Global Solutions of some Chemotaxis and Angiogenesis Systems in high space dimensions

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Abstract

We consider two simple conservative systems of parabolic-elliptic and parabolic-degenerate type arising in modeling chemotaxis and angiogenesis. Both systems share the same property that when the $L^{\frac{d}{2}}$ norm of initial data is small enough, where $d \geq 2$ is the space dimension, then there is a global (in time) weak solution that stays in all the L^p spaces with $\max\{1; \frac{d}{2} - 1\} \leq p < \infty$. This result is already known for the parabolic-elliptic system of chemotaxis, moreover blow-up can occur in finite time for large initial data and Dirac concentrations can occur. For the parabolic-degenerate system of angiogenesis in two dimensions, we also prove that weak solutions (which are equi-integrable in L^1) exist even for large initial data. But break-down of regularity or propagation of smoothness is an open problem.

Key words. Chemotaxis, angiogenesis, degenerate parabolic equations, global weak solutions, blow-up.

AMS subject classifications. 35B60; 35Q80; 92C17; 92C50.

1 Introduction

This paper is concerned with two systems of parabolic-degenerate and parabolic-elliptic equations arising in modeling some examples of chemotaxis process. *Chemotaxis* is a biological phenomenon describing the change of motion of a population densities or of single particles (such as amoebae, bacteria, endothelial cells, any cell, animals, etc.) in response (*taxis*) to an external chemical stimulus spread in the environment where they reside. The chemical signal can be secreted by the species itself or supplied to it by an external source. As a consequence, the species changes its movement toward (*positive chemotaxis*) or away from (*negative chemotaxis*) a higher concentration of the chemical substance. A possible fascinating issue of a positive chemotactical movement is the aggregation of the organisms involved to form a more complex organism or body. Moreover, the degradation of the chemical substances can also occur.

When a population density is involved in a chemotaxis process, two different levels of description of the process itself have been considered from a Partial Differential Equation viewpoint; the full population or macroscopic level and a more local (say individual) or mesoscopic level. For the mathematical analysis of the mesoscopic approach and the derivation of the macroscopic models from the mesoscopic ones, we refer to [39], [38], [43], [12], [27], [17] and the references therein. This approach involves kinetic (Boltzmann type) equations with nonlinear scattering kernels which are based upon a detailed knowledge of the motion at the cell level (*Escherichia Coli* bacterium for instance is known to move with a sequence of runs and tumbles, see Alt [1]).

Here we are interested in the macroscopic description of the chemotaxis. At this level, different models have been suggested and analyzed all of them of (degenerate) parabolic type. Anyway, we are surely not far from truth if we claim that the most studied model is the now "classical" chemotaxis system

$$\begin{cases} \partial_t u = \nabla \cdot [\kappa(u,v)\nabla u - \chi(u,v)\nabla v], \quad t > 0,\ x \in \Omega, \\ \\ \partial_t v = \eta \Delta v + \beta(u,v)u - \gamma(u,v)v, \quad t > 0,\ x \in \Omega, \\ \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x). \end{cases} \tag{1}$$

provided with boundary conditions, whenever Ω is a bounded domain of $\mathbb{R}^d$. Here u is the population density, v the density of the chemical substance and $\beta \geq 0$ and $\gamma \geq 0$ the production and decay rate of the chemical, respectively. The function χ is the chemotactic sensitivity of the species. Generally, it takes the analytical form $\chi(u,v) = u\chi(v)$ and it plays a crucial role in the chemotaxis process. Indeed, when κ is a positive diffusion coefficient, $\chi(v) > 0$ leads to positive chemotaxis, while $\chi(v) < 0$ leads to negative chemotaxis. Moreover, in a mathematical language, χ measures the balance between the diffusion effect of the Laplace term and the hyperbolic effect of the drift term.

System (1) is a general formulation of the Keller-Segel model proposed by these authors in [30] (see also [40] for an earlier model and [29], [31], [32] for various assessments) to describe the aggregation of the slime mold amoebae *Dictyostelium discoideum* as the result of a positive chemotaxis. Actually, the original Keller-Segel model is a four equations system taking into account other reactions observed in the chemotactic process. But the authors themselves proposed a simplified version with only two equations in the form of (1). Since the publication of [30], a huge quantity of papers have been written about the mathematical, biological and medical aspects of chemotaxis as described by (1), thus showing the importance of the problem and the great interest that different kind of scientists carry on it. We recommend the reference [26] where the author gives a nice survey of the mathematical problems encountered in the study of the Keller-Segel model and also a wide bibliography, including references on other types of models describing chemotaxis.

The present paper deals with two simple examples of systems of type (1), apparently different, nevertheless showing interesting analytical analogies.

The first one (which we will call sometimes the chemotaxis system) is the following parabolic-elliptic system

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \chi \nabla \cdot [n \nabla c], \quad t > 0, \; x \in \Omega, \\ \\ -\Delta c = n - \alpha c, \quad t > 0, \; x \in \Omega, \\ \\ n(0,x) = n_0(x), \qquad x \in \Omega. \end{cases} \tag{2}$$

where $\alpha \geq 0$. Here, the diffusion coefficient κ of the population density n is constant and the chemotactic sensitivity function χ is constant with respect to the chemical density c. Let us remark that in the case of the whole space $\mathbb{R}^d$ and with the ad hoc decay conditions at infinity on n and c, the chemical density can be represented exactly by

$$c(t,x) = \int_{\mathbb{R}^d} E_d(x-y) n(t,y) dy \tag{3}$$

where E_d is the fundamental solution of $(-\Delta + \alpha I)$. The equation on c in (2) is obtained assuming in (1) the production and the decay rate functions β and γ to be positive constants, with β and the diffusion coefficient η of order $\frac{1}{\varepsilon}$, ε small, and γ of order 1 or $\frac{1}{\varepsilon}$. Therefore, the parabolic-elliptic system (2) is obtained as the singular limit as $\varepsilon \to 0$ of the parabolic system (1). When $\alpha = 0$, the validity of (2) in the framework of chemotaxis is supported by some experiments on the *Escherichia Coli* bacterium (see [10], [5] and the references therein), even if this model does not seem to reproduce some of the observed chemotactic movement ([10]). Moreover, in this case ($\alpha = 0$) system (2) has other interesting physical interpretations. For example it arises in astrophysics and in statistical mechanics (see [6], [7], [8] and the references therein). When $\alpha > 0$, system (2) was extensively studied by many authors and a huge quantity of mathematical results on the existence of global in time solutions and on the blow-up of local in time solutions, have been produced. We refer to [26] for a quite complete bibliography. Let us just mention that system (2) has a conserved energy (see for example [9], [21] and [25]) given by

$$\frac{d}{dt} \int \left\{ \kappa n (\ln n - 1) - \frac{\chi}{2} \left(\alpha c^2 + |\nabla c|^2 \right) \right\} dx = - \int n |\nabla (\kappa \ln n - \chi c)|^2 dx.$$

The second system we consider here is another kind of chemotaxis model (although we sometimes refer to it as the angiogenesis system) that has been much less studied than system (2), except in one space dimension, i.e. the parabolic-degenerate system

$$\begin{cases} \frac{\partial}{\partial t} n = \kappa \Delta n - \nabla \cdot [n \chi(c) \nabla c], \quad t > 0, \; x \in \Omega, \\ \\ \frac{\partial}{\partial t} c = -c^m n, \quad t > 0, \; x \in \Omega, \\ \\ n(0,x) = n_0(x), \qquad c(0,x) = c_0(x), \qquad x \in \Omega. \end{cases} \tag{4}$$

where m is a positive parameter ($m \geq 1$ in our results). In this case, the diffusion coefficient κ of the density n is still constant while the sensitivity function $\chi(c)$ is a given positive function on $\mathbb{R}_+$, generally chosen as a decreasing function since sensitivity is lower for higher concentrations of the chemical because of saturation effects. In particular, constant χ, $\chi(c) = c^{-\alpha}$, $0 < \alpha < 1$, and $\chi(c) = \frac{\beta}{\alpha+\beta c}$ with $\alpha, \beta > 0$, are allowed in our results. Moreover, when $\chi(c)$ is such that

$$\mu := \frac{1}{2} \inf_{c \geq 0} \{ \frac{c\, \chi'}{\chi} + m \} > 0, \tag{5}$$

system (4) has an energy structure (see [15]) given by

$$\tfrac{d}{dt}\mathcal{E}(t) \leq - \textstyle\int_\Omega n \Big[\kappa |\nabla \ln(n)|^2 + \mu c^{m-1} |\nabla \Phi(c)|^2 \Big] \leq 0 \ , \tag{6}$$

where

$$\mathcal{E}(t) := \int_\Omega [\frac{1}{2} |\nabla \Phi(c)|^2 + n \ln(n)] \quad \text{ and } \quad \Phi'(c) = \sqrt{\frac{\chi}{c^m}}.$$

