h_0(y) + (1+i\delta)\bar{v}_2 h_2(y) + i\hat{v}_0 h_0(y)),$$

we end-up with

$$w(y,s) = e^{i\mu logs}\left[\kappa - (1+i\delta)\frac{\kappa}{8\sqrt{p(p+1)}}\frac{y^2-2}{\sqrt{s}} + O(\frac{1}{s^{5/4}})\right] \text{ with } \mu = \frac{\delta}{8p}$$

Remark: This expansion is valid in L^2_ρ and uniformly on compact sets by parabolic regularity. However, for bounded y, we see no shape: the expansion is assymptotically a constant (in modulus).

Idea: What if $z = \frac{y}{s^{1/4}}$ is the relevant space variable for the solution's shape?

Outer expansion

To have a shape, following the inner expansion, (valid for bounded $|y|$),

$$w(y,s) = e^{i\mu logs}\left[\kappa + (1+i\delta)\left(-\frac{\kappa}{8\sqrt{p(p+1)}}z^2 + \frac{2}{8\sqrt{p(p+1)}\sqrt{s}}\right) + o(\frac{1}{\sqrt{s}})\right]$$

let us look for a solution of the following form (valid for $|z|$ bounded):

$$w(y,s) = e^{i\mu logs}\left[f(z) + \frac{a}{\sqrt{s}} + O(\frac{1}{s^\nu})\right], \ \nu > 1/2,$$

with $z = \frac{y}{s^{1/4}}$, $f(0) = \kappa$ and f bounded.
Plugging this ansatz in the equation satisfied by w, and keeping only the main order, we get

$$-\frac{1}{2}zf'(z) - \frac{1+i\delta}{p-1}f(z) + (f(z))^p = 0,$$

hence, $f(z) = \kappa^{-i\delta}(p-1+b_c|z|^2)^{-\frac{1+i\delta}{p-1}}$, for some constant $b_c > 0$.

Matching asymptotics

For y bounded, both the inner expansion (valid for $|y|$ bounded):

$$w(y,s) = e^{i\mu logs}[\kappa - (1+i\delta)\frac{\kappa}{8\sqrt{p(p+1)}}\frac{y^2-2}{\sqrt{s}} + o(\frac{1}{s^{1/2}})]$$

and the outer expansion (valid for $|z|$ bounded):

$$w(y,s) = f(z) + \frac{a}{\sqrt{s}} + O(\frac{1}{s^{\nu}}),\ \nu > 1/2,\ z = \frac{y}{s^{1/4}}$$

with

$$f(z) = \kappa^{-i\delta}(p-1+b_c|z|^2)^{-\frac{1+i\delta}{p-1}} = \kappa - (1+i\delta)\frac{\kappa b_c}{(p-1)^2}z^2 + O(z^4),$$

have to agree.

Therefore

$$b_c = \frac{(p-1)^2}{8\sqrt{p(p+1)}},\ a = \frac{\kappa}{4\sqrt{p(p+1)}} \text{ and } \mu = \frac{\delta}{8p}.$$

Conclusion of the formal approach

We have just derived the blow-up profile for $|y| < Ks^{1/4}$

$$\begin{aligned}\varphi(y,s) &= e^{i\mu\log s}\left(f\left(\frac{y}{s^{1/4}}\right)+\frac{a}{s^{1/2}}\right)\\ &= e^{i\mu\log s}\left(\kappa^{-i\delta}(p-1+b_c\frac{|y|^2}{s^{1/2}})^{-\frac{1+i\delta}{p-1}}+\frac{a}{s^{1/2}}\right),\end{aligned}$$

where

$$b_c=\frac{\kappa(p-1)^2}{8\sqrt{p(p+1)}},\ a=\frac{\kappa}{4\sqrt{p(p+1)}} \text{ and } \mu=\frac{\delta}{8p}.$$

Strategy of the proof

We follow the strategy used by Bressan (1992), Bricmont and Kupiainen (1994), then Merle and Zaag (1997) for the semilinear heat equation, based on:

- The reduction of the problem to a finite-dimensional one;
- The solution of the finite-dimensional problem thanks to the degree theory.

