# $L^p$ bounds and uniqueness for a chemotaxis model

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# The classical model of Patlak / Keller-Segel

$$\begin{aligned}\partial_t n &= \mathrm{div}[\kappa(n,c)\nabla n - \chi(n,c)\nabla c], \;\; t > 0, \; x \in \Omega,\\ \partial_t c &= \eta \Delta c + \beta(n,c)n - \gamma(n,c)c, \;\; t > 0, \; x \in \Omega,\\ n(0,t) &= n_0(x), \;\; c(0,x) = c_0(x).\end{aligned}$$

$n =$ population density, $c =$ density of the chemical.
$\chi =$ chemtactic sensitivity. Generally, $\chi(n,c) = n\chi(c)$.
$\chi(c) > 0$ (decreasing): attraction, positive chemotaxis.
$\chi(c) < 0$ (increasing): repulsion, negative chemotaxis.
$\beta$ and $\gamma$: production and decay rate of the chemical.
Boundary condition to have the mass conservation:

$$\int_\Omega n(x,t)dx \equiv \int_\Omega n_0(x)dx.$$

Maximum principle: $n_0 \geq 0$, $c_0 \geq 0 \Longrightarrow n \geq 0$, $c \geq 0$.

## Angiogenesis version (parabolic-ode or P.O.)

A Model for the development of capillary vascular vessels near a cancer tumor.

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \nabla \cdot [n \chi(c) \nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ \frac{\partial}{\partial t} c = -c^m\, n, & t > 0,\ x \in \mathbb{R}^d, \\ n(0,x) = n_0(x), \qquad c(0,x) = c_0(x), & x \in \mathbb{R}^d. \end{cases}$$

where $m > 0$,
$n$: endothelial cells, $c$: the tumor angiogenic factor,
and $\chi(c) \geq 0$, the chemotactic sensitivity.

$c^{1-m}(x,t) = (m-1) \int_0^t n(x,\tau) d\tau + c_0^{1-m}(x)$ if $m \neq 1$ and

$c(x,t) = c_0(x) e^{-\int_0^t n(x,\tau) d\tau}$ if $m = 1$.

# The questions

1- Existence of global solutions in $L^p(\mathbb{R}^d)$ for small initial data.

2- $L^\infty$ bound.

# An existence result for the P.O. model

**Th PO1 (Global existence)**

*Assume* $d \geq 2$, $m \geq 1$ *and* $\chi(c)$ *a positive say continuous function on* $[0, \infty)$.

*Consider some* $n_0 \in L^1(\mathbb{R}^d)$ *and* $c_0 \in L^\infty(\mathbb{R}^d)$ *such that* $n_0 \geq 0$ *and* $c_0 \geq 0$.

*There exists a constant* $K_0(\kappa, \chi, d, \|c_0\|_{L^\infty(\mathbb{R}^d)})$ *such that if* $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq K_0$, *then the* **P.O** *model has a global (in time) weak solution* $(n, c)$ *such that*

$$n \in L^\infty(\mathbb{R}^+, L^1 \cap L^{\frac{d}{2}}(\mathbb{R}^d)), \quad c \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$$

*and* $\forall \max\{1; \frac{d}{2} - 1\} \leq p^* < \infty$,

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq C(t, K_0, p^*, \|n_0\|_{L^p(\mathbb{R}^d)}), \qquad \forall \max\{1; \frac{d}{2} - 1\} \leq p \leq p^*.$$

# An $L^\infty$ bound for the P.O. model

**Th PO2 (An $L^\infty$ bound)**

Moreover, if $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq \frac{K_0}{2}$ and $n_0 \in L^\infty(\mathbb{R}^d)$, then

$$\|n(t)\|_{L^\infty} \leq C\left(\kappa, \chi, \|n_0\|_{L^d(\mathbb{R}^d)}\right) \max\left\{\|n_0\|_{L^\infty}, \|n_0\|_{L^1}\right\}.$$

**Remark**: In both results, the smallness condition is only on the critical norm $L^{d/2}$.

# A twin result for the chemotaxis version (parabolic-elliptic or P.E.)

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \chi \nabla \cdot [n \nabla c], & t > 0,\ x \in \mathbb{R}^d, \\ -\Delta c = n - \alpha c, & t > 0,\ x \in \mathbb{R}^d, \\ n(0,x) = n_0(x), & x \in \mathbb{R}^d. \end{cases}$$

$n =$ population density, $c =$ density of the chemical.

$\kappa =$ diffusion coefficient.

$\chi =$ constant chemotactic sensitivity.

$\alpha \geq 0$ degradation of the chemical.

**Comment**: Unlike the P.O. system, the results we present for the P.E. system were known before. However, 3 facts in the P.E. results are new (see later).

