Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view

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1 Introduction

1.1 Presentation of the results

In this paper, we consider the following parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u + \mid u \mid^{p-1} u, \;\; u_0 = u(0) \tag{1}$$

$u(t) \in L^\infty, \;\; u : \mathbb{R}^N \times [0, T) \mapsto \mathbb{R}$.
We will assume $1 < p$, $(N-2)p < N+2$. We are interested in blow-up solutions $u(t)$ of (1), that is solutions of (1) for which there exists T such that

$$\|u(t)\|_{L^\infty} \to +\infty \text{ as } t \to T$$

(see [Bal77] and [Lev73] for the existence of such solutions). Note that from the regularizing effect of the heat equation, the blow-up time does not depend on the space where the Cauchy problem is solved. The point a is a blow-up point if and only if there exists $(a_n, t_n) \to (a, T)$ as $n \to +\infty$ such that $|u(a_n, t_n)| \to +\infty$. Note that in the case $p < \frac{3N+8}{3N-4}$ or $u(0) \geq 0$, we know from [MZ] that an equivalent definition could be a point $a \in \mathbb{R}^N$ such that

$$\mid u(x,t) \mid \underset{(x,t)\to(a,T)}{\longrightarrow} +\infty.$$

The blow-up set $S \subset \mathbb{R}^N$ at time T is the set of all blow-up points. From [Mer92] and [MZ], under some restrictions on p there exists $u^* \in \mathcal{C}^2_{\text{loc}}(\mathbb{R}^N / S)$

such that

$$u(x,t) \underset{t\to T}{\longrightarrow} u^*(x) \text{ in } \mathcal{C}^2_{\text{loc}}.$$

u^* can have different types of behavior close to a blow-up point a. For example, from [HV92], [Vel92], [BK94] and [MZ], we have solutions such that

$$u^*(x) \underset{x\to a}{\sim} U_1(x-a), \text{ where } U_1(x) = \Big(\frac{8p \mid \log \mid x \mid\mid}{(p-1)^2 \mid x \mid^2}\Big)^{\frac{1}{p-1}} \tag{2}$$

and other solutions such that for some $k \geq 2$,

$$u^*(x) \underset{x\to a}{\sim} CU_k(x-a), \text{ where } U_k(x) = |x|^{-\frac{2k}{p-1}} \text{ and } C \neq 0$$

(there are also non-symmetric profiles but they are suspected to be unstable).

We will be interested in proving stability properties of the behavior (2). The question is the following : consider initial data $\tilde{u}_0$ such that $\tilde{u}(x,t)$ blows-up in finite time $\tilde{T}$ at a unique point $\tilde{a}$ with $\tilde{u}(x,t) \underset{t\to \tilde{T}}{\longrightarrow} \tilde{u}^*(x)$ for all $x \neq \tilde{a}$, $\tilde{u}^* \in \mathcal{C}^2_{\text{loc}}(\mathbb{R}^N/\{\tilde{a}\})$ and $\tilde{u}^* \underset{x\to\tilde{a}}{\sim} U_1(x-a)$. Is such a behavior stable under a perturbation of the initial data? Note that the other types of behavior are supposed to be unstable. In particular, when there are at least two blow-up points or when the blow-up profile is different from (2).

To clarify the situation, let us introduce similarity variables (see [GK89]). Let u be a solution of (1) which blows-up at time T in point a. We introduce

$$y = \frac{x-a}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad w_{a,T}(y,s) = (T-t)^{1/(p-1)}u(x,t). \tag{3}$$

Then $w_{a,T}(y,s)$ satisfies the following equation

$$\partial_s w_{a,T} = \Delta w_{a,T} - \frac{1}{2}y\cdot\nabla w_{a,T} + \mid w_{a,T} \mid^{p-1} w_{a,T} - \frac{1}{p-1}w_{a,T}, \tag{4}$$

for all $(y,s) \in \mathbb{R}^N \times [-\log T, +\infty)$.
Giga and Kohn, under some more assumptions on the power p, proved that

$$\exists C_0 > 0, \quad \forall s \geq -\log T, \quad \|w_{a,T}(s)\|_{L^\infty} \leq C_0 \tag{5}$$

$$\text{and } w_{a,T}(y,s) \underset{s\to+\infty}{\longrightarrow} \pm\kappa. \tag{6}$$

where $\kappa = (p-1)^{-1/(p-1)}$ and the convergence is uniform on compact sets of $\mathbb{R}^N$. Let us give the following definition :

Definition 1.1 (Type I blow-up solutions) *A solution $u(t)$ of (1) is called a blow-up solution of type I if there exists $C > 0$ such that $\forall t \in [0, T)$,*

$$\|u(t)\|_{L^\infty} \leq C(T-t)^{-\frac{1}{p-1}}.$$

Let us assume in the following $w_{a,T}(y,s) \underset{s\to+\infty}{\longrightarrow} +\kappa$ (if not consider $-w_{a,T}$). The result then has been specified ([FK92], [HV93]) at least in the positive case. We have two possibilities (up to an orthogonal change of space variables) :

a) $w_{a,T}(y,s) - \kappa \sim \frac{\kappa}{2ps}(N - \frac{1}{2}\sum_{i=1}^{l} y_i^2)$ on compact sets, where $1 \leq l \leq N$,

b)$| w_{a,T}(y,s) - \kappa | \leq Ce^{-s/2}$ on compact sets.

In [MZ98b] (see also [Vel92]), the following equivalence is proved (not necessarily in the radial case) for type I blow-up solutions, we have

$$(P1) \Leftrightarrow (P2) \Leftrightarrow (P3)$$

where

(P1) $\forall R > 0, \ \sup_{|y|\leq R} | w_{a,T}(y,s) - \kappa - \frac{\kappa}{2ps}(N - \frac{|y|^2}{2}) | = o(\frac{1}{s})$,

(P2) $\|w_{a,T}(y,s) - f(\frac{y}{\sqrt{s}})\|_{L^\infty} \underset{s\to+\infty}{\longrightarrow} 0$ with

$$f(z) = \left(p - 1 + \frac{(p-1)^2}{4p}|z|^2\right)^{-\frac{1}{p-1}}, \tag{7}$$

(P3) $u^*(x) \sim U_1(x-a)$ as $x \to a$, $x \neq a$.

The first stability result for this problem is due to Merle and Zaag [MZ97] who prove the following. They exhibit a set of initial data $\tilde{u}_0$ such that $\tilde{u}(x,t)$ blows-up at one point $\tilde{a}$ and behaves like (P2). Equivalence was not known at that time. For **these initial data** constructed before, they proved the following stability result.

There exists a neighborhood $\mathcal{V}_0$ of $\tilde{u}_0$ such that for all $u_0 \in \mathcal{V}_0$, $u(x,t)$ solution to (1) with initial data u_0 blows-up at time T and has a unique blow-up point a such that $w_{a,T}$ satisfies (P2).

Unfortunately this result was proved only for initial data constructed as before and a priori does not hold for *all* initial data blowing-up at one point such that (P1), (P2) or (P3) is satisfied. In this paper, we extend the result to such functions. Note that the strategy will be completely different. The proof will be closely related to some uniform estimates with respect to initial data (following from a Liouville Theorem) and a solution of a dynamical system of ordinary differential equations. We claim the following :

Theorem 1 (Stability of the blow-up profile with respect to the initial data) *Let $\tilde{u}(t)$ be a type I blow-up solution of (1) with initial data $\tilde{u}_0$ which blows-up at time T at only one point $\tilde{a} = 0$. Assume that for some $R > 0$ and $M > 0$,*

$$\text{for all } |x| \geq R \text{ and } t \in [0, \tilde{T}), \quad |\tilde{u}(x,t)| \leq M. \tag{8}$$

and

$$\forall x \in \mathbb{R}^N \backslash \{0\}, \quad \tilde{u}(x,t) \underset{t \to \tilde{T}}{\longrightarrow} \tilde{u}^*(x) \text{ where } \tilde{u}^*(x) \underset{x \sim 0}{\sim} U_1(x).$$

Then, there is a neighborhood $\mathcal{V}$ in L^∞ of $\tilde{u}_0$ such that for all $u_0 \in \mathcal{V}$ the solution of (1) with initial data u_0 blows-up at time $T = T(u_0)$ at a unique point $a = a(u_0)$ and

$$\forall x \in \mathbb{R}^N \backslash \{a\}, \quad u(x,t) \underset{t \to T}{\longrightarrow} u^*(x) \text{ where } u^*(x) \underset{x \sim a}{\sim} U_1(x-a).$$

Moreover, $(a(u_0), T(u_0))$ goes to $(0, \tilde{T})$ as u_0 goes to $\tilde{u}_0$.

Using the equivalence between (P1), (P2) and (P3) we have equivalent formulations of Theorem 1. Let us write one which will be useful for the proof.

Theorem 2 *Let $\tilde{u}(t)$ be a type I blow-up solution of (1) with initial data $\tilde{u}_0$ which blows-up at $t = \tilde{T}$ at only one point $\tilde{a} = 0$. Assume that (8) holds and that the function $\tilde{w}_{0,\tilde{T}}(y,s)$ defined in (3) satisfies uniformly on compact sets of $\mathbb{R}^N$*

$$\tilde{w}_{0,\tilde{T}}(y,s) - \kappa \underset{s \to +\infty}{\sim} \frac{\kappa}{2ps}(N - \frac{|\, y \,|^2}{2}). \tag{9}$$

Then, there is a neighborhood $\mathcal{V}$ in L^∞ of $\tilde{u}_0$ such that for all $u_0 \in \mathcal{V}$ the solution of (1) with initial data u_0 blows-up at time $T = T(u_0)$ at a unique point $a = a(u_0)$ and the function $w_{a,T}(y,s)$ defined in (3) satisfies uniformly on compact sets of $\mathbb{R}^N$

$$w_{a,T}(y,s) - \kappa \underset{s \to +\infty}{\sim} \frac{\kappa}{2ps}(N - \frac{|\, y \,|^2}{2}).$$

Moreover, $(a(u_0), T(u_0))$ goes to $(0, \tilde{T})$ as u_0 goes to $\tilde{u}_0$.

Remark : The condition (8) means that $\tilde{u}$ does not blow-up at infinity (in space).

Remark : The hypothesis that $\tilde{u}(t)$ is of type I can be removed for we can show, by the techniques of [MZ98b], that (8) and (9) imply that $\tilde{u}(t)$ is of type I.
Remark : In [FKZ], the authors study the difference between two blow-up solutions of (1) with the same blow-up time and the same unique blow-up point. As a consequence of this and the stability result of [MZ97], they obtain the same stability result as in Theorem 1, under the restrictive condition $u(0) \geq 0$ or $(3N-4)p \leq 3N+8$ (which implies by [GK89] that all blow-up solutions are of type I).
Remark : For $\frac{N+2}{N-2} > p \geq \frac{3N+8}{3N-4}$ it is not known (with no positivity conditions on the initial data) that all blow-up solutions are of type I. In the proof, we show in fact that if a solution $u(t)$ is of type I, then in a neighborhood of $u(0)$, all the solutions blow-up (which was not known before) and are of type I. More precisely,

Theorem 3 *The set of initial data such that $u(t)$ is a blow-up solution of type I is open.*

Remark : It is still an open problem for $p \geq \frac{3N+8}{3N-4}$ to know if there are blow-up solutions which are not of type I.

Generalizations

As in [MZ98a], there are various generalizations of this result.

1) $\mid u \mid^{p-1} u$ can be replaced by $f(u)$ where $f(u) \sim \mid u \mid^{p-1} u$ in $\mathcal{C}^3$ as $\mid u \mid \to +\infty$.

2) $\mathbb{R}^N$ can be replaced by a convex domain.

3) Using techniques of [MZ] and [FM95], we can generalize the result to the case of the equation (1) where u is a vector-valued function $u : \mathbb{R}^N \mapsto \mathbb{R}^M$, and where $F(u) \sim \mid u \mid^{p-1} u$ as $|u|$ goes to infinity in $\mathcal{C}^3$.

1.2 Strategy of the proof

We consider a type I blow-up solution $\tilde{u}(t)$ of (1) with initial data $\tilde{u}_0$. We suppose that $\tilde{u}(t)$ blows-up at time $\tilde{T}$ with a unique blow-up point $\tilde{a} = 0$. We also assume that

$$\left\| \tilde{w}_{0,\tilde{T}}(s) - \left\{ \kappa + \frac{\kappa}{2ps} \left(N - \frac{|y|^2}{2} \right) \right\} \right\|_{L^2_\rho} = o\left(\frac{1}{s}\right). \tag{10}$$

Therefore, from the equivalence of the properties (Pi), $i = 1, 2, 3$ (which holds for type I solutions), we have

$$\tilde{u}(x,t) \underset{t \to \tilde{T}}{\longrightarrow} \tilde{u}^*(x), \quad \tilde{u}^*(x) \underset{x \sim 0}{\sim} U_1(x).$$

We shall consider initial data u_0 in a neighborhood of $\tilde{u}_0$ in $\mathcal{C}^3$. We shall note with a $\tilde{}$ the items associated with $\tilde{u}(t)$ such as $\tilde{w}_{0,\tilde{T}}$ while those associated with $u(t)$ will not present some $\tilde{}$, for example $w_{a,T}$ for some $a \in \mathbb{R}^N$ and $T \in \mathbb{R}$.

From the fact that if we have $u_n(0) \to u(0)$ in L^∞ as n goes to infinity then for all $\epsilon \in [0,T)$, $u_n(\epsilon) \to u(\epsilon)$ in $\mathcal{C}^3$ as n goes to infinity, we are reduced to prove the stability for the $\mathcal{C}^3$ topology.

