Regularity for two models of chemotaxis and angiogenesis

Hatem Zaag
CNRS École Normale Supérieure

Rio, February 21, 2005

In collaboration with

L. Corrias (University of Evry) and B. Perthame (ENS).

Outline of the talk

- An example: the Dictyostellium Discoideum

- the general Patlak / Keller-Segel model

- A first model: Chemotaxis (existence of L^p solutions)

- Chemotaxis : A blow-up criterion

- A second model: Angiogenesis (existence of L^p solutions)

An example: chemotaxis. Case of the amoeba Dictyostellium Discoideum

Chemotaxis (definition): movement of bacteria, amoebas, cells, under the attraction of some chemical (the chemoattractant).

1st movie (source: dictybase.org)

Aggregation of amoebas D. Discoideum towards a source point of the chemoattractant cAMP (cyclo Adenosine Monophosphate).

Time in minutes and seconds.

Experience by G. Gerisch, Max Planck Institut für Biochemie, Martinsried, Germany.

2nd movie (source: dictybase.org)

Chemotaxis of one amoeba towards a source point of cAMP.

Time in minutes and seconds.

Experience by G. Gerisch, Max Planck Institut für Biochemie, Martinsried, Germany.

3rd movie (source: dictybase.org)

Aggregation of amoebas.

Time interval between two steps: 6 minutes.

Experience by P. Devreotes, Johns Hopkins Medical Institutions, Baltimore, USA.

Explanation (from a paper by D. Horstmann)

1- D. Discoideum is a unicellular organism that reproduces by cellular division, as long as resources are available.

Explanation (from a paper by D. Horstmann)

1- D. Discoideum is a unicellular organism that reproduces by cellular division, as long as resources are available.

2- When there are no more resources, amoebas are everywhere.

Explanation (from a paper by D. Horstmann)

1- D. Discoideum is a unicellular organism that reproduces by cellular division, as long as resources are available.

2- When there are no more resources, amoebas are everywhere.

3- One amoeba secretes cAMP which attracts the other amoebas.

Explanation (from a paper by D. Horstmann)

1- D. Discoideum is a unicellular organism that reproduces by cellular division, as long as resources are available.

2- When there are no more resources, amoebas are everywhere.

3- One amoeba secretes cAMP which attracts the other amoebas.

4- Amoebas move towards the “founding” amoeba, and secrete cAMP (3rd movie).

5- Aggregation and beginning of differentiation.

5- Aggregation and beginning of differentiation.

6- Formation of a pseudoplasmoid (multicellular body).

5- Aggregation and beginning of differentiation.

6- Formation of a pseudoplasmoid (multicellular body).

7- The pseudoplasmoid moves towards light sources.

5- Aggregation and beginning of differentiation.

6- Formation of a pseudoplasmoid (multicellular body).

7- The pseudoplasmoid moves towards light sources.

8- Formation of a fruiting body and spreading of spores,

5- Aggregation and beginning of differentiation.

