Non characteristic points for the semilinear wave equation

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The equation

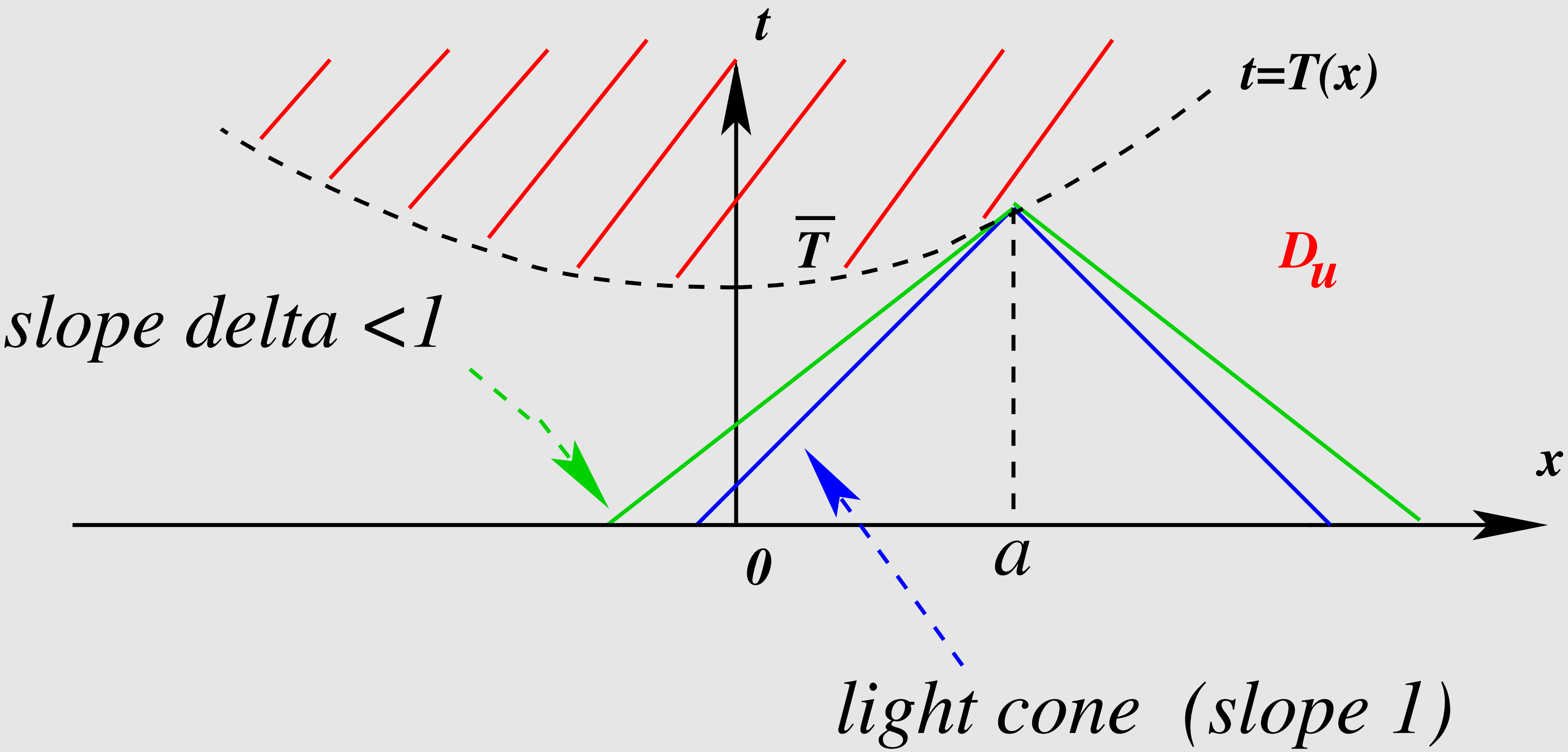
$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $p > 1$,
$u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$,
$u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R})$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$
and

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})} = \sup_{a\in\mathbb{R}} \left(\int_{a-1}^{a+1} |v(x)|^2 dx \right)^{1/2}.$$

Definition: Non characteristic points and characteristic points

A point a is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



The point is said *characteristic* if not.

