Regularity of the blow-up set for the semilinear heat equation

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Three contributions:

- Z., *Annales IHP, Analyse Non Linéaire*, 2002.

- Z., *Comm. Math. Phys.*, 2002.

- Z., 2004, preprint.

Motivation: singularities in PDEs

Solutions which are regular at $t = 0$, may become “infinite” in finite time T. Example: heat, Schrödinger, wave, generalized KdV, geometric flows, etc...

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Common questions:

- Find the asymptotic behavior(s) near the singularity.
- Discuss their stability.
- Obtain **uniforms** estimates / initial data, etc..
- Understand interactions between regular and singular regions.

The semilinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ \\ u(0) = u_0, \end{cases}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$ and

$$1 < p < \frac{N+2}{N-2} \text{ if } N \geq 3.$$

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Rk. Possible generalizations.

The solution of the Cauchy problem exists:

- either on $[0, +\infty)$: there is **global existence**,

- or on $[0, T)$ with $T < +\infty$: there is **finite-time blow-up**.

In this case,

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A point a is a **blow-up point** if

$$|u(a,t)| \to +\infty \text{ as } t \to T.$$

We denote by $S_u \subset \mathbb{R}^N$ the **blow-up set**, i.e. the set of all blow-up points.

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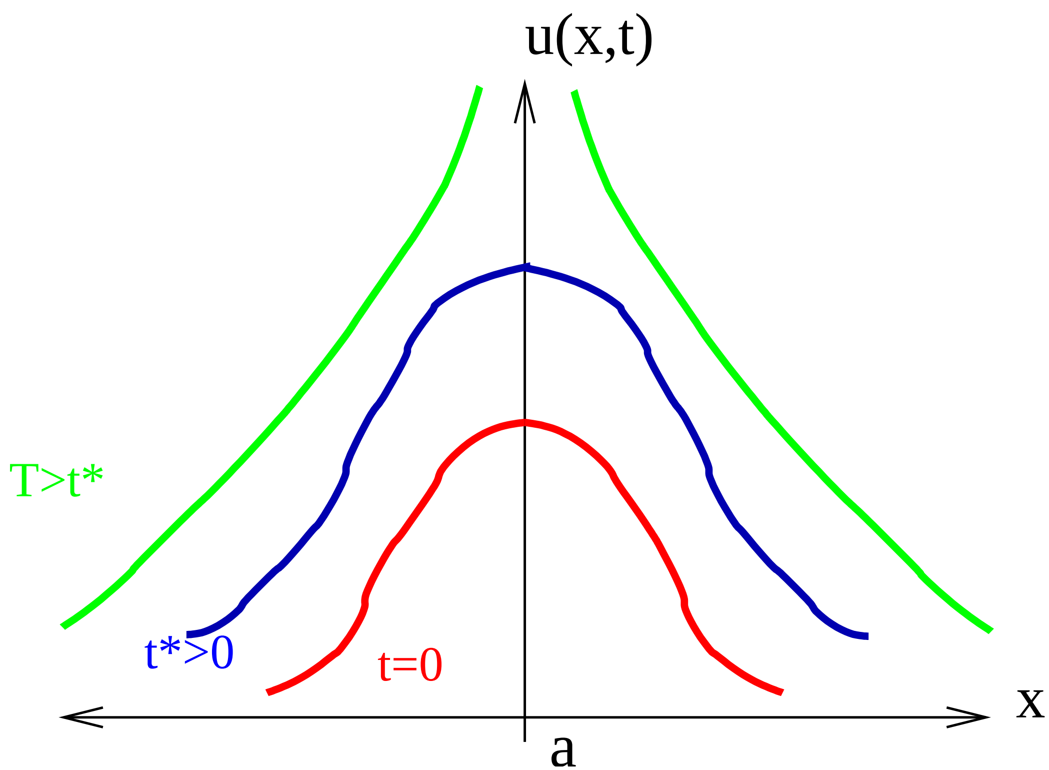
Goal : Study S_u.

