Une solution explosive en forme de croix pour une équation semilinéaire de la chaleur

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Séminaire MACS, Lyon - Saint-Étienne
Lyon, 6 décembre 2022

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Introduction

The equation:

$$\partial_t u = \Delta u + |u|^{p-1}u, \quad u_0 \in L^\infty(\mathbb{R}^N),$$

where:

- $u = u(x,t)$, $x \in \mathbb{R}^N$, $t \in [0,T)$, $T \leq +\infty$ is maximal;
- $p > 1$ is Sobolev subcritical:

$$(N-2)p < N+2.$$

Two cases arise:
- either $T = +\infty$: the solution is global;
- or $T < +\infty$: the solution blows up in finite time in the sense that

$$\|u(t)\|_{L^\infty} \to \infty \text{ as } t \to T.$$

Focus of the paper: blow-up solutions

We assume that the maximal existence time T is finite.

Definitions:

- T is the blow-up time;
- $a \in \mathbb{R}^N$ is a blow-up point if $|u(x_n, t_n)| \to +\infty$ with $x_n \to a$ and $t_n \to T$.

Literature: No list can be exhaustive (see the book by Souplet and Quittner and the references therein).

Two relevant questions about blow-up

Two relevant questions:

• Classification (a priori): If $u(x,t)$ blows up at time $T>0$ at some point $a\in\mathbb{R}^N$, can we determine the asymptotic behavior or *profile* of the solution near (a,T)?

• Construction of examples:

- If no classification is available, can we construct some examples of blow-up solutions?
- If there is some classification, does each modality appearing in the classification occur?

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Similarity variables

Consider $u(x,t)$ an *arbitrary* solution blowing up at time $T>0$ at some point $a\in\mathbb{R}^N$.

The asymptotic behavior of u near (a,T) is better expressed in *similarity variables*:

$$w_a(y,s)=(T-t)^{\frac{1}{p-1}}u(x,t) \text{ where } y=\frac{x-a}{\sqrt{T-t}} \text{ and } s=-\log(T-t).$$

From Giga and Kohn (80'), we know that up to replacing u by $-u$:

$$w_a(y,s)\to\kappa\equiv(p-1)^{-\frac{1}{p-1}} \text{ as } s\to\infty,$$

uniformly on compact sets.

Notion of blow-up profile

By linearization, if

$$w_a \not\equiv \kappa,$$

Herrero and Velázquez (90') (see also Filippas, Kohn and Liu (90')) give this refinement:

$$w_a(y,s) - \kappa \sim Q(y,s) \text{ as } s \to \infty$$

uniformly on compact sets, where $Q(y,s)$ is the (local) blow-up profile.

The situation is simpler when $N = 1$.

Blow-up profile when $N = 1$

We have

$$w_a(y,s) - \kappa \sim Q(y,s) \text{ as } s \to \infty$$

uniformly on compact sets, with:

$$\text{either } Q(y,s) = -\frac{\kappa}{4ps} h_2(y), \text{ or } Q(y,s) = -e^{-(\frac{m}{2}-1)s} C_m h_m(y) \tag{1}$$

for some even integer $m = m(a) \geq 4$ and $C_m > 0$, where h_m is the (rescaled) Hermite polynomial, eigenfunction of the linearized operator ($\lambda = 0$ or $\lambda = 1 - \frac{m}{2}$).

- Construction question: Do all the modalities in (1) occur?

- Answer: Yes, thanks to Bricmont and Kupiainen in 1994 who *constructed* examples for each modality (see also Herrero and Velázquez when $m = 4$).

Blow-up profile when $N \geq 2$

For simplicity, we take $N = 2$.
We still have

$$w_a(y,s) - \kappa \sim Q(y,s) \text{ as } s \to \infty$$

uniformly on compact sets, with:
- either

$$Q(y,s) = -\frac{\kappa}{4ps} \sum_{i=1}^{l} h_2(y_l),$$

where $l = 1$ or 2, after a rotation of coordinates;
- or

$$Q(y,s) = -e^{-(\frac{m}{2}-1)s} \sum_{j=0}^{m} C_{m,j} h_{m-j}(y_1) h_j(y_2)$$

for some even integer $m = m(a) \geq 4$ with

$$B(y) = \sum_{j=0}^{m} C_{m,j} y_1^{m-j} y_2^j \geq 0 \text{ and } \not\equiv 0.$$

Construction question for $N = 2$

- Question: Do all the described modalities occur?