- Notation: $\mathcal{R} \subset \mathbb{R}$ is the set of all *non* characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}$).

A Lyapunov functional and the blow-up rate

Selfsimilar transformation for all $x_0 \in \mathbb{R}$

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x-x_0}{T(x_0)-t},\ s = -\log(T(x_0)-t).$$

(x,t) in the light cone of vertex $(x_0, T(x_0)) \Longleftrightarrow (y,s) \in (-1,1) \times [-\log T(x_0), \infty)$.

Equation on $w = w_{x_0}$: For all $(y,s) \in (-1,1) \times [-\log T(x_0), \infty)$:

$$\partial^2_{ss} w - \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) + \frac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y\partial^2_{sy} w$$

where $\rho(y) = (1-|y|^2)^{\frac{2}{p-1}}$

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A Lyapunov functional (Antonini-Merle)

$$E(w) = \int_{-1}^{1} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy,$$

Thanks to a Hardy-Sobolev inequality, $E = E(w, \partial_s w)$ is well defined in the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(B) \;\middle|\; \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (\partial_y q_1)^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

Properties of the Lyapunov functional E

Lemma 1 (Monotonicity (Antonini-Merle)) *For all s_1 and s_2:*

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1}\int_{s_1}^{s_2}\int_{-1}^{1}(\partial_s w)^2(1-|y|^2)^{\frac{2}{p-1}-1}dyds.$$

Lemma 2 (A blow-up criterion) *Consider a solution W such that $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time $S > s_0$.*

An upper bound on the blow-up rate in selfsimilar variables

Th. For all $x_0 \in \mathbb{R}$ and $s \geq -\log T(x_0) + 1$,

$$\int_{-1}^{1} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2 (1 - |y|^2) + \frac{(p+1)}{(p-1)^2} w^2 + \frac{1}{p+1} |w|^{p+1} \right) \rho dy \leq K$$

where the constant K depends only on p and an upper bound on $T(x_0)$, $1/T(x_0)$ and $\|(u_0, u_1)\|$.

Getting rid of the weights

Reducing $(-1,1)$ to $(-\frac{1}{2}, \frac{1}{2})$, we get:

Cor. For all $x_0 \in \mathbb{R}$ and $s \geq -\log T(x_0) + 1$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left((\partial_s w)^2 + (\partial_y w)^2 + w^2 + |w|^{p+1} \right) dy \leq K.$$

Upper bound in the original $u(x,t)$ variables

Th. sup. *For all* $x_0 \in \mathbb{R}$ *and* $t \in [\frac{3}{4}T(x_0), T(x_0))$*:*

$$(T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{1/2}}$$

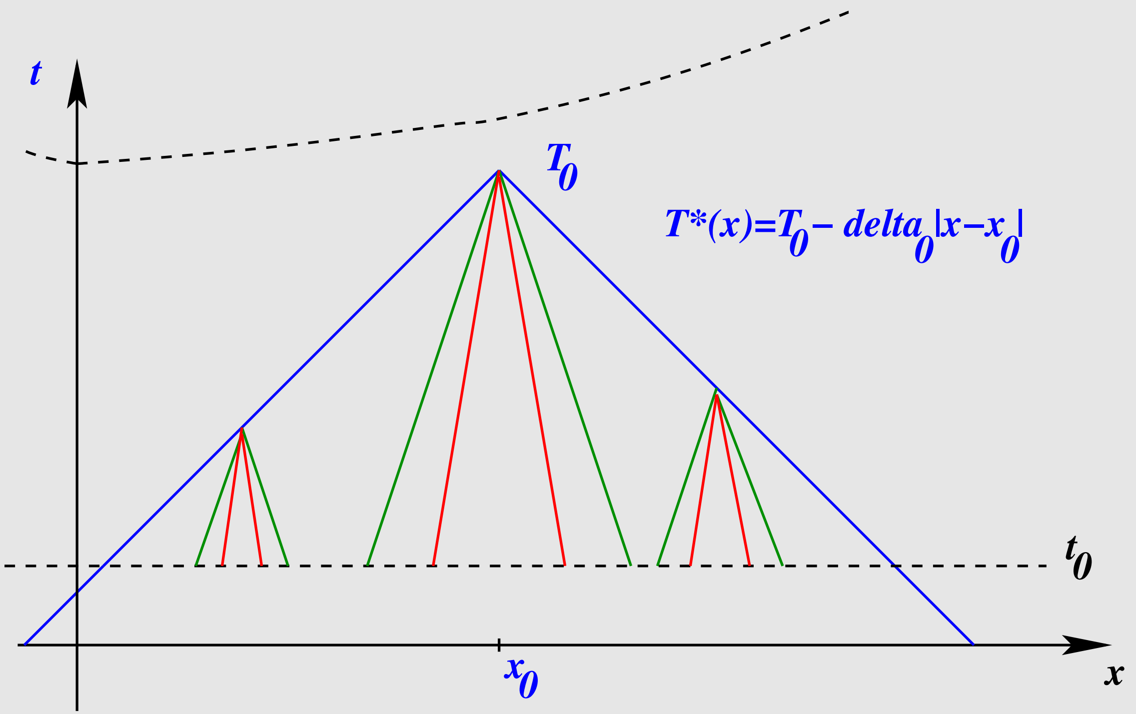
$$+(T(x_0)-t)^{\frac{2}{p-1}+1} \left(\frac{\|u_t(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{1/2}} + \frac{\|\partial_x u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{1/2}} \right) \le K.$$

