Profile of a touch-down solution to a nonlocal MEMS mode

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Definition

MEMS := "MicroElectroMechanical System" : An electronic device consisting in an elastic membrane hanging above a rigid plate connected to an electrical source and a capacitor.

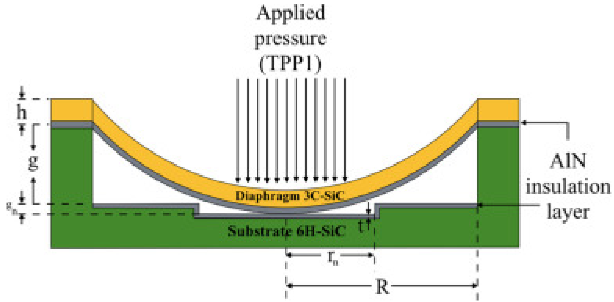

Figure: Mems diagram, *Courtesy of Jindal, Varma and Thukral, Microelectronics Journal, 2018*

MEMS are available in many electronic devices : microphones, transducers, sensors, etc.

References

For more information, see:

- Flores, Mercado, Pelesko, Smyth (SIAM J. Math. Anal 2007),
- Guo and Souplet (SIAM J. Math. Anal 2015),
- Kavallaris and Suzuki (2018 book),
- Guo and Hu (JDE 2018),
- Esteve and Souplet (ADE 2019, Nonlinearity 2018),...

General Model

We consider the (normalized) distance $u(x,t) \in [0,1)$ between the elastic membrane and the rigid plate.

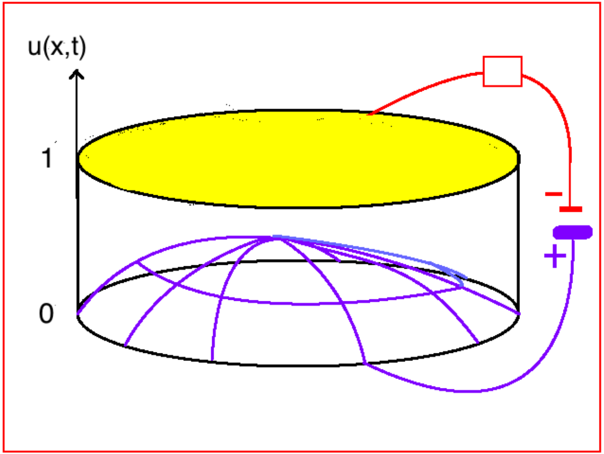

Figure: Courtesy of Carlos Esteve, Deusto University

Hyperbolic Equations

Remember that $u(x,t) \in [0,1)$.

$$\begin{cases} \varepsilon^2 \partial_{tt} u + \partial_t u = \Delta u + \dfrac{f(x,t)}{(1-u)^2 \left(1 + \gamma \displaystyle\int_\Omega \frac{1}{1-u} dx\right)^2}, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Parabolic limit

We take $\varepsilon = 0$.

We also take $f(x,t) \equiv 1$ and $\gamma > 0$.

This is our model:

$$\begin{cases} \partial_t u = \Delta u + \dfrac{1}{(1-u)^2 \left(1 + \gamma \displaystyle\int_\Omega \frac{1}{1-u} dx\right)^2}, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \quad (1)$$

Rk. This is a *non local* parabolic equation.

The “Touch-Down” phenomenon

Thanks to the Cauchy problem, we have two possibilities:

- either the solution is global,
- or there exists $T > 0$ such that $u(x,t) \in [0,1), \forall (x,t) \in \bar{\Omega} \times [0,T)$ and

$$\liminf_{t \to T} \left[\min_{x \in \bar{\Omega}} \{1 - u(x,t)\} \right] = 0. \quad (2)$$

This is finite-time quenching,
Or, in the MEMS context, this is “Touch-Down”, i.e. the membrane “touches down” the rigid plate: *the MEMS device is broken !!!*

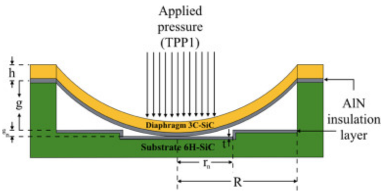

Figure: Mems diagram, *Courtesy of Jindal, Varma and Thukral, Microelectronics Journal, 2018*

Touch-down time and point

Def. 1. T is the touch-down time.