1.2.1 Formulation of the problem

We will adopt a dynamical system approach and work in the variable (y,s) in order to use the spectral properties of the operator

$$f \mapsto \mathcal{L}(f) = \Delta f - \frac{1}{2} y \cdot \nabla f + f$$

which appears in the equation of $w_{a,T}$.
Indeed, we have for all $s \in [-\log T,\ +\infty)$ and for all $y \in \mathbb{R}^N$,

$$\partial_s w_{a,T} = \Delta w_{a,T} - \frac{1}{2} y \cdot \nabla w_{a,T} + \mid w_{a,T} \mid^{p-1} w_{a,T} - \frac{1}{p-1} w_{a,T}.$$

Under the condition that a is a blow-up point, we have

$$\|w_{a,T}(s)\|_{L^\infty} \leq C, \tag{11}$$

(known if $p < \frac{3N+8}{3N-4}$ and unknown in general if $\frac{3N+8}{3N-4} \leq p < \frac{N+2}{N-2}$), proved by Giga and Kohn in [GK87]. Considering $-w$ if necessary, we can suppose

$$w_{a,T}(y,s) \underset{s \to +\infty}{\longrightarrow} \kappa,$$

on compact sets (see [GK89]). Let us now introduce

$$v = w_{a,T} - \kappa, \tag{12}$$

v satisfies the following equation

$$\partial_s v = \mathcal{L} v + f(v) \tag{13}$$

where $$f(v) = |v+\kappa|^{p-1}(v+\kappa) - \frac{\kappa}{p-1} - \frac{p}{p-1} v. \tag{14}$$

From (11), we obtain $| f(v) | \leq c | v |^2$. Therefore, f is a quadratic term.
Operator $\mathcal{L}$ is self-adjoint on L^2_ρ where $\rho(y) = \mathrm{e}^{-\frac{|y|^2}{4}} / (4\pi)^{N/2}$,
Spec $\mathcal{L} = \{1 - \frac{m}{2} \mid m \in \mathbb{N}\}$ and the eigenfunctions of $\mathcal{L}$ are derived from the Hermite polynomials. If $N = 1$, all the eigenvalues of $\mathcal{L}$ are simple. To $1 - \frac{m}{2}$ corresponds the eigenfunction

$$h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!} (-1)^n y^{m-2n}.$$

If $N \geq 2$, then the eigenfunctions corresponding to $1 - \frac{m}{2}$ are

$$H_\alpha(y) = h_{\alpha_1}(y_1) ... h_{\alpha_N}(y_N), \ \text{ with } \ \alpha = (\alpha_1, ..., \alpha_N) \ \text{ and } \ |\alpha| = m.$$

In particular,

1. 1 is an eigenvalue of multiplicity 1 and the corresponding eigenfunction is $H_0(y) = 1$,
2. $\frac{1}{2}$ is of multiplicity N and its eigenspace is generated by the orthogonal basis $\{y_i \mid i = 1, ..., N\}$,
3. 0 is of multiplicity $\frac{N(N+1)}{2}$ and its eigenspace is generated by the orthogonal basis

$$\{y_i y_j \mid i < j\} \cup \{y_i^2 - 2 \mid i = 1, ..., N\}. \tag{15}$$

Since the eigenfunctions of $\mathcal{L}$ constitute a total orthonormal family of L^2_ρ, we expand v as follows

$$v(y,s) = \sum_{m=0}^{2} v_m(y,s) + v_-(y,s) = v_2(y,s) + v_-(y,s) + v_+(y,s), \tag{16}$$

where $v_m(y,s)$ is the orthogonal projection of v on the eigenspace of $\lambda = 1 - \frac{m}{2}$, $v_-(y,s) = P_-(v)(y,s)$ and P_- is the projector on the negative subspace of $\mathcal{L}$ and $v_+(y,s) = v_0(y,s) + v_1(y,s)$. Let us define a $N \times N$ symmetric matrix $A(s)$ by

$$A_{ij}(s) = \int_{\mathbb{R}^N} v(y,s) \left(\frac{1}{4} y_i y_j - \frac{1}{2} \delta_{ij} \right) \rho(y) dy. \tag{17}$$

Then, from (16), (15) and the orthogonality between eigenfunctions of $\mathcal{L}$, we have

$$v_2(y,s) = \frac{1}{2} y^T A(s) y - \text{ tr } A(s). \tag{18}$$

It has been proved in [FK92], [FL93] and [Vel92] for a general blow-up solution as s goes to $+\infty$ that either $v \sim v_2$ (case a) or $v \sim v_-$ (case b) in L^2_ρ. In addition in the case a, there is a symmetric matrix Q and $l \in \{1, ..., N\}$ such that

$$v_2(Qy,s) \sim \frac{\kappa}{4ps}(2l - \sum_{i=1}^{l} y_i^2).$$

Thus, we now assume in accordance with (10) (see [FK92]) that v_2 is predominant and that $l = N$. Therefore,

$$\tilde{v}_2(y,s) \sim \frac{\kappa}{4ps}(2N - \mid y \mid^2) \text{ as } s \to +\infty. \tag{19}$$

We will prove that this behavior is stable under perturbation of the initial data, that is there is a neighborhood $\mathcal{V}$ of $\tilde{u}_0$ such that for all $u_0 \in \mathcal{V}$, $u(t)$ blows up at one point $a = a(u_0)$ at $T = T(u_0)$ and

$$v_{a,T}(y,s) \sim \frac{\kappa}{4ps}(2N - \mid y \mid^2) \text{ as } s \to +\infty.$$

1.2.2 Uniform L^∞ estimates on u and O.D.E. comparison.

Here, we use crucially a central argument (from [MZ]) in the proof of Theorem 1.

Proposition 1.2 (A Liouville Theorem for equation (1)) *Let $u(t)$ be a solution of (1) defined for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ such that for some $C > 0$,*

$$\mid u(x,t) \mid \leq \frac{C}{(T-t)^{\frac{1}{p-1}}}.$$

Then, there exist $T_1 \in [T, +\infty)$ and $\omega_0 \in \{-1, +1\}$ such that

$$u(x,t) = \omega_0 \kappa (T_1 - t)^{-\frac{1}{p-1}}.$$

This Theorem has an equivalent formulation for equation (4).

Proposition 1.3 (A Liouville Theorem for equation (4)) *Let w be a solution of (4) defined on $\mathbb{R}^N \times \mathbb{R}$ such that $w \in L^\infty(\mathbb{R}^N \times \mathbb{R})$. Then,*

$w \equiv 0$ or $w \equiv \pm\kappa$ or $w(y,s) \equiv \pm\theta(s+s_0)$ for a $s_0 \in \mathbb{R}$,

where $\theta(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$ and satisfies $\theta' = \theta^p - \frac{1}{p-1}\theta$, $\theta(-\infty) = \kappa$ and $\theta(+\infty) = 0$.

This allows Merle and Zaag [MZ] to prove for $u_0 \in \mathcal{C}^2$ the following :

Proposition 1.4 (Uniform ODE comparison of blow-up solutions of (1)) *Assume $1 < p < \frac{3N+8}{3N-4}$. If $\|u_0\|_{\mathcal{C}^2} \le C_0$ and $T < T_0$ then we have the following :*

i) Uniform estimates: $\exists C_1 = C_1(C_0, T_0)$ such that $\|u(t)\|_{L^\infty} \le \frac{C_1}{(T-t)^{\frac{1}{p-1}}}$.

ii) Uniform O.D.E. behavior: $\forall \epsilon > 0$, $\exists C = C(\epsilon, C_0, T_0)$ such that $\forall (x,t) \in \mathbb{R}^N \times [0,T)$,

$$| \partial_t u - | u |^{p-1} u | \le \epsilon | u |^p + C.$$

The purpose of this section is to prove without the condition $p < \frac{3N+8}{3N-4}$ the same result but for initial data only in a neighborhood of $\tilde{u}_0$, assuming that $\tilde{u}(t)$ is a type I blow-up solution. Indeed, in this case ($p \ge \frac{3N+8}{3N-4}$) the obstruction on the proof of ii) of Proposition 1.4 is the estimate $\|u(t)\|_{L^\infty} \le \frac{C}{(T-t)^{\frac{1}{p-1}}}$. Actually, we prove here that for all $\bar{u}_0$ such that $\bar{u}(t)$ is of type I, there is $\mathcal{V}_1$ neighborhood of $\tilde{u}_0$ such that i) and ii) of Proposition 1.4 is satisfied uniformly in $\mathcal{V}_1$ (see Theorem 3 and section 2). We first have the following :

Lemma 1.5 (Continuity of the blow-up time at $\tilde{u}_0$) *There exists $\mathcal{V}_0$ neighborhood of $\tilde{u}_0$ such that for all $u_0 \in \mathcal{V}_0$, $u(t)$ blows-up in finite time $T = T(u_0)$ and*

$$T(u_0) \longrightarrow \tilde{T} \text{ as } u_0 \to \tilde{u}_0.$$

Remark : The continuity of the blow-up time was known (see [Mer92]) in the case of bounded domains. Here, we will prove using an elementary but crucial blow-up result that the blow-up time is continuous and in particular finite at initial data such that $u(t)$ is a blow-up solution of type I. It is still open in the other cases.

Note that from the continuity of the blow-up time, we have the continuity of the blow-up profile in the following sense :

Corollary 1.6 (Continuity of the blow-up profile) *As $u_0 \to \tilde{u}_0$, we have $u^* \to \tilde{u}^*$ uniformly on compact sets of $\mathbb{R}^N \backslash S$.*

Proof : The proof of Proposition 2.3 in [Mer92] holds in the present case because we have the continuity of the blow-up time. ■

Proposition 1.7 (Uniform L^∞ bound, O.D.E. comparison) *There exist $\mathcal{V}_1$ a neighborhood of $\tilde{u}_0$, $C > 0$ and $\{C_\epsilon\}_\epsilon$ such that for all initial data u_0 in $\mathcal{V}_1$,*

i) $u(t)$ blows-up in T,

ii) $\forall t \in [0,T)$, $\|u(t)\|_{L^\infty} \leq C(T-t)^{-\frac{1}{p-1}}$,

iii) $\forall \epsilon > 0$, $\forall t \in [0,T)$, $\mid \partial_t u - \mid u \mid^{p-1} u \mid \leq \epsilon \mid u \mid^p + C_\epsilon$.

Corollary 1.8 (Constant sign property of $u(x,t)$ for x close to the blow-up point) *There exists $\mathcal{V}_2$ a neighborhood of $\tilde{u}_0$ and $\delta > 0$ such that for all initial data u_0 in $\mathcal{V}_2$,*

$$\forall t \in [T-\delta, T), \ \ \forall |x| \leq \delta, \ \ u(x,t) \geq 0.$$

1.2.3 Reduction to a finite dimensional problem

From continuity properties of the blow-up set and the blow-up time, we are able to control the unstable modes on the equation on v by the use of the geometric transformation (3), we note $v_{a,T} = w_{a,T} - \kappa$.

Proposition 1.9
i) There exists a neighborhood $\mathcal{V}_3$ of $\tilde{u}_0$ such that for all initial data u_0 in $\mathcal{V}_3$, there exists $a \in \mathbb{R}^N$, $T \in \mathbb{R}$ such that

$$v_{a,T} \underset{s \to +\infty}{\longrightarrow} 0 \ \text{in} \ C^{2,\alpha}_{\text{loc}}(\mathbb{R}^N).$$

ii) a goes to 0 and T goes to $\tilde{T}$ as u_0 goes to $\tilde{u}_0$ in $\mathcal{C}^3$.

Note that T is the blow-up time of $u(t)$ and a is a blow-up point of $u(t)$. At this stage of the proof, uniqueness of the blow-up point a is not known. We will see at the end of the proof the uniqueness of a. From now on, for each u_0 we fix a given a.

As a consequence of the Liouville Theorem, we have uniform convergence of $v_{a,T}$ with respect to the initial data, which allows us to compare uniformly the nonlinear problem with the linear problem (in fact with the quadratic problem when we deal with the neutral mode).

Proposition 1.10 (Uniform smallness of $v_{a,T}$) *There exists a neighborhood $\mathcal{V}_4$ of $\tilde{u}_0$ such that for all initial data u_0 in $\mathcal{V}_4$,* *i)* $\sup_{u_0\in\mathcal{V}_4} \|v_{a,T}(s)\|_{L^2_\rho} \underset{s\to+\infty}{\longrightarrow} 0,$

ii) $\forall R>0,\ \sup_{u_0\in\mathcal{V}_4} \big(\sup_{|y|\le R} \mid v_{a,T}(y,s)\mid\big) \underset{s\to+\infty}{\longrightarrow} 0.$

It follows from this Proposition and the fact that the dynamics on the neutral mode are slow for the quadratic approximation the following stability result. For simplicity we denote by v_i, $0\le i\le 2$ and v_- the components of the expansion of $v_{a,T}$ (see (16)).

Proposition 1.11 (Reduction to a finite dimensional problem) *There exist $\epsilon_0>0$ and a neighborhood $\mathcal{V}_5$ of $\tilde{u}_0$ such that for all $\epsilon\in(0,\epsilon_0)$, there is $s_0(\epsilon)\in\mathbb{R}$ such that for all initial data u_0 in $\mathcal{V}_5$,*

$$\forall s\ge s_0(\epsilon),\ \ \epsilon\|v_2(s)\|_{L^2_\rho}\ge\|v_-(s)\|_{L^2_\rho}+\|v_+(s)\|_{L^2_\rho}.$$

Note that the choice of the $(N+1)$ parameters (a,T) controls the $(N+1)$ unstable modes of v.

1.2.4 Solving of the finite dimensional problem

We study now $v_2(s)$ as s goes to infinity by using the matrix A introduced in (18). From (19), we have

$$\tilde{A}(s)\sim-\frac{\beta}{s}Id \tag{20}$$

as s goes to $+\infty$ with $\beta=\frac{\kappa}{2p}$. The question is now about the stability of behavior (20). Let us first give the form of the equation satisfied by $A(s)$.

Proposition 1.12 (Form of the finite dimensional problem : Finite dynamical system) *There exists a neighborhood $\mathcal{V}_6$ of $\tilde{u}_0$ such that for all $\epsilon>0$, there is $s_1(\epsilon)\in\mathbb{R}$ such that for all initial data u_0 in $\mathcal{V}_6$,*

$$\forall s\ge s_1(\epsilon),\ \ A'(s)=\frac{1}{\beta}A(s)^2+R(s), \tag{21}$$

where $\beta=\frac{\kappa}{2p}$ and $\mid R(s)\mid\le\epsilon\mid A(s)\mid^2$.

The stability result follows from the stability of the behavior $-\frac{\beta}{s}Id$ among solutions going to 0 for the ordinary differential equation $A'(s)=\frac{1}{\beta}A(s)^2$. Indeed,

Proposition 1.13 (Stability of $-\frac{\beta}{s}Id$ behavior) *There is a neighborhood $\mathcal{V}_7$ of $\tilde{u}_0$ such that for all $\epsilon > 0$, there exists $s_2(\epsilon)$ such that for all $u_0 \in \mathcal{V}_7$*

$$\forall s \geq s_2(\epsilon), \quad | A(s) + \frac{\beta}{s} Id | \leq \frac{\epsilon}{s}.$$

From this stability result and the uniform estimates of Proposition 1.7, we have the following Proposition which obviously implies Theorem 1 :

Proposition 1.14 (Uniform convergence of the different notions of profile in a neighborhood of $\tilde{u}_0$) *There is a neighborhood $\mathcal{V}_8$ of $\tilde{u}_0$ such that :*

i) $\forall \epsilon > 0$, $\exists s_3(\epsilon)$ *such that* $\forall s \geq s_3(\epsilon)$, $\forall u_0 \in \mathcal{V}_8$,

$$\left\| w_{a,T}(y,s) - \left\{ \kappa + \frac{\kappa}{2ps} \left(N - \frac{|\, y\,|^2}{2} \right) \right\} \right\|_{L^2_\rho} \leq \frac{\epsilon}{s},$$

ii) $\forall K_0 > 0$, $\forall \epsilon > 0$, $\exists s_4(K_0, \epsilon)$ *such that* $\forall s \geq s_4(K_0, \epsilon)$, $\forall u_0 \in \mathcal{V}_8$, $\sup_{|z| \leq 2K_0} | w_{a,T}(z\sqrt{s}, s) - f(z) | \leq \epsilon$ *where f is defined in (7),*

iii) For all $u_0 \in \mathcal{V}_8$ and $x \neq a$, $u(x,t) \to u^*(x)$ as $t \to T$ and

$$\sup_{u_0 \in \mathcal{V}_8} \left| \frac{u^*(a + x')}{U_1(x')} - 1 \right| \to 0 \text{ as } x' \to 0, x' \neq 0$$

where U_1 is defined in (2).

iv) For all $u_0 \in \mathcal{V}_8$, $u(t)$ blows-up at T with a unique blow-up point a, T goes to $\tilde{T}$ and a goes to 0 as u_0 goes to $\tilde{u}_0$.

2 Uniform ODE comparison and L^∞ bound

We prove Theorem 3, Lemma 1.5, Proposition 1.7 and Corollary 1.8 in this section.

Let us start by connecting some notions related to uniform ODE behavior and blow-up rate of type I. Consider a blow-up solution $u(t)$ with blow-up time T such that $\frac{T_0}{2} \leq T \leq T_0$ and $\|u_0\|_{C^2} \leq C_0$ for given T_0, C_0.