The equation on c is just an ordinary differential equation, expressing the consumption of the chemical. Indeed, it is now clear that the asymptotic behavior of n depends strongly on the coupling effects of the dynamics of c and the chemotactic sensitivity $\chi(c)$ and that is the reason why systems of type (4) have been analyzed in [33], [38], [41], [42], [44]. The way we write the right-hand side $-c^m n$ in the ODE in system (4) suggests that we consider nonnegative c. This is indeed reasonable and can be justified by saying that when we put $-|c|^{m-1} cn$ instead and take nonnegative initial data, c remains nonnegative. Furthermore, the omission of the diffusion of the chemical in the same ODE can be justified whenever the corresponding diffusion coefficient is small compared to the motility of the species of density n.

One more reason to consider system (4) is that it arises also in modeling the initiation of *angiogenesis*, a kind of chemotaxis process that occurs for example in the tumor growth ([13],[14], [15], [19], [35]). More specifically, angiogenesis is the formation of new capillary networks from a pre-existing vascular network. The density n is the density of the endothelial cells which form the lining of the different type of blood vessels. In the case of tumor growth, the chemical inducing the chemotactic movement of the endothelial cells toward the tumor is spread out by the tumor itself, in order to make up its own capillary network and to supply itself with the nourishment necessary for its development.

We have also to mention that degenerate parabolic systems with structure related to (4) arise in other applications as colloid transport in a porous medium [3]. The same system as (4), with additional zeroth order terms, arise in chemical degradation of a porous medium [23].

Finally, whenever Ω is a bounded domain of $\mathbb{R}^d$, systems (2) and (4) must be provided with boundary conditions. These can be the homogeneous Neumann boundary condition for n and c or the no-flux boundary condition

$$\kappa \frac{\partial n}{\partial \nu} - n \chi(c) \frac{\partial c}{\partial \nu} = 0 \qquad \text{on } \ \partial\Omega \tag{7}$$

and the homogeneous Dirichlet boundary condition for c in the case of system (2), and again the zero flux condition (7) in the case of system (4). Nevertheless, since the goal of this paper is to give a general idea of methods and results known for these systems, we shall consider the above two systems on the whole space $\mathbb{R}^d$, which simplifies the notations and opens some interesting questions. We then replace the boundary condition by a decay condition at infinity of the densities n and c. In any case (bounded Ω or $\Omega = \mathbb{R}^d$) the conservation of the initial mass has to be assured, as well as the positivity of the densities,

$$\int_\Omega n(t) = \int_\Omega n_0 =: M, \quad \forall t \geq 0, \qquad n(t,x) \geq 0, \quad c(t,x) \geq 0.$$

More specifically, our purpose is to show that some similar results and methods hold for both systems in any dimension $d \geq 2$, since in one space dimension global existence of smooth solutions has been established by several authors for both systems (see for example [19], [36], [41]). Concerning the chemotaxis system (2), the situation is well established and we recall the following results.

Theorem 1.1 *(Existence for the chemotaxis system (2)) Assume $d \geq 2$ and consider some $n_0 \in L^1(\mathbb{R}^d)$ such that $n_0 \geq 0$. There exists a constant $K_0(\kappa, \chi, d)$ such that if $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq K_0$, then system (2) has a global (in time) weak solution (n,c) such that for all $t > 0$*

$$\|n(t)\|_{L^1(\mathbb{R}^d)} = \|n_0\|_{L^1(\mathbb{R}^d)}, \qquad \|n(t)\|_{L^p(\mathbb{R}^d)} \leq \|n_0\|_{L^p(\mathbb{R}^d)}\ , \quad \max\{1; \frac{d}{2} - 1\} \leq p \leq \frac{d}{2},$$

and

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq C(t, K_0, \|n_0\|_{L^p(\mathbb{R}^d)}) \qquad \frac{d}{2} < p \leq \infty.$$

The results of this theorem were known before, except for one fact: we need a smallness condition only in the the critical norm $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$. We establish that fact in the proof of Theorem 1.1 given below. Let us remark that a smallness assumption is nevertheless necessary in the above theorem. Indeed, it is known since [28] that blow-up may occur in two space dimensions for large initial data. This result was extended to dimensions $d \geq 3$ in the case of radial symmetric solutions by Nagai [36] who shows that blow-up may arise whatever the initial mass is, depending on the x momentum of order d of n_0. Actually, the radial case is better understood and, in two space dimensions for large mass M (larger than the corresponding K_0 in Theorem 1.1), the type of blow-up has been specified. In [24] the authors proved that *chemotactic collapse* i.e. pointwise concentration as a Dirac mass occurs. Even in the non-radial case, blow-up and chemotactic collapse in two space dimensions are very close although there is no proof they are equivalent: indeed from the argument in [28], blow-up can occur only if solutions loose equicontinuity in L^1. In three dimensions, the extreme complexity of the behavior appears in the numerous blow-up modalities described in [9]. In order to give a hint of the main ideas, we establish the following result dealing with the full space.

Theorem 1.2 *(Blow-up for chemotaxis system (2)). For $d \geq 3$, assume that*

$$\int_{\mathbb{R}^d} \frac{|x|^2}{2} n_0(x) dx \leq C \left(\int_{\mathbb{R}^d} n_0 \right)^{\frac{d}{d-2}} \tag{8}$$

for some constant $C = C(\chi, \kappa, d) > 0$, and for $d = 2$, assume that $\int_{\mathbb{R}^d} \frac{|x|^2}{2} n_0(x)dx$ is finite and that $\int_{\mathbb{R}^d} n_0 \geq M_0$ for some $M_0(\chi, \kappa, d) > 0$. Then, the chemotaxis system (2) has no global smooth solution with fast decay.

Let us just point out that the assumption (8) is incompatible with the smallness assumption on $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ of Theorem 1.1 in view of the classical inequality $M^d \leq C\, (\int_{\mathbb{R}^d} \frac{|x|^2}{2} n_0(x)dx)^{d-2}\, \|n_0\|^2_{L^{\frac{d}{2}}(\mathbb{R}^d)}$. Except for radial solutions, it is an open question to replace assumption (8) by "$\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ large enough", as suggested in two dimensions radial solutions by the result of [24].

When considering system (4), it has been proved in [15] that for initial data with finite energy and $m \geq 1$, the energy (6) allows to prove the existence of weak solutions to (4) where the drift term $n\chi(c)\nabla c$ is well defined. This energy also provides equi-integrability for n, therefore solutions cannot exhibit concentrations. But the question of propagation of smoothness of the solutions is largely open. Here we prove the following result in this direction.