# The existence result for the P.E. model

**Th PE1 (Global existence)**
*Assume* $d \geq 2$ *and consider some* $n_0 \in L^1(\mathbb{R}^d)$ *such that* $n_0 \geq 0$.

*There exists a constant* $K_0(\kappa, \chi, d)$ *such that if* $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq K_0$, *then the* **P.E** *system has a global (in time) weak solution* $(n, c)$ *such that for all* $t > 0$

$$\begin{aligned} \|n(t)\|_{L^1(\mathbb{R}^d)} &= \|n_0\|_{L^1(\mathbb{R}^d)}, \\ \|n(t)\|_{L^p(\mathbb{R}^d)} &\leq \|n_0\|_{L^p(\mathbb{R}^d)}, \text{ when } \max\{1; \tfrac{d}{2} - 1\} \leq p \leq \tfrac{d}{2}, \\ \text{and } \|n(t)\|_{L^p(\mathbb{R}^d)} &\leq C(t, K_0, \|n_0\|_{L^p(\mathbb{R}^d)}) \text{ when } \tfrac{d}{2} < p < \infty. \end{aligned}$$

**Remark**: In the second case, the $L^p$ norm is decreasing, unlike the P.O. system (were weigthed $L^p$ norms decrease).

# The $L^\infty$ bound for the P.E. model

**Th PE2 ($L^\infty$ bound)**

*If* $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq \frac{K_0}{2}$ *and* $n_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$*, then*

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq \|n_0\|_{L^p(\mathbb{R}^d)}, \quad \max\{1; \frac{d}{2} - 1\} \leq p \leq d,$$

*and*

$$\|n(t)\|_{L^\infty} \leq C\left(\kappa, \chi, \|n_0\|_{L^d(\mathbb{R}^d)}\right) \max\left\{\|n_0\|_{L^1}, \|n_0\|_{L^\infty}\right\}.$$

# Uniqueness for the P.E. model with bounded intial data

**Th PE3 (Uniqueness)**

*Under the hypotheses of Th PE2, the solution is unique.*

**Remark**: We don't have a uniqueness result for the P.O. system. Unlike the global existence and the $L^\infty$ bound, the method for uniqueness is specific to the P.E. system.

# Comments

The results were known before for the P.E. model except fro three facts:

(i) the smallness condition for the global existence is needed only in the critical norm $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$,

(ii) the $L^\infty$ bound of $n$,

(iii) the uniqueness of the solution in the case of bounded initial data $n_0$.

# Idea of the proof

- The proofs of the global existence and the $L^\infty$ bound follow the same pattern in the P.O. and the P.E. systems.

- For the P.E, we handle $L^p$ norms with respect to the Lebesgue measure, whereas for the P.O., we handle $L^p$ norms with respect to weighted measures in the P.E.

- In the P.E., we have an additional result, the uniqueness, which is a new fact.

**Conclusion**: We present an outline of the proof for the P.E. system, though our result for the P.O. is our main contribution.

# Outline of the proof

- Part 1: Global existence for small (in the $L^{d/2}$ norm) initial data (Th PE1).

- Part 2: The $L^\infty$ bound (Th PE2).

- Part 3: The uniqueness result for initial data in $L^\infty$ (Th PE3).

# Part 1: Global existence for small (in the $L^{d/2}$ norm) initial data (Th PE1)

We recall the first equation:

$$\frac{\partial}{\partial t} n = \kappa \Delta n - \chi \nabla \cdot [n \nabla c].$$

We multiply it by $n^{p-1}$ and integrate to get:

$$\frac{d}{dt} \int_\Omega n^p + 4\kappa \frac{p-1}{p} \int_\Omega |\nabla n^{p/2}|^2 = \chi p(p-1) \int_\Omega n^{p-1} \nabla n \cdot \nabla c\,.$$

Since $-\Delta c = n - \alpha c$ with $\alpha \geq 0$, we integrate by parts to have

$$\frac{d}{dt} \int n^p + 4\kappa \frac{p-1}{p} \int |\nabla n^{p/2}|^2 \leq \chi (p-1) \int n^{p+1}.$$

Using Gagliardo-Nirenberg (condition : $p \geq \max\left(1, \frac{d}{2} - 1\right)$) :

$$\int n^{p+1} \leq C(d,p) \leq C(d) \|\nabla n^{p/2}\|_{L^2}^2 \, \|n\|_{L^{\frac{d}{2}}},$$

we obtain

$$\frac{d}{dt} \int n^p \leq (p-1) \|\nabla n^{p/2}\|_{L^2}^2 \left[ \chi \tilde{C}(d) \|n\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{p} \right] .$$

# Dimension $d = 2$

$\|n\|_{L^{\frac{d}{2}}} = \|n\|_{L^1} \equiv \|n_0\|_{L^1}$, therefore

$$\frac{d}{dt}\int n^p \leq (p-1)\|\nabla n^{p/2}\|_{L^2}^2 \left[\chi\tilde{C}(d)\|n_0\|_{L^1} - \frac{4\kappa}{p}\right] .$$

Hence, if

$$\chi\tilde{C}(d)\|n_0\|_{L^1} - \frac{4\kappa}{p^*} \leq 0 \, ,$$

then for all $p \leq p^*$, $\int n^p$ decreases and stays bounded.