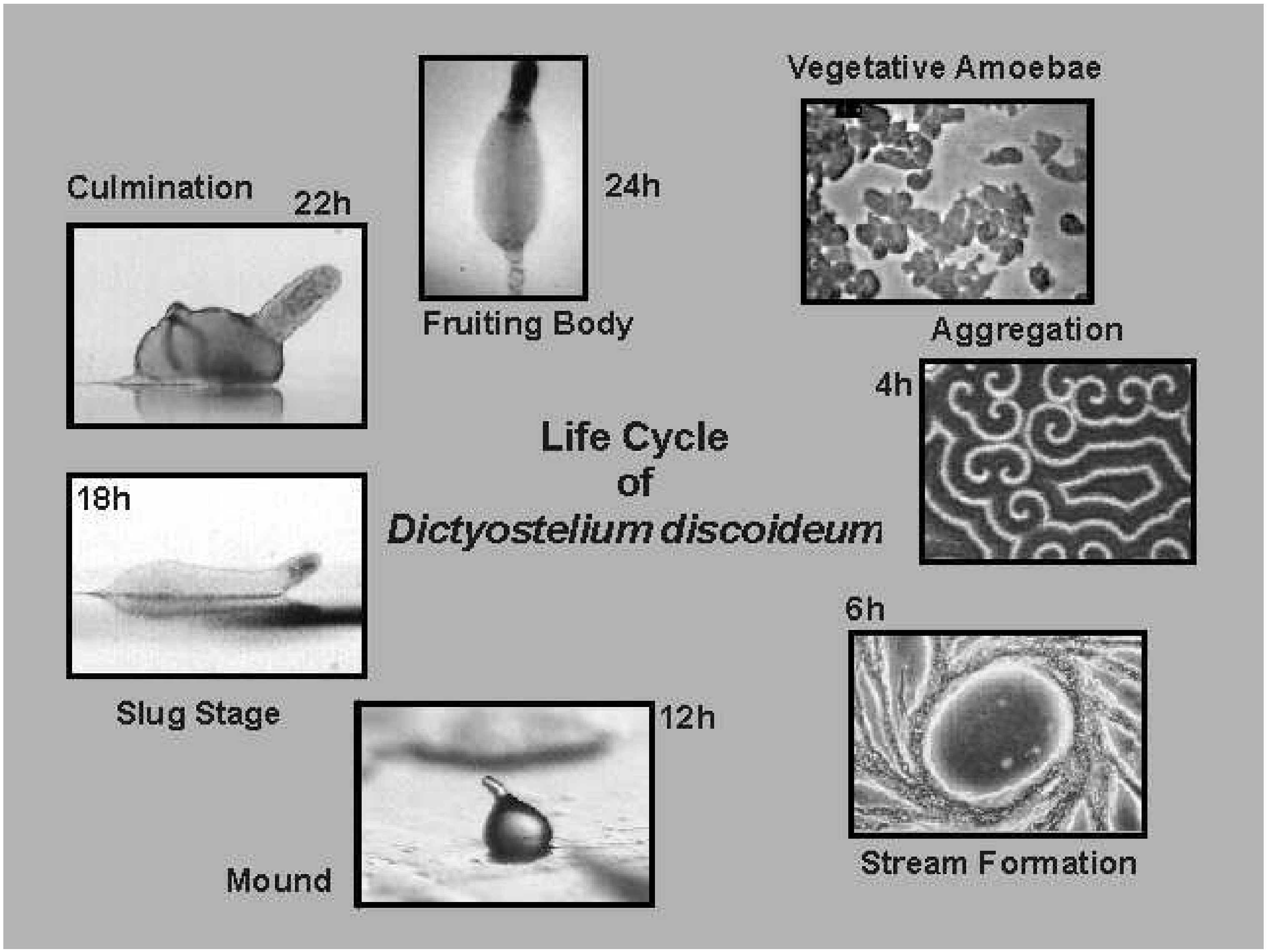
6- Formation of a pseudoplasmoid (multicellular body).

7- The pseudoplasmoid moves towards light sources.

8- Formation of a fruiting body and spreading of spores,

and the cycle restarts (birth of amiboes....).

A summary



Life Cycle
of
Dictyostelium discoideum
Vegetative Amoebae
Aggregation
4h
6h
Stream Formation
12h
Mound
Slug Stage
18h
Culmination
22h
24h
Fruiting Body

Interest of D. Discoideum for medical research

It is a simple model for the study of chemotaxis, which is involved in many processes in superior organisms (differentiation, cancer, etc...)

Example: Angiogenesis near a cancer tumor

At the beginning, the tumor takes directly the nutriments across its boundary. At some point, this is no longer enough.

Example: Angiogenesis near a cancer tumor

At the beginning, the tumor takes directly the nutriments across its boundary. At some point, this is no longer enough.

The tumor sends a chemical signal outside in order to attract endothelial cells (cells that make the interior of blood vessels), and then form a network of capillary vessels that will directly provide the tumor with nutriments.

The classical model of Patlak / Keller-Segel

$$\begin{aligned} \partial_t n &= \mathrm{div}[\kappa(n,c)\nabla n - \chi(n,c)\nabla c], \ \ t>0,\ x\in\Omega, \\ \partial_t c &= \eta\Delta c + \beta(n,c)n - \gamma(n,c)c, \ \ t>0,\ x\in\Omega, \\ n(0,t) &= n_0(x), \ \ c(0,x) = c_0(x). \end{aligned}$$

$n =$ population density, $c =$ density of the chemical.