Theorem 1.3 *(Existence for the angiogenesis system (4)). Assume $d \geq 2$, $m \geq 1$ and χ a positive say continuous function on $[0, \infty)$. Consider some $n_0 \in L^1(\mathbb{R}^d)$ and $c_0 \in L^\infty(\mathbb{R}^d)$ such that $n_0 \geq 0$ and $c_0 \geq 0$. There exists a constant $K_0(\kappa, \chi, d, \|c_0\|_{L^\infty(\mathbb{R}^d)})$ such that if $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq K_0$, then system (4) has a global (in time) weak solution (n, c) such that $n \in L^\infty(\mathbb{R}^+, L^1 \cap L^{\frac{d}{2}}(\mathbb{R}^d))$, $c \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ and for any fixed $\max\{1; \frac{d}{2} - 1\} \leq p^* < \infty$,*

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq C(t, K_0, p^*, \|n_0\|_{L^p(\mathbb{R}^d)}), \qquad \forall\ \max\{1; \frac{d}{2} - 1\} \leq p \leq p^*.$$

Remark. The same conclusion holds with less regular χ at $c = 0$. Indeed, one sees from the proof that χ can be continuous only on $(0, \infty)$ but integrable at $c = 0$ with the constant K_1 defined by (38) in the proof finite.

The derivation of an L^∞ bound for $n(t)$ is an open question as well as the relaxation of the smallness condition in dimension $d \geq 3$. However in two space dimensions we have the following

Theorem 1.4 *(L^p bound for the angiogenesis system (4) in two dimensions). Assume $d = 2$, $m \geq 1$, χ a positive say continuous differentiable function on $[0, \infty)$ and $\mu > 0$ in (5). Consider some non-negative initial data (n_0, c_0) with finite energy i.e. $\mathcal{E}(0) < \infty$, such that $\ln(1 + |x|)n_0 \in L^1(\mathbb{R}^d)$ and $c_0 \in L^\infty(\mathbb{R}^d)$. Then, there is a weak solution (n, c) of system (4) such that for any fixed $1 \leq p^* < \infty$,*

$$\|n(t)\|_{L^p(\mathbb{R}^d)} \leq C(t, p^*, \|n_0\|_{L^p(\mathbb{R}^d)}), \qquad \forall\ 1 \leq p \leq p^*\ .$$

The end of this paper is organized as follows. In the second section we consider the situation of the chemotaxis system (2) and we show how to adapt the arguments of [28], [6] to prove the theorems 1.1 and 1.2. Section 3 is devoted to the case of the angiogenesis system (3).

2 The parabolic-elliptic system.

In this section, we give a brief review of some of the most relevant results on the global existence of weak solutions of system (2) and we prove Theorems 1.1 and 1.2.

The main difficulty in studying (2) is the following. In order to obtain the continuation in time of local solutions, it is classical to attempt to control the L^p norm of these solutions. Following this idea, we found that in any dimension d it holds true that

$$\frac{d}{dt}\int_\Omega n^p + 4\kappa\frac{p-1}{p}\int_\Omega |\nabla n^{p/2}|^2 = \chi p(p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c\ . \tag{9}$$

Then, a natural and major issue arising from (9) is to prove that the drift velocity ∇c is bounded in x uniformly in t. Indeed, if this is the case, the right hand side of the identity (9) can be estimated with a simple Hölder's inequality that gives

$$\begin{aligned}\chi p(p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c &= 2\chi(p-1)\int_\Omega n^{p/2}\nabla n^{p/2}\cdot\nabla c\\ &\le 2\kappa\frac{p-1}{p}\int_\Omega|\nabla n^{p/2}|^2 + \frac{\chi^2 p(p-1)}{2\kappa}\,\|\nabla c\|^2_{L^\infty_{t,x}}\int_\Omega n^p,\end{aligned}$$

from which we deduce the inequality

$$\frac{d}{dt}\int_\Omega n^p + 2\kappa\frac{p-1}{p}\int_\Omega|\nabla n^{p/2}|^2 \le \frac{\chi^2 p(p-1)}{2\kappa}\,\|\nabla c\|^2_{L^\infty_{t,x}}\int_\Omega n^p\ , \tag{10}$$

implying the control of all the L^p norms of n, $1\le p\le +\infty$ (see [36], [25] and the references therein).

With this in mind and in the case of bounded Ω and homogeneous Neumann conditions on n and c, boundedness and global existence of solutions of (2) with $\alpha > 0$ have been proved in [36], in one space dimension and in two space dimensions in the radial symmetric case when the initial mass $\int_\Omega n_0$ is sufficiently small. The above mentioned results have been generalized in [6] to the case of a drift velocity $\nabla\psi(c)$, with strictly sublinear ψ. Moreover, when $\psi(c) = \log c$, global existence of solutions of (2) is proved in [6] in any space dimension $d\ge 2$, using an ad hoc Hölder inequality to estimate the right hand side of (9), (see also [37]).

Unfortunately, the technique in [6] depends strongly on the structure of the function $\psi(c) = \log c$ and it seems that it cannot be applied to a different $\psi(c)$. It seems also quite difficult to extend the results of [36] to dimension higher than 2. Therefore, we need to keep forward in (9) in order to obtain the estimation of the L^p norm of n. Nevertheless, since c is given by the Green formula (3), whenever n belongs to some L^p, $p > d$, Young's inequality gives us directly the L^∞ bound on ∇c and (10) can be applied. The rest of the section is devoted to the proof of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1.
First step : a priori estimate of the L^p norms of n. In this first step, we show that for $p\ge\max\{1;\frac{d}{2}-1\}$, the L^p norm of n is bounded for all $t > 0$ provided that the $L^{\frac{d}{2}}$ of n_0 is less than a constant that depends on p. This provides a proof of the theorem under a rough smallness assumption.

First of all, let us observe that with the decay at infinity condition on n and c, the initial mass $M = \int_{\mathbb{R}^d} n_0(x)\, dx$ is conserved, while by the maximum principle $n > 0$ and $c \geq 0$ if $n_0 \geq 0$. Next, using the second equation of (2), we write (9) as

$$\begin{aligned} \tfrac{d}{dt} \int_{\mathbb{R}^d} n^p + 4\kappa \tfrac{p-1}{p} \int_{\mathbb{R}^d} |\nabla n^{p/2}|^2 &= \chi(p-1) \int_{\mathbb{R}^d} \nabla n^p \cdot \nabla c \\ &= \chi(p-1) \int_{\mathbb{R}^d} n^{p+1} - \alpha\chi(p-1) \int_{\mathbb{R}^d} n^p c \\ &\leq \chi(p-1) \int_{\mathbb{R}^d} n^{p+1} \ . \end{aligned} \tag{11}$$

On the other hand, in order to estimate the L^{p+1} norm of n, we use standard interpolation and the Gagliardo-Nirenberg-Sobolev inequality (see [11], [18]). Hence, in space dimension $d > 2$ we get for any $p \geq \frac{d}{2} - 1$ (so that : $\frac{d}{2} \leq p + 1 \leq \frac{dp}{d-2}$)

$$\int_{\mathbb{R}^d} n^{p+1} \leq \|n\|^p_{L^{\frac{pd}{d-2}}(\mathbb{R}^d)} \|n\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} = \|n^{p/2}\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \|n\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq C(d) \|\nabla n^{p/2}\|^2_{L^2(\mathbb{R}^d)} \|n\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \ . \tag{12}$$

In dimension $d = 2$, following [28] (see also [11]), we have in a similar way for any $p > 0$

$$\begin{aligned} \int_{\mathbb{R}^2} n^{p+1} &\leq C \|\nabla n^{\frac{p+1}{2}}\|^2_{L^1(\mathbb{R}^2)} = C \left(\tfrac{p+1}{p}\right)^2 \left(\int_{\mathbb{R}^2} n^{\frac{1}{2}} |\nabla n^{p/2}|\right)^2 \\ &\leq C \left(\tfrac{p+1}{p}\right)^2 \|\nabla n^{p/2}\|^2_{L^2(\mathbb{R}^2)} \|n\|_{L^1(\mathbb{R}^2)} \ . \end{aligned} \tag{13}$$

Inequalities (12) and (13), together with (11), lead to the main differential inequality in any dimension $d \geq 2$

$$\frac{d}{dt} \int_{\mathbb{R}^d} n^p \leq (p-1) \|\nabla n^{p/2}\|^2_{L^2(\mathbb{R}^d)} \left[\chi \tilde{C}(d) \|n\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{4\kappa}{p} \right] , \qquad \forall \ \max\{1; \frac{d}{2} - 1\} \leq p < \infty \ . \tag{14}$$

In dimension $d = 2$, (14) means that if the initial mass M is sufficiently small, in terms of $p \geq 1$, then the $\|n(t)\|_{L^p(\mathbb{R}^2)}$ norm (for the same p) decreases for all times $t \geq 0$.