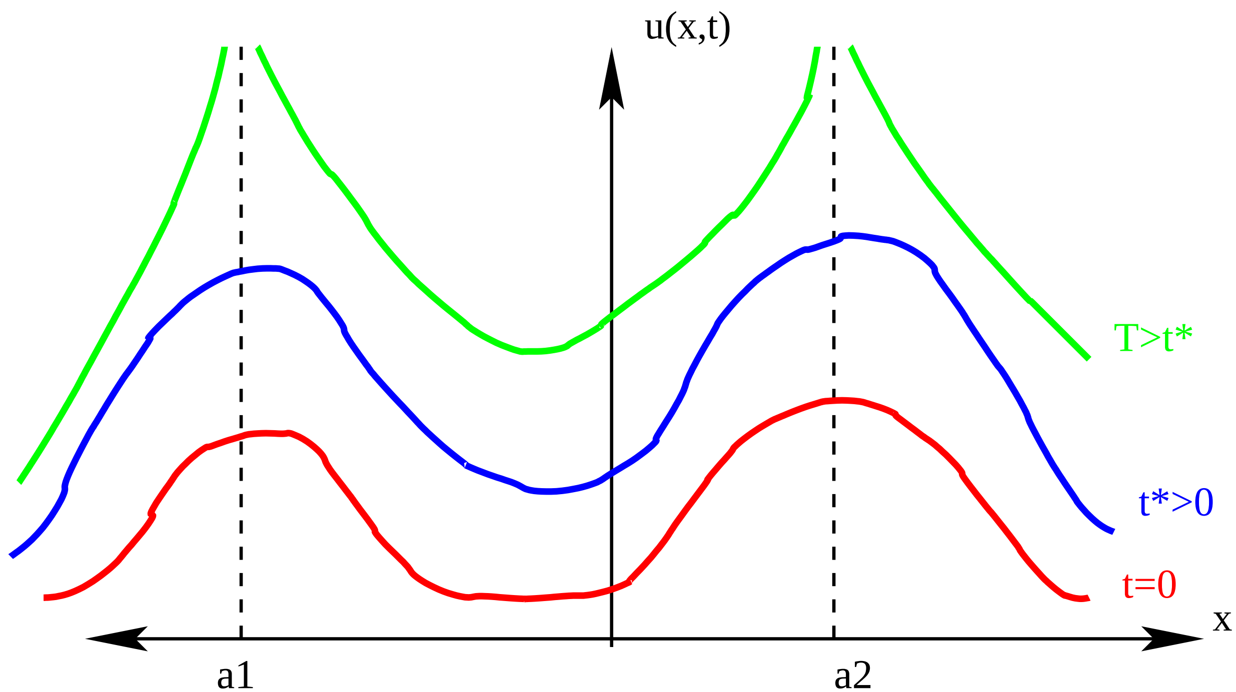
Example 1: Single-point blow-up



Rk. Sorry, this is not a simulation!

Rk. The only blow-up point is a. The other points are called "regular points".

