Blowup solutions for two non-variational semilinear parabolic systems

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CSMT 2018
Tabarka, March 19-22, 2018

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The non-variational semilinear parabolic system

$$\begin{cases} \partial_t u = \Delta u + F(v), \\ \partial_t v = \mu\Delta v + G(u), \end{cases} \quad \mu > 0, \qquad \textbf{(Sys)}$$

where $(u,v)(t) : x \in \mathbb{R}^N \to (u,v)(x,t) \in \mathbb{R}^2$, and

$$F(v) = |v|^{p-1}v, \quad G(u) = |u|^{q-1}u, \quad p, q > 1 \qquad \textbf{(C1)}$$

or

$$F(v) = e^{pv}, \quad G(u) = e^{qu}, \quad p, q > 0 \qquad \textbf{(C2)}$$

- (**Sys**) + (**C1**): Friedman-Giga '87, Escobedo-Herrero '91, Caristi and Mitidieri '97, Andreucci-Herrero-Velázquez '97, Deng '96, Fila-Souplet '01, Z. '01, Mahmoudi-Souplet-Tayachi '15, ...
- (**Sys**) + (**C2**): Friedman-Giga '87, Souplet-Tayachi '16, ...

Definition (Finite time blowup solution)

The Cauchy problem in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$:

- either global existence in time,
- or existence on $[0, T)$ with $T < +\infty$ and

$$\lim_{t \to T} \Big(\|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty} \Big) = +\infty.$$

$\Longrightarrow$ *finite time blowup solution*, T is the blowup time.

- A point $a \in \mathbb{R}^N$ is a *blowup point* of $(u, v)(x, t) \Longleftrightarrow \exists (a_n, t_n) \to (a, T)$ such that $|u(a_n, t_n)| + |v(a_n, t_n)| \to +\infty$ as $n \to +\infty$.
- Note that u and v blow up simultaneously in some finite time.

Definition (Type I and Type II blowup)

- Type I (for System C1):

$$\|u(t)\|_{L^\infty} \le C\bar{u}(t), \quad \|v(t)\|_{L^\infty} \le C\bar{v}(t),$$

where $(\bar{u}, \bar{v})$ is the positive blowup solution of the associated ODEs, namely that

$$\bar{u}(t) = \Gamma(T-t)^{-\frac{p+1}{pq-1}}, \quad \bar{v}(t) = \gamma(T-t)^{-\frac{q+1}{pq-1}} \qquad (\textbf{ODEsol-C1})$$

and (for System C2):

$$\|e^{qu}(t)\|_{L^\infty} + \|e^{pv(t)}\|_{L^\infty} \le \frac{C}{T-t},$$

.

- Type II: otherwise.

Goal:

- Construct a finite time blowup solution for (**Sys**) + (**C1**) or (**C2**) satisfying some prescribed blowup behavior.
- Prove its stability (with respect to perturbations of initial data).

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Theorem 1 (Ghoul-Nguyen -Z. '16).

Type I blowup solutions for (**Sys**) + (**C1**): $\exists (u_0, v_0) \in L^\infty \times L^\infty$ such that:

- u and v blow up simultaneously in finite time T only at the origin, and

$$\left\| (T-t)^{\frac{p+1}{pq-1}} u(x,t) - \Phi_0(\xi) \right\|_{L^\infty} + \left\| (T-t)^{\frac{q+1}{pq-1}} v(x,t) - \Psi_0(\xi) \right\|_{L^\infty} \to 0$$

as $t \to T$, where

$$\Phi_0(\xi) = \Gamma (1 + b|\xi|^2)^{-\frac{p+1}{pq-1}}, \quad \Psi_0(\xi) = \gamma (1 + b|\xi|^2)^{-\frac{q+1}{pq-1}},$$

and

$$\xi = \frac{x}{\sqrt{(T-t)|\ln(T-t)|}}, \quad b = \frac{(pq-1)(2pq+p+q)}{4pq(p+1)(q+1)(\mu+1)}.$$

- $\forall x \neq 0$, $(u,v)(x,t) \to (u^*, v^*)(x)$ as $t \to T$, where

$$u^*(x) \sim \Gamma \left(\frac{b|x|^2}{2|\ln|x||} \right)^{-\frac{p+1}{pq-1}} \textit{ and } v^*(x) \sim \gamma \left(\frac{b|x|^2}{2|\ln|x||} \right)^{-\frac{q+1}{pq-1}} \textit{ as } x \to 0.$$

Theorem 2 (Ghoul- Nguyen- Z. '17).