In dimension $d > 2$ and for $p = \frac{d}{2}$, (14) gives us that whenever we have initially

$$\chi \tilde{C}(d) \|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{8\kappa}{d} \leq 0 \ , \tag{15}$$

the $\|n(t)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ norm decreases for all times $t \geq 0$. As a consequence, whenever (15) holds true, all the $\|n(t)\|_{L^p(\mathbb{R}^d)}$ norms, with $\max\{1; \frac{d}{2} - 1\} \leq p \leq \frac{d}{2}$, decrease for all times $t \geq 0$. In the same way, if the initial density n_0 satisfies

$$\chi \tilde{C}(d) \|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{4\kappa}{p^*} \leq 0 \ , \tag{16}$$

for $\frac{d}{2} < p^* < \infty$, then (15) follows again and all the $\|n(t)\|_{L^p(\mathbb{R}^d)}$ norms with $\max\{1; \frac{d}{2} - 1\} \leq p \leq p^*$ decrease for all times $t \geq 0$.

In conclusion, a smallness condition depending on $\max\{1; \frac{d}{2} - 1\} \leq p < \infty$ on the initial density n_0 is sufficient to get a decreasing L^p norm of n, in any dimension $d \geq 2$.

Finally, if the smallness condition

$$\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < \frac{4\kappa}{d} \ , \qquad (d \geq 2), \tag{17}$$

holds true, we propagate all the norms $\|n(t)\|_{L^p(\mathbb{R}^d)}$ with $1 \leq p \leq p^*$, for some $d < p^* < \infty$ for which (16) follows. Young's inequality for that p^* gives us $\nabla c \in L^\infty((0,T) \times \mathbb{R}^d)$ for all $T > 0$ and using (10) we also control all the other L^p norms.

Second step : a refined smallness condition on the $L^{\frac{d}{2}}$ norm of n_0. Here we want to obtain a condition on the initial density n_0 better than (17) and sufficient for the control of all the $\|n(t)\|_{L^p(\mathbb{R}^d)}$ norms, namely

$$\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < \frac{8\kappa}{d} \ , \qquad (d \geq 2), \tag{18}$$

which corresponds to the "smaller possible" value of p^* in (16). Hence, we consider the L^p norm of the function $(n(t) - K)_+$ with $K > 1$ (which is irrelevant for the sequel) and we obtain

$$\begin{aligned}
\tfrac{d}{dt}\textstyle\int_{\mathbb{R}^d}(n-K)_+^p \quad &+4\kappa\tfrac{p-1}{p}\textstyle\int_{\mathbb{R}^d}|\nabla(n-K)_+^{p/2}|^2 \\
&= \chi p(p-1)\textstyle\int_{\mathbb{R}^d}(n-K)_+^{p-2} n\nabla n\cdot\nabla c \\
&= \chi(p-1)\textstyle\int_{\mathbb{R}^d}\nabla(n-K)_+^p\cdot\nabla c + \chi p\ K\textstyle\int_{\mathbb{R}^d}\nabla(n-K)_+^{p-1}\cdot\nabla c \\
&= \chi(p-1)\textstyle\int_{\mathbb{R}^d}(n-K)_+^{p+1} + K\chi(2p-1)\textstyle\int_{\mathbb{R}^d}(n-K)_+^p + K^2\chi p\textstyle\int_{\mathbb{R}^d}(n-K)_+^{p-1} \\
&\quad -\alpha\chi(p-1)\textstyle\int_{\mathbb{R}^d}(n-K)_+^p c - \alpha\chi pK\textstyle\int_{\mathbb{R}^d}(n-K)_+^{p-1}c \ .
\end{aligned} \tag{19}$$

Then, we use again (12) and (13) to estimate the L^{p+1} norm $\int_{\mathbb{R}^d}(n-K)_+^{p+1}$ and the argument below to estimate the L^{p-1} norm $\int_{\mathbb{R}^d}(n-K)_+^{p-1}$

$$\int_{\mathbb{R}^d}(n-K)_+^{p-1} \leq \int_{\mathbb{R}^d} n = M \ , \qquad 1 \leq p \leq 2$$

and

$$\int_{\mathbb{R}^d}(n-K)_+^{p-1} \leq \left(\int_{\mathbb{R}^d}(n-K)_+\right)^{\frac{1}{p-1}}\left(\int_{\mathbb{R}^d}(n-K)_+^p\right)^{1-\frac{1}{p-1}} \leq M^{\frac{1}{p-1}}\left(\int_{\mathbb{R}^d}(n-K)_+^p\right)^{1-\frac{1}{p-1}} \ , \qquad p > 2,$$

to obtain in any dimension $d \geq 2$

$$\begin{aligned}
\tfrac{d}{dt}\textstyle\int_{\mathbb{R}^d}(n-K)_+^p \leq (p-1)\|\nabla(n-K)_+^{p/2}\|_{L^2(\mathbb{R}^d)}^2 &\left[\chi\tilde{C}(d)\|(n-K)_+\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \tfrac{4\kappa}{p}\right] \\
&+K\chi(2p-1)\textstyle\int_{\mathbb{R}^d}(n-K)_+^p + K^2\chi pM \ ,
\end{aligned} \tag{20}$$

if $\max\{1; \frac{d}{2}-1\} \le p \le 2$ and

$$\frac{d}{dt}\int_{\mathbb{R}^d}(n-K)_+^p \le (p-1)\|\nabla(n-K)_+^{p/2}\|^2_{L^2(\mathbb{R}^d)}\left[\chi\tilde{C}(d)\|(n-K)_+\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{4\kappa}{p}\right]$$
$$+K\chi(2p-1)\int_{\mathbb{R}^d}(n-K)_+^p + K^2\chi p M^{\frac{1}{p-1}}\left(\int_{\mathbb{R}^d}(n-K)_+^p\right)^{1-\frac{1}{p-1}}\ , \tag{21}$$

if $\max\{2; \frac{d}{2}-1\} < p < \infty$.

Next, let us observe that for any fixed $\max\{1; \frac{d}{2}-1\} \le p < \infty$ and under condition (18), we can find $K(p)$ sufficiently large and independent of t such that

$$\chi\tilde{C}(d)\|(n(t)-K)_+\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \le \frac{4\kappa}{p}\ , \quad \forall\, t \ge 0\ . \tag{22}$$

Indeed, if (18) holds true we may choose a $p_0 > \frac{d}{2}$ such that $\chi\tilde{C}(d)\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \le \frac{4\kappa}{p_0}$ and thus $\|n(t)\|_{L^{p_0}(\mathbb{R}^d)} \le \|n_0\|_{L^{p_0}(\mathbb{R}^d)}$, $\forall t \ge 0$, thanks to the first step. Since $|\{x : n(t) \ge K\}| \le \frac{M}{K}$, (22) follows by

$$\|(n(t)-K)_+\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \le \|n(t)\|_{L^{p_0}(\mathbb{R}^d)}\left(\frac{M}{K}\right)^{\frac{2}{d}-\frac{1}{p_0}} \le \|n_0\|_{L^{p_0}(\mathbb{R}^d)}\left(\frac{M}{K}\right)^{\frac{2}{d}-\frac{1}{p_0}}.$$