- Answer:
- for the first modality (with $1/s$), yes, (Bricmont and Kupiainen when $l = 2$, trivial $1d$ examples such as $u(x_1, t)$) ;
- for the second modality (with $e^{(1-\frac{m}{2})s}$): no answer, apart from the trivial $1d$ examples.

Rk. Not even radial solutions obeying the second modality, up to our knowledge.

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Matano's question

Is there a blow-up solution such that

$$w_0(y,s) - \kappa \sim -e^{-s} h_2(y_1) h_2(y_2) \text{ as } s \to \infty, \tag{2}$$

uniformly on compact sets?

Rk. According to the classification by Velázquez, if a solution with (2) exists, then we have the following *extended profile*:
For any $K > 0$,

$$\sup_{|y| \le K e^{\frac{s}{4}}} \left| w_0(y,s) - \left(p - 1 + \frac{(p-1)^2}{\kappa} e^{-s} y_1^2 y_2^2 \right)^{-\frac{1}{p-1}} \right| \to 0 \text{ as } s \to \infty.$$

Since the maximum of this profile is attained on the axes $y_1 = 0$ and $y_2 = 0$ and not just at the origin, we are in the *degenerate* case.

A twin open (easier) question

Is there a blow-up solution such that

$$w_0(y,s) - \kappa \sim -e^{-s}[h_4(y_1) + 10h_4(y_2)] \text{ as } s \to \infty, \tag{3}$$

uniformly on compact sets?

Rk. This is a *non-degenerate* case, in the sense that the extended profile attains its maximum only at the origin. Indeed, if a solution with (3) exists, then we know from the classification by Velázquez that for any $K > 0$,

$$\sup_{|y| \le K e^{\frac{s}{4}}} \left| w_0(y,s) - \left(p - 1 + \frac{(p-1)^2}{\kappa} e^{-s}[y_1^4 + 10y_2^4]\right)^{-\frac{1}{p-1}} \right| \to 0 \text{ as } s \to \infty.$$

Rk. Our focus is on the degenerate case, which is much more difficult.

Our result: A solution with a cross-shaped blow-up profile

Thm. *There exists a solution $u(x,t)$ which blows up in finite time T only at the origin, with:*
(i) (Inner profile)

$$w_0(y,s) - \kappa \sim -e^{-s} h_2(y_1) h_2(y_2) \text{ as } s \to \infty,$$

uniformly on compact sets.
(ii) (Intermediate profile): For any $K > 0$, it holds that

$$\sup_{e^{-s} y_1^2 y_2^2 + \delta e^{-2s}(y_1^6 + y_2^6) < K} |w(y,s) - \Phi(y,s)| \to 0 \text{ as } t \to T,$$

where

$$\Phi(y,s) = \left[p - 1 + \frac{(p-1)^2}{\kappa}\left(e^{-s} y_1^2 y_2^2 + \delta e^{-2s}(y_1^6 + y_2^6)\right)\right]^{-\frac{1}{p-1}} \text{ and } \delta > 0,$$

(iii) (Final profile): For any $x \neq 0$, $u(x,t)$ converges to a finite limit $u(x,T)$ uniformly on compact sets of $\mathbb{R}^2 \backslash \{0\}$ as $t \to T$, with

$$u(x,T) \sim \left[\frac{(p-1)^2}{\kappa}(x_1^2 x_2^2 + \delta(x_1^6 + x_2^6))\right]^{-\frac{1}{p-1}} \text{ as } x \to 0.$$

The cross shape

Recall the *intermediate profile*:

$$\Phi(y,s) = \left[p - 1 + \frac{(p-1)^2}{\kappa}\left(e^{-s}y_1^2y_2^2 + \delta e^{-2s}(y_1^6 + y_2^6)\right)\right]^{-\frac{1}{p-1}} \text{ and } \delta > 0,$$