# Dimension $d = 3$

$\|n\|_{L^{\frac{d}{2}}}$ is not conserved, but with $p = \frac{d}{2}$, we write

$$\frac{d}{dt}\int n^{\frac{d}{2}} \leq (\frac{d}{2}-1)\|\nabla n^{\frac{d}{4}-\frac{1}{2}}\|_{L^2}^2 \left[\chi\tilde{C}(d)\|n\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{\frac{d}{2}}\right].$$

therefore, if

$$\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{\frac{d}{2}} \leq 0,$$

then $\|n\|_{L^{\frac{d}{2}}}$ decreases.

Hence,

$$\|n(t)\|_{L^{\frac{d}{2}}} \leq \|n_0\|_{L^{\frac{d}{2}}}$$

and we write for any other $p$:

$$\frac{d}{dt}\int n^p \leq (p-1)\|\nabla n^{p/2}\|_{L^2}^2 \left[\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}} - \frac{4\kappa}{p}\right].$$

Therefore, (like in 2 dimensions, but with the $L^{\frac{d}{2}}$ norm instead of the mass), if

$$\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}} \leq \min\left(\frac{4\kappa}{\frac{d}{2}}, \frac{4\kappa}{p^*}\right)$$

then, for all $p \leq p^*$, $\int n^p$ decreases and stays bounded.

We actually would like a uniform condition $p \in \left[\max\left(1, \frac{d}{2} - 1\right), +\infty\right)$ without the restriction $p \leq p^*$. For this, we work with $(n - K)_+$ instead of $n$, and we take $K$ sufficietly large.

After that, we regularize the system by introducing

$$-\Delta c_e = n_\varepsilon \star \rho_\varepsilon - \alpha c_\varepsilon$$

where $\rho_\varepsilon$ is a regularizing kernel. We obtain Th PE1 that I recall here:

# The existence result for the P.E. model

**Th PE1 (Global existence)**
*Assume $d \geq 2$ and consider some $n_0 \in L^1(\mathbb{R}^d)$ such that $n_0 \geq 0$.*

*There exists a constant $K_0(\kappa, \mathcal{X}, d)$ such that if $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq K_0$, then the* **P.E** *system has a global (in time) weak solution $(n, c)$ such that for all $t > 0$*

$$
\begin{aligned}
\|n(t)\|_{L^1(\mathbb{R}^d)} &= \|n_0\|_{L^1(\mathbb{R}^d)}, \\
\|n(t)\|_{L^p(\mathbb{R}^d)} &\leq \|n_0\|_{L^p(\mathbb{R}^d)}, \text{ when } \max\{1; \tfrac{d}{2} - 1\} \leq p \leq \tfrac{d}{2}, \\
\text{and } \|n(t)\|_{L^p(\mathbb{R}^d)} &\leq C(t, K_0, \|n_0\|_{L^p(\mathbb{R}^d)}) \text{ when } \tfrac{d}{2} < p < \infty.
\end{aligned}
$$

# Part 2: The $L^\infty$ bound

We assume here that

$$n_0 \in L^1 \cap L^\infty(\mathbb{R}^d).$$

We will show that if

$$\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq \frac{K_0}{2},$$

then

$$\|n(t)\|_{L^\infty} \leq C\left(\kappa, \chi, \|n_0\|_{L^d(\mathbb{R}^d)}\right) \max\left\{\|n_0\|_{L^1}, \|n_0\|_{L^\infty}\right\}.$$

Let us prove now that it is sufficient to control the $L^d$ norm of $n$ in order to obtain the $L^\infty$ bound of $n$.

# Part 2, Step 1: the main differential inequality

Recall the following:

$$\frac{d}{dt}\int n^p + 4\kappa\frac{p-1}{p}\int |\nabla n^{p/2}|^2 \le \chi(p-1)\int n^{p+1}.$$

By interpolation and the Gagliardo-Nirenberg-Sobolev inequality, we have:
- in dimension $d = 2$:

$$\int_{\mathbb{R}^d} n^{p+1} \le \|n\|_{L^2}\|n^{\frac{p}{2}}\|_{L^4}^2 \le C\|n\|_{L^2}\|n^{\frac{p}{2}}\|_{L^2}\|n^{\frac{p}{2}}\|_{H^1}\ ,$$