Def. 2. $x_0 \in \Omega$ is a touch-down point, if there exists (x_n, t_n) such that

$$u(x_n, t_n) \to 1 \text{ as } n \to 0,$$

with $(x_n, t_n) \to (x_0, T)$ as $n \to +\infty$.

Our aim

- Construct a solution of equation (1) with only one touch-down point $x_0 \in \Omega$, i.e.

$$u(x_0, t) \to 1 \text{ quand } t \to T.$$

- Describe the shape of the solution around x_0 at the touch-down time.

$$\Downarrow$$

Find a "profile" $\varphi(x)$ such that

$$1 - u(x, T) \sim \varphi(x) \text{ as } x \to x_0.$$

Some earlier results

- *Guo, Hu and Wang (Quart. Appl. Math. 2009)*: They give a sufficient touch-down condition, and provide a lower bound on the touch-down profile:

$$\varphi(x) \geq C(\beta)|x|^{\beta}, \beta \in \left(\frac{2}{3}, 1\right).$$

- *Guo and Kavallaris (Discrete Contin. Dyn. Syst. 2012)*: They prove touch-down under the following sufficient condition:

$$|\Omega| < \frac{1}{2}, \gamma > 0.$$

- *Guo and Hu (J. Diff. Eqs. 2018)*: They estimate the "Touch-Down Rate":

$$\inf_{x\in\Omega} 1 - u(x,t) \sim C(T-t)^{\frac{1}{3}} \text{ as } t \to T.$$

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Existence of a Touch-Down solution

Th. 1[Duong-Z., Math. Models Meth. Appl. Sc., 2019]

There exists a Touch-Down solution, with only one touch-down point 0, at time T, such that:

(i) *Intermediate profile* ($0 < t < T$):

$$\frac{(T-t)^{\frac{1}{3}}}{1-u(x,t)} \sim \theta^* \left(3 + \frac{9}{8}\frac{|x|^2}{(T-t)|\ln(T-t)|}\right)^{-\frac{1}{3}}$$

for some $\theta^* > 0$.

(ii) *Final profile* ($t = T$): $u(x,t) \to u^*(x)$ as $t \to T$ with

$$1 - u^*(x) \sim \theta^* \left[\frac{9}{16}\frac{|x|^2}{|\ln|x||}\right]^{\frac{1}{3}} \text{ as } x \to 0.$$

Some comments on Th. 1

Rk. 0: The final profile is a cusp (not C^1).
Rk. 1: By a simple translation, we may make the solution touch down at any point $x_0 \in \Omega$.

Rk. 2: Thanks to Merle (CPAM 1992), we may construct a solution which touches down at any arbitrary given $x_1, \dots, x_k$.

Rk. 3: The Touch-Down rate is given at the Touch-Down point:

$$u(0,t) = 1 - \frac{\sqrt[3]{3}}{\theta^*}(T-t)^{\frac{1}{3}} + o((T-t)^{\frac{1}{3}}), \text{ as } t \to T.$$

Rk. 4 (An open question): Can we construct a Touch-Down solution with a prescribed θ^* ?

Conjecture: yes, for any

$$\theta^* \in \left((1+\gamma|\Omega|)^{\frac{2}{3}}, +\infty\right).$$

Stability of the Touch-Down profile

Th. 2 [Duong-Z., Math. Mod. Meth. Appl. Sc., 2019]

From Th. 1, we have a solution $\hat{u}$, with initial data $\hat{u}_0$, a Touch-Down time $\hat{T}$ and a Touch-Down point $\hat{a}$ and a profile parameter $\hat{\theta}^*$.

Then, for any nearby u_0, the solution $u(x,t)$ Touches Down at time T_{u_0} at some point a_{u_0} with the *same profile* showing a profile parameter $\theta^*_{u_0}$, such that

$$(a_{u_0}, T_{u_0}, \theta^*_{u_0}) \to (\hat{a}, \hat{T}, \hat{\theta}^*) \text{ as } u_0 \to \hat{u}_0.$$

Proof: The existence result uses a reduction to a $(N+1)-$dimensional problem, which is the dimension of the geometric features of the problem:

- The Touch-Down time: 1 dimension;
- The Touch-Down point: N dimensions.

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A step by step reduction of the problem

The difficulty: This is a non-local PDE.