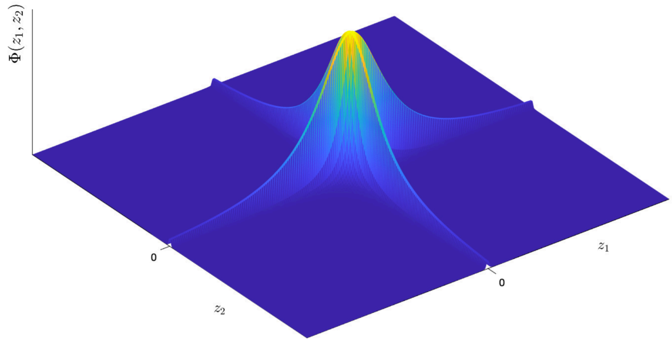

Figure: Courtesy of V.T. Nguyen

Rk. It is indeed cross-shaped !!!

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(Profile seen from 60 degrees)

Profile seen from 60 degrees (V.T. Nguyen)

Rk. Note that the speed is larger on the axes.

Some remarks

Recall the *intermediate profile*:

$$\Phi(y,s) = \left[p - 1 + \frac{(p-1)^2}{\kappa}\left(e^{-s}y_1^2y_2^2 + \delta e^{-2s}(y_1^6 + y_2^6)\right)\right]^{-\frac{1}{p-1}} \text{ and } \delta > 0,$$

Rk. Our description is stronger than Velázquez':
- Ours: For any $K > 0$, it holds that

$$\sup_{e^{-s}y_1^2y_2^2+\delta e^{-2s}(y_1^6+y_2^6)<K} |w(y,s) - \Phi(y,s)| \to 0 \text{ as } t \to T,$$

- Velázquez': For any $K > 0$,

$$\sup_{|y|\le Ke^{\frac{s}{4}}} \left| w_0(y,s) - \left(p - 1 + \frac{(p-1)^2}{\kappa}e^{-s}y_1^2y_2^2\right)^{-\frac{1}{p-1}} \right| \to 0 \text{ as } s \to \infty.$$

Stability

We suspect this solution to be unstable with respect to perturbations in initial data.

Unfortunately, we have no proof for that.

Rk. In fact, Herrero and Velázquez *claim* that the following profile is *generic*:

$$w_a(y,s) \sim \left[p - 1 + \frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right]^{-\frac{1}{p-1}}. \quad (4)$$

(note that (4) implies that a is necessarily an isolated blow-up point).

Hence, the profile (4) *should be* the only stable one.

Proof of genericity:

- $N = 1$: published in 1996.
- $N \geq 2$: claimed but never published.

Generalization

Our method applies to the construction of a large variety of *single-point blow-up* solutions, both with non-degenerate and degenerate blow-up profiles.

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A naive approach for the proof

Recall our goal: *Construct a solution such that*

$$w_0(y,s) - \kappa \sim -e^{-s} h_2(y_1) h_2(y_2) \text{ as } s \to \infty. \tag{5}$$

Question: Can we follow the approach used by Bricmont and Kupiainen when $N = 1$ to construct a blow-up solution such that

$$w_0(y,s) - \kappa \sim -e^{-s} h_4(y)?$$

There, they linearize the equation around the extended profile

$$\Phi_1(y,s) = \left(p - 1 + \frac{(p-1)^2}{\kappa} e^{-s} y^4\right)^{-\frac{1}{p-1}}$$

and crucially use the fact that $\Phi_1(y,s) \to 0$ as $|y| \to \infty$.

A naive approach (continued)

Answer: No. When $N = 2$, by analogy, we tried to linearize around the extended profile given in Velázquez' classification:

$$\Phi_2(y,s) = \left(p - 1 + \frac{(p-1)^2}{\kappa} e^{-s} y_1^2 y_2^2\right)^{-\frac{1}{p-1}}.$$

That profile was the only one available before our work.

Unfortunately, that profile doesn't decrease to 0 at infinity (problem on the axes $y_1 = 0$ or $y_2 = 0$).

Our approach

A key idea: Replace the extended profile from Velázquez' classification by a sharper profile which decreases to 0 at infinity.

How to proceed? In fact, the profile is an approximate solution. In order to construct it, let us start from the goal

$$w_0(y,s)-\kappa \sim -e^{-s}h_2(y_1)h_2(y_2) \text{ as } s\to\infty, \tag{6}$$

and refine it, in order to get a sharper profile with the decreasing property.