The classical model of Patlak / Keller-Segel

$$
\begin{aligned}
\partial_t n &= \mathrm{div}[\kappa(n,c)\nabla n - \chi(n,c)\nabla c], \;\; t > 0, \; x \in \Omega, \\
\partial_t c &= \eta \Delta c + \beta(n,c)n - \gamma(n,c)c, \;\; t > 0, \; x \in \Omega, \\
n(0,t) &= n_0(x), \;\; c(0,x) = c_0(x).
\end{aligned}
$$

$n =$ population density, $c =$ density of the chemical.

$\chi =$ chemtactic sensitivity. Generally, $\chi(n,c) = n\chi(c)$.

$\chi(c) > 0$ (decreasing): attraction, positive chemotaxis.

$\chi(c) < 0$ (increasing): repulsion, negative chemotaxis.

The classical model of Patlak / Keller-Segel

$$\begin{aligned}
\partial_t n &= \text{div}[\kappa(n,c)\nabla n - \chi(n,c)\nabla c], \;\; t>0,\ x\in\Omega,\\
\partial_t c &= \eta\Delta c + \beta(n,c)n - \gamma(n,c)c, \;\; t>0,\ x\in\Omega,\\
n(0,t) &= n_0(x), \;\; c(0,x) = c_0(x).
\end{aligned}$$

$n =$ population density, $c =$ density of the chemical.

$\chi =$ chemtactic sensitivity. Generally, $\chi(n,c) = n\chi(c)$.

$\chi(c) > 0$ (decreasing): attraction, positive chemotaxis.

$\chi(c) < 0$ (increasing): repulsion, negative chemotaxis.

β and γ: production and decay rate of the chemical.

The classical model of Patlak / Keller-Segel

$$\begin{aligned} \partial_t n &= \text{div}[\kappa(n,c)\nabla n - \chi(n,c)\nabla c], \;\; t > 0,\ x \in \Omega, \\ \partial_t c &= \eta \Delta c + \beta(n,c) n - \gamma(n,c) c, \;\; t > 0,\ x \in \Omega, \\ n(0,t) &= n_0(x), \;\; c(0,x) = c_0(x). \end{aligned}$$

$n =$ population density, $c =$ density of the chemical.

$\chi =$ chemtactic sensitivity. Generally, $\chi(n,c) = n\chi(c)$.

$\chi(c) > 0$ (decreasing): attraction, positive chemotaxis.

$\chi(c) < 0$ (increasing): repulsion, negative chemotaxis.

β and γ: production and decay rate of the chemical.

Boundary condition to have the mass conservation:

$$\int_\Omega n(x,t)dx \equiv \int_\Omega n_0(x)dx.$$

The classical model of Patlak / Keller-Segel

$$\begin{aligned}
\partial_t n &= \mathrm{div}[\kappa(n,c)\nabla n - \chi(n,c)\nabla c], \;\; t>0,\; x\in\Omega,\\
\partial_t c &= \eta\Delta c + \beta(n,c)n - \gamma(n,c)c, \;\; t>0,\; x\in\Omega,\\
n(0,t) &= n_0(x), \;\; c(0,x) = c_0(x).
\end{aligned}$$

$n =$ population density, $c =$ density of the chemical.

$\chi =$ chemtactic sensitivity. Generally, $\chi(n,c) = n\chi(c)$.

$\chi(c) > 0$ (decreasing): attraction, positive chemotaxis.

$\chi(c) < 0$ (increasing): repulsion, negative chemotaxis.

β and γ: production and decay rate of the chemical.

Boundary condition to have the mass conservation:

$$\int_\Omega n(x,t)dx \equiv \int_\Omega n_0(x)dx.$$

Maximum principle: $n_0 \geq 0$, $c_0 \geq 0 \Longrightarrow n \geq 0$, $c \geq 0$.