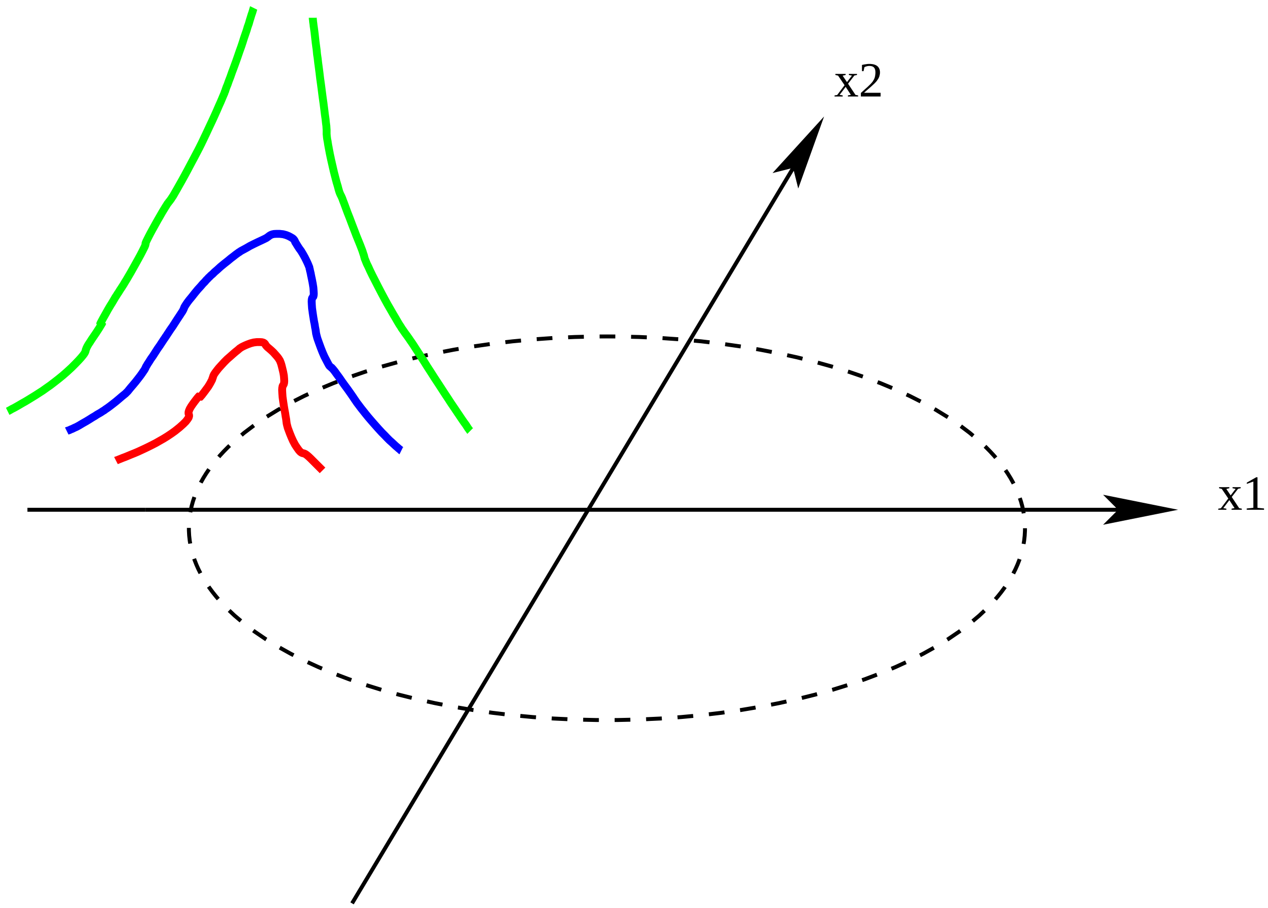
Example 1 bis : Two blow-up points (both isolated)



Rk. This is still not a simulation (by the way, blowing-up at 2 points is *unstable* and hard to get on a computer!)

Rk. Imagine the same picture with k points and in N dimensions.

Example 2: S_u is a sphere (radial sol., picture for $N = 2$).



Rk. Here, all blow-up points are *non isolated* in S_u.

Goal of the talk:

- Study of the blow-up set S_u ($\subset \mathbb{R}^N$).

Two questions arise: **the construction** and **the description**.

The construction : Given a set $\widehat{S} \subset \mathbb{R}^N$, is there a solution $\hat{u}$ of $u_t = \Delta u + |u|^{p-1}u$ that blows up at some finite time T such that $S_{\hat{u}} = \widehat{S}$?

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The answer is YES in the following cases:

- an isolated point (Herrero-Velázquez, Bricmont-Kupiainen, Weissler...),
- k points (Merle),
- a sphere (radial solution, Giga-Kohn).

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In all the other cases, the question remains open (the ellipse for example).

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known information:

- S_u is a closed set (by definition).
- S_u is bounded, if u_0 is small at infinity (Giga-Kohn 1989).
- The Hausdorff dimension of S_u is $\leq N-1$ (Velázquez 1992).

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Open questions: Is S_u locally connected? Is it C^1, C^∞,..?

Main result of Z. IHP 2002:

case of non isolated blow-up points ($C^0 \Longrightarrow C^1$)

Th. ($N = 2$) Consider u a solution of $u_t = \Delta u + |u|^{p-1}u$ and $\hat{a}$ a non isolated blow-up in S_u such that:

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1/ ($S_u \supset$ Continuum)

$\exists a \in C((-1,1), \mathbb{R}^2)$, $a(0) = \hat{a}$ and $\operatorname{Im} a \subset S_u$.