Proposition 2.1 (Equivalent properties for type I blow-up solutions) *Consider the following properties.*

Property i): *For all $\epsilon_1 > 0$, there is a constant $C_1 > 0$ such that*

$$|\frac{\partial u}{\partial t} - |u|^{p-1}u| = |\Delta u| \le \epsilon_1 |u|^p + C_1 \text{ on } \mathbb{R}^N \times [0,T). \tag{22}$$

Property ii): *There is a constant $C_2 > 0$ such that*

$$|\Delta u| \le \frac{1}{2}|u|^p + C_2 \text{ on } \mathbb{R}^N \times [0,T). \tag{23}$$

Property iii): *There is a constant $C_3 > 0$ such that*

$$\|u(t)\|_{L^\infty} \le C_3 (T-t)^{-\frac{1}{p-1}} \text{ on } [0,T). \tag{24}$$

We claim that

$$\textit{Property i)} \iff \textit{Property ii)} \iff \textit{Property iii)} . \tag{25}$$

with constants depending only on T_0, C_0, that is $C_2 = C_2(C_1)$, $C_3 = C_3(T_0, C_0, C_2)$ and $C_1 = C_1(\epsilon_1, T_0, C_0, C_3)$.

Remark : Under these properties, the solution is a blow-up solution of type I.

Proof of Proposition 2.1 :

Property i) $\Longrightarrow$ Property ii) : From the definitions.

Property ii) $\Longrightarrow$ Property iii) : Define for a given $x \in \mathbb{R}^N$, $z(t) = |u(x,t)|$. We have from Property ii),

$$z'(t) \ge \frac{1}{2} z(t)^p - C_2, \forall t \in [0,T) \text{ and } z(t) \text{ defined on } [0,T). \tag{26}$$

Let us prove that

$$\forall t \in [0,T), \frac{1}{z(t)^{p-1}} \ge \frac{1}{(4C_2)^{\frac{p-1}{p}}} + \frac{(p-1)(T-t)}{4}. \tag{27}$$

If $\forall t_0 \in [0,T), z(t_0)^p < 4C_2$, then it is done. If $z(t_0)^p \ge 4C_2$ for some $t_0 \in [0,T)$, then we have by a priori estimates, $\forall s \in [t_0,T), z'(s) \ge 0$, and $\forall s \in [t_0,T)$, $z(s)^p \ge 4C_2$. Therefore, $\forall s \in [t_0,T)$, $z'(s) \ge \frac{1}{4} z(s)^p$ and $(-z^{1-p})'(s) \ge \frac{p-1}{4}$.

By integration in time of this identity, we have

$$\frac{1}{z(t)^{p-1}} \ge \frac{(p-1)(T-t)}{4}. \tag{28}$$

which concludes the proof. Therefore

$$\forall t \in [0,T), \forall x \in \mathbb{R}^N, |u(x,t)| \le \frac{C''}{(T-t)^{\frac{1}{p-1}} + C'},$$

where $C' = C'(C_2)$ and $C'' = C''(C_2)$. This concludes the proof since the constants do not depend on $x \in \mathbb{R}^N$.

Property iii) $\Longrightarrow$ *Property i)* : The proof follows exactly the one of [MZ] (see also [MZ98a]) since in fact only Property i) was used there to prove the result. This concludes the proof of Proposition 2.1. ■

We now prove Lemma 1.5.

Proof of Lemma 1.5 : Since $\tilde{u}(t)$ is a type I blow-up solution, Lemma 1.5 follows directly from the following :

Lemma 2.2 *The blow-up time is continuous at $\bar{u}_0$ where the associated solution $\bar{u}(t)$ is a blow-up solution of type I.*

Remark : Note that no condition of decay at infinity in space is made.

Proof of Lemma 2.2 : Assume that $\bar{u}(t)$ is a blow-up solution of type I ($\bar{T}$ is the blow-up time). Let us consider a sequence u_n of initial data converging to $\bar{u}_0$. Let $u_n(t)$ and T_n be the associated solution and blow-up time.

By local wellposedness in time of the Cauchy problem it is classical to have $\liminf_{n\to+\infty} T_n \ge \bar{T}$.

Let us show that $\limsup_{n\to+\infty} T_n \le \bar{T}$. This will follow from two facts :

- **Fact 1** : From [MZ] (the results were in fact proved under the assumption that $\bar{u}(t)$ is a blow-up solution of type I), we have

$$\|\nabla \bar{w}(s)\|_{L^\infty} \to 0 \text{ and } \|\bar{w}(s)\|_{L^\infty} \to \kappa \text{ as } s \to +\infty \tag{29}$$

where $\bar{w}(y,s) = e^{-\frac{s}{p-1}}\bar{u}(ye^{-\frac{s}{2}}, T - e^{-s})$. In particular, there is a sequence $\hat{T}_m$ going to $\bar{T}$, $\hat{x}_m \in \mathbb{R}^N$, $\hat{s}_m$ going to $+\infty$ such that

$$\begin{aligned} &\|\nabla \bar{w}_{\hat{T}_m,\hat{x}_m}(\hat{s}_m)\|_{L^\infty} \to 0, \ \ \|\bar{w}_{\hat{T}_m,\hat{x}_m}(\hat{s}_m)\|_{L^\infty} \le 2.2^{\frac{1}{p-1}}\kappa \\ \text{and} \quad &\bar{w}_{\hat{T}_m,\hat{x}_m}(.,\hat{s}_m) \to 2^{\frac{1}{p-1}}\kappa \text{ in } L^\infty_{\text{loc}} \text{ as } m \to +\infty, \end{aligned} \tag{30}$$

where $\bar{w}_{\hat{T}_m,\hat{x}_m}$ is defined from $\bar{u}(t)$ in (3). Indeed, consider any sequence $\hat{s}_m$ going to infinity, from (29) we have then the existence of $\hat{x}_m$ such that

$$\bar{w}_{\bar{T},\hat{x}_m}(0,\hat{s}_m) \to \kappa.$$

Again from (29),

$$\|\nabla \bar{w}_{\bar{T},\hat{x}_m}(\hat{s}_m)\|_{L^\infty} \to 0, \ \ \|\bar{w}_{\bar{T},\hat{x}_m}(\hat{s}_m)\|_{L^\infty} \leq 2\kappa \text{ and } w_{\bar{T},\hat{x}_m}(\hat{s}_m) \to \kappa \text{ in } L^\infty_{\text{loc}} \tag{31}$$

as $m \to +\infty$.
Take now $\hat{T}_m$ such that $2(\bar{T} - \hat{t}_m) = \hat{T}_m - \hat{t}_m$ where $\hat{s}_m = -\log(\bar{T} - \hat{t}_m)$. From the definition (3), we have

$$\bar{w}_{\hat{T}_m,\hat{x}_m}(y, \hat{s}_m - \log 2) = 2^{\frac{1}{p-1}} \bar{w}_{\bar{T},\hat{x}_m}(y\sqrt{2}, \hat{s}_m).$$

It is easy then to check that $\bar{w}_{\hat{T}_m,\hat{x}_m}(\hat{s}_m - \log 2)$ satisfies (30).

- **Fact 2** : In [MZ], there is a blow-up criterion in the w variable (sharp for functions close to constants in L^∞_{loc}) which is the following. Consider w a solution of equation (4). Assume in addition that for some s_1 we have

$$I(w(s_1)) > 0$$

where for $w, \nabla w \in L^\infty$,

$$I(w) = -2E(w) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^N} |w(y)|^2 \rho(y) dy\right)^{\frac{p+1}{2}} \tag{32}$$

and

$$E(w) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla w(y)|^2 + \frac{1}{2(p-1)}|w(y)|^2 - \frac{1}{p+1}|w(y)|^{p+1}\right)\rho(y) dy. \tag{33}$$

Then, the solution $w(s)$ blows-up in finite time.

We claim that $\limsup_{n\to+\infty} T_n \leq \hat{T}_m$ for all m, which will conclude the proof of the Lemma. Indeed, from (30) and (32), we have

$$I(\bar{w}_{\hat{T}_m,\hat{x}_m}(\hat{s}_m)) \to I(2^{\frac{1}{p-1}}\kappa) > 0 \text{ as } m \text{ goes to infinity .} \tag{34}$$

We have by continuity of the solution with respect to the initial data that for a given m,

$$w_{\hat{T}_m,\hat{x}_{m_n}}(\hat{s}_m) \to \bar{w}_{\hat{T}_m,\hat{x}_m}(\hat{s}_m) \text{ in } W^{1,\infty} \text{ as } n \text{ goes to infinity and}$$

$$\|w_{\hat{T}_m,\hat{x}_{m_n}}(\hat{s}_m)\|_{W^{1,\infty}} \text{ is uniformly bounded in } n$$

where $w_{\hat{T}_m,\hat{x}_{m_n}}$ is defined from $u_n(t)$ by (3).

Therefore for n and m large, $I(w_{\hat{T}_m,\hat{x}_{m_n}}(\hat{s}_m) \geq \frac{I(2^{\frac{1}{p-1}}\kappa)}{2} > 0$. Thus $w_{\hat{T}_m,\hat{x}_{m_n}}$ blows up in finite time and for n large $T_n \leq \hat{T}_m$. Therefore, $\limsup_{n\to+\infty} T_n \leq \hat{T}_m$. This concludes the proof of Lemma 2.2 since $\hat{T}_m$ goes to $\bar{T}$ as $m \to +\infty$. ■

Proof of Theorem 3 and Proposition 1.7 : We now claim the following Lemma which concludes the proof of Theorem 3 and Proposition 1.7 (from the fact that Property i) $\Longleftrightarrow$ Property ii) $\Longleftrightarrow$ Property iii)). We still consider $\bar{u}(t)$ a blow-up solution of (1) of type I. From Proposition 2.1, we know that there is a constant C_2 such that

$$|\Delta\bar{u}| \leq \frac{1}{2}|\bar{u}|^p + C_2 \text{ on } \mathbb{R}^N \times [0,\bar{T}). \tag{35}$$

Lemma 2.3 (Uniform O.D.E. behavior in a neighborhood of a type I blow-up solution) *There exists $\mathcal{V}_0$ a neighborhood of $\bar{u}_0$ such that for all $u_0 \in \mathcal{V}_0$, $u(t)$ blows-up in finite time $T = T(u_0)$ and*

$$|\Delta u| \leq \frac{1}{2}|u|^p + 2C_2 \textit{ on } \mathbb{R}^N \times [0,T).$$

Proof of Lemma 2.3 : Let us argue by contradiction. Let us consider u_n a solution of (1) with initial data $u_{0n} \to \bar{u}_0 \in \mathcal{C}^2$ such that $u_n(t)$ blows-up at time $T_n \to \bar{T}$ and the statement

$$|\Delta u_n| \leq \frac{1}{2}|u_n|^p + 2C_2 \text{ on } \mathbb{R}^N \times [0,T_n)$$

is not valid. Therefore, there is $(x_n,t_n) \in \mathbb{R}^N \times [0,T_n)$ such that

$$|\Delta u_n(x_n,t_n)| - \frac{1}{2}|u_n(x_n,t_n)|^p \to 2C_2 \tag{36}$$

$$\text{and } |\Delta u_n| = \left|\frac{\partial u_n}{\partial t} - |u_n|^{p-1}u_n\right| \leq \frac{1}{2}|u_n|^p + 2C_2 \text{ on } \mathbb{R}^N \times [0,t_n]. \tag{37}$$

In the first step, we find a subsequence of (u_n) which tends (up to a translation) to a solution $\hat{u}$ of (1) blowing-up at $\bar{T}$ in 0. Then in a second step we use this limit object to find a contradiction.

First step : Behavior of $u_n(t_n)$ as $n \to +\infty$

a) $t_n \to \bar{T}$.

From the fact that $u_n(t_0) \to \bar{u}(t_0)$ in $\mathcal{C}^2$ for all $t_0 < \bar{T}$, there is for each $t_0 < \bar{T}$, a n_0 such that for all $n \geq n_0$,

$$|\Delta u_n| \leq \frac{1}{2}|u_n|^p + \frac{3}{2}C_2 \text{ on } \mathbb{R}^N \times [0, t_0].$$

Therefore, we have

$$\bar{T} - t_n \to 0 \text{ and } T_n - t_n \to 0 \text{ as } n \to +\infty.$$

b) Compactness procedure on the limit object $\bar{u}$.

Note that in the case where initial data is decaying at infinity, the blow-up set is compact, and this step is not needed. In this situation, take $\hat{u} = \bar{u}$. In the general case, from Proposition 2.1, we have

$$\forall t \in [0, \bar{T}), \quad \|\bar{u}(t)\|_{L^\infty} \leq \frac{C_1}{(\bar{T} - t)^{\frac{1}{p-1}}} \tag{38}$$

where $C_1 = C_1(C_2, \bar{T}, \|\bar{u}(0)\|_{C^2})$. Considering $\bar{u}(x_n + x, t)$ and using a compactness procedure based on (38), we can assume that

$$\bar{u}(x_n + x, t) \to \hat{u}(x, t) \text{ in } C^{2,1}_{\text{loc}}(\mathbb{R}^N \times [0, \bar{T})) \tag{39}$$

where $\hat{u}$ is also a solution of (1). We now claim the following :

c) The point 0 is a blow-up point of $\hat{u}$ and $|u_n(x_n, t_n)| \to \infty$.

i) Proof of the fact that 0 is a blow-up point of $\hat{u}$ and $\bar{T}$ is its blow-up time.

We proceed by contradiction. Assume that there are $\delta > 0$ and $M > 0$ such that $\forall (x, t) \in B_\delta \times [0, \bar{T})$,

$$|\hat{u}(x, t)| + |\Delta \hat{u}(x, t)| \leq M.$$

From the fact that

$$u_n(x_n + x, t) \to \hat{u}(x, t) \text{ in } C^{2,1}_{\text{loc}}(\mathbb{R}^N \times [0, \bar{T})) \tag{40}$$

(which follows from (39) and the fact that $u_n \to \bar{u}$ in $L^\infty_{\text{loc}}([0, \bar{T}), C^2(\mathbb{R}^N)))$ and the ODE property for $t < t_n$ stated in (37), an easy calculation shows that for n large enough, $\forall (x, t) \in B(x_n, \frac{\delta}{2}) \times [0, t_n)$,

$$|u_n(x, t)| \leq 2M.$$

Let us now recall a useful Lemma on parabolic regularization.

Lemma 2.4 (Parabolic regularity for equation (1)) *Assume h is a solution of (1) defined for $(\xi,\tau) \in D = B(0,\eta)\times[0,t^*]$ and satisfying $\|h\|_{L^\infty} \le M$. Consider $t_1 \in (0,t^*)$. Then, there exist $\alpha(p)\in(0,1)$ and $K(t_1,\eta,M)$ such that*

$$\|h\|_{C^{2,1}(D')} + |\nabla^2 h|_{\alpha,D'} + |\frac{\partial h}{\partial t}|_{\alpha,D'} \le K$$

where $D' = B(0,\frac{\eta}{2})\times[t_1,t^*]$, $\|h\|_{C^{2,1}(D')} = \|h\|_{L^\infty(D')} + \|\nabla h\|_{L^\infty(D')} + \|\nabla^2 h\|_{L^\infty(D')} + \|\frac{\partial h}{\partial t}\|_{L^\infty(D')}$ and

$$|a|_{\alpha,D} = \sup_{(\xi,\tau),(\xi',\tau')\in D} \frac{|a(\xi,\tau)-a(\xi',\tau')|}{\left(|\xi-\xi'|+|\tau-\tau'|^{1/2}\right)^\alpha}. \tag{41}$$

Proof : see Lemmas 2.8 and 2.10 in [MZ98b]. ■

Thus, there is $M^*(\bar T,\delta,M)$ such that for n large,

$$|\Delta u_n|_{\alpha,D_n} + |\frac{\partial u_n}{\partial t}|_{\alpha,D_n} \le M^*,$$

where $D_n = B(x_n,\frac{\delta}{4})\times[\frac{\bar T}{2},t_n)$. In particular, Δu_n and $\frac{\partial u_n}{\partial t}$ are uniformly continuous on D_n, with a constant of continuity independent of n.
We claim now that $\forall (x,t)\in D_n$,

$$|\Delta u_n(x,t)| \le \frac{1}{2}|u_n(x,t)|^p + \frac{3}{2}C_2. \tag{42}$$

Indeed, since $t_n\to\bar T$ and the constant of uniform continuity of Δu_n and $\frac{\partial u_n}{\partial t}$ on D_n is independent of n, there is a t_0 such that for n large, $\forall (x,t)\in D_n$, we have

$$\begin{array}{l}\text{either } t\le t_0 \text{ or } |\Delta u_n(x,t)-\Delta u_n(x,t_0)| \le \frac{C_2}{16}\\ \text{and } ||u_n(x,t)|^{p-1}u_n(x,t) - |u_n(x,t_0)|^{p-1}u_n(x,t_0)| \le \frac{C_2}{16}.\end{array} \tag{43}$$

From (40) and the identity

$$|\Delta \hat u(x,t)| \le \frac{1}{2}|\hat u(x,t)|^p + C_2 \text{ on } \mathbb{R}^N\times[0,\bar T) \tag{44}$$

(which follows from (35) and (39)), we have for n large and for all $(x,t)\in B(x_n,\frac{\delta}{4})\times[\frac{\bar T}{2},t_0]$,

$$|\Delta u_n(x,t)| \le \frac{1}{2}|u_n(x,t)|^p + \frac{17}{16}C_2. \tag{45}$$

It follows then from (43) and (45) that $\forall (x,t) \in D_n$,

$$|\Delta u_n(x,t)| \leq \frac{1}{2}|u_n(x,t)|^p + \frac{3}{2}C_2, \tag{46}$$

which is a contradiction. Therefore 0 is a blow-up point of $\hat{u}$.

ii) Proof of the fact that $|u_n(x_n,t_n)| \to +\infty$.