Type I blowup solutions for (**Sys**) + (**C2**): $\exists (u_0, v_0) \in \mathcal{H}_a \equiv \{(u,v) \in (\bar\phi, \bar\psi) + L^\infty \times L^\infty, q\bar\phi = p\bar\psi = -\ln(1+a|x|^2)\}$ such that:

- e^{qu} and e^{pv} blow up simultaneously in finite time T only at the origin, and

$$\left\|(T-t)e^{qu(x,t)} - \Phi_0(\xi)\right\|_{L^\infty} + \left\|(T-t)e^{pv(x,t)} - \Psi_0(\xi)\right\|_{L^\infty} \to 0$$

as $t \to T$, where

$$p\Phi_0(\xi) = q\Psi_0(\xi) = (1+b|\xi|^2)^{-1}$$

and

$$\xi = \frac{x}{\sqrt{(T-t)|\ln(T-t)|}}, \quad b = \frac{1}{2(\mu+1)}.$$

- $\forall x \neq 0$, $(u,v)(x,t) \to (u^*, v^*)(x)$ as $t \to T$, where

$$u^*(x) \sim \frac{1}{q}\ln\left(\frac{2b}{p}\frac{|\ln|x||}{|x|^2}\right) \textit{ and } \; v^*(x) \sim \frac{1}{p}\ln\left(\frac{2b}{q}\frac{|\ln|x||}{|x|^2}\right) \textit{ as } x \to 0.$$

Theorem 3 (Ghoul- Nguyen- Z. '17).

The constructed solution is stable with respect to perturbations of initial data in $L^\infty \times L^\infty$ for (**Sys**) + (**C1**) and in $\mathcal{H}_a$ (a special affine space) for (**Sys**) + (**C2**).

- **Remark:** Other profiles are possible, but they are suspected to be unstable.

Idea of the proof:

- Implementation of the constructive proof developed by Bricmont-Kupiainen '94, Merle-Z. '97 for the standard semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u, \quad p > 1.$$

- The method relies on two arguments:

- Reduction of the problem to a finite dimensional one ($N + 1$ parameters),
- Solving the finite dimensional problem thanks to a topological argument based on index theory.

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This kind of method has been successfully applied for various problems:

- The semilinear heat equation involving a nonlinear gradient term

$$\partial_t u = \Delta u + |u|^{p-1}u + \mu|\nabla u|^q, \quad p > 1,\ 0 \le q \le q_c = \frac{2p}{p+1},$$

... Ebde-Z. '11 (for $q < q_c$), Tayachi-Z. '15 (for $q = q_c$), Ghoul-Nguyen-Z. '16 (for $u^p \to e^u$, $q = 2$), Bressan '92 (for $u^p \to e^u$ and $\mu = 0$).

- The complex Ginzburg-Landau equation

$$\partial_t u = (1 + \imath\beta)\Delta u + (1 + \imath\delta)|u|^{p-1}u - \gamma u,$$

... Z. '98 (for $\beta = \gamma = 0$), Masmoudi-Z. '08 (for $p - \delta^2 - \beta\delta(p+1) > 0$),
... Nouaili-Z. '17 (for $\beta = 0, \delta = \pm\sqrt{p}$).

- The energy critical and super-critical semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1}u,$$

... Schweyer '12 (for $p = \frac{N+2}{N-2}, N = 4$),
... Collot '16 (for $p > p_{JL} = 1 + \frac{4}{N-4-2\sqrt{N-1}}, N \ge 11$).

- The energy critical and super-critical semilinear wave equation

$$\partial_{tt} u = \Delta u + |u|^{p-1} u,$$

... Hillairet-Raphaël '12 (for $p = \frac{N+2}{N-2}$, $N = 4$), Collot '16 (for $p > p_{JL}, N \geq 11$).

- The energy critical and super-critical nonlinear Schrödinger equation

$$\imath \partial_t u + \Delta u + |u|^{p-1} u = 0,$$

... Merle '90, Merle-Raphaël '05, Merle-Raphaël-Rodnianski '15 (for $p > p_{JL}$, $N \geq 11$).