2/ ($\hat{a}$ is not an endpoint).

3/ (A “reasonable” technical condition).

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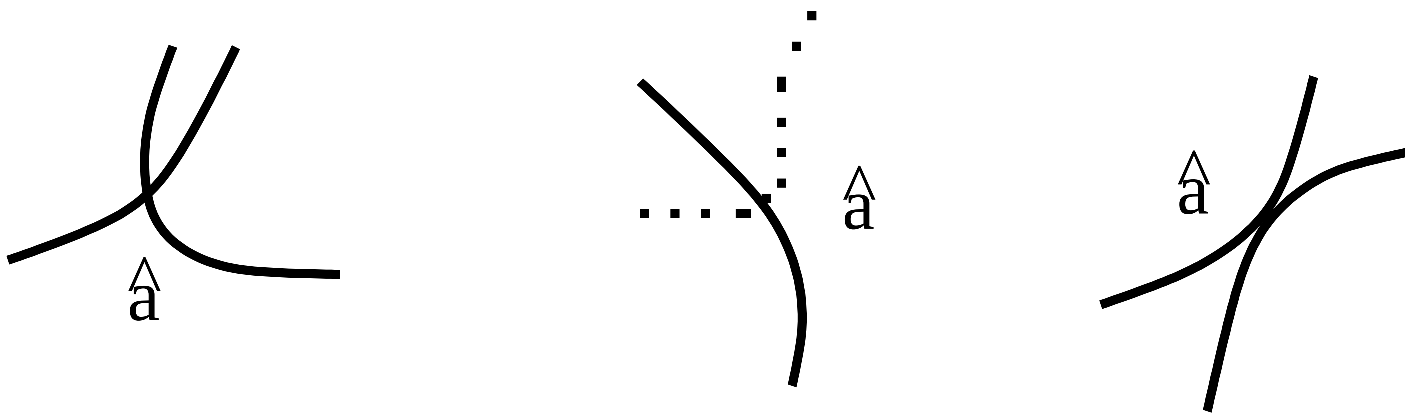
Conclusion: Locally near $\hat{a}$, S_u is the graph of a C^1 function.

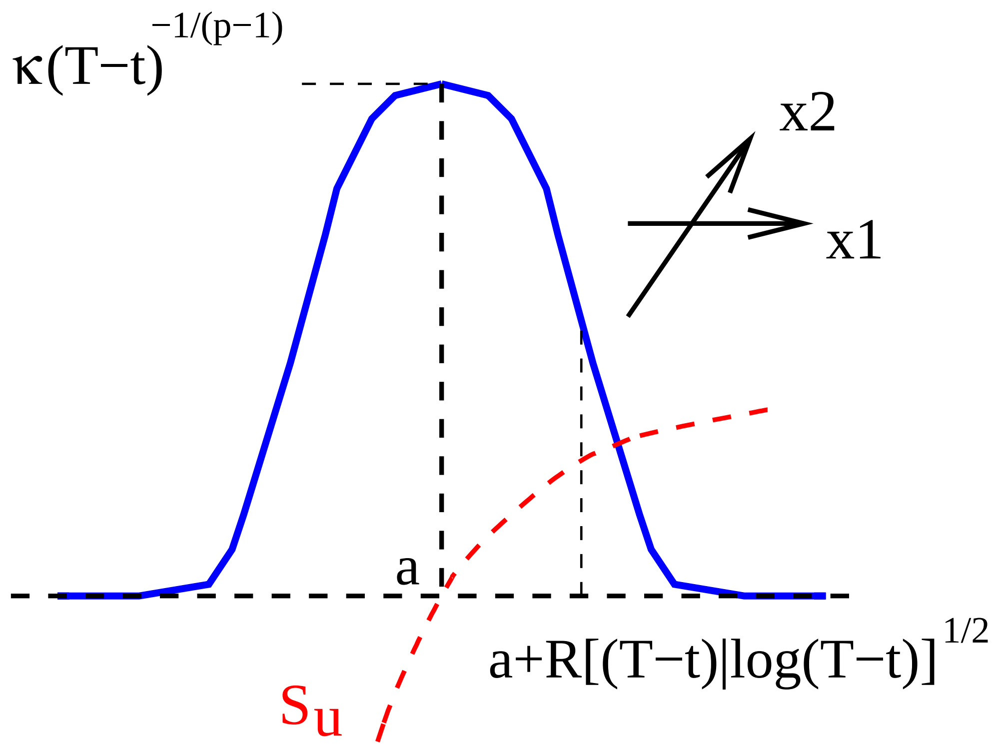
Rk. Valid in any dimension.