We argue by contradiction. Assume that there is $M > 0$ such that $|u_n(x_n,t_n)| \leq M$. Integrating the ODE (37) backwards in time, there is a $M^* > 0$ such that $\forall t \in [0,t_n], |u_n(x_n,t)| \leq M^*$. Fix any $t_0 < \bar{T}$. For n large we have $|u_n(x_n,t_0)| \leq M^*$. Let n go to infinity, we obtain $|\hat{u}(0,t_0)| \leq M^*$. Thus for all $t < \bar{T}$,

$$|\hat{u}(0,t)| \leq M^*$$

which is a contradiction with the fact that 0 is a blow-up point of $\hat{u}$.

Second step : Conclusion of the proof using the Liouville Theorem.

We now follow an argument in [MZ98b] except that we will use the O.D.E. approximation to obtain key estimates (see [MZ98b] for more details). We consider two cases.

Case 1 : $\frac{u_n(x_n,t_n)}{\|u_n(t_n)\|_{L^\infty}} \not\longrightarrow 0$ *(the very singular region).*

Let us assume that $\frac{\kappa}{2}\frac{u_n(x_n,t_n)}{\|u_n(t_n)\|_{L^\infty}} \to \kappa_1 \in (0, \frac{\kappa}{2}]$, and consider

$$v_n(\xi,\tau) = M_n^{\frac{1}{p-1}} u_n(x_n + \xi\sqrt{M_n}, t_n + \tau M_n) \tag{47}$$

where

$$M_n^{\frac{1}{p-1}}\|u_n(t_n)\|_{L^\infty} = \frac{\kappa}{2}.$$

From the Liouville Theorem stated for equation (1) (Proposition 1.2), we show that the nonlinear term is "subcritical" on compact sets of $\mathbb{R}^N \times (-\infty, 0]$. In particular, we show that $v_n(\xi,\tau) \to v(\tau)$ where $v' = v^p$ and $v(0) = \kappa_1$, uniformly on compact sets of $\mathbb{R}^N \times (-\infty, 0]$ (Note that

$$v(\tau) = \kappa\left(\left(\frac{\kappa}{\kappa_1}\right)^{p-1} - \tau\right)^{-\frac{1}{p-1}} \tag{48}$$

and $v(1) < +\infty$ since $2\kappa_1 \leq \kappa$).

We have from the definition of v_n that v_n is defined for all $\tau \in [\tau_n, 0]$ where

$\tau_n = -\frac{t_n}{M_n} \to -\infty$ (since $M_n \to 0$ from the fact that $|u_n(x_n,t_n)| \to +\infty$ and $t_n \to \bar{T}$) and satisfies

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + |v_n|^{p-1} v_n.$$

Moreover, v_n satisfies the following O.D.E. estimate from (47) and (37) : for all $(\xi,\tau) \in \mathbb{R}^N \times [\tau_n, 0]$,

$$\frac{\partial |v_n|}{\partial \tau} \geq \frac{1}{2}|v_n|^p - 2M_n^{\frac{p}{p-1}} C_2. \qquad (49)$$

Therefore, for all $\tau \in [\tau_n, 0]$, $\|v_n(\tau)\|_{L^\infty} \leq \max\left((4C_2)^{\frac{1}{p}} M_n^{\frac{1}{p-1}}, \kappa\right)$. Indeed, if not, we have by integration $\|v_n(0)\|_{L^\infty} \geq \kappa$ which is a contradiction.
By the wellposedness of the Cauchy problem in L^∞, there is $\tau_0 \in (0,1)$ such that $v_n(\tau)$ is uniformly bounded on $[0,\tau_0]$ in L^∞. By a classical compactness procedure and up to extracting a subsequence, we can assume $v_n \to v$ in $C^{2,1}_{loc}(\mathbb{R}^N \times (-\infty, \tau_0])$ where

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \Delta v + |v|^{p-1} v \\ v(0,0) &= \kappa_1. \end{aligned}$$

Moreover, letting n go to infinity in (49), we obtain that for all $\tau \in (-\infty, 0]$,

$$\frac{\partial |v|}{\partial \tau} \geq \frac{1}{2}|v|^p \text{ and } \|v(0)\|_{L^\infty} \leq \frac{\kappa}{2}. \qquad (50)$$

Therefore, by integration, we have

$$\forall \tau \leq \tau_0, \quad \|v(\tau)\|_{L^\infty} \leq \frac{C'}{(1-\tau)^{\frac{1}{p-1}}} \text{ for some } C' > 0.$$

From Proposition 1.2, that is using in some sense the Liouville Theorem in the very singular region, we have $v(\xi,\tau) = v(\tau)$, defined in (48). By a direct calculation and from (47) and (36), we have for n large
$|\Delta v_n(0,0)| = M_n^{\frac{p}{p-1}} |\Delta u_n(x_n,t_n)| \geq \frac{1}{2}|u_n(x_n,t_n)|^p M_n^{\frac{p}{p-1}} = \frac{1}{2}|v_n(0,0)|^p$. Letting $n \to +\infty$, we obtain

$$0 \geq \frac{1}{2} v(0)^p > 0,$$

which is a contradiction.

Case 2 : $\frac{u_n(x_n,t_n)}{\|u_n(t_n)\|_{L^\infty}} \longrightarrow 0$ *(the singular region).*

Define

$$M_n(t) = \left(\frac{\|u_n(t)\|_{L^\infty}}{\kappa}\right)^{1-p} \text{ and } M(t) = \left(\frac{\|\hat{u}(t)\|_{L^\infty}}{\kappa}\right)^{1-p}. \qquad (51)$$

From the Liouville Theorem, we prove a useful Lemma relating $M_n(t)$ and $T_n - t$.

Lemma 2.5 *Consider* $(t_{n,1})$ *a subsequence tending to* $\bar{T}$ *and assume that* $t_{n,1} \le t_n$*. Then*

$$\frac{T_n - t_{n,1}}{M_n(t_{n,1})} \to 1 \text{ as } n \to +\infty.$$

Proof : Since u_n blows-up at time T_n, we have by the maximum principle $\|u_n(t_{n,1})\|_{L^\infty} \ge \kappa(T_n - t_{n,1})^{-\frac{1}{p-1}}$. Therefore,

$$\forall n \in \mathbb{N}, \ \ \frac{T_n - t_{n,1}}{M_n(t_{n,1})} \ge 1. \qquad (52)$$

We now claim that

$$\limsup_{n\to+\infty} \frac{T_n - t_{n,1}}{M_n(t_{n,1})} \le 1. \qquad (53)$$

This estimate follows from the Liouville Theorem and the blow-up criterion which is sharp near constants. Let us consider

$$\tilde{h}_n(\xi,\tau) = M_n(t_{n,1})^{\frac{1}{p-1}} u_n\left(x_{n,1} + \xi\sqrt{M_n(t_{n,1})}, t_{n,1} + \tau M_n(t_{n,1})\right). \qquad (54)$$

where $x_{n,1}$ is chosen such that $|u_n(x_{n,1},t_{n,1})| M_n(t_{n,1})^{\frac{1}{p-1}} \to \kappa$.
From the Liouville Theorem stated for equation (1) (Proposition 1.2), we will show that $\tilde{h}_n(\xi,\tau) \to \tilde{h}(\tau)$, $\tilde{h}' = \tilde{h}^p$ and $\tilde{h}(0) = \kappa$ uniformly on compact sets of $\mathbb{R}^N \times (-\infty, 0]$. Note that

$$\tilde{h}(\tau) = \kappa\,(1-\tau)^{-\frac{1}{p-1}}. \qquad (55)$$

Since $\bar{T} - t_{n,1} \to 0$, we have $M_n(t_{n,1}) \to 0$. Indeed, if not, then we have $\|u_n(t_{n,1})\|_{L^\infty} \le C^*$ for some $C^* > 0$ and some subsequence. Therefore, from the wellposedness of the Cauchy problem for equation (1), $T_n \ge t_{n,1} + \tau^*$ for some $\tau^*(C^*) > 0$. As $n \to +\infty$, we obtain $\bar{T} \ge \bar{T} + \tau^*$ which is a

contradiction.
As in Case 1 and up to a subsequence, we have from the definition of $\tilde{h}_n$ that $\tilde{h}_n$ is defined for all $\tau \in [\tau_n, 0]$ where $\tau_n \to -\infty$ (since $\bar{T} - t_{n,1} \to 0$ and $M_n(t_{n,1}) \to 0$) and satisfies

$$\begin{array}{l} \frac{\partial \tilde{h}_n}{\partial \tau} = \Delta \tilde{h}_n + |\tilde{h}_n|^{p-1}\tilde{h}_n, \\ \tilde{h}_n(0,0) \to \kappa \text{ and } \|\tilde{h}_n(0)\|_{L^\infty} \leq \kappa. \end{array} \tag{56}$$

By the same techniques as in the previous case and from the integration of the O.D.E. (37) backwards (note that $t_{n,1} \leq t_n$), we have for all $\tau \in [\tau_n, 0]$, $\|\tilde{h}_n(\tau)\|_{L^\infty} \leq \max\left((4C_2)^{\frac{1}{p}} M_n(t_{n,1})^{\frac{1}{p-1}}, \kappa\right)$. Thus, by parabolic regularity and a compactness procedure, we can assume that $\tilde{h}_n \to h$ in $C^{2,1}_{loc}(\mathbb{R}^N \times (-\infty, 0])$ where

$$\begin{array}{l} \frac{\partial h}{\partial \tau} = \Delta h + |h|^{p-1} h \\ h(0,0) = \kappa \text{ and } \|h(0)\|_{L^\infty} \leq \kappa. \end{array}$$

As in Case 1, if we integrate backwards the limit ODE (obtained form (37) when $n \to +\infty$), we obtain $\|h(\tau)\|_{L^\infty} \leq C'(1-\tau)^{-\frac{1}{p-1}}$ for all $\tau \in (-\infty, 0]$. From Proposition 1.2, (that is using in some sense the Liouville Theorem in the very singular region), we have $\tilde{h}(\xi, \tau) = \tilde{h}(\tau)$, where $\tilde{h}$ is defined in (55). Let us note that from this argument, we have for all $\theta > 1$ and n large

$$I\left(\theta^{\frac{1}{p-1}} \tilde{h}_n(.\sqrt{\theta}, 0)\right) > \frac{1}{2} I\left(\theta^{\frac{1}{p-1}} \tilde{h}(0)\right) = \frac{1}{2} I(\theta^{\frac{1}{p-1}} \kappa) > 0$$

where I is defined in (32). Therefore, for n large enough, $I(\tilde{w}_n(0)) > 0$ where $\tilde{w}_n(y,s) = e^{-\frac{s}{p-1}} \theta^{\frac{1}{p-1}} \tilde{h}_n\left(ye^{-\frac{s}{2}}\sqrt{\theta}, (1-e^{-s})\,\theta\right)$ is defined from $\theta^{\frac{1}{p-1}} \tilde{h}_n(.\sqrt{\theta}, .\theta)$ by (3). Thus, from the blow-up criterion, $\tilde{w}_n$ blows-up in finite time, hence, $\tilde{h}_n$ blows-up before the time θ. This implies by (54) that for n large enough, $\frac{T_n - t_{n,1}}{M_n t_{n,1}} \leq \theta$, hence (53) follows.
(52) and (53) finish the proof of Lemma 2.5. ∎

Let us recall the following result which asserts that the smallness of the following weighted energy (related to the energy $E(w_a)$ defined in (33))

$$\begin{aligned} \mathcal{E}_{a,t}(u) \;&=\; t^{\frac{2}{p-1} - \frac{N}{2} + 1} \int \left[\frac{1}{2}|\nabla u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right] \rho(\frac{x-a}{\sqrt{t}}) dx \\ &+\; \frac{1}{2(p-1)} t^{\frac{2}{p-1} - \frac{N}{2}} \int |u(x)|^2 \rho(\frac{x-a}{\sqrt{t}}) dx \end{aligned}$$

implies an L^∞ bound on $u(x,t)$ locally in space-time.

Proposition 2.6 (Local energy smallness result) *There exists $\sigma_0 > 0$ such that for all $\delta' > 0$ and $\theta' > 0$, the following property holds . If h_n is a solution of (1) which blows-up at time θ_n and satisfies $\mathcal{E}_{\xi,\theta_n-\tau'}(h_n(\tau')) \le \sigma_0$ for some $\tau' \in [0, \theta_n - \theta']$ and for all $\xi \in B(0, 2\delta')$, then*
- $\forall |\xi| \le \delta'$, $\forall \tau \in [\frac{\tau'+\theta_n}{2}, \theta_n)$, $|h_n(\xi,\tau)| \le C\sigma_0^{\theta}(\theta_n - \tau)^{-\frac{1}{p-1}}$
- Moreover, if $\forall |\xi| \le \delta'$, $|h_n(\xi, \frac{\tau'+\theta_n}{2})| \le M'$ then $\forall |\xi| \le \frac{\delta'}{2}$, $\forall \tau \in [\frac{\tau'+\theta_n}{2}, \theta_n)$, $|h_n(\xi,\tau)| \le M^*$ where $M^* = M^*(M', \delta', \theta')$.