Rk. The non-decreasing property can already be seen in the goal (6):
- First, take $|y|$ large, hence, keep only the main term in the polynomials:

$$w_0(y,s)-\kappa = -e^{-s}y_1^2y_2^2+o(e^{-s}).$$

- Then, staying on the axes (i.e. when $y_1y_2=0$), we get

$$w_0(y,s)-\kappa = o(e^{-s})$$

and we see no decreasing.

Our idea: refining the goal

Recall our goal: *Construct a solution such that*

$$w_0(y,s) - \kappa \sim -e^{-s} h_2(y_1) h_2(y_2) \text{ as } s \to \infty. \tag{7}$$

For simplicity, we seek to construct a solution which is *symmetric with respect to the axes and the bissectrices*.

If such a solution exists, using the PDE satisfied by w_0:
$\forall y \in \mathbb{R}^2, \forall s \geq -\log T$,

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w,$$

we can refine (7) and get the following *refined goal*:

Our refined goal

Construct a solution such that

$$w_0(y,s) = \kappa - e^{-s}h_2h_2 + e^{-2s}\Big\{-\frac{32p}{3\kappa}h_0h_0 - \frac{16p}{\kappa}(h_2h_0 + h_0h_2) - \frac{4p}{\kappa}(h_4h_0 + h_0h_4) - \frac{32p}{\kappa}h_2h_2 + C_{6,0}(h_6h_0 + h_0h_6) + (\frac{4p}{\kappa}s + C_{6,2})(h_4h_2 + h_2h_4) + \frac{p}{2\kappa}h_4h_4\Big\} + O\left(s^2e^{-3s}\right)$$

for some constants $C_{6,0}$ and $C_{6,2}$, where $h_ih_j = h_i(y_1)h_j(y_2)$.

Question: *Can the order* e^{-2s} *allow any decreasing on the axes?*

Towards the decreasing property on the axes

Method: As with term of order e^{-s}:

- First, take $|y|$ large, hence, keep only the main terms of the polynomials,

$$w_0(y,s) = \kappa - e^{-s}y_1^2y_2^2 + e^{-2s}\left\{-\frac{32p}{3\kappa} - \frac{16p}{\kappa}(y_1^2+y_2^2) - \frac{4p}{\kappa}(y_1^4+y_2^4)\right.$$
$$\left. -\frac{32p}{\kappa}y_1^2y_2^2 + C_{6,0}(y_1^6+y_2^6) + (\frac{4p}{\kappa}s + C_{6,2})(y_1^4y_2^2+y_1^2y_2^4) + \frac{p}{2\kappa}y_1^4y_2^4\right\} + \dots$$

- Then, go on the axes, for example, $y_2 = 0$:

$$w_0(y,s) = \kappa + e^{-2s}\left\{-\frac{32p}{3\kappa} - \frac{16p}{\kappa}y_1^2 - \frac{4p}{\kappa}y_1^4 + C_{6,0}y_1^6\right\} + \dots$$

- For large $|y_1|$, we get

$$w_0(y,s) = \kappa + e^{-2s}C_{6,0}y_1^6 + \dots$$

Taking

$$C_{6,0} = -\delta < 0,$$

we get the decreasing property. Thus, we suggest the following correction to the goal:

Corrected goal

- Initial goal :

$$w_0(y,s)-\kappa \sim -e^{-s}h_2(y_1)h_2(y_2).$$

- Bad profile :

$$\left(p-1+\frac{(p-1)^2}{\kappa}e^{-s}y_1^2y_2^2\right)^{-\frac{1}{p-1}}$$

- Corrected goal :

$$w_0(y,s)-\kappa = -e^{-s}h_2(y_1)h_2(y_2)-\delta e^{-2s}[h_6(y_1)+h_6(y_2)]+\dots$$

- Corrected profile (keeping only the leading terms of the polynomials):

$$\left(p-1+\frac{(p-1)^2}{\kappa}\left[e^{-s}y_1^2y_2^2+\delta e^{-2s}(y_1^6+y_2^6)\right]\right)^{-\frac{1}{p-1}}$$

for some (large) $\delta>0$.