Rk. (Z. CMP 2002) If $\operatorname{codim} S_u = 1$, then S_u is $C^{1,\frac{1}{2}}$.

Some impossible cases for the blow-up set



Second main result of Z. IHP 2002: The blow-up profile

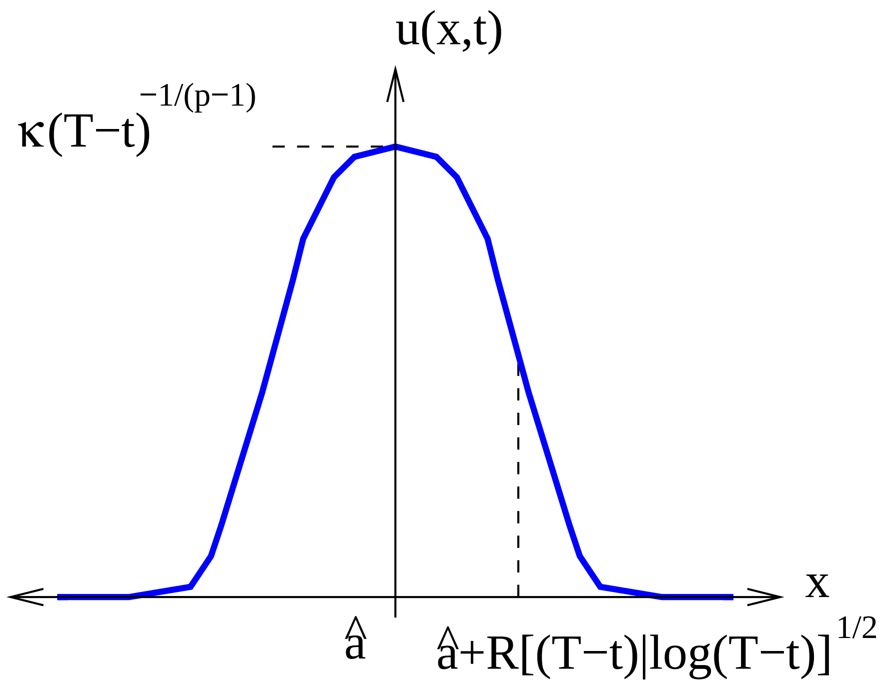


$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} f\left(\frac{d(x,S_u)}{\sqrt{(T-t)|\log(T-t)|}}\right)$$

where $f(z) = \left(p - 1 + b(p)z^2\right)^{-\frac{1}{p-1}}$.

Rk. Only the one-dimensional variable $d(x, S_u)$ (orthogonal to S_u) is responsible of the size of u at blow-up.

Rk. f is the generic profile in dimension 1.



$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} f\left(\frac{|x-\hat{a}|}{\sqrt{(T-t)|\log(T-t)|}}\right)$$

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Rk. In this case $|x - \widehat{a}| = d(x, S_u)$.

Hence, in all cases (isolated points or not),

$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} f\left(\frac{d(x,S_u)}{\sqrt{(T-t)|\log(T-t)|}}\right)$$

$\Longrightarrow$ **Universality**.

Liouville (or rigidity) theorem (Merle, Z.)

$$1 < p < \frac{N+2}{N-2}.$$

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$$\forall (x,t) \in \mathbb{R}^N \times (-\infty, T), \;\; |u(x,t)| \leq C(T-t)^{-\frac{1}{p-1}}.$$

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Then,

either $u \equiv 0$,

of there exists $T^* \ge T$ such that

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Rk. This result yields blow-up estimates which are *uniform* (with respect to initial data, blow-up point, etc...)

Most recent contribution (preprint 2004)

The blow-up set is in fact C^2, and we can explicitly compute its curvature (which is a geometric invariant).

In one word, $C^0 \Longrightarrow C^2$.