References

- Patlak, *Bull. Math. Biophys.*, 1953.

- Keller and Segel, *J. Theor. Biol.*, 1970 and 1971 (aggregation of Dictyostellium Discodeum).

- Horstmann, *Jahresber. Deutsch. Math.-Verein*, 2004 (survey)

- Stevens, *SIAM J. Appl. Math.*, 2000.

Other models

- hyperbolic models (Preziosi *et al.*, initiation of angiogenesis

- kinetic models (Filbet, Laurençot, Perthame).

A first model: a parabolic-elliptic system of chemotaxis

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \chi \nabla \cdot [n \nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ -\Delta c = n - \alpha c, & t > 0,\ x \in \mathbb{R}^d, \\ n(0,x) = n_0(x), & x \in \mathbb{R}^d. \end{cases}$$

$\chi(n,c) = \chi n$ where the "sensitivity" χ is constant.

A first model: a parabolic-elliptic system of chemotaxis

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \chi \nabla \cdot [n \nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ -\Delta c = n - \alpha c, & t > 0,\ x \in \mathbb{R}^d, \\ n(0, x) = n_0(x), & x \in \mathbb{R}^d. \end{cases}$$

$\chi(n, c) = \chi n$ where the "sensitivity" χ is constant.

E_d the fundamental solution of $-\Delta + \alpha Id$, $\alpha \geq 0$:

$$c(x, t) = \int_{\mathbb{R}^d} E_d(x - y) n(y, t) dy.$$

If $\alpha = 0$, then

$$\nabla c(x, t) = -C(d) \int_{\mathbb{R}^d} \frac{(x - y)}{|x - y|^d} n(y, t) dt, \quad d \geq 2.$$

If $\alpha \geq 0$, then: $n(t) \in L^p(\mathbb{R}^d)$, $p > d \implies \nabla c(t) \in L^\infty(\mathbb{R}^d)$.

A 2nd model : a parabolic-ode system of angiogenesis.

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \nabla \cdot [n \chi(c) \nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ \frac{\partial}{\partial t} c = -c^m\, n, & t > 0,\ x \in \mathbb{R}^d, \\ n(0,x) = n_0(x), \qquad c(0,x) = c_0(x), & x \in \mathbb{R}^d. \end{cases}$$

where $m > 0$,

n: endothelial cells, c: the tumor angiogenic factor,

A 2nd model : a parabolic-ode system of angiogenesis.

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \nabla \cdot [n \chi(c) \nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ \frac{\partial}{\partial t} c = -c^m \, n, & t > 0,\ x \in \mathbb{R}^d, \\ n(0,x) = n_0(x), \qquad c(0,x) = c_0(x), & x \in \mathbb{R}^d. \end{cases}$$

where $m > 0$,

n: endothelial cells, c: the tumor angiogenic factor,

and $\chi(c) = c^{-\alpha}$, $0 < \alpha < 1$ or $\chi(c) = \frac{\beta}{\alpha + \beta c}$,

A 2nd model : a parabolic-ode system of angiogenesis.

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \nabla \cdot [n\chi(c)\nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ \frac{\partial}{\partial t} c = -c^m n, & t > 0,\ x \in \mathbb{R}^d, \\ n(0,x) = n_0(x), \qquad c(0,x) = c_0(x), & x \in \mathbb{R}^d. \end{cases}$$

where $m > 0$,

n: endothelial cells, c: the tumor angiogenic factor,

and $\chi(c) = c^{-\alpha}$, $0 < \alpha < 1$ or $\chi(c) = \frac{\beta}{\alpha + \beta c}$,

$c^{1-m}(x,t) = (m-1)\int_0^t n(x,\tau)d\tau + c_0^{1-m}(x)$ if $m \neq 1$ and

$c(x,t) = c_0(x)e^{-\int_0^t n(x,\tau)d\tau}$ if $m = 1$.

A first model: a parabolic-elliptic system of chemotaxis

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \chi \nabla \cdot [n \nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ -\Delta c = n - \alpha c, & t > 0,\ x \in \mathbb{R}^d, \\ n(0,x) = n_0(x), & x \in \mathbb{R}^d. \end{cases}$$

$\chi(n,c) = \chi n$ where the "sensitivity" χ is constant.