Proof : See [GK89] and [Mer92] (Proposition 2.5). ■
From the fact that 0 is a blow-up point of $\hat{u}$, we are able to choose a suitable scaling parameter connecting $(0, t_n)$ and the "very singular region" of u_n. Consider $\kappa_0 \in (0, 1)$ a constant such that $\mathcal{E}_{0,1}(\kappa_0) \le \frac{\sigma_0}{2}$ ($\mathcal{E}_{0,1}(0) = 0$ yields the existence of such a κ_0).
Since 0 is a blow-up point of $\hat{u}$ and $\hat{u}$ is a blow-up solution of type I (this follows from (44) and Proposition 2.1), the results of Giga and Kohn [GK89] and those of [MZ] apply and we have as t goes to $\bar{T}$ (up to a sign change),

$$\hat{u}(0,t)(\bar{T} - t)^{\frac{1}{p-1}} \to \kappa \text{ and } \frac{M(t)}{\bar{T} - t} \to 1. \tag{57}$$

We claim the following : we have the existence of $\tilde{t}_n \in [0, t_n]$ going to $\bar{T}$ such that $|u_n(x_n, \tilde{t}_n)|M_n(\tilde{t}_n)^{\frac{1}{p-1}} = \kappa_0$ and $\forall t \in (\tilde{t}_n, t_n]$, $|u_n(x_n, t)|M_n(t)^{\frac{1}{p-1}} < \kappa_0$. Indeed, from (40) and the fact that $T_n \to \bar{T}$, we have

$$u_n(x_n, t) \to \hat{u}(0, t) \text{ and } \frac{\bar{T} - t}{T_n - t} \to 1$$

uniformly on compact subsets of $[0, \bar{T})$. Thus, from a diagonal process and up to a subsequence, there is a sequence $t_{n,2} \to \bar{T}$ such that

$$t_{n,2} \le t_n, \ \ \frac{\bar{T} - t_{n,2}}{T_n - t_{n,2}} \to 1 \text{ and } \frac{u_n(x_n, t_{n,2})}{\hat{u}(0, t_{n,2})} \to 1. \tag{58}$$

Therefore, we have from (57), (58) and Lemma 2.5,
$u_n(x_n, t_{n,2})M_n(t_{n,2})^{\frac{1}{p-1}} =$

$$\frac{u_n(x_n, t_{n,2})}{\hat{u}(0, t_{n,2})} \left(\frac{M_n(t_{n,2})}{T_n - t_{n,2}}\right)^{\frac{1}{p-1}} \left(\frac{T_n - t_{n,2}}{\bar{T} - t_{n,2}}\right)^{\frac{1}{p-1}} \hat{u}(0, t_{n,2}) \left(\bar{T} - t_{n,2}\right)^{\frac{1}{p-1}} \to \kappa. \tag{59}$$

Since we assume $\frac{u_n(x_n,t_n)}{\|u_n(t_n)\|_{L^\infty}} \to 0$, we have from (51) and continuity arguments the existence of $\tilde{t}_n \in [t_{n,2}, t_n]$ such that $|u_n(x_n, \tilde{t}_n)|M_n(\tilde{t}_n)^{\frac{1}{p-1}} = \kappa_0$

and $\forall t \in (\tilde{t}_n, t_n]$, $|u_n(x_n,t)|M_n(t)^{\frac{1}{p-1}} < \kappa_0$.
Note that we have $\tilde{t}_n \to \bar{T}$ from the fact that $\tilde{t}_n \geq t_{n,2}$, thus $M_n(\tilde{t}_n) \to 0$. Indeed, if for some $C^* > 0$ and a subsequence, $\|u_n(\tilde{t}_n)\|_{L^\infty} \leq C^*$, then we have $T_n \geq \tilde{t}_n + \tau^*$ for some $\tau^*(C^*) > 0$, from the wellposedness of the Cauchy problem for equation (1). As $n \to +\infty$, we obtain $\bar{T} \geq \bar{T} + \tau^*$ which is a contradiction.
Let us now consider

$$h_n(\xi,\tau) = M_n(\tilde{t}_n)^{\frac{1}{p-1}} u_n\left(x_n + \xi\sqrt{M_n(\tilde{t}_n)}, \tilde{t}_n + \tau M_n(\tilde{t}_n)\right). \qquad (60)$$

From the Liouville Theorem stated for equation (1) (Proposition 1.2), we will show now that $h_n(\xi,\tau) \to h(\tau)$, $h' = h^p$ and $h(0) = \kappa_0$ uniformly on compact sets of $\mathbb{R}^N \times (-\infty, 1)$. Note that

$$h(\tau) = \kappa\left(\left(\frac{\kappa}{\kappa_0}\right)^{p-1} - \tau\right)^{-\frac{1}{p-1}}. \qquad (61)$$

As in Case 1 and up to a subsequence, we have from the definition of h_n that h_n is defined for all $\tau \in [\tau_n, 0]$ where $\tau_n \to -\infty$ (since $\bar{T} - \tilde{t}_n \to 0$ and $M_n(\tilde{t}_n) \to 0$) and satisfies

$$\begin{aligned} &\frac{\partial h_n}{\partial \tau} = \Delta h_n + |h_n|^{p-1}h_n, \\ &h_n(0,0) = \kappa_0 \text{ and } \|h_n(0)\|_{L^\infty} \leq \kappa. \end{aligned} \qquad (62)$$

By the same techniques as in Case 1 and from the integration of the O.D.E. (37) backwards, we have for all $\tau \in [\tau_n, 0]$,
$\|h_n(\tau)\|_{L^\infty} \leq \max\left((4C_2)^{\frac{1}{p}} M_n(\tilde{t}_n)^{\frac{1}{p-1}}, \kappa\right)$. Since $\|h_n(0)\|_{L^\infty} \leq \kappa$, we have from the wellposedness for the Cauchy problem of (62), h_n is well defined for all $\tau \in [0,1)$ and for all $\tau \in [0,1)$, $\|h_n(\tau)\|_{L^\infty} \leq \kappa(1-\tau)^{-\frac{1}{p-1}}$. Thus, by parabolic regularity and a compactness procedure, we can assume that $h_n \to h$ in $C^{2,1}_{loc}(\mathbb{R}^N \times (-\infty, 1))$ where

$$\begin{aligned} &\frac{\partial h}{\partial \tau} = \Delta h + |h|^{p-1}h \\ &h(0,0) = \kappa_0 \text{ and } \forall \tau \in [0,1), \quad \|h(\tau)\|_{L^\infty} \leq \kappa(1-\tau)^{-\frac{1}{p-1}}. \end{aligned}$$

As in Case 1, if we integrate backwards the limit ODE obtained from (37) as $n \to +\infty$, we obtain $\|h(\tau)\|_{L^\infty} \leq C'(1-\tau)^{-\frac{1}{p-1}}$ for all $\tau \in (-\infty, 1)$. From Proposition 1.2, (that is using in some sense the Liouville Theorem in the very singular region), we have $h(\xi,\tau) = h(\tau)$, where h is defined in (61).

Thanks to this result, we are again in subcritical estimates and we can conclude as in Case 1. Define $\tau_n' = \frac{t_n - \tilde{t}_n}{M_n(\tilde{t}_n)}$ and $\theta_n = \frac{T_n - \tilde{t}_n}{M_n(\tilde{t}_n)}$ the blow-up time of h_n. From Lemma 2.5, we have $\theta_n \to 1$. Therefore, we have uniformly with respect to $|\xi| \le 2$,

$$\mathcal{E}_{\xi,\theta_n}(h_n(0)) \to \mathcal{E}_{\xi,1}(h(0)) = \mathcal{E}_{0,1}(\kappa_0) \le \frac{\sigma_0}{2}.$$

Thus, for n large, $\forall |\xi| \le 2$, $\mathcal{E}_{\xi,\theta_n}(h_n(0)) \le \sigma_0$, $|h_n(\xi, \frac{\theta_n}{2})| \le 2h(\frac{3}{4})$, and by Proposition 2.6, $\forall |\xi| \le \frac{1}{2}$, $\forall \tau \in [\frac{\theta_n}{2}, \theta_n)$, $|h_n(\xi,\tau)| \le M^*$. Therefore, $\forall (\xi,\tau) \in B(0,\frac{1}{2}) \times [-1,\theta_n)$, $|h_n(\xi,\tau)| \le M_1^*$.
By Lemma 2.4, this implies that for n large, $\|h_n\|_{C^{2,1}(B(0,\frac{1}{4})\times[0,\theta_n))} \le M^{**}$. Since for all $\xi \in B(0,\frac{1}{4})$, $\Delta h_n(\xi,0) \to \Delta h(\xi,0) = 0$, we obtain by a classical parabolic estimate

$$\sup_{(\xi,\tau)\in B(0,\frac{1}{8})\times[0,\theta_n)} |\Delta h_n(\xi,\tau)| \to 0 \text{ as } n \to +\infty. \tag{63}$$

Since h_n is a solution of (1) and $h_n(0,0) = \kappa_0$, a continuity argument for ODEs shows that

$$\sup_{\tau\in[0,\theta_n)} |h_n(0,\tau) - h(\tau)| \to 0 \text{ as } n \to +\infty. \tag{64}$$

Since $\tau_n' \in [0,\theta_n)$, we have from (60) and (36), $|\Delta h_n(0,\tau_n')|$
$= M_n(\tilde{t}_n)^{\frac{p}{p-1}} |\Delta u_n(x_n,t_n)| \ge \frac{1}{2}|u_n(x_n,t_n)|^p M_n(\tilde{t}_n)^{\frac{p}{p-1}} = \frac{1}{2}|h_n(0,\tau_n')|^p$. Letting $n \to +\infty$, we obtain

$$0 \ge \frac{1}{2}\left(\min_{\tau\in[0,1]} h(\tau)\right)^p \ge \frac{1}{2}\kappa_0^p > 0$$

which is a contradiction. This concludes the proof of Lemma 2.3, Theorem 3 and Proposition 1.7 too. ■

Proof of Corollary 1.8 : By Proposition 1.7 applied to $\epsilon = \frac{1}{2}$, there exists $C_{\frac{1}{2}} > 0$ such that

$$\forall u_0 \in \mathcal{V}_1,\ \ \forall (x,t) \in \mathbb{R}^N \times [0,T),\ \ \frac{\partial u}{\partial t} \ge |u|^{p-1}u - |u|^p - C_{\frac{1}{2}}. \tag{65}$$

We choose $A > 0$ such that

$$\frac{1}{2}A^p - C_{\frac{1}{2}} > 0. \tag{66}$$

From (10) (see also Property (P1)), we have $\tilde{u}(0,t) \to +\infty$ as $t \to \tilde{T}$. Therefore, there exist $\delta > 0$ and $\delta' > 0$ such that for all $x \in \bar{B}(0,\delta)$, $\tilde{u}(x,\tilde{T}-\delta') > 2A$. Then, from continuity arguments applied to equation (1) and the continuity of the blow-up time (Lemma 1.5), there exists $\mathcal{V}_2 \subset \mathcal{V}_1$, a neighborhood of $\tilde{u}_0$ such that

$$\forall u_0 \in \mathcal{V}_1', \;\; \forall x \in \bar{B}(0,\delta), \;\; u(x,T-\delta') > A. \tag{67}$$

Thanks to (66), we can prove from (65) and (67) by a priori estimates that $u(x,t) > A > 0$, for all $u_0 \in \mathcal{V}_2$ and $(x,t) \in \bar{B}(0,\delta) \times [T-\delta', T)$, which concludes the proof of Corollary 1.8. ■

3 Reduction to a finite dimensional problem

In this section we reduce the initial problem to a finite dimensional one by proving Propositions 1.9, 1.10 and 1.11.

Proposition 1.9 is a crucial step in the proof of Proposition 1.10. It asserts that for all u_0 in some neighborhood of $\tilde{u}_0$, there exist $a \in \mathbb{R}^N$ and $T \in \mathbb{R}$ such that

$$v_{a,T} \to 0 \text{ as } s \to +\infty.$$

In Proposition 1.10, we crucially use this fact and the Liouville Theorem to prove that this convergence is uniform with respect to u_0. Let us prove Proposition 1.9.

Proof of Proposition 1.9 :

i) We know from Proposition 1.7 that $u(t)$ blows-up at time T, for all $u_0 \in \mathcal{V}_1$. The following Lemma asserts that $u(t)$ does not blow-up at infinity in space, and allows us to conclude.

Lemma 3.1 (No blow-up at infinity) *There exists $M_1 > 0$ and $\mathcal{V}_3^*$ a neighborhood of $\hat{u}_0$ such that for all $u_0 \in \mathcal{V}_3^*$,*

$$\forall |x| \geq \delta, \;\; \forall t \in [0,T), \;\; |u(x,t)| \leq M_1$$

where δ is introduced in Corollary 1.8.

indeed, let $\mathcal{V}_3 = \mathcal{V}_3^* \cap \mathcal{V}_1 \cap \mathcal{V}_2$ where $\mathcal{V}_2$ in introduced in Corollary 1.8 and consider $u_0 \in \mathcal{V}_3$. Since $u(t)$ blows-up in finite time, it has a blow-up point $a = a(u_0) \in \bar{B}(0,\delta)$ by the previous Lemma. For this a, (6) holds, namely in $\mathcal{C}^{k,\alpha}$,

$$w_{a,T} \underset{s \to +\infty}{\longrightarrow} \pm\kappa, \tag{68}$$

uniformly on compact sets. Since $|a| \leq \delta$, the case $w_{a,T} \to -\kappa$ is ruled out by Corollary 1.8. It remains for us to prove Lemma 3.1.

Proof of Lemma 3.1 : The Lemma follows from the fact that $\tilde{u}(t)$ does not blow-up at infinity and the ODE comparison. From (8) and the fact that $\tilde{u}(t)$ blows-up only at the point 0, there is $M_2 > 0$ such that

$$\forall |x| \geq \delta \text{ and } t \in [0, \tilde{T}), \quad |\tilde{u}(x,t)| \leq M_2. \tag{69}$$

From Proposition 1.7, we know that for all $u_0 \in \mathcal{V}_1$ and $(x,t) \in \mathbb{R}^N \times [0,T)$,

$$|\partial_t u - |u|^{p-1}u| \leq |u|^p + C(1).$$

By a priori estimates, there is $\eta_0 > 0$ such that for all $v \in C^1([0,\eta_0))$ satisfying

$$\begin{array}{l} |v' - |v|^{p-1}v| \leq |v|^p + C(1), \\ |v(0)| \leq 2M_2, \end{array} \tag{70}$$

we have $\forall \tau \in [0,\eta_0)$, $|v(\tau)| \leq 3M_2$.
From the continuity of the solution to the Cauchy problem of (1) and the continuity of the blow-up time stated in Lemma 1.5, there is a neighborhood $\mathcal{V}_3^* \subset \mathcal{V}_1$ such that for all $u_0 \in \mathcal{V}_3$

$$\forall (x,t) \in \mathbb{R}^N \times [0, T-\eta_0], \quad |u(x,t) - \tilde{u}(x,t)| \leq M_2.$$

Using (69), we obtain

$$\forall |x| \geq \delta, \quad \forall t \in [0, T-\eta_0], \quad |u(x,t)| \leq 2M_2. \tag{71}$$

Therefore, if $|x| \geq \delta$, then $v(\tau) = u(x, T-\eta_0+\tau)$ satisfies (70) and

$$\forall |x| \geq \delta. \ \forall t \in [T-\eta_0, T), \quad |u(x,t)| \leq 3M_2. \tag{72}$$

(71) and (72) finish the proof of Lemma 3.1. ■

ii) is a consequence of Lemma 1.5 and Corollary 1.6. This closes the proof of Proposition 1.9. ■

Now, we prove Proposition 1.10.

Proof of Proposition 1.10 :
i) The proof crucially uses the Liouville Theorem of Proposition 1.3. Let us suppose that we cannot find any neighborhood of $\tilde{u}_0$ such that i) holds.