- in dimension $d \ge 3$:

$$\int_{\mathbb{R}^d} n^{p+1} \le \|n^{\frac{p}{2}}\|_{L^{\frac{2d}{d-2}}}\|n^{\frac{p}{2}+1}\|_{L^{2\frac{d}{d+2}}} \le C\|\nabla n^{\frac{p}{2}}\|_{L^2}\|n\|_{L^p}^{\frac{p}{2}}\|n\|_{L^d}\ .$$

Therefore,

$$\frac{d}{dt}\int_{\mathbb{R}^d} n^p + 2\kappa\frac{p-1}{p}\int_{\mathbb{R}^d} |\nabla n^{p/2}|^2 \leq p(p-1)C(\chi,\kappa,\|n\|^2_{L^d})\int_{\mathbb{R}^d} n^p.$$

Finally, taking $n_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ and the smallness condition on $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ so that

$$\|n(t)\|_{L^d} \leq \|n_0\|_{L^d},$$

we have **the main differential inequality**

$$\frac{d}{dt}\int_{\mathbb{R}^d} n^p + 2\kappa\frac{p-1}{p}\int_{\mathbb{R}^d} |\nabla n^{p/2}|^2 \leq C_0 p(p-1)\int_{\mathbb{R}^d} n^p$$

with

$$C_0 = C_0(\chi,\kappa,\|n_0\|^2_{L^d})$$

independent of $p$.

# Part 2, Step 2: Argument of Alikakos

Taking $p = 2^k$ and using again the Gagliardo-Nirenberg-Sobolev inequality, we get as $k \to \infty$ the following $L^\infty$ bound of $n$

$$\sup_{t \geq 0} \|n(t)\|_{L^\infty} \leq C \max\{\|n_0\|_{L^1}; \|n_0\|_{L^\infty}\}.$$

# Part 3: the uniqueness for the P.E. system

**Remark**: Unlike Part 1 and 2, this does not work for the P.O. system.

We follow an idea of Gajewski and Zacharias.

We assume that initial data is in $L^\infty$ and that $\|n_0\|_{L^{d/2}}$ is small enough to have $\|n(t)\|_{L^\infty}$ bounded uniformly in $t$.

Consider $(n_1, c_1)$ and $(n_2, c_2)$ two solutions of the P.E. system with the same initial data.

Consider the function

$$f(n_1, n_2) = n_1 \ln n_1 + n_2 \ln n_2 - (n_1 + n_2) \ln(\frac{n_1 + n_2}{2}),$$

which satisfies

$$f(n_1, n_2) \geq \frac{1}{4}(\sqrt{n_1} - \sqrt{n_2})^2.$$

Then, using the first equation in the P.E. system that we recall:

$$\frac{\partial}{\partial t} n = \kappa \Delta n - \chi \nabla \cdot [n \nabla c],$$

and Cauchy-Schwarz, we compute

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^d} f(n_1, n_2) \\
= \quad & -\kappa \int_{\mathbb{R}^d} \frac{n_1 n_2}{n_1 + n_2} |\nabla \ln(\frac{n_1}{n_2})|^2 + \int_{\mathbb{R}^d} \frac{n_1 n_2}{n_1 + n_2} \nabla \ln(\frac{n_1}{n_2}) \cdot [\nabla c_1 - \nabla c_2] \\
\leq \quad & -\frac{\kappa}{2} \int_{\mathbb{R}^d} \frac{n_1 n_2}{n_1 + n_2} |\nabla \ln(\frac{n_1}{n_2})|^2 + \frac{1}{8\kappa} \|n_1 + n_2\|_{L^\infty} \int_{\mathbb{R}^d} |\nabla (c_1 - c_2)|^2.
\end{aligned}$$

Using the second equation in the P.E. system that we recall:

$$-\Delta c = n - \alpha c,$$

we estimate

$$\int_{\mathbb{R}^d} |\nabla(c_1 - c_2)|^2 \leq C \int_{\mathbb{R}^d} |n_1 - n_2|^2 = C\|\sqrt{n_1} - \sqrt{n_2}\|_{L^2}^2.$$

Therefore, we get

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(n_1, n_2) \leq C\|n_1 + n_2\|_{L^\infty} \|\sqrt{n_1} - \sqrt{n_2}\|_{L^2}^2.$$

Recall that

$$f(n_1, n_2) \geq \frac{1}{4}(\sqrt{n_1} - \sqrt{n_2})^2. \tag{1}$$

Therefore, we get

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(n_1, n_2) \leq C \|n_1 + n_2\|_{L^\infty} \int_{\mathbb{R}^d} f(n_1, n_2).$$

Since

$$\|n_1 + n_2\|_{L^\infty} \leq C_0,$$

and $\int_{\mathbb{R}^d} f(n_1(0), n_2(0)) = 0$, we get by Gronwall's inequality $\int_{\mathbb{R}^d} f(n_1(t), n_2(t)) = 0$, hence by (1), $n_1 \equiv n_2$.