E_d the fundamental solution of $-\Delta + \alpha Id$, $\alpha \geq 0$:

$$c(x,t) = \int_{\mathbb{R}^d} E_d(x-y) n(y,t) dy.$$

If $\alpha = 0$, then

$$\nabla c(x,t) = -C(d) \int_{\mathbb{R}^d} \frac{(x-y)}{|x-y|^d} n(y,t) dt, \quad d \geq 2.$$

If $\alpha \geq 0$, then: $n(t) \in L^p(\mathbb{R}^d)$, $p > d \Longrightarrow \nabla c(t) \in L^\infty(\mathbb{R}^d)$.

How to obtain global weak solutions?

A classical idea: (for example), control the L^p norm of n for all t. Computation gives:

$$\frac{d}{dt}\int_\Omega n^p + 4\kappa\frac{p-1}{p}\int_\Omega |\nabla n^{p/2}|^2 = \chi p(p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c\ .$$

If $\nabla c(x,t)$ is uniformly bounded in x and t, then we are done (technique of Nagai and Hortsmann). Indeed, We simply estimate the RHS as follows:

$$\chi p(p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c = 2\chi(p-1)\int_\Omega n^{p/2}\nabla n^{p/2}\cdot\nabla c$$

$$\leq\ 2\kappa\frac{p-1}{p}\int_\Omega |\nabla n^{p/2}|^2 + \frac{\chi^2 p(p-1)}{2\kappa}\|\nabla c\|^2_{L^\infty_{t,x}}\int_\Omega n^p,$$

which gives

$$\frac{d}{dt}\int_\Omega n^p + 2\kappa\frac{p-1}{p}\int_\Omega |\nabla n^{p/2}|^2 \leq \frac{\chi^2 p(p-1)}{2\kappa}\|\nabla c\|^2_{L^\infty_{t,x}}\int_\Omega n^p\ ,$$

$\Longrightarrow$ control of all L^p norms of n, $1\leq p\leq +\infty$.

- This method worked only in 1 dimension or in the radial case in 2 dimensions, for Nagai (who obtained the L^∞ bound on ∇c).

- Generalization to other sensitivity functions (non handeled by us) by Biler.

A new idea. $\Omega = \mathbb{R}^d$, dimension $d \geq 2$. I recall the system...

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \chi \nabla \cdot [n \nabla c], & t > 0,\ x \in \Omega, \\ -\Delta c = n - \alpha c, & t > 0,\ x \in \Omega, \\ n(0, x) = n_0(x), & x \in \Omega. \end{cases}$$

... and the equation on the L^p norm:

$$\begin{aligned} \frac{d}{dt} \int n^p + 4\kappa \frac{p-1}{p} \int |\nabla n^{p/2}|^2 &= \chi (p-1) \int \nabla n^p \cdot \nabla c \\ &= -\chi (p-1) \int n^p \cdot \Delta c. \end{aligned}$$

Since $-\Delta c = n - \alpha c$, we have

$$\frac{d}{dt}\int n^p + 4\kappa\frac{p-1}{p}\int |\nabla n^{p/2}|^2 \leq \chi(p-1)\int n^{p+1}.$$

Gagliardo-Nirenberg (condition : $p \geq \max\left(1, \frac{d}{2}-1\right)$) :

$$\int n^{p+1} \leq C(d,p) \leq C(d)\|\nabla n^{p/2}\|_{L^2}^2 \ \|n\|_{L^{\frac{d}{2}}},$$

and

$$\frac{d}{dt}\int n^p \leq (p-1)\|\nabla n^{p/2}\|_{L^2}^2\left[\chi\tilde{C}(d)\|n\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{p}\right] \ .$$

Dimension $d = 2$: $\|n\|_{L^{\frac{d}{2}}} = \|n\|_{L^1} \equiv \|n_0\|_{L^1}$, therefore

$$\frac{d}{dt}\int n^p \leq (p-1)\|\nabla n^{p/2}\|_{L^2}^2 \left[\chi \tilde{C}(d)\|n_0\|_{L^1} - \frac{4\kappa}{p}\right] \ .$$

Hence, if

$$\chi \tilde{C}(d)\|n_0\|_{L^1} - \frac{4\kappa}{p^*} \leq 0 \ ,$$

then for all $p \leq p^*$, $\int n^p$ decreases and stays bounded.