- The simplified energy critical and super-critical harmonic heat flow

$$\partial_t u = \partial_{rr} u + \frac{N-1}{r} \partial_r u - \frac{N-1}{2r^2} \sin(2u),$$

... Raphaël-Schweyer '14 (for $N = 2$), Ibrahim-Ghoul-Nguyen '16 (for $N \geq 7$).

- The simplified energy critical and super-critical wave maps

$$\partial_{tt} u = \partial_{rr} u + \frac{N-1}{r} \partial_r u - \frac{N-1}{2r^2} \sin(2u),$$

... Raphaël-Rodnianski '12 (for $N = 2$), Ibrahim-Ghoul-Nguyen '17 (for $N \geq 7$).

How to reduce the problem to a finite dimensional one?

Two general approaches:

- **Energy-type estimates** applied for problems that Lyapunov functionals are known.
For examples: the semilinear heat and wave equations, the nonlinear Schrödinger equation, the harmonic heat flow or the wave maps problem.

- **Spectral analysis** applied for problems that spectral properties of the linearized operator is fairly understood (possibly applied for problems that Lyapunov functionals are known).
For examples: the semilinear heat equation involving a nonlinear gradient term, the complex Ginzburg-Landau equation.

- As for system (**Sys**) without a variational structure $\Longrightarrow$ **Spectral analysis**.

Similarity variables:

$$\begin{cases} \Phi(y,s) = (T-t)^{\frac{p+1}{pq-1}} u(x,t), & \Psi(y,s) = (T-t)^{\frac{q+1}{pq-1}} v(x,t) \quad \text{for } \mathbf{(C1)} \\ \Phi(y,s) = (T-t)e^{qu(x,t)}, & \Psi(y,s) = (T-t)e^{pv(x,t)} \quad \text{for } \mathbf{(C2)} \end{cases}$$

$$\text{where} \quad y = \frac{x}{\sqrt{T-t}}, \quad s = -\ln(T-t).$$

Then, $\forall s \geq -\ln T$ and $\forall y \in \mathbb{R}^N$,

$$\begin{cases} \partial_s \Phi &= \Delta\Phi - \frac{1}{2} y.\nabla\Phi - \frac{p+1}{pq-1}\Phi + |\Psi|^{p-1}\Psi, \\ \partial_s \Psi &= \mu\Delta\Psi - \frac{1}{2} y.\nabla\Psi - \frac{q+1}{pq-1}\Psi + |\Phi|^{q-1}\Phi, \end{cases} \qquad \textbf{(Sys-C1)}$$

and

$$\begin{cases} \partial_s \Phi &= \Delta\Phi - \frac{1}{2} y.\nabla\Phi - \Phi + q\Phi\Psi - \frac{|\nabla\Phi|^2}{\Phi}, \\ \partial_s \Psi &= \mu\Delta\Psi - \frac{1}{2} y.\nabla\Psi - \Psi + p\Phi\Psi - \mu\frac{|\nabla\Psi|^2}{\Psi}, \end{cases} \qquad \textbf{(Sys-C2)}$$

In the similarity variables setting, we reduce the proof of Theorem 1 & 2 to the following:

Theorem 4 (Equivalent formulatin of Theorem 1 & 2)**.**

There exist initial data such that system (**Sys-C1**) (or system (**Sys-C2**)) has a solution (Φ, Ψ) defined for all $(y, s) \in \mathbb{R}^N \times [s_0, +\infty)$ satisfying

$$\left\| \Phi(y,s) - \Phi_0\left(\frac{y}{\sqrt{s}}\right) \right\|_{L^\infty} + \left\| \Psi(y,s) - \Psi_0\left(\frac{y}{\sqrt{s}}\right) \right\|_{L^\infty} \longrightarrow 0 \quad \text{as } s \to +\infty.$$

Goal:

- Construct a global solution (Φ, Ψ) for (**Sys-C1**) or (**Sys-C2**).
- Determine the profile Φ_0 and Ψ_0.
- Prove the stability of the constructed solution (Φ, Ψ).