This strategy was later adapted for:

- the Heat equation with subcritical gradient exponent by Ebde and Zaag (2011), with critical power nonlinear gradient term by Tayachi and Zaag (2015);
- The complex heat equation by Nouaili and Zaag (2015).
- the Ginzburg-Landau equation: by Zaag (1998), then Masmoudi and Zaag (2008);
- the supercritical gKdV and NLS in Côte, Martel and Merle (2011);
- the semilinear wave equation in Côte and Zaag (2013), for the construction of a blow-up solution showing multi-solitons.

Construction of PDEs with prescribed behavior

More generally, we are in the framework of constructing a solution for some PDE with some prescribed behavior:

- NLS: Merle (1990), Martel and Merle (2006),
- Kdv (and gKdv): Martel (2005), Côte (2006,2007),
- water waves: Ming-Rousset-Tzvetko (2013),
- Schrodinger maps: Merle-Raphael-Rodiansky (2013),
- etc ..

The strategy of the proof ($N = 1$)

We recall our aim: To consruct a solution $w(y, s)$ of the equation in similarity variables:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{1+i\delta}{p-1} w + (1+i\delta)|w|^{p-1} w,$$

such that

$$w(y,s) \sim e^{i\mu \log s} \varphi(y,s)$$

where

$$\varphi(y,s) = \kappa^{-i\delta} (p - 1 + b \frac{y^2}{s^{1/2}})^{-\frac{1+i\delta}{p-1}} + (1+i\delta) \frac{a}{s^{1/2}}.$$

Idea:

We linearize aound φ, introducing $q(y,s)$ and $\theta(s)$

$$w(y,s) = e^{i(\mu \log s + \theta(s))}(\varphi(y,s) + q(y,s))$$

In that case, our aim becomes to find $\theta \in C^1([-\log T, \infty),)$ such that $q(y,s)$ is defined for all $(y,s) \in \times[-\log T, \infty)$ and

$$\|q(s)\|_{L^\infty} \to 0 \text{ as } s \to \infty$$

with a modulation condition (related to $\theta(s)$):

$$\int (\Im(v) - \delta\Re(v))\rho = 0.$$

This choice of $\theta(s)$ kills one neutral mode.

Decomposition of $q(y,s)$ into inner and outer parts

The variable $z = \frac{y}{s^{1/4}}$ plays a fundamental role. Thus we will consider the dynamics for the outer region $|z| > K$ and the inner region$|z| < 2K$.
Consider a cut-off function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{Ks^{1/4}}\right),$$

where $\chi_0 \in C^\infty([0,\infty),[0,1])$, s.t. $supp(\chi_0) \subset [0,2]$ and $\chi_0 \equiv 1$, in $[0,1]$.
Then, we introduce

$$q = q_{inner} + q_{outer}$$

$q(y,s)$ satisfies for all $s \geq s_0$ and $y \in \mathbb{R}$,

$$\partial_s q = \tilde{\mathcal{L}} q - i\left(\frac{\mu}{s} + \theta'(s)\right) q + V_1 q + V_2 \bar{q} + B(q,y,s) + R^*(\theta', y, s),$$

where

$$\begin{array}{lcl}
\tilde{\mathcal{L}} q & = & \partial_y^2 q - \frac{1}{2} y \cdot \partial_y q + (1+i\delta)\Re q, \\
V_1(y,s) & = & (1+i\delta)\frac{p+1}{2}\left(|\varphi|^{p-1} - \frac{1}{p-1}\right), \\
V_2(y,s) & = & (1+i\delta)\frac{p-1}{2}\left(|\varphi|^{p-3}\varphi^2 - \frac{1}{p-1}\right), \\
B(q,y,s) & = & (1+i\delta)\left(|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi - |\varphi|^{p-1} q \right. \\
 & & \left. - \frac{p-1}{2}|\varphi|^{p-3}\varphi(\varphi\bar{q} + \bar{\varphi} q\right), \\
R^*(\theta', y, s) & = & R(y,s) - i\left(\frac{\mu}{s} + \theta'(s)\right)\varphi, \\
R(y,s) & = & -\partial_s \varphi + \partial_y^2 \varphi - \frac{1}{2} y \cdot \partial_y \varphi - \frac{(1+i\delta)}{p-1}\varphi + (1+i\delta)|\varphi|^{p-1}\varphi
\end{array}$$