Introducing

$$v = 1 - u \text{ and } \alpha(v) = \frac{1}{\left(1 + \gamma \int_\Omega \frac{1}{v} dx\right)^2},$$

equation (1) becomes

$$\partial_t v = \Delta v - \frac{\alpha(v)}{v^2}. \tag{3}$$

and our goal becomes to construct a solution to (3) such that

$$v \to 0 \text{ as } t \to T.$$

Rk. The behavior of $\alpha(v(t))$ is crucial in the study.

The simple case when $\alpha(v)$ is replaced by a constant $\equiv \alpha_0 > 0$

In this case, we have the following PDE, with no integral terms.

$$\partial_t v = \Delta v - \frac{\alpha_0}{v^2}.$$

From Merle-Z. (Nonlinearity 1997), we have a solution such that

(i) *Intermediate profile* $(0 < t < T)$:

$$v(x,t) \sim (\alpha_0(T-t))^{\frac{1}{3}} \left(3 + \frac{9}{8} \frac{|x|^2}{(T-t)|\ln(T-t)|}\right)^{\frac{1}{3}}$$

(ii) *Final profile* $(t = T)$: $v(x,t) \to v^*(x)$ as $t \to T$ with

$$v^*(x) \sim \left[\frac{9\alpha_0}{16} \frac{|x|^2}{|\ln|x||}\right]^{\frac{1}{3}} \text{ as } x \to 0.$$

Another simple case : $\alpha(v)$ is replaced by $\alpha(t) \to \alpha_0$

Here again, $\alpha = \alpha(t)$ doesn't depend on the solution $v(x,t)$.

If

$$\alpha(t) \to \alpha_0 \text{ as } t \to T,$$

then, this case is a perturbation of the simple case

$$\alpha(t) \equiv \alpha_0.$$

Our case seen as a perturbation of the simple case

We write our equation as a system

$$\begin{cases} \partial_t v &= \Delta v - \frac{\theta(t)}{v^2}, \\ \theta(t) &= \frac{1}{\left(1+\gamma \int_\Omega \frac{1}{v} dx\right)^2}. \end{cases}$$

The idea: We try to make $\theta(t)$ converge to some $\theta_0 > 0$, so that we reduce to the simple case.

The difficulty: $\theta(t) = \alpha(v(t))$! It depends on the solution itself !

Idea: Let us introduce

$$V = \theta(t)^{-\frac{1}{3}} v.$$

The reduced problem

We obtain the following equation for $V(x,t)$:

$$\begin{cases} \partial_t V &= \Delta V - \frac{1}{V^2} - \frac{\theta'}{3\theta} V, \\ \theta(t) &= \frac{1}{\left(1+\gamma\theta(t)^{-\frac{1}{3}} \int_\Omega \frac{1}{V} dx\right)^2}. \end{cases} \tag{4}$$

and the following new goal: to construct a solution for (4) such that

$$\theta(t)^{\frac{1}{3}} V(x,t) \to 0.$$

Rk. : We see that $\lambda(t) \equiv \theta(t)^{-\frac{1}{3}}$ satisfies a polynomial equation, with $\int_\Omega \frac{1}{V(x,t)} dx$ being one of the coefficients:

$$\left(1 + \gamma\lambda(t) \int_\Omega \frac{1}{V} dx\right)^2 = \lambda(t)^3.$$

Solution of the reduced problem

Let us first recall the reduced problem

$$\begin{cases} \partial_t V &= \Delta V - \frac{1}{V^2} - \frac{\theta'}{3\theta} V, \\ \theta(t) &= \frac{1}{\left(1+\gamma\theta(t)^{-\frac{1}{3}} \int_\Omega \frac{1}{V} dx\right)^2}. \end{cases}$$

and the goal:

$$\theta(t)^{\frac{1}{3}} V(x,t) \to 0.$$

Idea: If we stick to our idea to get

$$\theta(t) \to \theta_0 > 0 \text{ (and } \theta'(t) \to 0) \text{ as } t \to T, \qquad \text{(H)}$$

then, our goal becomes to have

$$V(x,t) \to 0.$$

Hence, we may think that

$$\left| \frac{\theta'}{3\theta} V \right| \ll \frac{1}{V^2} \text{ as } t \to T,$$

so, we end-up with the following question:

Approximate problem

$$\begin{cases} \partial_t V & \simeq \Delta V - \frac{1}{V^2}, \\ \left(1 + \gamma\lambda(t) \int_\Omega \frac{1}{V} dx\right)^2 & = \lambda(t)^3 \text{ where } \lambda(t) = \theta(t)^{-\frac{1}{3}}. \end{cases}$$

Good news, this system in *decoupled*, and we can solve the first equation (PDE), then the second (polynomial).