As a consequence, (20) and (21) become respectively

$$\frac{d}{dt}\int_{\mathbb{R}^d}(n-K)_+^p \le K\chi(2p-1)\int_{\mathbb{R}^d}(n-K)_+^p + K^2\chi p M\ , \tag{23}$$

if $\max\{1; \frac{d}{2}-1\} \le p \le 2$, and

$$\frac{d}{dt}\int_{\mathbb{R}^d}(n-K)_+^p \le K\chi(2p-1)\int_{\mathbb{R}^d}(n-K)_+^p + K^2\chi p M^{\frac{1}{p-1}}\left(\int_{\mathbb{R}^d}(n-K)_+^p\right)^{1-\frac{1}{p-1}} \tag{24}$$

if $\max\{2; \frac{d}{2}-1\} < p < \infty$. Finally, Gronwall inequality gives us for $E(t) = \int_{\mathbb{R}^d}(n-K)_+^p\, dx$ that

$$E(t) \le e^{KC_1 t}E(0) + KC_2[e^{KC_1 t}-1]\ , \qquad \max\{1; \frac{d}{2}-1\} \le p \le 2, \tag{25}$$

and

$$E^{\frac{1}{p-1}}(t) \le e^{K\overline{C}_1 t}E^{\frac{1}{p-1}}(0) + K\overline{C}_2[e^{K\overline{C}_1 t}-1]\ , \quad \max\{2; \frac{d}{2}-1\} < p < \infty\ , \tag{26}$$

for some constants C_i and $\overline{C}_i$ depending on M, p, χ. Since $n(t) \in L^1(\mathbb{R}^d)$, for any fixed $\max\{1; \frac{d}{2}-1\} \le p^* < \infty$ and for $K = K(p^*)$, from (25)-(26) follows a global in time control of the norms $\|n(t)\|_{L^p(\mathbb{R}^d)}$, $\max\{1; \frac{d}{2}-1\} \le p \le p^*$, under condition (18) in any dimension $d \ge 2$. The global in time control of all the L^p norms of n with $\max\{1; \frac{d}{2}-1\} \le p \le \infty$ is again a consequence of (3).

Third step : regularization of system (2). Here we briefly comment how to regularize system (2) in order to prove the existence of a solution satisfying the a priori estimates obtained in the first and second step. Indeed, we regularize the equation on c by a convolution as follows

$$-\Delta c_\varepsilon = n_\varepsilon \star \rho_\varepsilon - \alpha c_\varepsilon, \tag{27}$$

for some regularizing kernel $\rho_\varepsilon = \frac{1}{\varepsilon^d}\rho(\frac{x}{\varepsilon})$ with $\rho \in \mathcal{D}^+(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \rho = 1$. Then, following the standard parabolic theory, the system of the drift-diffusion equation on $(n_\varepsilon, c_\varepsilon)$ admits a unique smooth solution, with fast decay in space, when the initial data is regularized and truncated. Next, we notice that the above a priori estimates (14), (20) and (21) still hold true for n_ε uniformly in ε. Indeed, the only modification in (11) consists in the additional argument

$$\int_{\mathbb{R}^d} \nabla n_\varepsilon^p \cdot \nabla c_\varepsilon \le \int_{\mathbb{R}^d} n_\varepsilon^p (n_\varepsilon \star \rho_\varepsilon) \le \left(\int_{\mathbb{R}^d} n_\varepsilon^{p+1}\right)^{p/(p+1)} \left(\int_{\mathbb{R}^d} (n_\varepsilon \star \rho_\varepsilon)^{p+1}\right)^{1/(p+1)} \le \int_{\mathbb{R}^d} n_\varepsilon^{p+1}. \tag{28}$$

and the same type of argument can be used to obtain (19) from which (20) and (21) follow.

As a consequence, under condition (18) for n_0 and hence for its regularization, we have a global in time control of the norms $\|n_\varepsilon(t)\|_{L^p(\mathbb{R}^d)}$, $p \ge \max\{1; \frac{d}{2} - 1\}$, in any dimension $d \ge 2$. Finally, we can pass to the limit as $\varepsilon \to 0$ because of the space gradient estimate

$$\int_0^\infty \|\nabla n^{p/2}\|^2_{L^2(\mathbb{R}^d)} \le C(K_0, \|n_0\|_{L^p(\mathbb{R}^d)}) \tag{29}$$

that also follows from the a priori estimate (14) and because of Lions-Aubin lemma which provides the necessary time compactness. In the limit, we obtain the results of Theorem 1.1.

Remark 1. The estimate (14) has been obtained also in [9] with $p = \frac{d}{2}$, for system (2) with $\alpha = 0$.

Remark 2. It is easy to see that Theorem 1.1 holds true also in the case of nondecreasing sensitivity functions $\chi(c)$, since in this case we have an additional negative term in the right hand side of (11) and (19).

Proof of Theorem 1.2. Here we use the standard quantity

$$I(t) = \int_{\mathbb{R}^d} \frac{|x|^2}{2} n(t,x)dx$$

and the formula (3) in the case $\alpha = 0$ (otherwise one readily checks that the corresponding formula does not change the proof), since we deal with a smooth solution (n, c) of (2) with fast decay at infinity. Hence

$$\nabla c(t,x) = -\lambda_d \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} n(t,y)dy, \qquad \lambda_d > 0.$$

Next, following [22], we compute for $d \geq 3$

$$\begin{aligned}
\tfrac{d}{dt} I(t) &= \kappa d \int_{\mathbb{R}^d} n_0(x)dx + \chi \int_{\mathbb{R}^d} n(t,x)\; x\cdot \nabla c(t,x)\; dx = \kappa d M - \lambda_d \chi \int_{\mathbb{R}^d\times\mathbb{R}^d} n(t,x)n(t,y) \tfrac{x\cdot(x-y)}{|x-y|^d}\; dxdy \\
&= \kappa d M - \tfrac{\lambda_d}{2}\chi \int_{\mathbb{R}^d\times\mathbb{R}^d} n(t,x)n(t,y) \tfrac{1}{|x-y|^{d-2}}\; dxdy \\
&\leq \kappa d M - \tfrac{\lambda_d}{2R^{d-2}}\chi \int_{|x-y|\leq R} n(t,x)n(t,y)\; dxdy \\
&= \kappa d M - \tfrac{\lambda_d \chi}{2R^{d-2}} M^2 + \tfrac{\lambda_d}{2R^{d-2}}\chi \int_{|x-y|\geq R} n(t,x)n(t,y)\; dxdy \\
&\leq \kappa d M - \tfrac{\lambda_d \chi}{2R^{d-2}} M^2 + \tfrac{\lambda_d}{2R^{d}}\chi \int_{\mathbb{R}^d\times\mathbb{R}^d} n(t,x)n(t,y)|x-y|^2\; dxdy \\
&\leq \kappa d M - \tfrac{\lambda_d \chi}{2R^{d-2}} M^2 + \tfrac{2\lambda_d}{R^{d}}\chi M \int_{\mathbb{R}^d} |x|^2 n(t,x)\; dx = \kappa d M - \tfrac{\lambda_d \chi}{2R^{d-2}} M^2 + \tfrac{4\lambda_d}{R^{d}}\chi M\; I(t).
\end{aligned}$$

Choosing $R = \mu M^{\frac{1}{d-2}}$ with μ small enough we find

$$\frac{d}{dt} I(t) \leq M \; \left[\frac{C}{M^{\frac{d}{d-2}}} I(t) - 1 \right].$$

When the second x momentum of n_0, $I(0)$, is too small compared to $M^{\frac{d}{d-2}}$, then $I(t)$ decreases for all times and

$$\frac{d}{dt} I(t) \leq M \; \left[\frac{C}{M^{\frac{d}{d-2}}} I(0) - 1 \right] < 0 \quad \forall t \geq 0.$$

This leads to a contradiction after the time $T^* = I(0)M^{-1} \left[1 - \frac{C}{M^{\frac{d}{d-2}}} I(0) \right]^{-1}$ since $I(t)$ cannot be negative for smooth solutions.

For $d = 2$, the situation is simpler because we arrive directly at the identity

$$\frac{d}{dt} I(t) = 2\kappa M - \frac{\lambda_d}{2}\chi \int_{\mathbb{R}^d\times\mathbb{R}^d} n(t,x)n(t,y)\; dxdy = 2\kappa M - \frac{\lambda_d}{2}\chi M^2,$$

which leads directly to the same contradiction as before after the time $T^* = I(0)M^{-1}[\frac{\lambda_d}{2}\chi M - 2\kappa]^{-1}$.