Rk. We have a lower bound of the same size when x_0 is non characteristic (see Part 4 on profiles near a non characteristic point).

Covering technique at a *non characteristic* point

If x_0 is non characteristic point, then we can recover the estimate in whole section of the light-cone (or in the y variable, on the whole interval $(-1,1)$):

Prop. *If $x_0 \in \mathcal{R}$, then for all $s \geq -\log T(x_0)$, we have* $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-1,1)} \leq C_0$.



Blue: slope $\delta_0 < 1$, Green: slope 1 (light-cone), Red: slope 2.

Asymptotic behavior at a *non characteristic* point

Take $x_0 \in \mathbb{R}$ **non characteristic**. Using the energy structure, we obtain that $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-1,1)}$ is bounded.

Question: Does $w_{x_0}(y,s)$ have a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$).

Remark: In the context of Hamiltonian systems, **this question is delicate**, and there is no natural reason for such a convergence, since **the wave equation is time reversible.**

See for similar difficulty and approach, results for

- the **critical KdV** (Martel and Merle),
- **NLS** (Merle and Raphaël).

Stationary solutions.

We look for solutions of

$$\frac{1}{\rho}\left(\rho(1-y^2)w'\right)' - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w = 0.$$

We work in $\mathcal{H}_0$, the (**stationary energy space**) defined by

$$\mathcal{H}_0 = \{r \in H^1_{loc}(-1,1) \mid \|r\|^2_{\mathcal{H}_0} \equiv \int_{-1}^{1} \left(r'^2(1-y^2) + r^2\right) \rho dy < +\infty\}.$$

Prop. Consider a stationary solution in $\mathcal{H}_0$. Then, either $w \equiv 0$ or there exist $d \in (-1,1)$ and $e = \pm 1$ such that $w(y) = e\kappa(d,y)$ where

$$\forall (d,y) \in (-1,1)^2,\ \ \kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ and } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}.$$

Remark: We have 3 connected components. $E(0) = 0 < E(\pm\kappa(d)) = E(\kappa_0)$.

Blow-up profile near a non characteristic point

Th. *There exist* $C_0 > 0$ *and* $\mu_0 > 0$ *such that*
if x_0 *is* **non characteristic**, *then there exist* $d(x_0) \in (-1,1)$, $e(x_0) = \pm 1$ *and* $s^*(x_0) \geq -\log T(x_0)$ *such that :*
(i) For all $s \geq s^*(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \begin{pmatrix} \kappa(d(x_0), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}$$

and $E(w_{x_0}(s) \to E(\kappa_0)$ *where the energy space*

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \;\mid\; \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

(ii) $d(x_0) = T'(x_0)$.

Rk. We have exp. fast convergence (hence, C^{1,μ_0} regularity of $\mathcal{R}$, see Nouaili).

Rk. $\|w_{x_0}(y,s) - e(x_0)\kappa(d(x_0), y)\|_{L^\infty(-1,1)} \to 0$.