Solving the first equation (PDE)

$$\partial_t V \simeq \Delta V - \frac{1}{V^2}.$$

This is a perturbation of the PDE solved by Merle and Z. in Nonlinearity 1997. With some care about the perturbative terms, we can construct a solution such that

(i) *Intermediate profile* ($0 < t < T$):

$$V(x,t) \sim (T-t)^{\frac{1}{3}} \left(3 + \frac{9}{8} \frac{|x|^2}{\sqrt{(T-t)|\ln(T-t)|}}\right)^{\frac{1}{3}}$$

(ii) *Final profile* ($t = T$): $V(x,t) \to V^*(x)$ as $t \to T$ with

$$V^*(x) \sim \left[\frac{9}{16} \frac{|x|^2}{|\ln|x||}\right]^{\frac{1}{3}} \text{ as } x \to 0.$$

With such a solution, we move to the polynomial equation satisfied by $\lambda(t) = \alpha(t)^{-\frac{1}{3}}$:

The second equation (polynomial)

$$\left(1+\gamma\lambda(t)\int_\Omega \frac{1}{V}dx\right)^2 = \lambda(t)^3 \text{ where } \lambda(t) = \theta(t)^{-\frac{1}{3}}. \qquad (5)$$

Now, since we have the final profile, we may estimate $\int_\Omega \frac{1}{V(x,t)}dx$ as follows:

$$\int_\Omega \frac{1}{V(x,t)}dx \sim \int_\Omega \frac{1}{V^*(x)}dx \sim \int_\Omega \left[\frac{9}{16}\frac{|x|^2}{|\ln|x||}\right]^{-\frac{1}{3}} dx < +\infty.$$

Since this coefficient is finite, we see from (5) that $\lambda(t) \to \lambda_0$ positive and finite, hence

$$\theta(t) \to \theta_0 > 0 \text{ as } \ t \to T.$$

Since

$$1-u(x,t) = v(x,t) = \theta(t)^{\frac{1}{3}} V(x,t),$$

We get our solution AND its profile.

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Strategy of the proof

We follow *the constructive existence proof* used by Bressan (1990), Bricmont-Kupiainen (1994), Merle-Z. (1997) for the *standard semilinear heat equation*.

That method is based on two parts:

- The construction of an approximate solution (candidate for the "profile"). This was (*the formal approach*);
- A perturbative argument, where we linearize around the approximate solution and show that the linearized PDE has a solution converging to zero. This is *a rigorous proof*.

  Here, two steps are needed:

  - The reduction of the problem to a finite-dimensional one ($N+1$ parameters);
  - The solution of the finite-dimensional problem thanks to the degree theory.

Earlier (non exhaustive) literature

We are in the framework of constructing a solution to some PDE with some *prescribed behavior* (Courtesy of Van Tien NGUYEN, NYU Abu Dhabi):

- heat equation:
  - Bressan, Indiana 1990;
  - Bricmont and Kupiainen, Nonlinearity 1994;
  - Merle and Z., Duke1998,
  - Ghoul- Nguyen - Zaag, AIHP 2018, JDE 2019 (Type I blowup, system)
  - Schweyer, JFA 2012 (Type II, N = 4, energy critical);
  - Mahmoudi, Nouaili and Z. (periodic),
  - del Pino - Musso - Wei, arXiv 2019 (Type II, N=5, energy critical),
  - del Pino - Musso - Wei, APDE 2020 (Infinite time blowup, N = 3, energy critical)
  - del Pino - Musso- Wei - Zhang - Zhang, arXiv 2020 (Type II, N = 3, energy critical)
  - del Pino - Musso - Wei - Zhou, DCDS-A 2020 (Type II, N = 4, energy critical)
  - Cortázar - del Pino - Musso, JEMS 2020 (Infinite time blowup, energy critical)
  - Collot, APDE 2017 (Type II, energy supercritical)
  - Tayachi and Z., TAMS 2019,
  - Collot-Merle-Raphael, JAMS 2020 (Type II anisotropic, energy supercritical)
  - Merle - Raphaël - Szeftel, IMRN 2020 (Type I anisotropic)
  - Harada, AIHP 2020 (Type II, N =5, energy critical)
  - Seki, JDE 2020 (Type II, energy supercritical, Lepin exponent)