The corrected goal in similarity variables

Construct a solution $w_0(y,s)$ *of the similarity variables version:*

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w.$$

such that

$$\|w(y,s) - \Phi(y,s)\|_{L^\infty(\mathbb{R}^2)} \to 0 \text{ as } s \to \infty,$$

where the corrected profile is

$$\Phi(y,s) = \left[p - 1 + \frac{(p-1)^2}{\kappa}\left(e^{-s} y_1^2 y_2^2 + \delta e^{-2s}(y_1^6 + y_2^6)\right)\right]^{-\frac{1}{p-1}} \text{ with } \delta > 0.$$

The rigorous proof

A natural idea : Linearize the equation around the corrected profile, by introducing

$$q(y,s) = w_0(y,s) - \Phi(y,s).$$

This is the equation satisfied by $q(y,s)$:

$$\partial_s q = (\mathscr{L} + V(y,s))q + B(y,s,q) + R(y,s), \tag{8}$$

with

$$\mathscr{L} q = \Delta q - \frac{1}{2} y \cdot \nabla q + q, \quad V(y,s) = p\Phi(y,s)^{p-1} - \frac{p}{p-1},$$

$$B(y,s,q) = |\Phi(y,s) + q|^{p-1}(\Phi(y,s) + q) - \Phi(y,s)^p - p\Phi(y,s)^{p-1} q,$$

$$R(y,s) = -\partial_s \Phi(y,s) + (\mathscr{L} - 1)\Phi(y,s) - \frac{\Phi(y,s)}{p-1} + \Phi(y,s)^p.$$

New goal: *Construct a solution of equation* (8) *such that*

$$\|q(s)\|_{L^\infty} \to 0 \text{ as } s \to \infty.$$

Structure of the equation

- The linear operator is self-adjoint in $L^2_\rho(\mathbb{R}^2)$ where $\rho(y) = e^{-|y|^2/4}$, and the (rescaled) Hermite polynomials $h_i(y_1)h_j(y_2)$ are its eigenfunctions.
  $\Longrightarrow$ The Hilbert space $L^2_\rho(\mathbb{R}^2)$ is a natural space for the construction.
- The remainder term $R(y,s)$ measures how far is the corrected profile $\Phi(y,s)$ from being a solution.

  Since $\Phi(y,s)$ is indeed a (very) good approximate solution, $R(y,s)$ is small.

  $\Longrightarrow$ Center manifold theory would be the good tool for the construction !

Unfortunately, the nonlinear term $B(q,y,s)$ ($= q^2$ if $p=2$ and $w \geq 0$) is not quadratic in the space $L^2_\rho(\mathbb{R}^2)$....

Control of the nonlinear term

Making the following (reasonable) a priori estimate:

$$\|w_0(s)\|_{L^\infty(R^2)} \leq M, \qquad (9)$$

we can use parabolic regularity techniques to derive the following delay estimate:

$$\|q(s)^2\|_{L^2_\rho} \leq C(M)\|q(s-s^*)\|^2_{L^2_\rho}$$

and (largely adapted) Center Manifold Theory works.

Last step: It remains to prove the a priori estimate (9).

Reduction

We need to prove that:

$$\|w_0(s)\|_{L^\infty(R^2)} \leq M. \tag{10}$$

Recalling the similarity variables transformation:

$$w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t) \text{ where } y = \frac{x-a}{\sqrt{T-t}} \text{ and } s = -\log(T-t),$$

we see that

$$w_a(y,s) = w_0(y + ae^{\frac{s}{2}}, s) \text{ and } \nabla w_a(y,s) = \nabla w_0(y + ae^{\frac{s}{2}}, s).$$

Thus, the estimate in (10) becomes equivalent to

$$\forall a \in \mathbb{R}^2, \quad |w_a(0,s)| \leq M.$$

Making the following second a priori estimate:

$$\|\nabla w_0(s)\|_{L^\infty} \leq \eta_0$$

for some small $\eta_0 > 0$, we replace the first a priori estimate by:

$$\forall a \in \mathbb{R}^2, \quad \|w_a(s)\|_{L^2_\rho} \leq M.$$

Proof of the gradient a priori estimate

This is a consequence of the following Liouville Theorem proved with F. Merle in 1998 and 2000, and which is valid *only* for Sobolev subcritical exponent p :

$$(N-2)p < N+2. \tag{11}$$

This is the statement:

Proposition (A Liouville theorem for ancient solutions)

Under condition (11), *consider* $W(y,s)$ *a solution of the similarity variables' equation*

$$\partial_s W = \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{p-1} + |W|^{p-1} W,$$

defined and uniformly bounded for all $(y,s) \in \mathbb{R}^N \times (-\infty, \bar{s})$ *for some* $\bar{s} \leq +\infty$. *Then, either* $W \equiv 0$, *or* $W \equiv \pm\kappa$ *or* $W(y,s) = \pm\kappa(1 \pm e^{s-s^*})^{-\frac{1}{p-1}}$ *for all* $(y,s) \in \mathbb{R}^N \times (-\infty, \bar{s}]$ *and for some* $s^* \in \mathbb{R}$.
In all cases, it holds that $\nabla W \equiv 0$.

Control of $\|w_a(s)\|_{L^2_\rho}$

Recall that $w_a(y,s)$ satisfies the same equation, independently from a:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w.$$

This equation has only 2 stationary solutions, in the nonnegative case (see Giga and Kohn 1980'):

- $w \equiv 0$, which is stable ;
- $w \equiv \kappa = (p-1)^{-\frac{1}{p-1}} > 0$, which has both stable and unstable directions.

It also has the following heteroclinic orbit:

$$w(y,s) = \psi(s) \equiv \kappa(1+e^s)^{-\frac{1}{p-1}},$$

which turns to be stable too.

Control of $\|w_a(s)\|_{L^2_\rho}$ (continued)

Here, comes the choice of initial data (depending on parameters suitable for the Center Manifold Theory technique).

Roughly speaking, Initial data at $s = s_0$ for $w_0(y, s)$ will be equal to the profile

$$\Phi(y, s_0) = \left(p - 1 + \frac{(p-1)^2}{\kappa}\left[e^{-s}y_1^2y_2^2 + \delta e^{-2s}(y_1^6 + y_2^6)\right]\right)^{-\frac{1}{p-1}}.$$

In particular, it connects κ at the origin to 0 at infinity.

Recalling the following relation

$$w_a(y, s_0) = w_0(y + ae^{\frac{s_0}{2}}, s_0),$$

we see that whenever $|a|$ increases, $w_a(s_0)$ will travel from κ to 0, going through some intermediate region, where it will be close to the heteroclinic orbit $\psi(s)$.

Control of $\|w_a(s)\|_{L^2_\rho}$ (continued)

Therefore, 3 scenarios occur:

- If $|a|$ is large, then, $\|w_a(s_0)\|_{L^2_\rho}$ is small. By stability of the zero solution, $\|w_a(s)\|_{L^2_\rho}$ remains small for $s \geq s_0$.
- If $|a|$ is "intermediate", then, $w_a(y, s_0)$ is in the vicinity of the heteroclinic orbit, which is stable and bounded. Therefore, $w_a(y, s)$ will remain in the vicinity of the heteroclinic orbit, hence, it will remain bounded.
- If $|a|$ is small, this is the most delicate case. Since initial data are close to the profile $\Phi(y, s)$, we can integrate the equation in similarity variables, in a very delicate way, to show that after some time, $w_a(y, s)$ will be in the vicinity of the heteroclinic orbit. By stability, it will remain bounded.

Rk. This delicate integration technique is at the origin of our result concerning the geometry of the blow-up set near a non isolated blow-up set (IMRN 2021).

Thank you for your attention.

Example 1 with a *degenerate* profile when $N = 2$

A non orthogonal cross:

- $w_0(y,s) \sim -e^{-s} h_2(y_1) h_2(y_2 + a_0 y_1)$;
- $u(x,T) \sim \left[C_0 \left(x_1^2 (x_1 + a_0 x_2)^2 + \delta(x_1^6 + x_2^6)\right)\right]^{-\frac{1}{p-1}}$;
- Conditions: $N = 2$, $\delta \geq \delta_0$, $a_0 \in \mathbb{R}$.