Remark 3. One can examine further the type of blow-up that follows from the above computation. We claim that it disproves the existence after T^* of a distributional solution with the properties (8) and

$$n(t) \geq 0, \qquad \int_{\mathbb{R}^d} n(t,x)\; dx = M, \quad x\cdot\nabla c \in L^\infty.$$

Indeed, we may repeat the same above computation with the quantity $\int_{\mathbb{R}^d} \frac{|x|^2}{2}\varphi(\frac{|x|}{R}) n(t,x)dx$ instead of $I(t)$, for some test function $\varphi \in \mathcal{D}(\mathbb{R}_+)$ with $\varphi(r) = 1$ for $0 \leq r \leq 1$. We then obtain

$$\begin{aligned}
\tfrac{d}{dt} \int_{\mathbb{R}^d} \tfrac{|x|^2}{2}\varphi\left(\tfrac{|x|}{R}\right) n(t,x)dx = &\; \kappa \int_{\mathbb{R}^d} \Delta \left[\tfrac{|x|^2}{2}\varphi(\tfrac{|x|}{R}) \right] n(t,x)\; dx \\
&+ \chi \int_{\mathbb{R}^d} n(t,x) \nabla \left[\tfrac{|x|^2}{2}\varphi\left(\tfrac{|x|}{R}\right) \right] \cdot \nabla c(t,x)\; dx\;.
\end{aligned}$$

Since $\varphi(\frac{|x|}{R}) \to 1$ as $R \to \infty$, we have

$$\int_{\mathbb{R}^d} \Delta[\frac{|x|^2}{2}\varphi(\frac{|x|}{R})]n(t,x)dx \to dM \ ,$$

and

$$\begin{aligned} \textstyle\int_{\mathbb{R}^d} n(t,x)\ \nabla[\frac{|x|^2}{2}\varphi(\frac{|x|}{R})]\cdot\nabla c\ dx &= -\lambda_d \textstyle\int_{\mathbb{R}^d\times\mathbb{R}^d} n(t,x)n(t,y)\nabla[\frac{|x|^2}{2}\varphi(\frac{|x|}{R})]\cdot\frac{(x-y)}{|x-y|^d} \\ &\to -\lambda_d \textstyle\int_{\mathbb{R}^d\times\mathbb{R}^d} n(t,x)n(t,y)x\cdot\frac{(x-y)}{|x-y|^d}\ . \end{aligned}$$

Then, the end of the argument is justified entirely.

3 The parabolic-degenerate system.

As mentioned in the introduction, it is interesting to analyze systems of type (4), where the equation on c is a simple ODE, in order to understand the mechanism of the competition between the chemotactic effects due to the sensitivity function $\chi(c)$ and the dynamic of c. In this optics, parabolic-degenerate systems of type (4) have been considered, with the particular choice of $\chi(c) = \delta c^{-\alpha}$, $\delta > 0$, and

$$\frac{\partial}{\partial t}c = \sigma\, c^m\, n \tag{30}$$

where $\sigma = \pm 1$. Indeed, with the above $\chi(c)$ and when $\alpha \neq 1$, the density n evolves according to

$$\frac{\partial}{\partial t}n = \kappa\Delta n - \delta\nabla\cdot\left[n\,\nabla\left(\frac{c^{1-\alpha}}{1-\alpha}\right)\right], \quad t>0,\ x\in\Omega, \tag{31}$$

while from (30) the velocity $\nabla\left(\frac{c^{1-\alpha}}{1-\alpha}\right)$ of the drift species of density n evolves according to

$$\frac{\partial}{\partial t}\left(\frac{c^{1-\alpha}}{1-\alpha}\right) = \sigma\, c^{m-\alpha}\, n\ . \tag{32}$$

Hence, the above velocity is expected to remain bounded when $m-\alpha \geq 0$. Moreover, when $m = \alpha$ the c variable can be eliminated. When $\alpha = 1$, the same argument holds true with drift velocity $\nabla \log c$.

On the other hand and independently to the expression of $\chi(c)$, when $m \neq 1$ equation (30) is equivalent to

$$c^{1-m}(t,x) = \sigma(1-m)\int_0^t n(\tau,x)\ d\tau\ + c_0^{1-m}(x) \tag{33}$$

so that the chemical density c can be zero or $+\infty$ in a finite time, even if the initial density c_0 is everywhere positive and bounded. When $m = 1$, the equation (33) must be replaced by

$$c(t,x) = c_0(x)e^{\sigma\int_0^t n(\tau,x)\ d\tau} \tag{34}$$

and the singularity $c = 0$ is reached at any time t exactly where the initial density is zero.

In the case of consumption of the chemical ($\sigma = -1$) and $m = \alpha \geq 1$ or $0 \leq \alpha < m$ and $m > 1$, the local in time existence and uniqueness of classical Hölder continuous solution of (31) has been proved in [41], when Ω is a smooth bounded domain in $\mathbb{R}^d$, n satisfies homogeneous Neumann boundary condition and c_0 is a positive constant. The continuation of the solution for all time has also been proved in one space dimension. When $m = \alpha = 1$, the same global existence result of a unique smooth solution in a bounded interval of $\mathbb{R}$ has been shown later in [19], with some simplifications in the proof due to the peculiar form of $\chi(c) = \frac{1}{c}$. Moreover, the authors in [19] proved also the convergence of non stationary solutions to a non constant stationary solution, i.e. an aggregation phenomena. System (4) with $m = 1$, $\chi(c) = \frac{b}{a+bc}$ and an additional source term, has been considered in [23] and global existence and uniqueness of smooth solutions has been proved in the full line. In more than one dimension, the only result we are aware of is our previous paper [15] . There, we prove the existence of global weak solutions in the natural entropy space $n_0 \log n_0 \in L^1(\mathbb{R}^d)$ of system (4) with $m = 1$ (but the generalization to the case $m > 1$ is straightforward) and $\chi(c)$ such that $c\chi(c)$ is strictly increasing (thus including the case $\chi(c) = c^{-\alpha}$, $0 \le \alpha < 1$). This follows from the Lyapunov functional (6).

In the case of production of the chemical ($\sigma = +1$) and $m = \alpha = 1$, the instability of the spatially homogeneous solution $(n_0, c_0 e^{n_0 t})$, with n_0 and c_0 positive constants, has been proved in [38]. This result together with numerical simulations suggested to the authors that blow-up must occur in one space dimension. An explicit family of solutions of such a problem that blows up has been constructed in [33], thus supporting the conjecture in [38]. Nevertheless, for the same problem, in [44] two kinds of solutions have been found, one that exists globally in time and another that blows up (see also [34]). Moreover, the instability of the system has been proved. Finally we would like to cite [20], where the authors consider for the evolution of c an ODE of the type

$$\frac{\partial c}{\partial t} = g(c, n)$$

under some hypotheses on g not satisfied in the case we consider in system (4).

In this section, our goal is again to control the L^p norm of the solution of system (4) with consumption of the chemical, since for the corresponding system in the case of production of the chemical ($\sigma = +1$) these estimates are trivial (see Remark 4). As for the chemotaxis system (2), the L^p norm cannot be estimated from the identity (9). Therefore our analysis is based on an inequality related to the main differential inequality (14) for system (2). In order to do that, it will be useful to derive from (4) the equation on the ratio $\frac{n}{\phi(c)}$, where ϕ is such that

$$\phi'(c) = \frac{1}{\kappa}\phi(c)\chi(c) \qquad c > 0, \qquad \phi(0) = 1,$$

that is

$$\frac{\partial}{\partial t}\left(\frac{n}{\phi(c)}\right) = \kappa \frac{1}{\phi(c)} \nabla \cdot \left[\phi(c) \nabla \left(\frac{n}{\phi(c)}\right)\right] + \frac{1}{\kappa}\left(\frac{n}{\phi(c)}\right)^2 \phi(c)\chi(c)\, c^m. \tag{35}$$

Actually, the change of variable $n \to \frac{n}{\phi}$ is the natural change of variable that puts the equation on n in divergence form and it is used by many authors, especially to prove the existence of classical solutions (see [19] and [23] for instance). Moreover, in terms of reinforced random walk, the function ϕ is the transition probability rate of n ([38], [35]).