Earlier literature

- NLS:
  - Merle, 1990,
  - Martel and Merle, 2006;
  - Martel - Raphaël, AENS 2018 (interacting blowup bubbles, mass critical)
  - Merle - Raphaël - Rodnianski, CJM 2015 (energy supercritical)
  - Merle, P. Raphaël, J. Szeftel, Duke 2014 (collapsing ring blowup, mass supercritical)
  - Merle -Raphael - Rodnianski - Szeftel, arXiv 2019 (blowup defocusing , energy supercritical, N >=5)
  - Raphael- Szeftel, CMP 2009 (standing ring blowup)
- Wave-type equations:
  - Côte and Z., CPAM 2013,
  - Ming-Rousset-Tzvetkov, 2013 (Water waves),
  - Collot, MEMS 2018 (type II blowup, energy supercritical wave equation, non radial)
  - Hillairet - Raphaël, APDE 2014 (type II blowup, energy critical wave equation, N = 4)
  - Krieger, W. Schlag, D. Tataru, Duke 2009 (Type II blowup, energy critical wave equation, N =3)
  - Ibrahim - Ghoul - Nguyen, JDE 2019 (type II blowup, energy supercritical (d >=7) wave maps)
  - Krieger - Schlag - Tataru, Invent. Math 2008 (type II blowup, energy critical (d = 2) wave maps)
  - Raphael - Rodnianski, IHES 2012 (Type II blowup, energy critical (d=2) wave maps)
  - Donninger and Schörkhuber, 2017 (supreconformal case)

Earlier literature

- Other equations:
  - Martel, 2005, Côte 2006, 2007, (KdV and gKdV),
  - Masmoudi and Z., 2008, Nouaili and Z., 2018 (Complex Ginzburg-Landau),
  - Merle-Raphaël-Rodniansky, 2013, (Schrödinger maps),
  - Ibrahim - Ghoul - Nguyen, APDE 2019 (Type II energy supercritical (d >=7) heat flow)
  - Schweyer - Raphael, APDE 2014 & CPAM 2013 (Type II energy critical (d= 2) heat flow)
  - Dávila- del Pino - Wei, Invent math 2020 (Type II energy critical (d=2) heat flow, non radial)
  - Collot - Ghoul - Masmoudi - Nguyen, arXiv 2019 (Type II, 2D Keller segel)
  - Schweyer - Raphael, MA 2014 (Type II, 2D Keller-Segel)
  - Collot, Ghoul, Ibrahim and Masmoudi, 2018, (Prandtl's system),
  - Hadzic, Raphaël, 2019 (Stefan problem),
  - Merle, Raphaël, Rodnianski, Szeftel, 2019 (Fluids)

Change of variables: A blow-up question

Introducing

$$\bar{u} = \frac{u}{1-u}, \bar{\theta}(t) = \left(1 + \gamma|\Omega| + \gamma \int_\Omega \bar{u}(x,t)dx\right)^{\frac{2}{3}}$$

and

$$U(x,t) = \frac{1}{\bar{\theta}(t)}\bar{u}(x,t),$$

we write the following equation for U:

$$\begin{cases} \partial_t U &= \Delta U - 2\frac{|\nabla U|^2}{U+\frac{1}{\bar{\theta}(t)}} + \left(U + \frac{1}{\bar{\theta}(t)}\right)^4 - \frac{\bar{\theta}'(t)}{\bar{\theta}(t)}U, \qquad x \in \Omega, t > 0, \\ \bar{\theta}(t) &= \left(1 + \gamma|\Omega| + \gamma\bar{\theta}(t)\int_\Omega U(x,t)dx\right)^{\frac{2}{3}}, \\ U(x,t) &= 0, \qquad x \in \partial\Omega, t > 0. \end{cases}$$

This way, the question of constructing a Touch-Down solution u $\Leftrightarrow$ the question of constructing a blow-up solution U.