Then there exist $\eta_0 > 0$, $s_n \to +\infty$ and a subsequence $u_{0n} \to \tilde{u}_0$ as $n \to +\infty$ of functions of $\mathcal{V}_1 \cap \mathcal{V}_3$ such that

$$\forall n \in \mathbb{N}, \quad || \, w_{n,a_n,T_n}(s_n) - \kappa \, ||_{L^2_\rho} > \eta_0, \tag{73}$$

where $a_n = a(u_{0n})$ and $T_n = T(u_{0n})$ are given by Proposition 1.9, and w_{n,a_n,T_n} is defined from u_n by (3). By Proposition 1.9, we know that $w_{n,a_n,T_n} \to \kappa$ as $s \to +\infty$ in $C^{k,\alpha}_{\text{loc}}(\mathbb{R}^N)$. Therefore, $E(w_{n,a_n,T_n}(s)) \to E(\kappa)$ as $s \to +\infty$, where E is defined in (33), and since E is a decreasing function of time, we have

$$E(w_{n,a_n,T_n}(s_n)) \geq E(\kappa). \tag{74}$$

From Proposition 1.9, we have $a_n \to 0$ as $n \to +\infty$. Since $s_n \to +\infty$, Corollary 1.8 implies that for n large,

$$w_{n,a_n,T_n}(0, s_n) = e^{-\frac{s_n}{p-1}} u_n(a_n, T_n - e^{-s_n}) \geq 0. \tag{75}$$

We introduce

$$W_n(y,s) = w_{n,a_n,T_n}(y, s + s_n). \tag{76}$$

Then, W_n satisfies the equation (4) and (73), (74) and (75) yield for n large,

$$E(W_n(0)) \geq E(\kappa), \quad W_n(0,0) \geq 0 \text{ and } \|W_n(.,0) - \kappa\|_{L^2_\rho} > \eta_0. \tag{77}$$

By Proposition 1.7, there exists $C > 0$ such that

$$\forall s \in [-\log T_n - s_n, +\infty), \quad \|W_n(s)\|_{L^\infty} \leq C. \tag{78}$$

By parabolic regularity and a compactness procedure, and since $s_n \to +\infty$, there exists $W(y,s)$ such that up to a subsequence

$$W_n \underset{n \to +\infty}{\longrightarrow} W \text{ in } C^{2,1}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}). \tag{79}$$

Moreover, W satisfies (4), and we have from (77) and (78),

$$\|W\|_{L^\infty} \leq C, \quad E(W(0)) \geq E(\kappa), \quad W(0,0) \geq 0 \text{ and } \|W(0) - \kappa\|_{L^2_\rho} > \eta_0. \tag{80}$$

Therefore, by using Proposition 1.3, we obtain

$$W \equiv \pm\kappa, \quad W \equiv 0 \text{ or } W(y,s) = \pm\theta(s + s_0), \tag{81}$$

for some $s_0 \in \mathbb{R}$. This is in contradiction with (80). Indeed, $W \equiv -\kappa$ contradicts $W(0,0) \geq 0$, $W \equiv \kappa$ contradicts $\|W(0)-\kappa\|_{L^2_\rho} > \eta_0$ and $W \equiv 0$ or $W = \pm\theta(s+s_0)$ contradicts $E(W(0)) \geq E(\kappa)$ (for $E(0) = 0 < E(\kappa)$ and $\forall s \in \mathbb{R}$, $E(\pm\theta(s)) < E(\kappa)$. This concludes the proof of i) of Proposition 1.10. Hence the neighborhood $\mathcal{V}_4 \subset \mathcal{V}_1 \cap \mathcal{V}_3$.

ii) It is well-known that for solutions to (1), L^2_ρ estimates on a time interval $[s, s+1]$ yield L^∞ estimates on compact subsets of $\mathbb{R}^N$ at time $s+1$. Let us check that the uniformity with respect to the initial data is preserved. From (13) and Proposition 1.7, there exists $C > 0$ such that for all $u_0 \in \mathcal{V}_4$ and $(y,s) \in \mathbb{R}^N \times [-\log T, +\infty)$, we have

$$\frac{\partial}{\partial s} \mid v_{a,T}(y,s) \mid^2 \leq (\mathcal{L} + C) \mid v_{a,T}(y,s) \mid^2 .$$

Let us introduce the kernel of the operator $\mathcal{L}$

$$\mathrm{e}^{\sigma\mathcal{L}}(y,x) = \frac{\mathrm{e}^{\sigma}}{\big(4\pi(1-\mathrm{e}^{-\sigma})\big)^{N/2}} \exp\big(-\frac{|y\mathrm{e}^{-\sigma/2}-x|^2}{4(1-\mathrm{e}^{-\sigma})}\big).$$

Therefore, for all $(y,s) \in \mathbb{R}^N \times [-\log T + 1, +\infty)$,
$|v_{a,T}(y,s)|^2 \leq C \int_{\mathbb{R}^N} \mathrm{e}^{\mathcal{L}}(y,x) |v_{a,T}(x,s-1)|^2 dx$
$\leq \|v_{a,T}(s-1)\|^2_{L^2_\rho} \sup_{x\in\mathbb{R}^N} \mathrm{e}^{\mathcal{L}}(y,x)\rho(x)^{-1} \leq C(N)\rho(y)^{-1}\|v_{a,T}(s-1)\|^2_{L^2_\rho}$.
Thus, ii) follows from i) of Proposition 1.10. This closes the proof of Proposition 1.10. ■

Our aim is to describe more precisely this uniform smallness of $v_{a,T}$ for initial data near $\tilde{u}_0$ by considering the components of $v_{a,T}$ on the eigenspaces of $\mathcal{L}$ introduced in (16). Let us note for some $k \in \mathbb{N}$,

$$x(s) = \|\, v_2(s) \,\|_{L^2_\rho} \tag{82}$$

$$y(s) = \|\, v_-(s) \,\|_{L^2_\rho} + \|\, |y|^{k/2} v \,\|_{L^2_\rho} \tag{83}$$

$$z(s) = \|\, v_+(s) \,\|_{L^2_\rho} \tag{84}$$

In [FK92], Filippas and Kohn proved the following

Proposition 3.2 *If u is a solution of (1) blowing-up at time T and the point a, and satisfies (P1) then,*

$$\forall \epsilon > 0,\ \ \exists s_0(\epsilon, u_0),\ \ \forall s \geq s_0(\epsilon),\ \ \epsilon x(s) \geq y(s) + z(s). \tag{85}$$

See Theorem A in [FK92]. Proposition 1.11 consists in proving that this fact holds uniformly with respect to the initial data u_0. By Proposition 1.11, we prove that this domination is uniform with respect to initial data. Therefore, we are led to study v_2 which reduces our initial problem to a finite dimensional problem.

Proof of Proposition 1.11 : We follow the proof of Theorem A in [FK92] with uniformity with respect to initial data. The first step consists in deriving ordinary differential inequalities describing the evolution of x, y and z. Then, in a second step, we prove that z is not the dominating component uniformly and in a third step, comparing x and y we prove that x is the dominating component uniformly with respect to the initial data.

First step : By using Proposition 1.7 and by following the proof of Theorem A in [FK92], we obtain the following claim.

Claim 1 *There exists* $\mathcal{V}_4' \subset \mathcal{V}_4$ *neighborhood of* $\tilde{u}_0$ *such that* $\forall \epsilon > 0, \ \ \exists s_1(\epsilon) \in \mathbb{R}, \ \ \forall s \geq s_1(\epsilon), \ \ \forall u_0 \in \mathcal{V}_4$,

$$\dot{z}(s) \geq \frac{1}{2}z(s) - \epsilon\big(x(s) + y(s) + z(s)\big), \tag{86}$$

$$|\ \dot{x}(s)\ | \leq \epsilon\big(x(s) + y(s) + z(s)\big), \tag{87}$$

$$\dot{y}(s) \leq -\frac{1}{2}y(s) + \epsilon\big(x(s) + y(s) + z(s)\big). \tag{88}$$

Remark : k introduced in (83) is fixed and depends only on C_0 given by Proposition 1.7 such that for all $u_0 \in \mathcal{V}_1$ and $t \in [0,T)$, $\|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} \leq C_0$.

Second Step : Uniform smallness of z.

Claim 2 : $\forall \epsilon > 0, \ \ \exists s_2(\epsilon) \in \mathbb{R}, \ \ \forall u_0 \in \mathcal{V}_4'$,

$$\forall s \geq s_2(\epsilon), \ \ \epsilon\big(x(s) + y(s)\big) \geq z(s). \tag{89}$$

Let us prove this claim by contradiction. We suppose that there exists $\epsilon_0 \in (0,1)$ such that for all $s_0^* > 0$, there exists some initial data u_0^* in $\mathcal{V}_4'$ and some $s^* > s_0^*$ such that

$$z(s^*) - \epsilon_0\big(x(s^*) + y(s^*)\big) > 0. \tag{90}$$

Take $\epsilon = \frac{\epsilon_0}{20}$ and $s_0^* = s_1(\epsilon)$ defined in Claim 1. Consider $u_0^* \in \mathcal{V}_4'$ and $s^* > s_0^*$ such that

$$\alpha(s^*) > 0, \tag{91}$$

where for all $s > s_0^*$, $\alpha(s) = z(s) - \epsilon_0\big(x(s)+y(s)\big)$. From Claim 1, we obtain for all $s > s_0^*$

$$\begin{aligned}\alpha'(s) &= z'(s) - \epsilon_0\big(x'(s)+y'(s)\big)\\ &\geq \frac{1}{2}z(s) - \epsilon\big(x(s)+y(s)+z(s)\big) - \epsilon_0\epsilon\big(x(s)+y(s)+z(s)\big)\\ &\quad +\frac{\epsilon_0}{2}y - \epsilon_0\epsilon\big(x(s)+y(s)+z(s)\big)\\ &\geq z(s)(\frac{1}{2} - \epsilon - 2\epsilon_0\epsilon) + y(s)(\frac{\epsilon_0}{2} - \epsilon - 2\epsilon\epsilon_0) - x(s)\epsilon(1+2\epsilon_0).\end{aligned}$$

Therefore, for s such that $\alpha(s) > 0$, we have

$$\alpha'(s) \geq z(s)\Big(\frac{1}{2} - \frac{\epsilon}{\epsilon_0}(1+2\epsilon_0^2+3\epsilon_0)\Big) + \frac{\epsilon_0}{2}y(s). \tag{92}$$

Since $1+2\epsilon_0^2+3\epsilon_0 > 1$ and $\frac{\epsilon_0}{\epsilon} = 20 > 3(1+2\epsilon_0^2+3\epsilon_0)$, we obtain

$$\frac{1}{2} - \frac{\epsilon}{\epsilon_0}(1+2\epsilon_0^2+3\epsilon_0) > 0. \tag{93}$$

Note that $y(s) = z(s) = 0$ contradicts $\alpha(s) > 0$. Therefore, if $\alpha(s) > 0$, then either $y(s) > 0$ or $z(s) > 0$, and (92) and (93) yield $\alpha'(s) > 0$. As a conclusion, we obtain

$$\forall s > s_0^*, \;\; \alpha(s) > 0 \Longrightarrow \alpha'(s) > 0. \tag{94}$$

By (91), we obtain for all $s > s^*$, $\alpha(s) > \alpha(s^*) > 0$, which contradicts $\alpha(s) \underset{s\to+\infty}{\longrightarrow} 0$ (this follows from Proposition 1.10) and closes the proof of the claim of the second step.

Step 3: Uniform predominance of x. Here, two arguments are mixed-up: the uniform stability of the dynamics where x is predominant and the fact that for initial data $\tilde{u}_0$, $\tilde{x}$ is predominant. The key Lemma of the proof is the following :

Lemma 3.3 (Uniform stability of the dynamic where x **is predominant)** *For all $C^* > 0$, there exists s^* such that for all initial data in $\mathcal{V}_4'$ and $s_0 \geq s^*$,*

$$\text{if } x(s_0) \geq C^* y(s_0), \text{ then } \forall s \geq s_0, \;\; x(s) \geq \frac{C^*}{2}y(s).$$

Proof of Lemma 3.3 : Considering Claim 1 and Claim 2 we have
$\forall \epsilon \in (0, \frac{1}{2}), \ \ \exists s_3(\epsilon) = \max\big(s_1(\epsilon), s_2(\epsilon)\big), \ \ \forall u_0 \in \mathcal{V}_4', \ \ \forall s \geq s_3(\epsilon)$

$$\mid x'(s) \mid \leq \frac{3}{2}\epsilon\big(x(s) + y(s)\big) \text{ and } y'(s) \leq -\frac{1}{2}y(s) + \frac{3}{2}\epsilon\big(x(s) + y(s)\big). \quad (95)$$

We argue by contradiction. Suppose that there exists $C > 0$, $s_0 > s_3(\epsilon)$ where $\epsilon = \frac{C}{3(2+C)^2}$ and u_0 in $\mathcal{V}_4'$ such that

$$x(s_0) \geq Cy(s_0) \ \text{ and } \ \exists s_0^* > s_0, \ \ x(s_0^*) < \frac{C}{2}y(s_0^*).$$

Consider $\gamma(s) = x(s) - \frac{C}{2}y(s)$, then $\gamma(s_0) \geq 0$ and $\gamma(s_0^*) < 0$. Therefore, there exists $s_2 \in [s_0, s_0^*)$ such that

$$\gamma(s_2) = 0 \ \text{ and } \ \gamma(s) < 0 \text{ for all } s \in [s_2, s_0^*[. \quad (96)$$

But we have

$$\gamma'(s) = x'(s) - \frac{C}{2}y'(s) \geq \frac{C}{4}y(s) - \frac{3}{2}\epsilon(1 + \frac{C}{2})\big(x(s) + y(s)\big). \quad (97)$$

Then, (96) and (97) yield

$$\gamma'(s_2) \geq \big(\frac{C}{4} - \frac{3}{2}\epsilon(1 + \frac{C}{2})^2\big)y(s_2). \quad (98)$$

Therefore, since $\epsilon = \frac{C}{3(2+C)^2}$ we have

$$\frac{C}{4} - \frac{3}{2}\epsilon(1 + \frac{C}{2})^2 > 0. \quad (99)$$

Moreover, if $y(s_2) = 0$ then $x(s_2) = 0$ by (96) and $z(s_2) = 0$ by (89). Therefore, $v_{a,T}(s_2) \equiv 0$ and the uniqueness of the solution to the Cauchy problem of (13) yields $v_{a,T}(s) \equiv 0$ for all $s \geq s_2$. Hence, a contradiction with $\gamma(s_0^*) < 0$. Therefore, $y(s_2) > 0$ and (98) and (99) yield $\gamma'(s_2) > 0$ which contradicts (96) and closes the proof of Lemma 3.3. ∎

We apply Lemma 3.3 to $C^* = 2$, hence some s^* such that

$$\forall u_0 \in \mathcal{V}_4', \ \textit{if } \exists s_0 \geq s^*, \ x(s_0) \geq 2y(s_0) \textit{ then } \forall s \geq s_0, \ \ x(s) \geq y(s). \quad (100)$$

Moreover, since $\tilde{u}(t)$ satisfies (P1), by Proposition 3.2 applied to $\epsilon = \frac{1}{3}$ there exists $s_0(\frac{1}{3}, \tilde{u}_0)$ such that

$$\forall s \geq s_0(\frac{1}{3}, \tilde{u}_0), \ \ \tilde{x}(s) \geq 3\big(\tilde{y}(s) + \tilde{z}(s)\big).$$

Let us note $\tilde{s}_0 = \max\big(s_0(\frac{1}{3}, \tilde{u}_0), s^*\big)$. By using continuity arguments applied to equation (1) we obtain the existence of a neighborhood $\mathcal{V}_4''$ of $\tilde{u}_0$ such that

$$\forall u_0 \in \mathcal{V}_4'', \;\; x(\tilde{s}_0) \geq 2\big(y(\tilde{s}_0) + z(\tilde{s}_0)\big) \geq 2y(\tilde{s}_0).$$

By (100) we obtain for all $u_0 \in \mathcal{V}_5 = \mathcal{V}_4'' \cap \mathcal{V}_4'$,

$$\forall s \geq \tilde{s}_0, \;\; x(s) \geq y(s). \tag{101}$$

Let $\epsilon > 0$ and $u_0 \in \mathcal{V}_5$. If necessary we shall restrict ϵ to small ones in the following. The first step, the second step and the work performed immediately above yield that if $s_4(\epsilon) = \sup\big(\tilde{s}_0, s_1(\epsilon), s_2(\epsilon)\big)$, then

$$\forall s \geq s_4(\epsilon), \;\; (86), \; (87), \; (88), \; (89), \; (95) \; \text{ and } \; (101) \text{ hold.}$$