The rest of the section is devoted to the proof of Theorem 1.3 and Theorem 1.4. In order to prove the first one, we proceed in three steps in the same way we did in the proof of Theorem 1.1, but here the first step is no longer enough (since (3) is no longer valid) and we need to go through the argument of the second step to obtain a smallness condition on $\|n_0\|_{L^{d/2}(\mathbb{R}^d)}$ independent of p and sufficient to control the norms $\|n(t)\|_{L^p(\mathbb{R}^d)}$, $\max\{1; \frac{d}{2}-1\} \le p < \infty$.

Proof of Theorem 1.3.

First step : a priori estimate of the L^p norms of n. In this step, we show that for any fixed $\max\{1; \frac{d}{2}-1\} \le p < \infty$, the L^p norm of n is bounded for all $t > 0$ provided that the $L^{\frac{d}{2}}$ of n_0 is less than a constant that depends on p.

First of all, by the maximum principle, $n > 0$ since $n_0 \ge 0$. Then, from (35) and for any twice differentiable function f on $\mathbb{R}_+$, we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} f\left(\frac{n}{\phi(c)}\right)\phi(c) &= \kappa \int_{\mathbb{R}^d} f'\left(\frac{n}{\phi(c)}\right)\nabla\cdot\left[\phi(c)\nabla\left(\frac{n}{\phi(c)}\right)\right] \\ &\quad + \frac{1}{\kappa}\int_{\mathbb{R}^d}\phi^2(c)\chi(c)c^m\frac{n}{\phi(c)}\left[\frac{n}{\phi(c)}f'\left(\frac{n}{\phi(c)}\right) - f\left(\frac{n}{\phi(c)}\right)\right] \\ &= -\kappa\int_{\mathbb{R}^d}\phi(c)f''\left(\frac{n}{\phi(c)}\right)\left|\nabla\left(\frac{n}{\phi(c)}\right)\right|^2 \\ &\quad + \frac{1}{\kappa}\int_{\mathbb{R}^d}\phi^2(c)\chi(c)c^m\frac{n}{\phi(c)}\left[\frac{n}{\phi(c)}f'\left(\frac{n}{\phi(c)}\right) - f\left(\frac{n}{\phi(c)}\right)\right] . \end{aligned} \tag{36}$$

For $f(x) = x^p$, $p \ge 1$, the identity (36) becomes

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}\right)^p\phi(c) = -4\kappa\frac{p-1}{p}\int_{\mathbb{R}^d}\phi(c)\left|\nabla\left(\frac{n}{\phi(c)}\right)^{p/2}\right|^2 + \frac{1}{\kappa}(p-1)\int_{\mathbb{R}^d}\phi^2(c)\chi(c)c^m\left(\frac{n}{\phi(c)}\right)^{p+1} \tag{37}$$

i.e. of type (11).

Next, let us observe that since $m \ge 1$ and the initial density c_0 is bounded, the density c remains bounded and $0 \le c(t,x) \le \|c_0\|_{L^\infty}$, as it follows by (33) and (34). Therefore,

$$K_1 := \sup_{0\le c\le\|c_0\|_{L^\infty}}\left(\phi^2(c)\chi(c)c^m\right) < +\infty. \tag{38}$$

Since $\phi \ge 1$, using the same argument as in the previous section, we obtain in any dimension $d \ge 2$

and for any $\max\{1; \frac{d}{2}-1\} \le p < \infty$:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}\right)^p\phi(c) &\le -4\kappa\frac{p-1}{p}\int_{\mathbb{R}^d}|\nabla(\frac{n}{\phi(c)})^{p/2}|^2 + \frac{1}{\kappa}(p-1)K_1\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}\right)^{p+1} \\
&\le (p-1)\|\nabla(\frac{n}{\phi(c)})^{p/2}\|^2_{L^2(\mathbb{R}^d)}\left[\frac{1}{\kappa}\tilde{C}(d)K_1\|\frac{n}{\phi(c)}\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{4\kappa}{p}\right] \qquad (39)\\
&\le (p-1)\|\nabla(\frac{n}{\phi(c)})^{p/2}\|^2_{L^2(\mathbb{R}^d)}\left[\frac{1}{\kappa}\tilde{C}(d)K_1\|\phi^{2/d}(c)\left(\frac{n}{\phi(c)}\right)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{4\kappa}{p}\right] .
\end{aligned}$$

Thus, in dimension $d = 2$, if the initial mass M is sufficiently small with respect to $p \ge 1$, the L^p norm of n decreases for all time $t \ge 0$. In dimension $d > 2$, if $\|\phi^{2/d}(c_0)\left(\frac{n_0}{\phi(c_0)}\right)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ is small enough with respect to p, then $\|\phi^{1/p}(c)\left(\frac{n}{\phi(c)}\right)\|_{L^p(\mathbb{R}^d)}$ decreases in time and $\|n(t)\|_{L^p(\mathbb{R}^d)}$ stays bounded for all $t > 0$.

Second step : a refined smallness condition on the $L^{\frac{d}{2}}$ norm of n_0. In this step, we show the boundedness of the L^p norm of n with $\max\{1; \frac{d}{2}-1\} \le p < \infty$ as before, under a smallness condition on the $L^{\frac{d}{2}}$ norm of the initial density n_0 independent of p, i.e.

$$\frac{1}{\kappa}\tilde{C}(d)K_1\|\phi^{2/d}(c_0)\left(\frac{n_0}{\phi(c_0)}\right)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < \frac{8\kappa}{d} , \quad (d \ge 2) . \qquad (40)$$

In order to do that, we can argue as done in the previous section for system (2). Indeed, it holds true for any $\max\{1, \frac{d}{2}-1\} \le p < \infty$ and in any dimension $d \ge 2$ that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}-K\right)_+^p\phi(c) &= -4\kappa\frac{p-1}{p}\int_{\mathbb{R}^d}\phi(c)|\nabla(\frac{n}{\phi(c)}-K)_+^{p/2}|^2 \\
&+\frac{1}{\kappa}(p-1)\int_{\mathbb{R}^d}\phi^2(c)\chi(c)c^m\left(\frac{n}{\phi(c)}-K\right)_+^{p+1} \qquad (41)\\
&+\frac{1}{\kappa}(2p-1)K\int_{\mathbb{R}^d}\phi^2(c)\chi(c)c^m\left(\frac{n}{\phi(c)}-K\right)_+^p + \frac{1}{\kappa}pK^2\int_{\mathbb{R}^d}\phi^2(c)\chi(c)c^m\left(\frac{n}{\phi(c)}-K\right)_+^{p-1} .
\end{aligned}$$

For $K > 1$ it follows

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}-K\right)_+^p\phi(c) \\
&\le (p-1)\|\nabla(\frac{n}{\phi(c)}-K)_+^{p/2}\|^2_{L^2(\mathbb{R}^d)}\left[\frac{1}{\kappa}\tilde{C}(d)K_1\|\phi^{2/d}(c)\left(\frac{n}{\phi(c)}-K\right)_+\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{4\kappa}{p}\right] \qquad (42)\\
&\quad +\frac{1}{\kappa}(2p-1)K\ K_1\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}-K\right)_+^p\phi(c) + \frac{1}{\kappa}pK^2K_1M
\end{aligned}$$

if $\max\{1, \frac{d}{2}-1\} \le p \le 2$ or

$$\begin{aligned}
&\tfrac{d}{dt}\int_{\mathbb{R}^d} \left(\tfrac{n}{\phi(c)} - K\right)_+^p \phi(c) \\
&\le (p-1)\|\nabla(\tfrac{n}{\phi(c)} - K)_+^{p/2}\|^2_{L^2(\mathbb{R}^d)} \left[\tfrac{1}{\kappa}\tilde{C}(d)K_1\|\phi^{2/d}(c)\left(\tfrac{n}{\phi(c)} - K\right)_+\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \tfrac{4\kappa}{p}\right] \\
&\quad + \tfrac{1}{\kappa}(2p-1)K\ K_1 \int_{\mathbb{R}^d}\left(\tfrac{n}{\phi(c)} - K\right)_+^p \phi(c) + \tfrac{1}{\kappa}pK^2\ K_1\ M^{\frac{1}{p-1}}\left(\int_{\mathbb{R}^d}\left(\tfrac{n}{\phi(c)} - K\right)_+^p \phi(c)\right)^{1-\frac{1}{p-1}}
\end{aligned} \tag{43}$$

if $\max\{2; \frac{d}{2}-1\} < p < \infty$.