Similarity variables framework

Thinking about the "twin" equation

$$\partial_t U = \Delta U + U^4,$$

we use the *similarity variables* first introduced by Giga and Kohn (CPAM, 1985):

$$W(y,s) = (T-t)^{-\frac{1}{3}} U(x,t), y = \frac{x}{\sqrt{T-t}} \text{ and } s = -\ln(T-t).$$

Equation in *similarity variables* (y,s)

$$
\begin{cases}
\partial_s W = \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{3} - 2\frac{|\nabla W|^2}{W + \frac{e^{-\frac{s}{3}}}{\theta(s)}} \\
\quad + \left(W + \frac{e^{-\frac{s}{3}}}{\theta(s)}\right)^4 - \frac{\theta'(s)}{\theta(s)} W, & y \in \Omega_s, s > -\ln T, \\
W(y,s) = 0, & y \in \partial\Omega_s, s > -\ln T,
\end{cases}
$$

where $\Omega_s = e^{\frac{s}{2}}\Omega$, $\theta(s) = \bar{\theta}(t(s)) = \bar{\theta}(T - e^{-s})$, and

$$
\bar{\theta}(t) = \left(1 + \gamma|\Omega| + \gamma\bar{\theta}(t) \int_\Omega U(x,t)dx\right)^{\frac{2}{3}}.
$$

A key remark: Given t and $U(x,t)$, $\bar{\theta}(t)$ solves an algebraic equation !

Main terms in the equation

We rewrite the equation:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{3} w - 2 \frac{|\nabla w|^2}{w + \frac{e^{-\frac{s}{3}}}{\theta(s)}} + \left(w + \frac{e^{-\frac{s}{3}}}{\theta(s)} \right)^4 - \frac{\theta'(s)}{\theta(s)} w.$$

As in the formal approach, we will control the red and blue terms to be small. This way, we reduce to the following equation

$$\partial_s w \simeq \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{3} w - 2 \frac{|\nabla w|^2}{w} + w^4,$$

already studied in Merle and Z., Nonlinearity, 1997.
In particular, we may find a solution such that

$$w(y,s) \sim \left(3 + \frac{9}{8} \frac{|y|^2}{s} \right)^{-\frac{1}{3}} + \frac{(3)^{-\frac{1}{3}} n}{4s} \text{ the intermediate profile.}$$

3 Regions with 3 Different controls for the solution of the PDE

We control the solution in 3 different regions:

- The inner region $P_1(t) = \left\{x \in \mathbb{R}^N \mid |x| \leq K_0\sqrt{(T-t)|\ln(T-t)|}\right\}$, with a control:

$$w(y,s) \sim \left(3 + \frac{9}{8}\frac{|y|^2}{s}\right)^{-\frac{1}{3}} + \frac{(3)^{-\frac{1}{3}}n}{4s}; \text{ intermediate profile}$$

- The intermediate region $P_2(t) = \left\{x \in \mathbb{R}^N \mid \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|} \leq |x| \leq \epsilon_0\right\}$, with a control:

$$U(x,t) \sim \left[\frac{9}{16}\frac{|x|^2}{|\ln|x||}\right]^{-\frac{1}{3}}; \text{ final profile}$$

- The outer region $P_3(t) = \left\{x \in \mathbb{R}^N \mid |x| \geq \frac{\epsilon_0}{4}\right\}$, with a control

$$U(x,t) \sim U_0(x,t); \text{ initial data}$$

(we will in fact take T small).

Control of the solution of the polynomial equation

We will have the following control:

$$|\theta'(s)| \le e^{-\eta s},$$

for some $\eta > 0$. This will imply that

$$\theta(s) \to \theta_0 > 0 \text{ as } s \to \infty.$$

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Control in the inner region P_1 (setting)

In P_1, we control w instead of U,

$$w(y,s) \sim \varphi(y,s) \Leftrightarrow \|w-\varphi\|_{L^\infty(\mathbb{R}^N)} \to 0.$$