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A formal approach to find the profile (Φ^*, Ψ^*) (N=1)

- The trivial solutions of systems (**Sys-C1**) and (**Sys-C2**) are (Γ, γ) and $(\frac{1}{q}, \frac{1}{p})$ respectively.
- Introducing

$$\begin{aligned} (\bar\Phi, \bar\Psi) &= (\Phi - \Gamma, \Psi - \gamma) && \text{for } (\textbf{Sys-C1}),\\ (\bar\Phi, \bar\Psi) &= (\Phi - \tfrac{1}{p}, \Psi - \tfrac{1}{q}) && \text{for } (\textbf{Sys-C2}) \end{aligned}$$

leading to the system

$$\partial_s \begin{pmatrix} \bar\Phi \\ \bar\Psi \end{pmatrix} = \left(\mathcal{H} + \mathcal{M}_1\right) \begin{pmatrix} \bar\Phi \\ \bar\Psi \end{pmatrix} + \begin{pmatrix} |\bar\Psi + \gamma|^{p-1}(\bar\Psi + \gamma) - p\gamma^p - \frac{p(p-1)}{2}\gamma^{p-1}\bar\Psi \\ |\bar\Phi + \Gamma|^{q-1}(\bar\Phi + \Gamma) - q\Gamma^q - \frac{q(q-1)}{2}\Gamma^{q-1}\bar\Phi \end{pmatrix},$$

and

$$\partial_s \begin{pmatrix} \bar\Phi \\ \bar\Psi \end{pmatrix} = \left(\mathcal{H} + \mathcal{M}_2\right) \begin{pmatrix} \bar\Phi \\ \bar\Psi \end{pmatrix} + \begin{pmatrix} q \\ p \end{pmatrix} \bar\Phi\bar\Psi - \begin{pmatrix} |\nabla\bar\Phi|^2(\bar\Phi + \frac{1}{p})^{-1} \\ \mu|\nabla\bar\Psi|^2(\bar\Psi + \frac{1}{q})^{-1} \end{pmatrix},$$

Spectral properties of the linearized operator (N=1)

- $\mathcal{H}$ and $\mathcal{M}_i$ are given by

$$\mathcal{H} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_\mu \end{pmatrix}, \quad \mathcal{L}_\eta = \eta\Delta - \frac{1}{2} y.\nabla.$$

$$\mathcal{M}_1 = \begin{pmatrix} -\frac{p+1}{pq-1} & p\gamma^{p-1} \\ q\Gamma^{q-1} & -\frac{q+1}{pq-1} \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} 0 & \frac{q}{p} \\ \frac{p}{q} & 0 \end{pmatrix}.$$

- Note that $\mathcal{L}_\eta$ is self-adjoint in $D(\mathcal{L}_\eta) \subset L^2_{\rho_\eta}(\mathbb{R}^N)$, where

$$L^2_{\rho_\eta}(\mathbb{R}^N) = \left\{ f \Big| \int_{\mathbb{R}^N} |f(y)|^2 \rho_\eta(y) dy < +\infty \right\}, \;\; \rho_\eta(y) = \frac{1}{(4\pi\eta)^{N/2}} e^{-\frac{|y|^2}{4\eta}}.$$

- The spectrum of $\mathcal{L}_\eta$ is explicitly given by

$$\text{Spec}(\mathcal{L}_\eta) = \left\{ -\frac{n}{2}, n \in \mathbb{N} \right\}, \quad \mathcal{L}_\eta h_n = -\frac{n}{2} h_n,$$

where the eigenfunctions h_n are (rescaled) Hermite polynomials.

Spectral properties of the linearized operator (N=1)

Lemma (Spectral properties of $\mathcal{H}+\mathcal{M}_i$)

$\exists \binom{f_n}{g_n}, \binom{\tilde f_n}{\tilde g_n}$, *which is a linear combination of Hermite polynomials of degree* $\leq n$, *such that*

$$\left(\mathcal{H}+\mathcal{M}_i\right)\begin{pmatrix} f_n \\ g_n \end{pmatrix} = \left(1-\frac{n}{2}\right)\begin{pmatrix} f_n \\ g_n \end{pmatrix}, \quad \left(\mathcal{H}+\mathcal{M}_i\right)\begin{pmatrix} \tilde f_n \\ \tilde g_n \end{pmatrix} = -\left(\lambda_i^+ + \frac{n}{2}\right)\begin{pmatrix} \tilde f_n \\ \tilde g_n \end{pmatrix},$$

where $\lambda_1^+ = \frac{(p+1)(q+1)}{pq-1}$ *and* $\lambda_2^+ = 1$.