Next, for any fixed $\max\{1, \frac{d}{2}-1\} \le p < \infty$ and under the smallness condition (40), we can find $K(p)$ sufficiently large and independent of t such that

$$\frac{1}{\kappa}\tilde{C}(d)K_1\|\phi^{2/d}(c)\left(\frac{n}{\phi(c)} - K\right)_+\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \le \frac{4\kappa}{p}\ , \quad \forall t > 0\ . \tag{44}$$

Therefore, (42) and (43) becomes respectively

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)} - K\right)_+^p \phi(c) \le \frac{1}{\kappa}(2p-1)K\ K_1 \int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)} - K\right)_+^p \phi(c) + \frac{1}{\kappa}pK^2\ K_1\ M \tag{45}$$

if $\max\{1, \frac{d}{2}-1\} \le p \le 2$, and

$$\begin{aligned}
&\tfrac{d}{dt}\int_{\mathbb{R}^d} \left(\tfrac{n}{\phi(c)} - K\right)_+^p \phi(c) \le \tfrac{1}{\kappa}(2p-1)K\ K_1 \int_{\mathbb{R}^d}\left(\tfrac{n}{\phi(c)} - K\right)_+^p \phi(c) \\
&\quad + \tfrac{1}{\kappa}pK^2\ K_1\ M^{\frac{1}{p-1}}\left(\int_{\mathbb{R}^d}\left(\tfrac{n}{\phi(c)} - K\right)_+^p \phi(c)\right)^{1-\frac{1}{p-1}}\ , \quad \max\{2; \tfrac{d}{2}-1\} < p < \infty\ .
\end{aligned} \tag{46}$$

Again, Gronwall inequality for $E(t) = \int_{\mathbb{R}^d}(\frac{n}{\phi(c)} - K)_+^p\phi(c)\ dx$ gives us

$$E(t) \le e^{KC_1 t}E(0) + KC_2[e^{KC_1 t} - 1]\ , \quad \max\{1, \frac{d}{2} - 1\} \le p \le 2\ , \tag{47}$$

and

$$E^{\frac{1}{p-1}}(t) \le e^{K\overline{C}_1 t}E^{\frac{1}{p-1}}(0) + K\overline{C}_2[e^{K\overline{C}_1 t} - 1]\ , \quad \max\{2; \frac{d}{2} - 1\} < p < \infty\ . \tag{48}$$

For any fixed $\max\{1, \frac{d}{2}-1\} \le p^* < \infty$ and for $K = K(p^*)$, a global in time control of all the norms $\|n(t)\|_{L^p(\mathbb{R}^d)}$, $\max\{1, \frac{d}{2}-1\} \le p \le p^*$, under condition (40) follows.

Third step : regularization of system (4). In this final step, we briefly comment how to regularize system (4) in order to prove the existence of a solution satisfying the a priori estimates obtained in the second step. Indeed, let us consider the regularization of the equation on c as follows

$$\frac{\partial}{\partial t}c_\varepsilon = -c_\varepsilon^m \phi(c_\varepsilon)\left(\frac{n_\varepsilon}{\phi(c_\varepsilon)} \star \rho_\varepsilon\right) \tag{49}$$

for some regularizing kernel $\rho_\varepsilon = \frac{1}{\varepsilon^d}\rho(\frac{x}{\varepsilon})$ with $\rho \in \mathcal{D}^+(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d}\rho = 1$. With regularized initial data this makes c_ε smooth for all times and again existence of classical solutions of (4) with fast decay at infinity follows. For these solutions, the a priori estimates hold true with a slight modification. Indeed, the source term in (35) changes into $\frac{1}{\kappa}(\frac{n_\varepsilon}{\phi(c_\varepsilon)})(\frac{n_\varepsilon}{\phi(c_\varepsilon)} \star \rho_\varepsilon)\phi(c_\varepsilon)\chi(c_\varepsilon)\ c_\varepsilon^m$ and consequently (36) becomes

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} f\left(\frac{n_\varepsilon}{\phi(c_\varepsilon)}\right)\phi(c_\varepsilon) &= -\kappa\int_{\mathbb{R}^d}\phi(c_\varepsilon)f''\left(\frac{n_\varepsilon}{\phi(c_\varepsilon)}\right)\left|\nabla\left(\frac{n_\varepsilon}{\phi(c_\varepsilon)}\right)\right|^2 \\ &\quad + \frac{1}{\kappa}\int_{\mathbb{R}^d}\phi^2(c_\varepsilon)\chi(c_\varepsilon)c_\varepsilon^m\left(\frac{n_\varepsilon}{\phi(c_\varepsilon)}\star\rho_\varepsilon\right)\left[\frac{n_\varepsilon}{\phi(c_\varepsilon)}f'\left(\frac{n_\varepsilon}{\phi(c_\varepsilon)}\right) - f\left(\frac{n_\varepsilon}{\phi(c_\varepsilon)}\right)\right] .\end{aligned}$$

Again, performing the same kind of interpolation as in (28) is enough to obtain (39), (42) and (43), with the same constant K_1 (independent of ε) as before. Finally, we can pass to the limit observing that (39) also provides an a priori bound

$$\int_0^\infty \|\nabla(\frac{n}{\phi(c)})^{p/2}\|^2_{L^2(\mathbb{R}^d)} \leq C(K_0, \|\phi^{2/d}(c_0)\left(\frac{n_0}{\phi(c_0)}\right)\|_{L^p(\mathbb{R}^d)}) \ . \tag{50}$$

Remark 4. We point out that when using (30) in the computation of (37), the second term in the right hand side of this identity comes with the factor $-\frac{\sigma}{\kappa}(p-1)$ in front of the integral. Therefore, when $\sigma = +1$ this term is negative and the identity (37) leads directly to strong estimates in $\frac{n}{\phi(c)} \in (L^1 \cap L^\infty)(\mathbb{R}^d)$, if $0 < m \leq 1$. The difficulty in our case ($\sigma = -1$) is similar to that in Section 2 and comes from the competition between the two terms in the right hand side of (37). Let us notice that the case $\sigma = +1$ and $m > 1$ may lead to finite time extinction for c (see (33)) and to a possible lack of the existence of a global positive solution c as a consequence. The same problem is encountered when $\sigma = -1$ and $0 < m < 1$.

We now pass to the proof of Theorem 1.4. It is a variant of the above proof using the energy structure (6).

Proof of Theorem 1.4. Arguing as in [15] and thanks to the energy control, we know that $\int_{\mathbb{R}^2} n|\ln(n)|dx \in L^\infty(0,T)$ for all $T > 0$. Therefore for $K = K(p)$ large enough, the bracket

$$\left[\frac{1}{\kappa}\tilde{C}K_1\|\phi(c)\left(\frac{n}{\phi(c)} - K\right)_+\|_{L^1(\mathbb{R}^2)} - \frac{4\kappa}{p}\right]$$

in inequalities (42) and (43) with $d = 2$, is negative without any condition on M. Hence we conclude again the L^p bound as in the second step of the proof of Theorem 1.3.

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