Example 2 with a *degenerate* profile when $N = 2$

A multiline cross, with higher-order polynomials:

- $w_0(y,s) \sim -e^{(1-\frac{k}{2})s} h_{k_1}(y_1)\dots h_{k_l}(a_l y_1 + b_l y_2)$;
- $u(x,T) \sim \left[C_0 \left(x_1^{k_1}\dots(a_l x_1 + b_l x_2)^{k_l} + \delta(x_1^{k+2} + x_2^{k+2})\right)\right]^{-\frac{1}{p-1}}$;
- Conditions: $N = 2$, $\delta \geq \delta_0$, $k = k_1 + \dots + k_l$, $k_i \geq 2$ is even, $(a_i, b_i) \neq (0,0)$, the straight lines $\{y_1 = 0\}$,...., $\{a_l y_1 + b_l y_2 = 0\}$ are distinct.

Example 3 with a *degenerate* profile when $N \geq 3$

- $w_0(y,s) \sim -e^{(1-\frac{k}{2})s} \prod_{i=1}^{m} \left(\sum_{j \in I_i} h_{2\theta_i}(y_j) \right)$;
- $u(x,T) \sim \left[C_0 \left(\prod_{i=1}^{m} \left(\sum_{j \in I_i} |x_j|^{2\theta_i} \right) + \delta(x_1^{k+2} + \cdots + x_N^{k+2}) \right) \right]^{-\frac{1}{p-1}}$;
- Conditions: $N \geq 3$, $\delta \geq \delta_0$, $k = \sum_{i=1}^{m} 2\theta_i$, and the set I_i make a partition of $\{1, \ldots, N\}$.

Example 4 with a *non-degenerate* profile when $N = 2$

- $w_0(y,s) - \kappa \sim -e^{(1-\frac{k}{2})s} \sum_{i=0}^{k} C_{k,i} h_{k-i}(y_1) h_i(y_2)$;
- $u(x,T) \sim [C_0 B(x)]^{-\frac{1}{p-1}}$ with $B(x) = \sum_{i=0}^{k} C_{k,i} x_1^{k-i} x_2^i$;
- Conditions: $N = 2$, $k \geq 4$ is even and $B(x) > 0$ for $x \neq 0$.

Rk. The multilinear form $B(x)$ need not be symmetric.

Example 5 with a *non-degenerate* profile when $N \geq 3$

- $w_0(y,s) - \kappa \sim e^{-(\frac{k}{2}-1)s} \sum_{j_1+\cdots+j_N=k} C_{k,j_2,,\dots,j_N} h_{j_1}(y_1)\dots h_{j_N}(y_N)$;
- $u(x,T) \sim [C_0 B(x)]^{-\frac{1}{p-1}}$ with $B(x) = \sum_{j_1+\cdots+j_N=k} C_{k,j_2,\dots,j_N} x_1^{j_1}\dots x_N^{j_N}$;
- Conditions: $N \geq 3$, $k \geq 4$ is even and $B(x) > 0$ for $x \neq 0$.

Rk. The multilinear form $B(x)$ need not be symmetric.

A related result : Geometry of the blow-up set

The present work shares delicate integration techniques of the PDE in similarity variables with our paper in IMRN 2021.

In that paper, we proved the following:
Thm. *When $N = 2$ and $p = 2$, we consider $u(x,t)$ a solution of*

$$\partial_t u = \Delta u + |u|^{p-1} u,$$

which blows up in finite time $T > 0$, with the origin being a non-isolated blow-up point. In addition, we assume that $\|w_0(s) - \kappa\|_{L^2_\rho} \sim C_0 e^{-s}$, for some $C_0 > 0$. Then, for any sequence of blow-up points $a_n \to 0$, it holds that $a_{n,1} \geq 0$ and $a_{n,2} \geq 0$, with

$$\text{either } a_{n,1} = o({a_{n,2}}^2), \text{ or } a_{n,1} \sim L{a_{n,2}}^2 \text{ or } a_{n,1} \sim L{a_{n,2}}^{3/2} \text{ with } L > 0,$$

up to extracting a subsequence (still denoted the same), and up to some rotation of coordinates and symmetry with respect to axes.
Rk. The result is valid for any $p > 1$ and for other speeds of $\|w_0(s) - \kappa\|_{L^2_\rho}$. Our method applies also in higher dimensions $N \geq 3$, with a more complicated statement.