Similarities in blow-up approaches between semilinear heat and wave equations

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This short note is intended to non specialists. We aim at showing a surprising fact : how a “parabolic” program initially developed for blow-up solutions of the *semilinear* heat equation works for blow-up solutions of the *semilinear* wave equation. We feel this fact surprising because for the *linear* equations, everything separates the heat and the wave equations.

For simplicity, all equations are considered on the whole space $\mathbb{R}^N$.

Differences between linear equations

Heat equation $\partial_t u = \Delta u$

- Regularizing effect
- Infinite speed of propagation
- Dissipation of the energy $\displaystyle\int |\nabla u|^2 dx$, non reversible equation

Wave equation $\partial_{tt}^2 u = \Delta u$

- No gain of regularity
- finite speed of propagation ($c = 1$)
- conservation of the energy $\displaystyle\int \left(|\partial_t u|^2 + |\nabla u|^2\right) dx$, reversible equation

Semilinear equations

the heat: $\partial_t u = \Delta u + |u|^{p-1}u$ where $p > 1$ is subcritical with respect to the Sobolev injection:

$$p < 1 + \frac{4}{N-2} \text{ if } N \geq 3.$$

the wave: $\partial^2_{tt} u = \Delta u + |u|^{p-1}u$ where $p > 1$ is subcritical with respect to the conformal invariance:

$$p \leq 1 + \frac{4}{N-1} \text{ if } N \geq 2.$$

Solution of the Cauchy problem and existence of blow-up solutions

The maximal solution either exists for all $t > 0$ (global solution) or on $[0, T)$ for some $T > 0$. In that case:

the heat: $\|u(t)\|_{L^\infty} \to \infty$ as $t \to T$,

the wave: $\|u(t)\|_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|u(t)\|_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|u(t)\|_{\mathrm{L}^2_{\mathrm{loc,u}}} \to \infty$ as $t \to T$ where $\mathrm{L}^2_{\mathrm{loc,u}}$ is the set of all v such that

$$\|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} \equiv \sup_{a \in \mathbb{R}^N} \int_{|x-a|<1} |v(x)|^2 dx < +\infty.$$

Remark: For the semilinear heat equation, no matter how weak is the initial regularity (let's stay in L^q spaces), blow-up occurs always in L^∞ due to the regularizing effect. See Weissler [10]. For the wave equation, there is no regularizing effect. We work with weak solutions and consider the case where u, $\partial_t u$ and ∇u are in $\mathrm{L}^2_{\mathrm{loc,u}}$.

Trivial solutions

When initial data do not depend on space, we just have to solve an ODE. We have the following solutions

the heat: $v' = v^p$ whose solution is $v(t) = \kappa_h (T-t)^{-\frac{1}{p-1}}$ where $\kappa_h = (p-1)^{-\frac{1}{p-1}}$ for any $T > 0$.

the wave: $v'' = v^p$ whose solution is $v(t) = \kappa_w (T-t)^{-\frac{2}{p-1}}$ where $\kappa_w = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}$ for any $T > 0$.

Question: Take an arbitrary solution which blows up at time T. Can we estimate its blow-up rate ? More precisely, can we find an equivalent of its norm in the Cauchy space ?

Answer (the same for the heat and the wave): the blow-up rate is given by the solution of the associated ODE which blows-up at the same time T.

More precisely,

Heat equation

Theorem (A result due to Giga and Kohn [2] and [3] and Giga, Matsui and Sasayama [4]). *Let u be a solution of $\partial_t u = \Delta u + |u|^{p-1}u$ where $p > 1$ and $p < 1 + 4/(N-2)$ if $N \geq 3$ which blows up at time $T > 0$. Then, for all $t \in [0, T)$,*

$$\kappa_h (T-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty} \leq C(T-t)^{-\frac{1}{p-1}}$$

where $C = C(\|u_0\|, T)$.

Near the blow-up time, we have this better estimate:

Theorem (Merle and Zaag [6] and [5], see also the note [7]). *Under the same hypotheses,*

$$\|u(t)\|_{L^\infty} \sim \kappa_h (T-t)^{-\frac{1}{p-1}} \text{ as } t \to T.$$

Wave equation

Theorem (Merle and Zaag [8] and [9]). *Let u be a solution of $\partial_{tt}^2 u = \Delta u + |u|^{p-1}u$ where $p > 1$ and $p \le 1 + 4/(N-1)$ if $N \ge 2$ which blows up at time $T > 0$. Then, for all $t > 0$,*

$$\begin{aligned}\epsilon_{N,p} \le\; & (T-t)^{\frac{2}{p-1}} \|u(t)\|_{\mathrm{L}^2_{\mathrm{loc,u}}} \\ & + (T-t)^{\frac{2}{p-1}+1} \left(\|\nabla u(t)\|_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\partial_t u(t)\|_{\mathrm{L}^2_{\mathrm{loc,u}}} \right) \le C.\end{aligned}$$

Remark: In both cases (heat and wave), lower bounds on the blow-up rate are trivial.

A common method for the proof: the self-similar change of variables

Let u be a solution that blows up at time $T > 0$. For each $a \in \mathbb{R}^N$, we introduce $w_a(y,s)$ defined by:

heat equation:

$$w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = \frac{x-a}{\sqrt{T-t}}, \quad s = -\log(T-t).$$

wave equation:

$$w_a(y,s) = (T-t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x-a}{T-t}, \quad s = -\log(T-t).$$

Remark: In both cases, w_a is the ratio between the blow-up solution u and its supposed blow-up rate (given by the ode). Hence, the goal is to show that for all $s \geq -\log T$,

$$1/C_0 \leq \|w(s)\| \leq C_0.$$

Remark: In both cases, studying the behavior of $u(x,t)$ when (x,t) approaches (a,T) is equivalent to the study of the long-time asymptotics of $w_a(y,s)$ when y is near 0 and the new time variable s goes to infinity.