It is well-known that if x, y and z satisfy (86), (87) and (88) then there exists $s_5'(\epsilon, u_0)$ such that for all $s \geq s_5'(\epsilon, u_0)$

$$\text{either } \; y(s) + z(s) \leq 4\epsilon x(s) \; \text{ or } \; x(s) + z(s) \leq 4\epsilon y(s), \tag{102}$$

(see Appendix A in [MZ98a]). In view of (101), x is necessarily the dominating component and

$$\forall s \geq s_5'(\epsilon, u_0), \;\; y(s) + z(s) \leq 4\epsilon x(s).$$

It remains for us to prove in some sense that $s_5'(\epsilon, u_0) - s_4(\epsilon)$ is bounded only in terms of ϵ, independently from u_0. For this, let us introduce for all $\epsilon > 0$ and $u_0 \in \mathcal{V}_5$,

$$s_5(\epsilon, u_0) = \inf\{s \geq s_4(\epsilon), \;\; \forall \sigma \geq s, \;\; y(\sigma) \leq 4\epsilon x(\sigma)\}. \tag{103}$$

Then, we claim the following : *There exists $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$ and if $s_4(\epsilon) < s_5(\epsilon, u_0)$ then*

$$\forall s \in [s_4(\epsilon), s_5(\epsilon, u_0)], \;\; y(s) \geq 4\epsilon x(s). \tag{104}$$

Indeed, if $s_4(\epsilon) < s_5(\epsilon, u_0)$, then there exists $s_n \underset{n \to +\infty}{\longrightarrow} s_5(\epsilon, u_0)$ with $s_n \in [s_4(\epsilon), s_5(\epsilon, u_0)]$ and $\beta(s_n) > 0$ where $\beta(s) = y(s) - 4\epsilon x(s)$. We argue by contradiction as in the proof of Lemma 3.3. If (104) does not hold, then we can construct $\sigma^* \in [s_4(\epsilon), s_5(\epsilon, u_0))$ such that

$$\beta(\sigma^*) = 0 \text{ and } \forall \sigma \in (\sigma^*, s_n], \;\; \beta(\sigma) > 0 \text{ for some } s_n. \tag{105}$$

But (95) yields $\beta'(\sigma^*) \leq (-\frac{1}{8} + 3\epsilon(1+2\epsilon))y(\sigma^*)$. Note that

$$\forall \epsilon < \epsilon_0, \quad -\frac{1}{8} + 3\epsilon(1+2\epsilon) < 0 \tag{106}$$

where ϵ_0 is a constant. As at the end of the proof of Lemma 3.3, we can not have $y(\sigma^*) = 0$, because otherwise $v_{a,T}(s) \equiv 0$ for all $s \geq \sigma^*$, and this gives $s_5(\epsilon, u_0) \leq \sigma^*$ which is a contradiction. Therefore, $y(\sigma^*) > 0$ and (106) implies that $\beta'(\sigma^*) < 0$, which contradicts (105) and then (104) holds. Note that (103) and (104) yield

$$\beta\big(s_5(\epsilon, u_0)\big) = 0. \tag{107}$$

Moreover, (89), (101) and (104) yield

$$\forall s \in [s_4(\epsilon), s_5(\epsilon, u_0)], \quad 4\epsilon x(s) \leq y(s) \leq x(s), \quad z(s) \leq \epsilon\big(x(s) + y(s)\big).$$

Therefore (87) and (88) become

$$y'(s) \leq (-\frac{1}{4} + \frac{5}{4}\epsilon + \epsilon^2)y(s) \text{ and } x'(s) \geq -3\epsilon(\epsilon+1)x(s).$$

Hence,

$$\begin{array}{rcl} y(s_5) & \leq & y(s_4)\exp\big((-\frac{1}{4} + \frac{5}{4}\epsilon + \epsilon^2)(s_5 - s_4)\big) \\ x(s_5) & \geq & x(s_4)\exp\big(-3\epsilon(\epsilon+1)(s_5 - s_4)\big). \end{array} \tag{108}$$

Again here, we can have neither $x(s_4) = 0$ nor $x(s_5) = 0$, otherwise in both cases $x(s_4) = y(s_4) = z(s_4) = 0$ and $v_{a,T}(s) \equiv 0$ for all $s \geq s_4$ and this implies $s_5(\epsilon, u_0) = s_4(\epsilon)$. Therefore, using (107) and (108), we obtain

$$\begin{array}{rcl} 4\epsilon & = & \dfrac{y(s_5)}{x(s_5)} \leq \dfrac{y(s_4)}{x(s_4)}\exp\big((-\dfrac{1}{4} + \dfrac{17}{4}\epsilon + 4\epsilon^2)(s_5 - s_4)\big) \\ & \leq & \exp\big((-\dfrac{1}{4} + \dfrac{17}{4}\epsilon + 4\epsilon^2)(s_5 - s_4)\big). \end{array}$$

Therefore, $\log(4\epsilon) \leq (-\frac{1}{4} + \frac{17}{4}\epsilon + 4\epsilon^2)(s_5 - s_4)$. Then, for some $\epsilon_0' > 0$ and for all $\epsilon < \epsilon_0'$, we have $1 - 17\epsilon - 16\epsilon^2 > 0$ and

$$s_5(\epsilon, u_0) - s_4(\epsilon) \leq \frac{4 \mid \log \epsilon \mid}{1 - 17\epsilon - 16\epsilon^2}.$$

As a conclusion, for $\epsilon < \epsilon_0'$ we have

$$\forall u_0 \in \mathcal{V}_5, \quad \forall s \geq s_4(\epsilon) + \frac{4 \mid \log \epsilon \mid}{1 - 17\epsilon - 16\epsilon^2}, \quad y(s) + z(s) \leq (4\epsilon + \epsilon(1+4\epsilon))\, x(s).$$

This closes the proof of Proposition 1.11. ■

4 Stability properties for the finite dimensional dynamical system and for the equation (1) at the profile (2)

At this level of the proof, we know that for all $u_0 \in \mathcal{V}_5$, neighborhood of $\tilde{u}_0$, the control of $v_{a,T}$ near zero as s goes to infinity reduces to the control of its component $v_2(y,s)$. Our aim in this section if to prove Proposition 1.14 which directly implies Theorem 1. We proceed in 2 Parts :

- In Part I, we write $v_2(y,s) = \frac{1}{2}y^T A(s)y - \mathrm{tr}A(s)$ as in (18) where $A(s)$ is a $N \times N$ matrix. From (20), we have $\tilde{A}(s) \sim -\frac{\beta}{s}Id$ as $s \to +\infty$. In this Part, we show that this behavior is stable with respect to u_0.

- In Part II, we use the stability of the behavior of $A(s)$ and the uniform estimates of Proposition 1.7 to conclude the proof of Proposition 1.14.

In the following v stands for $v_{a,T}$.

Part I : Stability for the finite dimensional problem

We show in this Part that the behavior of $A(s)$ at infinity is stable with respect to u_0. For this, we first show that $A(s)$ satisfies the following ODE

$$A'(s) = \frac{1}{\beta}A(s)^2 + R(s) \tag{109}$$

where $\beta = \frac{\kappa}{2p}$ and $R(s) = o\left(|A(s)|^2\right)$ as $s \to +\infty$, uniformly for u_0 near $\tilde{u}_0$ (this will prove Proposition 1.12). Then, we use the finite dimensional system (109) to show that the behavior $\tilde{A}(s) \sim -\frac{\beta}{s}Id$ as $s \to +\infty$ is stable with respect to u_0 (this will prove Proposition 1.13).

Step 1 : An ODE satisfied by $A(s)$

We prove Proposition 1.12 in this Step. Proposition 1.12 gives the same result as Theorem B in [FK92], except that there is no uniformity with respect to u_0 in the statement of Theorem B in [FK92]. However, the proof of [FK92] actually holds uniformly with respect to u_0. In the following, we recall briefly the main steps of the proof of [FK92] and expand only the parts where uniformity is not obvious. Let us consider $u_0 \in \mathcal{V}_1$ defined in Proposition 1.7. From (13), (17) and (18), we have for all $i, j \in \{1, ..., N\}$,

$$\forall s \geq -\log T, \;\; A'_{ij}(s) = \int f(v(y,s))\left(\frac{1}{4}y_i y_j - \frac{1}{2}\delta_{ij}\right)\rho(y)dy. \tag{110}$$

From the uniform estimate of Proposition 1.7, we have for all $(y,s) \in \mathbb{R}^N \times [-\log T, +\infty)$, $|v(y,s)| \leq C$. Therefore, we expand $f(v)$ defined in (14) as follows

$$\left| f(v) - \frac{1}{4\beta} v^2 \right| \leq C|v|^3$$

and use (16) to write

$$A'_{ij}(s) = I + II + III \tag{111}$$

where $I = \frac{1}{4\beta} \int v_2(y,s)^2 \left(\frac{1}{4} y_i y_j - \frac{1}{2}\delta_{ij}\right) \rho(y) dy$,
$II = \frac{1}{4\beta} \int \left(v^2 - v_2^2\right) \left(\frac{1}{4} y_i y_j - \frac{1}{2}\delta_{ij}\right) \rho(y) dy$ and
$|III| \leq C \int |v(y,s)|^3 (1+|y|^2) \rho(y) dy$.
From (18) and straightforward calculations, it is easy to see that

$$I = \frac{1}{\beta} \left(A^2\right)_{ij}(s). \tag{112}$$

The control of II and III is possible thanks to the following Lemma (which is the uniform version of Lemma 5.1 in [FK92]).

Lemma 4.1 *There exist a neighborhood $\mathcal{V}_6^*$ of $\tilde{u}_0$, $\delta_0 > 0$ and an integer $k > 4$ with the following property : for all $\delta \in (0, \delta_0)$, there exists a time $s^*(\delta)$ such that for all $u_0 \in \mathcal{V}_6^*$ and $s \geq s^*$,*

$$\int_{\mathbb{R}^N} v(y,s)^2 |y|^k \rho(y) dy \leq c_0(k) \delta^{2-\frac{k}{2}} \int_{\mathbb{R}^N} v_2(y,s)^2 \rho(y) dy.$$

Proof : See Appendix A. ■

From Propositions 1.7, 1.10 and 1.11, we claim that the following holds uniformly with respect to $u_0 \in \mathcal{V}_1 \cap \mathcal{V}_4 \cap \mathcal{V}_5 \cap \mathcal{V}_6^*$:

- For all $(y,s) \in \mathbb{R}^N \times [-\log T, +\infty)$, $|v_{a,T}(y,s)| \leq C$,
- $\forall R > 0$, $\sup_{|y|<R} |v_{a,T}(y,s)| \to 0$ as $s \to +\infty$,
- v_2 dominates v_- and v_+ in L^2_ρ as $s \to +\infty$.

With these facts and Lemma 4.1, one can check straightforwardly that the proof of Theorem B in [FK92] (page 850) gives $|II|+|III| = o\left(\|v_2(y,s)\|^2_{L^2_\rho}\right)$ $= o\left(|A(s)|^2\right)$ as $s \to +\infty$ uniformly for $u_0 \in V_1 \cap \mathcal{V}_4 \cap \mathcal{V}_5 \cap \mathcal{V}_6^*$. Combined with (110), (111) and (112), this yields Proposition 1.12. ■

Step 2 : Stability of the $-\frac{\beta}{s} Id$ behavior for the ODE (21)

We prove Proposition 1.13 in this Step. Proposition 1.13 follows from the following :

Lemma 4.2 *There exists s_2^* such that for any initial data u_0 in $\mathcal{V}_6$,*

$$if\ \ for\ \ some\ \ s_0 \geq s_2^*,\ \ -2\frac{\beta}{s_0}Id \leq A(s_0) \leq -\frac{\beta}{2s_0}Id,$$

$$then\ \ \forall s \geq s_0,\ \ -\frac{3\beta}{s}Id \leq A(s) \leq -\frac{\beta}{3s}Id.$$

Remark : If A and B stand for two $N \times N$ symmetric matrices, then the notation $A \leq B$ means that the matrix $B - A$ is positive.
Indeed, from (20), there exists $s_0 \geq s_2^*$ such that

$$-\frac{3\beta}{2s_0}Id \leq \tilde{A}(s_0) \leq -\frac{2\beta}{3s_0}Id.$$

From continuity arguments applied to equation (1), there exists a neighborhood $\mathcal{V}_7 \subset \mathcal{V}_6$ of $\tilde{u}_0$ such that for all $u_0 \in \mathcal{V}_7$,

$$-\frac{2\beta}{s_0}Id \leq A(s_0) \leq -\frac{\beta}{2s_0}Id.$$

Hence, by Lemma 4.2, we have

$$\forall s \geq s_0,\ \ -\frac{3\beta}{s}Id \leq A(s) \leq -\frac{\beta}{3s}Id. \tag{113}$$

Since $A(s)$ is a C^1 symmetric matrix, it is more convenient to work with its eigenvalues. We have the following result :

Lemma 4.3 *Suppose that $A(s)$ is a $N \times N$ symmetric and continuously differentiable matrix-function in some interval I. Then, there exist continuously differentiable functions $\lambda_1(s)$,...,$\lambda_N(s)$ in I such that for all $j \in \{1, ..., N\}$,*

$$A(s)\phi^{(j)}(s) = \lambda_j(s)\phi^{(j)}(s)$$

for some (properly chosen) orthonormal system of vector functions $\phi^{(1)}(s)$,...,$\phi^{(N)}(s)$.

Proof : See (for instance) Kato [Kat95]. ■

Since $\sum_{i=1}^{N} |\lambda_i|$ is an equivalent norm for $A(s)$, we see from this Lemma and Proposition 1.12 that $\forall \epsilon > 0$,

$$\forall i \in \{1, ..., N\},\ \ \forall s \geq s_1(\epsilon),\ \ \left|\lambda_i'(s) - \frac{1}{\beta}\lambda_i^2(s)\right| \leq \epsilon C(N) \left(\sum_{j=1}^{N} |\lambda_j(s)|\right)^2 \tag{114}$$

for some $C(N) > 0$. Hence, from (113), we see that

$$\forall s \geq s_0, \ \ \lambda_i'(s) = \frac{1}{\beta}\lambda_i^2(s)(1+\gamma_i(s))$$

where $\forall s \geq \max(s_0, s_1(\epsilon))$, $|\gamma_i(s)| \leq C\epsilon$. Therefore, one can straightforwardly solve this differential equation and prove the existence of some $s_2(\epsilon)$ such that for all $s \geq s_2(\epsilon)$ and $u_0 \in \mathcal{V}_7$, $|\lambda_i(s) + \frac{\beta}{s}| \leq C\frac{\epsilon}{s}$, which concludes the proof of Proposition 1.13. It remains for us to prove Lemma 4.2.