- The linearized operator $\mathcal{H}+\mathcal{M}_i$ has two positive eigenvalues 1 and $\frac{1}{2}$, a zero eigenvalue and an infinite many discrete negative spectrum.

A formal approach to find the profile (Φ^*, Ψ^*) (N=1)

- Expand $(\bar\phi, \bar\psi)$ according to the eigenfunctions of $\mathcal{H} + \mathcal{M}_i$:

$$\begin{pmatrix} \bar\Phi \\ \bar\Psi \end{pmatrix}(y,s) = \sum_{n\in\mathbb{N}} \left[\theta_n(s) \begin{pmatrix} f_n \\ g_n \end{pmatrix} + \tilde\theta_n(s) \begin{pmatrix} \tilde f_n \\ \tilde g_n \end{pmatrix} \right].$$

- Since $\binom{f_n}{g_n}$ for $n \geq 3$ and $\binom{\tilde f_n}{\tilde g_n}$ for $n \geq 0$ correspond to the negative eigenvalues of $\mathcal{H} + \mathcal{M}_i$, assuming that $(\bar\Phi, \bar\Psi)$ is even in y, we may consider

$$\begin{pmatrix} \bar\Phi \\ \bar\Psi \end{pmatrix}(y,s) = \theta_0(s) \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} + \theta_2(s) \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}.$$

- Projecting on $\binom{f_0}{g_0}$ and $\binom{f_2}{g_2}$ yields

$$\begin{cases} \theta_0' = \theta_0 + \mathcal{O}(\theta_0^2 + \theta_2^2), \\ \theta_2' = c_2\theta_2^2 + \mathcal{O}(|\theta_0\theta_2| + |\theta_2|^3 + |\theta_0|^3), \end{cases} \quad \text{for some } c_2 = c_2(p,q,\mu) > 0.$$

A formal approach to find the profile (Φ^*, Ψ^*) (N=1)

- Assuming that $|\theta_0(s)| \ll |\theta_2(s)|$ for $s \gg 1$, we end up with

$$\theta_2(s) = -\frac{1}{c_2 s} + \mathcal{O}\left(\frac{\ln s}{s^2}\right), \quad \theta_0(s) = \mathcal{O}\left(\frac{1}{s^2}\right).$$

- The expansion for $|y|$ bounded:

$$\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}(y,s) = \begin{pmatrix}\Gamma\\ \gamma\end{pmatrix} - \frac{1}{c_2 s}\begin{pmatrix}f_2(y)\\ g_2(y)\end{pmatrix} + \mathcal{O}\left(\frac{\ln s}{s^2}\right) \qquad (\textbf{asym-C1})$$

and

$$\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}(y,s) = \begin{pmatrix}1/p\\ 1/q\end{pmatrix} - \frac{1}{c_2 s}\begin{pmatrix}f_2(y)\\ g_2(y)\end{pmatrix} + \mathcal{O}\left(\frac{\ln s}{s^2}\right). \qquad (\textbf{asym-C2})$$

- These expansions are asymptotically constant for $|y|$ bounded. However, this suggests the relevant space variable for the blowup profile

$$\xi = \frac{y}{\sqrt{s}} = \frac{x}{\sqrt{(T-t)|\ln(T-t)|}}.$$

A formal approach to find the profile (Φ^*, Ψ^*) (N=1)

- To have a shape, we look for a solution of the form

$$\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}(y,s) = \begin{pmatrix}\Phi_0\\ \Psi_0\end{pmatrix}(\xi) + \frac{1}{s}\begin{pmatrix}\Phi_1\\ \Psi_1\end{pmatrix}(\xi) + \cdots, \quad \xi = \frac{y}{\sqrt{s}}.$$

- Plugging this ansatz in the system and keeping only the main order term, we get

$$-\frac{\xi}{2}\Phi_0' - \frac{p+1}{pq-1}\Phi_0 + \Psi_0^p = 0, \quad -\frac{\xi}{2}\Psi_0' - \frac{q+1}{pq-1}\Psi_0 + \Phi_0^q = 0 \qquad \textbf{(ODEs-C1)}$$

and

$$-\frac{\xi}{2}\Phi_0' - \Phi_0 + q\Phi_0\Psi_0 = 0, \quad -\frac{\xi}{2}\Psi_0' - \Psi_0 + p\Phi_0\Psi_0 = 0. \qquad \textbf{(ODEs-C2)}$$