Rk. The parameter of the profile $d(x_0)$ has a geometrical interpretation ($T'(x_0)$).

Difficulties of the proof of convergence

- The set of non zero stationary solutions is made up of non isolated solutions (one parameter family):
  $\longrightarrow$ we need a **modulation technique**.
- The linearized operator around a non zero stationary solution **is non self-adjoint**:
  $\longrightarrow$ we need to use dispersive properties coming from the Lyapunov functional to control the negative part of the spectrum.

The aim of the talk: the proof the convergence

Consider $x_0 \in \mathcal{R}$ and write w instead of w_{x_0}. We proceed in 3 parts:

- Part 1: Approaching the set of (non zero) stationary solutions.

- Part 2: Study of the linearized operator around a stationary solution and decomposition of the solution.

- Part 3: Convergence to a stationary solution.

Part 1: Approaching the set of stationary solutions

We claim the following

Prop. *For some* $d_0 \in (-1,1)$ *and* $e_0 = \pm 1$, *we have*

$$\inf_{|d|<d_0} \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - e_0 \begin{pmatrix} \kappa(d) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-1,1)} \to \text{ as } s \to \infty.$$

Consider the set of stationary solutions $Stat = \{0, \pm\kappa(d) \mid |d| < 1\}$. Since

- for s large, $0 < \epsilon_0(p) \le \|(w(s), \partial_s w(s))\|_{H^1 \times L^2(-1,1)} \le C_0$,
- $\|\kappa(d)\|_{H^1(-1,1)} \to +\infty$ as $|d| \to 1$,
- $Stat$ is made of 3 connected components $\{0\}$, $\{\kappa(d) \mid |d| < 1\}$ and $\{-\kappa(d) \mid |d| < 1\}$,

it is enough to prove that

$$\inf_{w^* \in Stat} \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \begin{pmatrix} w^* \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-1,1)} \to \text{ as } s \to \infty.$$

The proof of Part 1

We proceed in 2 steps:

- In Step 1, we use compactness to prove the convergence in $L^\infty(-1,1)$.

- In Step 2, we use the energy localization in the u variable to gain control in $H^1(-1,1)$.

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Step 1: Compactness and convergence in $L^\infty(-1,1)$

Consider an arbitrary sequence $s_n \to \infty$. We will show that for some $w^* \in S$ and up to a subsequence, we have

$$\left\| \begin{pmatrix} w(s_n) \\ \partial_s w(s_n) \end{pmatrix} - \begin{pmatrix} w^* \\ 0 \end{pmatrix} \right\|_{H^1\times L^2(-1,1)} \to \text{ as } n \to \infty.$$

From the bound

$$\|(w(s), \partial_s w(s))\|_{H^1\times L^2(-1,1)} \leq C_0$$

for s large enough and compactness, we see that for some $(w^*, v^*) \in H^1 \times (-1,1)$ and up to a subsequence,

$$\begin{pmatrix} w(s_n) \\ \partial_s w(s_n) \end{pmatrix} \to \begin{pmatrix} w^* \\ v^* \end{pmatrix} \text{ weakly in } H^1 \times L^2(-1,1) \text{ as } n \to \infty.$$

and

$$\|w(s_n) - w^*\|_{L^\infty(-1,1)} \to 0 \text{ as } n \to \infty.$$

Step 1: Compactness and convergence in $L^\infty(-1,1)$ (cont.)

From the dissipation of the Lyapunov functional:

$$\int_{-\log T(x_0)}^{\infty}\int_{-1}^{1}\frac{\partial_s w(y,s)^2}{1-y^2}\rho(y)dy \le E(w(-\log T(x_0))) \le C_0,$$

we prove that

$$v^* \equiv 0,\ \ w^* \in Stat \text{ and } w(y,s_n+s) \to w^*(y) \text{ as } n \to \infty$$

uniformly for $|y|<1$ and $|s| \le M$, for any $M>0$.