Proof of Lemma 4.2 : We argue by contradiction. Let us consider $\hat{\epsilon} > 0$ to be fixed in terms of N and p later, $s_0 \geq s_1(\hat{\epsilon})$ and $u_0 \in \mathcal{V}_6$ (where $\mathcal{V}_6$ and $s_1(\hat{\epsilon})$ are introduced in Proposition 1.12) such that

$$-\frac{2\beta}{s_0} Id \leq A(s_0) \leq -\frac{\beta}{2s_0} Id$$

and for some $s_* > s_0$ and $i_0 \in \{1, ..., N\}$,

$$\forall s \in [s_0, s_*), \ \ \forall i = 1, ..., N, \ \ -\frac{3\beta}{s} \leq \lambda_i(s) \leq -\frac{\beta}{3s} \tag{115}$$

and $\lambda_{i_0}(s_*) = -\frac{3\beta}{s_*}$ or $\lambda_{i_0}(s_*) = -\frac{\beta}{3s_*}$. Let us treat for example the case

$$\lambda_{i_0}(s_*) = -\frac{3\beta}{s_*} \tag{116}$$

(the other case being quite similar).
On one hand, we have from (115) and (116),

$$\lambda_{i_0}'(s_*) \leq \left(-\frac{3\beta}{s}\right)'_{|s=s_*} = \frac{3\beta}{s_*^2}. \tag{117}$$

On the other hand, (114) and (115) yield

$$\begin{aligned} \lambda_{i_0}'(s_*) &\geq \frac{1}{\beta}\lambda_{i_0}^2(s_*) - \hat{\epsilon}C(N)\left(\sum_{i=1}^{N}|\lambda_i(s_*)|\right)^2 \\ &\geq \frac{9\beta}{s_*^2} - \hat{\epsilon}C(N)N^2\frac{9\beta^2}{s_*^2} \geq \frac{6\beta}{s_*^2} \end{aligned}$$

if $\hat{\epsilon}$ is fixed lower than $\left(3\beta N^2 C(N)\right)^{-1}$, which contradicts (117). Taking $s_2^* = s_1(\hat{\epsilon})$, we conclude the proof of Lemma 4.2. ■

Part II : Stability of the behavior (2) with respect to initial data

In this Part, we prove Proposition 1.14 which implies directly the stability result of Theorem 1.

Proof of Proposition 1.14 :

i) Let us first introduce $\mathcal{V}_8' = \mathcal{V}_1 \cap \mathcal{V}_5 \cap V_7$. From Propositions 1.11 and 1.13, (12), (16) and (18), we have :

For all $\epsilon > 0$, $s \geq \max(s_0(\epsilon), s_2(\epsilon))$ *and* $u_0 \in \mathcal{V}_8'$,

$$\left\| w_{a,T}(y,s) - \left\{ \kappa + \frac{\kappa}{2ps}\left(N - \frac{|y|^2}{2}\right)\right\}\right\|_{L^2_\rho} \leq \frac{\epsilon}{s},$$

which yields i) of Proposition 1.14.

ii) We claim the following :

Proposition 4.4 (Uniform L^∞ estimates) *There exist positive constants C_1, C_2 and C_3 such that for all $\epsilon > 0$, there exists $s_0(\epsilon) \in \mathbb{R}$ such that for all $s \geq s_0(\epsilon)$, $u_0 \in \mathcal{V}_1$ and $i \in \{1,2,3\}$,*

$$\|w_{a,T}(s)\|_{L^\infty} \leq \kappa + \left(\frac{N\kappa}{2p} + \epsilon\right)/s, \quad \|\nabla^i w_{a,T}(s)\|_{L^\infty} \leq C_i s^{-i/2}.$$

Remark : The notation $\nabla^i w_{a,T}$ stands for the differential of order i of $w_{a,T}$.
Proof : From Proposition 1.7, we have for all $u_0 \in \mathcal{V}_1$,

$$\forall t \in [0,T), \quad \|u(t)\|_{L^\infty} \leq C_0(T-t)^{-\frac{1}{p-1}}$$

for some $C_0 > 0$. With this uniform estimate, Theorem 4 of [MZ] applies and gives estimates independent of u_0. ∎

Remark : Theorem 4 of [MZ] follows from the Liouville Theorem and the techniques of [MZ98b].

The previous Proposition provides us with a uniform estimate on $\Delta w_{a,T}$ which allows us to consider (4) as a perturbation of a hyperbolic equation. Using the characteristic method, we propagate the estimate of i) of Proposition 4.4 (which gives informations on compact sets of $\mathbb{R}^N$) to sets of the type $|y| \leq K_0\sqrt{s}$. We claim the following :

Proposition 4.5 (Convergence extension to space-time parabolas) *For all $K_0 > 0$ and $\epsilon > 0$, there exists $s_4(K_0, \epsilon)$ such that for all $s \geq s_4$ and $u_0 \in \mathcal{V}_8'$, $\sup_{|z|\leq K_0} \left|w_{a,T}(z\sqrt{s}, s) - f(z)\right| \leq \epsilon$ where f is defined in (7).*

Proof : The proof of Proposition 3.1 in [MZ98b] holds here and there is no problem with uniformity with respect to initial data, because i) of Proposition 1.14 holds uniformly with respect to u_0 (in L^2_ρ and in $L^\infty(B(0,R))$ for all $R > 0$ thanks to classical parabolic estimates) and Proposition 4.4 provides us with an L^∞ estimate on the Laplacian, uniform with respect to u_0. ■

This yields ii) of Proposition 1.14.

iii) Let us fix some $K_0 > 0$ and introduce for each $u_0 \in \mathcal{V}'_8$ and $x \in B(a, K_0 e^{-1/2})$, a time $t(x, u_0) \in [0, T)$ defined by

$$|x - a| = K_0 \sqrt{(T - t(u_0, x))|\log(T - t(u_0, x))|}. \tag{118}$$

We shall write simply $t(x)$ in the following.

We now introduce $v(x, \xi, \tau)$ defined for all $(\xi, \tau) \in \mathbb{R}^N \times [-\frac{t(x)}{T-t(x)}, 1)$ by

$$v(x, \xi, \tau) = (T - t(x))^{\frac{1}{p-1}} u(x + \xi\sqrt{T - t(x)}, t(x) + \tau(T - t(x))). \tag{119}$$

From ii) in Proposition 1.14 and Proposition 1.7, we have (through the transformations (119) and (3)) for all $\epsilon > 0$, $u_0 \in \mathcal{V}'_8$, $x \in B(a, \eta(K_0))$, $\xi \in B(0, 2)$ and $\tau \in [-\frac{1}{2}, 1)$,

$$\begin{cases} \left|\left(\partial_\tau v - |v|^{p-1}v\right)(x, \xi, \tau)\right| \le \epsilon|v(x, \xi, \tau)|^p + C_\epsilon(T - t(x))^{\frac{p}{p-1}}, \\ |v(x, \xi, 0) - f(K_0 + \xi\,|\log(T - t(x))|^{-1/2})| \le \epsilon \end{cases} \tag{120}$$

for some $\eta(K_0) > 0$. Let us introduce for all $\tau \in (-\infty, 1]$,

$$v_*(\tau) = \left[\frac{(p-1)^2}{4p} K_0^2 + (p-1)(1-\tau)\right]^{-\frac{1}{p-1}} \tag{121}$$

which is the solution of

$$\begin{cases} v_*'(\tau) = v_*(\tau)^p, \\ v_*(0) = f(K_0). \end{cases} \tag{122}$$

We claim that

$$\sup_{u_0 \in \mathcal{V}'_8} \sup_{\tau \in [0,1)} |v(a + x', 0, \tau) - v_*(\tau)| \to 0 \text{ as } x' \to 0. \tag{123}$$

Indeed, if not, then there exists $\epsilon_0 > 0$ and sequences $x'_n \to 0$, $u_{0n} \in \mathcal{V}'_8$ and $\tau_n \in [0, 1)$ such that

$$|v(a_n + x'_n, 0, \tau_n) - v_*(\tau_n)| \ge \epsilon_0, \tag{124}$$

where a_n is the blow-up point associated with u_{0n}. Let us denote $v(a_n + x'_n,.,.)$ by v_n.
Since v_* is bounded on $(-\infty, 1]$, it follows from a comparison principle and (120) that for n large, we have

$$\forall u_0 \in \mathcal{V}'_8,\ \ \forall (\xi,\tau) \in B(0,2) \times [-\frac{1}{2}, 1),\ \ |v_n(\xi,\tau)| \le C_0(K_0).$$

From (119), we see that v_n solves (1). Therefore, applying Lemma 2.4, we find

$$\|v_n\|_{L^\infty(D)} + \|\partial_\tau v_n\|_{L^\infty(D)} + |\partial_\tau v_n|_{\alpha,D} \le C'_0$$

where $D = B(0,1) \times [0,1)$, $\alpha \in (0,1)$ and $|.|_{\alpha,D}$ is defined in (41). Using a compactness argument, we find $\hat{v} \in C^1([0,1),\mathbb{R})$ and a subsequence (still denoted by v_n) such that $v_n(0,.) \to \hat{v}$ in $C^1_{\rm loc}([0,1),\mathbb{R})$. Since $\|\partial_\tau v_n\|_{L^\infty(D)} \le C'_0$, we actually have

$$v_n(0,.) \to \hat{v} \text{ in } C^1([0,1),\mathbb{R}). \tag{125}$$

Using (120) with $v_n(0,\tau) = v(x_n, 0, \tau)$ and letting $n \to +\infty$ and then $\epsilon \to 0$, we obtain $\hat{v} \equiv v_*$ on $[0,1)$, in view of (122). Together with (125), this contradicts (124). Thus, (123) holds.
Let us remark that from (118), we see that for all $x' \neq 0$, $t(a + x')$ depends only on x' and is equal to $t_0(x')$ defined by

$$|x'| = K_0 \sqrt{(T - t_0(x'))|\log(T - t_0(x'))|}. \tag{126}$$

Using (119), we rewrite (123) as the following :

$$\sup_{u_0 \in \mathcal{V}'_8} \sup_{t \in [t_0(x'),T)} \left| (T - t_0(x'))^{\frac{1}{p-1}} u(a + x', t) - v_* \left(\frac{t - t_0(x')}{T - t_0(x')} \right) \right| \underset{x' \to 0}{\longrightarrow} 0. \tag{127}$$

Since v_* is continuous and $v_*(1) > 0$, this implies the following :

Lemma 4.6 *There exists $\delta_0 > 0$ such that for all $x' \in B(0,\delta_0)\backslash\{0\}$, for all $u_0 \in \mathcal{V}'_8$, $a + x'$ is not a blow-up point of $u(t)$, and $u(a + x', t) \to u^*(a + x')$ as $t \to T$ where u^* is the limiting profile of $u(t)$.*

Proof : From (127) and (126), there exists $\delta_0 > 0$ such that for all $x' \in B(0,\delta_0)\backslash\{0\}$, for all $u_0 \in \mathcal{V}'_8$, $|u(a + x', t)|$ can not go to $+\infty$ as $t \to T$. Therefore, by [GK89], $a + x'$ is not a blow-up point, and by [Mer92] (for instance), $u(a + x', t) \to u^*(a + x')$ as $t \to T$. ∎

If for all $u_0 \in \mathcal{V}_8'$, we let $t \to T$ in (127), we find

$$\sup_{u_0 \in \mathcal{V}_8'} \left| (T - t_0(x'))^{\frac{1}{p-1}} u^*(a + x') - v_*(1) \right| \to 0 \text{ as } x' \to 0, \;\; x' \neq 0$$

on one hand. On the other hand, one can see from (126), (121) and (2) that

$$(T - t_0(x'))^{\frac{1}{p-1}} U_1(x') \to v_*(1) \text{ as } x' \to 0, \;\; x' \neq 0.$$

Thus,

$$\sup_{u_0 \in \mathcal{V}_8'} \left| \frac{u^*(a + x')}{U_1(x')} - 1 \right| \to 0 \text{ as } x' \to 0, \;\; x' \neq 0.$$

This yields iii) of Proposition 1.16.

iv) Since $\tilde{u}(t)$ blows-up only at the origin, Corollary 1.6 implies the existence of a neighborhood of $\tilde{u}_0$, $\mathcal{V}_8 \subset \mathcal{V}_8'$ such that for all $u_0 \in \mathcal{V}_8$, all the blow-up points of $u(t)$ are in $B(0, \frac{\delta_0}{3})$. Now, using Lemma 4.6, we see that there is no blow-up point in $B(a(u_0), \delta_0) \backslash \{a(u_0)\}$, where $a(u_0)$ is the blow-up point associated with u_0. Thus, $a(u_0)$ is the only blow-up point of $u(t)$. Therefore, the limiting profile u^* is defined for all $x \neq a(u_0)$. The continuity of $a(u_0)$ and $T(u_0)$ follows from Lemma 1.5 and Corollary 1.6. This concludes the proof of Proposition 1.14. ■

A Proof of Lemma 4.1

Let us introduce

$$K(s)^2 = \int_{\mathbb{R}^N} v(y,s)^2 |y|^k \rho(y) dy \text{ and } x(s)^2 = \int_{\mathbb{R}^N} v_2(y,s)^2 \rho(y) dy.$$

Arguing exactly as in the proof of Lemma 5.1 in [FK92], we claim the following :

Claim 1 *There exist a neighborhood $\hat{\mathcal{V}}_6$ of $\tilde{u}_0$, $\delta_1 > 0$ and an integer $k > 4$ with the following property : for all $\delta \in (0, \delta_1]$, there exists a time $\hat{s}(\delta)$ such that for all $u_0 \in \hat{\mathcal{V}}_6$,*

i) $\forall s \geq \hat{s}(\delta)$, $\begin{cases} K' \leq -K + d(k,N)\delta^{2-\frac{k}{2}} x \\ |x'| \leq CM\delta^{k/2} I + 2C\delta x \end{cases}$,

ii) If for some $s_0 \geq \hat{s}(\delta)$ we have $K(s_0) - 2d\delta^{2-\frac{k}{2}} x(s_0) < 0$, then $\forall s \geq s_0$, $K(s) - 2d\delta^{2-\frac{k}{2}} x(s) \leq 0$, where $d(k,N) = k(k + N - 2)$.

Let us fix $k > 4$ as given by Claim 1. The conclusion follows from $ii)$ of this claim and the following :

Claim 2 *There exist $\delta_2 > 0$ and s_{10} such that $\forall s \geq s_{10}$, $\tilde{K}(s) \leq d\delta_2^{2-\frac{k}{2}}\tilde{x}(s)$.*

Indeed, let $\delta_0 = \min(\delta_1, \delta_2)$ and $s_0 = \max(s_{10}, \hat{s}(\delta_0))$. From Claim 2 and the continuity with respect to initial data in equation (1), there is a neighborhood $\mathcal{V}_6^* \subset \hat{\mathcal{V}}_6$ of $\tilde{u}_0$ such that for all $u_0 \in \mathcal{V}_6^*$,

$$K(s_0) < 2\delta_0^{2-\frac{k}{2}}x(s_0).$$

From ii) of Claim 1, we have $\forall \delta \in (0, \delta_0)$, $\forall s \geq s_0$, $\forall u_0 \in \mathcal{V}_6^*$,

$$K(s) \leq 2\delta_0^{2-\frac{k}{2}}x(s) \leq 2\delta^{2-\frac{k}{2}}x(s),$$

which is the conclusion of the proof of Lemma 4.1. It remains for us to prove Claim 2.

Proof of Claim 2 : From Cauchy-Schwartz's inequality, we have

$$\tilde{K}(s) = \left(\int \tilde{v}(y,s)^2|y|^k\rho dy\right)^{1/2} \leq \left(\int \tilde{v}(y,s)^4\rho dy\right)^{1/4}\left(|y|^{2k}\rho dy\right)^{1/4}. \qquad (128)$$

We now use the following regularity estimate with a delay time shown by Herrero and Velázquez in [HV93]. Although they proved their result in the case $N = 1$, their proof holds in higher dimensions.

Claim 3 *There exist $s_0^* > 0$ and $C > 0$ such that for all $s \geq -\log\tilde{T}$,*

$$\left(\int \tilde{v}^4(y, s + s_0^*)\rho dy\right)^{1/4} \leq C\left(\int \tilde{v}^2(y,s)\rho dy\right)^{1/2}.$$

For s large enough, we have
$\left(\int \tilde{v}^4(y,s)\rho dy\right)^{1/4} \leq C_1\left(\int \tilde{v}(y, s-s_0^*)^2\rho dy\right)^{1/2} \leq \frac{C_2}{s-s_0^*}$ by (19)
$\leq \frac{C_3}{s} \leq C_4\left(\int \tilde{v}_2(y,s)^2\rho dy\right)^{1/2}$ by (19).
Using (128), we conclude the proof of Claim 2. This concludes the proof of Lemma 4.1. ■

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