- Solving these ODEs yields

$$\Phi_0(\xi) = \Gamma(1 + b|\xi|^2)^{-\frac{p+1}{pq-1}}, \quad \Psi_0(\xi) = \gamma(1 + b|\xi|^2)^{-\frac{q+1}{pq-1}} \qquad \textbf{(Sol-ODEs-C1)}$$

and

$$p\Phi_0(\xi) = q\Psi_0(\xi) = (1 + b|\xi|^2)^{-1}. \qquad \textbf{(Sol-ODEs-C2)}$$

$b > 0$ needs to be determined !!!

Value of b: Matching asymptotics (given only for the power case)

- Working in L^2_ρ (or uniformly on compact sets $|y| \leq R$, (a smaller zone)), we found

$$\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}(y,s) = \begin{pmatrix}\Gamma\\ \gamma\end{pmatrix} - \frac{1}{c_2 s}\begin{pmatrix}f_2(y)\\ g_2(y)\end{pmatrix} + \mathcal{O}\left(\frac{\ln s}{s^2}\right) \qquad \textbf{(asym-C1)}$$

where $f_2(y)$ and $g_2(y)$ are polynomials of order 2.

- Working uniformly for $|y| \leq K\sqrt{s}$ (a larger zone), we found

$$\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}(y,s) \sim \begin{pmatrix}\Phi_0\\ \Psi_0\end{pmatrix}(\xi), \quad \xi = \frac{y}{\sqrt{s}}$$

with

$$\Phi_0(\xi) = \Gamma(1+b|\xi|^2)^{-\frac{p+1}{pq-1}}, \quad \Psi_0(\xi) = \gamma(1+b|\xi|^2)^{-\frac{q+1}{pq-1}} \qquad \textbf{(Sol-ODEs-C1)}$$

- Matching the two estimates, we get the value of b, hence Φ_0 and Ψ_0

Conclusion of the formal approach

The value of Φ_1 and Ψ_1 can be easily derived (start the induction,...)

Recalling the ansatz:

$$\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}(y,s) = \begin{pmatrix}\Phi_0\\ \Psi_0\end{pmatrix}(\xi) + \frac{1}{s}\begin{pmatrix}\Phi_1\\ \Psi_1\end{pmatrix}(\xi) + \cdots, \quad \xi = \frac{y}{\sqrt{s}},$$

and given that we have just determined Φ_0 and Ψ_0, we have the candidate for the profile :

$$\Phi^*(y,s) = \Phi_0\left(\frac{y}{\sqrt{s}}\right) + \frac{1}{s}\Phi_1(0);$$
$$\Psi^*(y,s) = \Psi_0\left(\frac{y}{\sqrt{s}}\right) + \frac{1}{s}\Psi_1(0).$$

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A sketch of the existence proof

- Introducing

$$\begin{pmatrix} \Lambda \\ \Upsilon \end{pmatrix} = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} - \begin{pmatrix} \Phi^* \\ \Psi^* \end{pmatrix}$$

and (Λ, Υ) satisfies the system

$$\partial_s \begin{pmatrix} \Lambda \\ \Upsilon \end{pmatrix} = \left(\mathcal{H} + \mathcal{M}_i + \mathcal{V}_i\right) \begin{pmatrix} \Lambda \\ \Upsilon \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \text{"quadratic term"}. \qquad (\star)$$

- Construct for $(\star)$ a solution (Λ, Υ) such that

$$\|\Lambda(s)\|_{L^\infty} + \|\Upsilon(s)\|_{L^\infty} \longrightarrow 0 \quad \text{as} \quad s \to +\infty.$$

- The linear part has two fundamental properties:

- for $|y| \geq K\sqrt{s}$: $\mathcal{H} + \mathcal{M}_i + \mathcal{V}_i$ has a negative spectrum for $K \gg 1$.

$\Longrightarrow$ *Control of* $\binom{\Lambda}{\Upsilon}$ *for* $|y| \geq K\sqrt{s}$ *is easy.*

- for $|y| \leq K\sqrt{s}$: the potential $\mathcal{V}_i$ is regarded as a perturbation of $\mathcal{H} + \mathcal{M}_i$.

A sketch of the existence proof (cont.)