Effect of the different terms

- The linear term $\tilde{\mathcal{L}}$: Its spectrum is given by $\{1 - \frac{m}{2} | m \in \mathbb{N}\}$ and its eigenfunctions are Hermite polynomials $(1 + i\delta)h_m$ and ih_m.
- The potential terms V_1 and V_2: $V_1 + V_2 \to 0$ in $L^2_\rho(\mathbb{R})$ as $s \to \infty$.
  The effect of $q \mapsto V_1 q + V_2 \bar{q}$ in the blow-up area is regarded as a perturbation of the effect of $\tilde{\mathcal{L}}$.
  Interestingly, V_1 and V_2 converge to negative constants as $s \to \infty$, making the spectrum of $q \mapsto \tilde{\mathcal{L}} q + V_1 q + V_2 \bar{q}$ negative for $|y| \geq K s^{1/4}$.
  Good news: q_{outer} is easily controlled!!!
- The nonlinear term B: It is quadratic $|B(q)| \leq C|q|^2$
- The rest term R^*: It is small $\|R^*(., s)\|_{L^\infty} \leq \frac{C}{\sqrt{s}} + |\theta'(s)|$.

Decomposition of the inner part

We decompose q_{inner} according to the sign of the eigenvalues of $\tilde{\mathcal{L}}$

$$q_{inner} = \sum_0^2 \tilde{q}_n(1+i\delta)h_n + \left(\sum_3^M \tilde{q}_n(1+i\delta)h_n + \sum_0^M \hat{q}_n i h_n\right) + q_-(y,s)$$

- We choose $M \geq 4(\sqrt{1+\delta^2} + 1 + 2 \max_{y\in\mathbb{R}, s\geq 1, i=1,2} |V_i(y,s)|)$, then q_- is easily controlled.
- From the modulation condition (which kills a neutral mode), we get the smalness of the modulation parameter $\theta(s)$:

$$|\theta'(s)| \leq \frac{C}{s^{\frac{3}{2}}}, \text{ where } C > 0.$$

- It remains then to control $\tilde{q}_0$, $\tilde{q}_1$ and $\tilde{q}_2$.

Control of $\tilde{q}_2$

This is delicate, because it corresponds to the direction $(1+i\delta)h_2(y)$, the null mode of the linear operator $\tilde{\mathcal{L}}$.
We need to refine the contribition of the potentials $q \mapsto V_1 q + V_2 \bar{q}$, the nonlinear term B and the rest term R^*.
This is delicate because we are studying a critical problem.
Adding the contributions of all the terms, we obtain:

$$\tilde{q}_2' = -\frac{2}{s}\tilde{q}_2 + O(\frac{1}{s^2})$$

which shows a negative eigenvalue (in the slow variable $\tau = -\log s$).

Conclusion: $\tilde{q}_2$ can be controlled as well.

The finite dimentional problem $\tilde{q}_0$ and $\tilde{q}_1$

The remaining components correspond respectively to the projection along $(1+i\delta)h_0$ and $(1+i\delta)h_1$, the positive direction of $\tilde{\mathcal{L}}$.
Projecting the equation:

$$\partial_s q = \tilde{\mathcal{L}} q - i\left(\frac{\mu}{s} + \theta'(s)\right) q + V_1 q + V_2 \bar{q} + B(q, y, s) + R^*(\theta', y, s),$$

we obtain:

$$\tilde{q}_0^{'} = \tilde{q}_0 + O\left(\frac{1}{s^{3/2}}\right),$$

$$\tilde{q}_1^{'} = \frac{1}{2}\tilde{q}_1 + O\left(\frac{1}{s^{3/2}}\right).$$

with given initial data at s_0 by $\tilde{q}_0 = d_0, \ \ \tilde{q}_1 = d_1$.
This problem can be easily solved by contradiction, using **index Theory**.
There exist a particular $(d_0, d_1) \in \mathbb{R}^2$ such that the problem has a solution $(\tilde{q}_0(s), \tilde{q}_1(s))$ which converges to $(0, 0)$ as $s \to \infty$.

Thank you for your attention.