Dimension $d = 3$: $\|n\|_{L^{\frac{d}{2}}}$ is not conserved, but with $p = \frac{d}{2}$, we write

$$\frac{d}{dt}\int n^{\frac{d}{2}} \leq (\frac{d}{2} - 1)\|\nabla n^{\frac{d}{4}-\frac{1}{2}}\|_{L^2}^2 \left[\chi\tilde{C}(d)\|n\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{\frac{d}{2}}\right] .$$

therefore, if

$$\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{\frac{d}{2}} \leq 0,$$

then $\|n\|_{L^{\frac{d}{2}}}$ decreases and we write for any other p:

$$\frac{d}{dt}\int n^p \leq (p-1)\|\nabla n^{p/2}\|_{L^2}^2 \left[\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{p}\right] \ .$$

Therefore, (like in 2 dimensions, but with the $L^{\frac{d}{2}}$ norm instead of the mass), if

$$\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}} \leq \min\left(\frac{4\kappa}{\frac{d}{2}}, \frac{4\kappa}{p^*}\right)$$

then, for all $p \leq p^*$, $\int n^p$ decreases and stays bounded.

We actually would like a uniform condition $p \in [\max\left(1, \frac{d}{2} - 1\right), +\infty)$ without the restriction $p \leq p^*$. For this, we work with $(n-K)_+$ instead of n, and we take K sufficietly large.

After that, we regularize the system by introducing

$$-\Delta c_e = n_\varepsilon \star \rho_\varepsilon - \alpha c_\varepsilon$$

where ρ_ε is a regularizing kernel. We obtain the following theorem (I first recall the system):

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \chi \nabla \cdot [n \nabla c], & t > 0, \ x \in \mathbb{R}^d, \\ -\Delta c = n - \alpha c, & t > 0, \ x \in \mathbb{R}^d, \\ n(0, x) = n_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Theorem (Existence for the chemotaxis system) *($d \geq 2$)*

If $n_0 \geq 0$, $n_0 \in L^1(\mathbb{R}^d)$ and $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq K_0(\kappa, \chi, d)$, then the system has a global weak solution such that for all $t > 0$

$$\|n(t)\|_{L^1(\mathbb{R}^d)} = \|n_0\|_{L^1(\mathbb{R}^d)},$$

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq \|n_0\|_{L^p(\mathbb{R}^d)} \ , \quad \max\{1; \tfrac{d}{2} - 1\} \leq p \leq \tfrac{d}{2},$$

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq C\left(t, K_0, \|n_0\|_{L^p(\mathbb{R}^d)}\right) \qquad \tfrac{d}{2} < p \leq \infty.$$

Theorem (Blow-up criterion for the chemotaxis system) *For $d \geq 3$ assume that*

$$\int_{\mathbb{R}^d} \frac{|x|^2}{2} n_0(x)dx \leq C(\chi, \kappa, d) \left(\int_{\mathbb{R}^d} n_0\right)^{\frac{d}{d-2}}$$

and for $d = 2$ assume that $\int_{\mathbb{R}^d} \frac{|x|^2}{2} n_0(x)dx$ is finite and that $\int_{\mathbb{R}^d} n_0 \geq M_0$ for some $M_0(\chi, \kappa, d) > 0$. Then, the chemotaxis system has no global smooth soution with fast decay at infinity.

The classical system of chemotaxis

$$
\begin{cases}
\partial_t n = \nabla \cdot [\kappa(n,c)\nabla n - \chi(n,c)\nabla c], & t > 0,\ x \in \Omega, \\
\partial_t c = \eta \Delta c + \beta(n,c) n - \gamma(n,c) c, & t > 0,\ x \in \Omega, \\
n(0,x) = n_0(x), \quad c(0,x) = c_0(x). &
\end{cases}
$$

$+$ boundary conditions, $\Omega \subset \mathbb{R}^d$,

n is the density of the population,

c is the density of the chemo-attractant,

χ is the sensitivity of the the chemo-attractant.