- For $|y| \leq K\sqrt{s}$, we decompose

$$\begin{pmatrix}\Lambda\\ \Upsilon\end{pmatrix} = \sum_{n\in\mathbb{N}} \left(\theta_n \begin{pmatrix} f_n\\ g_n\end{pmatrix} + \tilde{\theta}_n \begin{pmatrix}\tilde{f}_n\\ \tilde{g}_n\end{pmatrix}\right) \equiv \sum_{n=0}^{2} \theta_n \begin{pmatrix} f_n\\ g_n\end{pmatrix} + \begin{pmatrix}\Lambda_-\\ \Upsilon_-\end{pmatrix},$$

where $\binom{\Lambda_-}{\Upsilon_-} = \Pi_- \binom{\Lambda}{\Upsilon}$ with Π_- being the projection on the subspace associated to the negative eigenvalues of $\mathcal{H} + \mathcal{M}_i$.
$\Longrightarrow$ $\binom{\Lambda_-}{\Upsilon_-}$ *is controllable to zero.*

- **Control of θ_2 is delicate.** Projecting the system satisfied by $\binom{\Lambda}{\Upsilon}$ on $\binom{f_2}{g_2}$, we need to refine the term $\mathcal{V}_i\binom{\Lambda}{\Upsilon}$ (and a nonlinear gradient term for the (**C2**) case)

$$\left|\frac{d\theta_2(s)}{ds} + \frac{2}{s}\theta_2(s)\right| \leq \frac{C}{s^3}.$$

Making the change of variable $\tau = \ln s$ yields

$$\frac{d\theta_2(\tau)}{d\tau} = -2\theta_2(\tau) + \mathcal{O}\left(e^{-2\tau}\right),$$

which shows a negative eigenvalue $\Longrightarrow$ *θ_2 is controllable to zero.*

A sketch of the existence proof (cont.)

- It remains to control θ_0 and θ_1, the positive directions of the linear operator $\mathcal{H} + \mathcal{M}_i$ (*this is the finite dimensional problem*).

Projecting the PDE on these directions, we find the following (*finite dimensional problem*) ODE system:

$$\theta_0' = \theta_0 + O\left(\frac{1}{s^2}\right),$$
$$\theta_1' = \frac{1}{2}\theta_1 + O\left(\frac{1}{s^2}\right).$$

with given initial data at s_0 by $\theta_0 = d_0 \in \mathbb{R}, \ \ \theta_1 = d_1 \in \mathbb{R}^N$.
This problem can be easily solved by contradiction, using **index Theory**: *There exist a particular* $(d_0, d_1) \in \mathbb{R}^N$ *such that the problem has a solution* $(\theta_0(s), \theta_1(s))$ *which converges to* $(0,0)$ *as* $s \to \infty$.

For the full *infinite dimensional problem*, we consider the initial data depending on $(d_0, d_1) \in \mathbb{R}^{1+N}$:

$$\begin{pmatrix} \Lambda \\ \Upsilon \end{pmatrix}(y, s_0) = \frac{A}{s_0^2}\left(d_0 \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} + d_1 . \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}\right)\chi(y, s_0).$$

Idea of the stability proof

It follows from the existence proof, through the interpretation of the parameters (d_0, d_1) of the finite dimensional problem in terms of the blowup time and the blowup point thanks to the space-time translation invariance of the problem.

Conclusion

- **Conclusion:** We exhibit Type I blowup for system (**Sys**) coupled with (**C1**) or (**C2**) through the spectral analysis. The constructed solution is stable with respect to perturbations of initial data.

- **Conjecture:** The blowup profile (Φ^*, Ψ^*) given in Theorem 1 or 2 is **generic**. *The only available result is due to Herrero-Velázquez '92 for the standard semilinear heat equation in one dimensional case; and they announced the same for higher dimensional cases, but they have never published !*

- **Interesting question:** Are there Type II blowup solutions for (**Sys**)? *To our knowledge, the existence of Type II blowup solutions satisfying some prescribed behavior have been rigorously proved through energy-type methods, and the blowup profile is "generally" given by the (rescaled) stationary solution. Since system (**Sys**) has no variational structure, we expect an implementation of the spectral analysis for Type I blowup would be applicable for the construction of Type II blowup solutions.*

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Thanks!

Thank you for your attention.