Therefore, we introduce $q = w - \varphi$, and we try to control

$$\|q\|_{L^\infty(\mathbb{R}^N)} \to 0.$$

Control in the inner region P_1 (equation on q)

$$\partial_s q = (\mathcal{L} + V)q + T(q) + B(q) + N(q) + R(y,s),$$

$$\begin{aligned}
\mathcal{L} &= \Delta - \frac{1}{2} y \cdot \nabla + Id, V(y,s) = 4\left(\varphi^3(y,s) - \frac{1}{3}\right), \\
T(q,\theta(s)) &= -2\frac{|\nabla q + \nabla \varphi|^2}{q + \varphi + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}} + 2\frac{|\nabla \varphi|^2}{\varphi + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}}, \\
B(q) &= \left(q + \varphi + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}\right)^4 - \varphi^4 - 4\varphi^3 q, \\
R(y,s) &= -\partial_s \varphi + \Delta \varphi - \frac{1}{2} y \cdot \nabla \varphi - \frac{\varphi}{3} + \varphi^4 - 2\frac{|\nabla \varphi|^2}{\varphi + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}}, \\
N(q) &= -\frac{\theta'(s)}{\theta(s)}(q + \varphi).
\end{aligned}$$

+ The blue and red terms are small ; The linear part driven by $\mathcal{L} + V$ remains to be controlled.

The inner region P_1 (spectral properties)

- The operator $\mathcal{L}+V$ has 2 important properties:

- On $\{|y| \geq K\sqrt{s}\}$, $\mathcal{L}+V$ has a negative spectrum, provided that $K \gg 1$: Control of q in this region is easy.
- On $\{|y| \leq K\sqrt{s}\}$ (the inner region), the potential V can be considered as a perturbation of $\mathcal{L}$.

- This justifies the introduction of a Cut-Off function χ defined by

$$\chi(y,s) = 1, \forall |y| \leq K\sqrt{s} \text{ et } \chi(y,s) = 0, \forall |y| \geq 2K\sqrt{s}.$$

- We will decompose q as follows:

$$q = \chi q + (1-\chi)q \equiv q_b + q_e.$$

- Since $\text{supp}(q_e) \subset \{|y| \geq K\sqrt{s}\}$, by the first property of the potential, the control of q_e is easy.

- It remains to control q_b.

The inner region P_1 (spectral properties)

- The operator $\mathcal{L} = \Delta - \frac{1}{2}y \cdot \nabla + Id$ is self-adjoint in $L^2_\rho(\mathbb{R}^N)$ with

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{N}{2}}}.$$

Its spectrum is given by

$$Spec\mathcal{L} = \{1 - \frac{m}{2}, m \geq 0\}.$$

The eigenspace E_m corresponding to the eigenvalue $1 - \frac{m}{2}$ is given by

$$\mathcal{E}_m = \langle h_{m_1}(y_1).h_{m_2}(y_2)....h_{m_N}(y_N) \mid m_1 + ... + m_N = m \rangle ,$$

where h_{m_i} is the (rescaled) Hermite polynomial.

- Thus, we decompose q_b, according to the sign of eigenvalues:

$$q_b(y,s) = q_0(s) + q_1(s) \cdot y + y^T \cdot q_2(s) \cdot y - 2\,\mathrm{Tr}(q_2(s)) + q_-(y,s).$$

The inner region P_1 (control of non-positive directions

- Control of q_- : This is the negative part of the spectrum ; Easily controlled.

- Control of $q_2(s)$: This corresponds to the eigenvalue $\lambda = 0$ o $\mathcal{L} \Rightarrow$: It is delicate. We need the Potential and also the Gradient term to derive this ODE:

$$q_2'(s) = -\frac{2}{s} q_2(s) + O\left(\frac{1}{s^2}\right).$$

Introducing the slow variable $\tau = \ln s$, this yields

$$\frac{q_2}{d\tau}(\tau) = -2q_2(\tau) + O(e^{-2\tau}),$$

we see a negative eigenvalue $\Rightarrow$ control of q_2 is easy.

- We are left with the two nonnegative directions q_0, q_1.

The inner region P_1 (A topological argument)

This is the ODE satisfied by (q_0, q_1):

$$\begin{aligned} q_0' &= q_0 + O\left(\frac{1}{s^2}\right), \\ q_1' &= \frac{1}{2} q_1 + O\left(\frac{1}{s^2}\right). \end{aligned}$$

Proceeding by contradiction and using the degree theory, we find initial data $(q_0, q_1)(s_0)$ such that

$$(q_0, q_1)(s) \to 0 \text{ quand } s \to +\infty.$$

Finally, we conclude that all the components can be controlled, hence

$$\|q\|_{L^\infty(\mathbb{R}^N)} \to 0.$$

Thank you for your attention !