Step 2: Convergence in H^1 through energy localization in the u variable

Going to the $u(x,t)$ variable, writing a Duhamel formulation in the light cone and coming back to the $w(y,s)$ variable, we get a Duhamed formulation in w, for $s \in [s_n - M, s_n]$, yielding

$$\left\| \begin{pmatrix} w(s_n) \\ \partial_s w(s_n) \end{pmatrix} - \begin{pmatrix} w^* \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-1,1)} \leq C(M) \|w(s_n - M) - w^*\|_{L^\infty(-1,1)} + C_0 e^{-\frac{2M}{p-1}}.$$

Fixing M then n large enough, we get to the conclusion which we recall:

Prop. *For some $d_0 \in (-1,1)$ and $e_0 = \pm 1$, we have*

$$\inf_{|d|<d_0} \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - e_0 \begin{pmatrix} \kappa(d) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-1,1)} \to \text{ as } s \to \infty.$$

Part 2: Linearization around a possible limit

Let us assume that $w(y,s) \to e_0\kappa(d,y)$ in the energy space $\mathcal{H}$. Let

$$q(y,s) = w(y,s) - e_0\kappa(d,y).$$

To simplify the notation, we assume that

$$e_0 = 1, \quad p = 2 \text{ and } w \geq 0.$$

Part 2: Linearization around a possible limit (cont.)

For all $s \geq -\log T(x_0)$,

$$\frac{\partial}{\partial s}\left(\begin{array}{c} q_1 \\ q_2 \end{array}\right) = L_d\left(\begin{array}{c} q_1 \\ q_2 \end{array}\right) + \left(\begin{array}{c} 0 \\ q_1^2 \end{array}\right)$$

where

$$L_d\left(\begin{array}{c} q_1 \\ q_2 \end{array}\right) = \left(\begin{array}{l} q_2 \\ \mathcal{L}q_1 + \psi(d,y)q_1 - 5q_2 - 2yq_2' \end{array}\right),$$

$$\mathcal{L}q_1 = \frac{1}{\rho}\partial_y\left(\rho(1-y^2)\partial_y q_1\right),$$

$$\psi(d,y) = 6\left(2\frac{(1-d^2)}{(1+dy)^2} - 1\right)$$

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Properties of the linear operator

- The operator L_d is not self-adjoint in the energy space.
- Its spectrum is given by

$$\lambda_n = 1 - n \text{ and } \mu_n = -6 - n, \quad n \in \mathbb{N}.$$

In particular, it has $\lambda = 1$ and $\lambda = 0$ as eigenvalues. The others are negative.

Two problems :

- ▷ How to control the zero eigenvalue? *By modulation*.
- ▷ How to control the negative part? *By a linear version of the Lyapunov functional*.

Decomposition of the solution

For $\lambda = 1$ or 0, we introduce the eigenfunction $F_\lambda^d(y)$ such that

$$L_d F_\lambda^d = \lambda F_\lambda^d$$

and the projector π_λ^d on F_λ^d.
Rk. We have

$$F_0^d(y) = C(d) \begin{pmatrix} \partial_d \kappa(d,y) \\ 0 \end{pmatrix}$$

and F_1^d is coming from the choice of the scaling time in the definition of $w = w_{x_0}$.
We decompose q as

$$q(y,s) = \pi_1^d(q(s)) F_1^d(y) + \pi_0^d(q(s)) F_0^d(y) + q_-(y,s).$$

Control of the negative part

Let us introduce the symmetric bilnear form

$$\varphi_d\left(q,r\right) = \int_{-1}^{1}\left(-q_1\left(\mathcal{L}r_1 + \psi(d,y)r_1\right) + q_2 r_2\right)\rho dy$$

where $\mathcal{L}r_1 + \psi(d,y)r_1$ already apperas in the definition of L_d:

$$L_d\left(\begin{array}{c} r_1 \\ r_2 \end{array}\right) = \left(\begin{array}{l} r_2 \\ \mathcal{L}r_1 + \psi(d,y)r_1 - 5r_2 - 2yr_2' \end{array}\right).$$

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Control of the negative part (cont.)

We recall the decomposition:

$$r(y) = \pi_1^d(r)F_1^d(y) + \pi_0^d(r)F_0^d(y) + r_-(y).$$

We claim the following:

Prop.