Remark: In both cases, the new space variable y is a time dependent zoom of the old one x near the point a. This zoom becomes sharper as $t \to T$ (that is $s \to \infty$). However, y is not the same in the heat and the wave setting, since in the former, a derivative in space is like half a derivative in time, whereas in the latter, space and time play the same role.

Equations satisfied by $w_a(y,s)$ (or $w(y,s)$ for simplicity)

For all $y \in \mathbb{R}^N$ and $s \geq -\log T$,

the heat:

$$\partial_s w = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w,$$

the wave:

$$\partial^2_{ss} w - \operatorname{div}\left(\nabla w - (y.\nabla w)y\right) + \frac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y.\nabla \partial_s w.$$

Remark: Surprisingly, the new wave equation is dissipative, unlike the original. This means that we are unveiling a new structure in the problem.

A new structure derived from the self-similar transformation: a Lyapunov functional: the heat (Giga and Kohn [2])

If

$$E_h(w) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)} - \frac{1}{p+1}|w|^{p+1} \right) \exp(-|y|^2/4) dy,$$

then

$$\frac{d}{ds} E_h(w(s)) = -\int_{\mathbb{R}^N} (\partial_s w)^2 \exp(-|y|^2/4) dy.$$

A new structure derived from the self-similar transformation: a Lyapunov functional: the wave (Antonini and Merle [1])

If

$$E_w(w) = \int_{B(0,1)} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}|\nabla w|^2 - \frac{1}{2}(y.\nabla w)^2 + \frac{(p+1)}{(p-1)^2}w^2 - \frac{1}{p+1}|w|^{p+1}\right)(1-|y|^2)^\alpha dy$$

where $\alpha = \frac{2}{p-1} - \frac{N-1}{2} \geq 0$, then

$$\frac{d}{ds}E_w(w(s)) = -2\alpha \int_{B(0,1)} (\partial_s w)^2 (1-|y|^2)^{\alpha-1} \text{ when } p < 1 + \frac{4}{N-1}.$$

Remark: The natural space domain in the wave setting is the unit ball which corresponds in the (x, t) variable to the backward light cone with vertex (a, T), a notion adapted to the finite speed of propagation $(c = 1)$.

Remark: For the wave equation, when $p = 1 + \frac{4}{N-1}$ (critical case), the dissipation of E_w becomes degenerate and is supported in the boundary of the unit ball.

Remark: Still for the wave equation, please note that this Lyapunov functional is not the energy in conformal coordinates.

The same blow-up criterion in similarity variables

Prop. *If a solution W satisfies $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time $S^* > s_0$.*

Proof: For the heat, this is classical. For the wave, see Antonini and Merle [1]).

Since all $w_a(y, s)$ are defined for all $s \geq -\log T$ by construction, they never blow-up. More precisely, we have the following:

Corollary. *For all $a \in \mathbb{R}^N$ and $s \geq -\log T$,*

$$E(w_a(s)) \geq 0.$$

Remark: E stands for E_h or E_w here.

Control of the energy

Because of the blow-up criterion and the monotonicity of E, it holds that

$$\forall a \in \mathbb{R}^N, \quad \forall s \geq -\log T, \quad 0 \leq E(w_a(s)) \leq E(w_a(-\log T)) \leq C_0(T, \|u_0\|).$$

End of the proof:

Since the energy is bounded, one has to use interpolation is Sobolev spaces and show that each term in the energy is bounded, uniformly with respect to the scaling point a. See Giga and Kohn [3] and Giga, Matsui and Sasayama [4] for the heat; see Merle and Zaag [8] and [9] for the wave equation.

References

[1] C. Antonini and F. Merle. Optimal bounds on positive blow-up solutions for a semilinear wave equation. *Internat. Math. Res. Notices*, (21):1141–1167, 2001.

[2] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.*, 38(3):297–319, 1985.

[3] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36(1):1–40, 1987.

[4] Y. Giga, S. Matsui, and S. Sasayama. Blow up rate for semilinear heat equations with subcritical nonlinearity. *Indiana Univ. Math. J.*, 53(2):483–514, 2004.

[5] F. Merle and H. Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2):139–196, 1998.

[6] F. Merle and H. Zaag. A Liouville theorem for vector-valued nonlinear heat equations and applications. *Math. Annalen*, 316(1):103–137, 2000.

[7] F. Merle and H. Zaag. O.D.E. type behavior of blow-up solutions of nonlinear heat equations. *Discrete Contin. Dyn. Syst.*, 8(2):435–450, 2002. Current developments in partial differential equations (Temuco, 1999).

[8] F. Merle and H. Zaag. Determination of the blow-up rate for the semilinear wave equation. *Amer. J. Math.*, 125:1147–1164, 2003.

[9] F. Merle and H. Zaag. Determination of the blow-up rate for a critical semilinear wave equation. *Math. Annalen*, 2004. to appear.

[10] F. B. Weissler. Local existence and nonexistence for semilinear parabolic equations in L^p. *Indiana Univ. Math. J.*, 29(1):79–102, 1980.