A 2nd model : a parabolic-ode system of angiogenesis.

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \nabla \cdot [n \chi(c) \nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ \frac{\partial}{\partial t} c = -c^m n, & t > 0,\ x \in \mathbb{R}^d, \\ n(0,x) = n_0(x), \qquad c(0,x) = c_0(x), & x \in \mathbb{R}^d. \end{cases}$$

where $m > 0$,

n: endothelial cells, c: the tumor angiogenic factor,

and $\chi(c) = c^{-\alpha}$, $0 < \alpha < 1$ or $\chi(c) = \frac{\beta}{\alpha + \beta c}$,

$c^{1-m}(x,t) = (m-1) \int_0^t n(x,\tau) d\tau + c_0^{1-m}(x)$ if $m \neq 1$ and

$c(x,t) = c_0(x) e^{-\int_0^t n(x,\tau) d\tau}$ if $m = 1$.

Conservation of mass: $\int n(x,t)dx = \int n_0(x)dx$.

Maximum principle: $n(x,t) \geq 0$ and $0 \leq c(x,t) \leq \|c_0\|_{L^\infty}$.

Divergence form: Let $v = \frac{n}{\phi(c)}$, where ϕ is defined by

$$\phi'(c) = \frac{1}{\kappa}\phi(c)\chi(c) \qquad c > 0, \qquad \phi(0) = 1.$$

Then,

$$\frac{\partial}{\partial t}\left(\frac{n}{\phi(c)}\right) = \kappa \frac{1}{\phi(c)} \nabla \cdot \left[\phi(c) \nabla \left(\frac{n}{\phi(c)}\right)\right] + \frac{1}{\kappa}\left(\frac{n}{\phi(c)}\right)^2 \phi(c)\chi(c)\ c^m.$$

A fundamental differential inequality $(\Omega = \mathbb{R}^d)$:

For all $p \geq \max(1, \frac{d}{2} - 1)$,

$$\frac{d}{dt}\int \left(\frac{n}{\phi(c)}\right)^p \phi(c)$$

$$\leq (p-1)\|\nabla(\tfrac{n}{\phi(c)})^{p/2}\|^2_{L^2}\left[\frac{1}{\kappa}\tilde{C}(d)K_1\|\phi^{2/d}(c)\left(\frac{n}{\phi(c)}\right)\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{p}\right] ,$$

analogous to the inequality for the parabolic-elliptic system:

$$\frac{d}{dt}\int n^p \leq (p-1)\|\nabla n^{p/2}\|^2_{L^2}\left[\chi\tilde{C}(d)\|n\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{p}\right] .$$

We do as before, and we obtain the following theorem (I recall the system first):

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \nabla \cdot [n \chi(c) \nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ \frac{\partial}{\partial t} c = -c^m\, n, & t > 0,\ x \in \mathbb{R}^d, \\ n(0,x) = n_0(x), \qquad c(0,x) = c_0(x), & x \in \mathbb{R}^d. \end{cases}$$

où $m > 0$.

Theorem (Existence for the angiogenesis system). *If* $d \geq 2$, $m \geq 1$, $n_0 \in L^1(\mathbb{R}^d)$, $c_0 \in L^\infty(\mathbb{R}^d)$, $n_0 \geq 0$, $c_0 \geq 0$ *and* $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq K_0\left(\kappa, \chi, d, \|c_0\|_{L^\infty(\mathbb{R}^d)}\right)$, *then the angiogenesis system has a global weak solution* (n, c) *such that* $n \in L^\infty(\mathbb{R}^+, L^1 \cap L^{\frac{d}{2}}(\mathbb{R}^d))$, $c \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ *and for all* $p^* \geq \max\{1; \frac{d}{2} - 1\}$,

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq C(t, K_0, p^*, \|n_0\|_{L^p(\mathbb{R}^d)}),$$

$$\forall \ \max\{1; \frac{d}{2} - 1\} \leq p \leq p^*.$$