(i) If $r \in \mathcal{H}$ and $\pi_1^d(r) = \pi_0^d(r) = 0$, then

$$\frac{1}{C_0}\|r\|_{\mathcal{H}}^2 \le \varphi_d(r,r) \le C_0\|r\|_{\mathcal{H}}^2.$$

(ii) If $r \in \mathcal{H}$, then,

$$\frac{1}{C_0}\|r\|_{\mathcal{H}}^2 \le \varphi_d(r_-,r_-) + \sum_{\lambda=0}^{1} |\pi_\lambda^d(r)|^2 \le C_0\|r\|_{\mathcal{H}}^2.$$

Modulation technique

I recall that we know that

$$\inf_{|d|<d_0} \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \begin{pmatrix} \kappa(d) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-1,1)} \to \text{ as } s \to \infty$$

for some $d_0 \in (-1,1)$. We want to prove the convergence to some $\kappa(d^*,y)$.
We introduce

$$q(y,s) = w(y,s) - \kappa(d(s),y)$$

where $d(s) \in (-1,1)$ is chosen so that

$$\pi_0^{d(s)}(q(s)) = 0 \text{ in } q(y,s) = \pi_1^{d(s)}(q(s))F_1^{d(s)}(y) + \pi_0^{d(s)}(q(s))F_0^{d(s)}(y) + q_-(y,s).$$

This is possible because $F_0^d(y) = C(d) \begin{pmatrix} \partial_d \kappa(d,y) \\ 0 \end{pmatrix}$.

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Modulation technique (cont.)

The decomposition becomes

$$q(y,s) = \pi_1^{d(s)}(q(s))F_1^{d(s)}(y) + 0 + q_-(y,s)$$

and if we define

$$\alpha_1(s) = \pi_1^{d(s)}(q(s)) \text{ and } \alpha_-(s) = \sqrt{\varphi_{d(s)}(q_-,q_-)},$$

then

$$\frac{1}{C_0}\|q(s)\|_{\mathcal{H}} \le |\alpha_1(s)| + |\alpha_-(s)| \le C_0\|q(s)\|_{\mathcal{H}}.$$

Projection of the equation on the components of q

We recall the equation:

$$\frac{\partial}{\partial s}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = L_d \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 \\ q_1^2 \end{pmatrix} - d'(s) \begin{pmatrix} \partial_d \kappa(d) \\ 0 \end{pmatrix}.$$

If $d(s) = \tanh \zeta(s)$, then

$$\begin{aligned}
|\zeta'(s)| &\leq C\|q(s)\|_{\mathcal{H}}^2, \\
|\alpha_1'(s) - \alpha_1(s)| &\leq C\|q(s)\|_{\mathcal{H}}^2, \\
(\frac{1}{2}\alpha_-(s)^2 + R_-)' &\leq -4\int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + C\|q(s)\|_{\mathcal{H}}^3 \\
\text{with } |R_-(s)| &\leq C\|q(s)\|_{\mathcal{H}}^3 \\
\frac{d}{ds}\int_{-1}^{1} q_1 q_2 \rho &\leq -\frac{4}{5}\alpha_-(s)^2 + C\int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + C\alpha_1^2.
\end{aligned}$$

Decreasing of the function

If

$$f(s) = \alpha_-(s)^2 + 2R_-(s) + \eta \int_{-1}^{1} q_1 q_2 \rho$$

and $\eta > 0$ is small, then

$$\begin{aligned} f'(s) &\leq -2\mu f(s), \\ \frac{1}{C_0}\|q(s)\|_{\mathcal{H}}^2 &\leq f(s) \leq C_0\|q(s)\|_{\mathcal{H}}^2. \end{aligned}$$

Thus,

$\|q(s)\|_{\mathcal{H}}^2 \leq Ce^{-\mu s}$ with $q(y,s) = w(y,s) - \kappa(d(s),y)$.

But does $w(y,s)$ converge?

Convergence of the modulation parameter

Recalling that $|\zeta'(s)| \le C\|q(s)\|_{\mathcal{H}}^2$, we see that $\zeta(s)$ converges and so does $d(s) = \tanh \zeta(s)$.

Finally, we see that

$$\|w(y,s) - \kappa(d^*,y)\|_{\mathcal{H}} \le Ce^{-\mu s}$$

for some $d^* \in (-1,1)$.