Boundedness till blow-up of the difference between two solutions to a semilinear heat equation

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1 Introduction

In this paper, we are concerned with the following semilinear equation :

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u \\ u(.,0) &= u_0 \in H, \end{aligned} \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$ and Δ stands for the Laplacian in $\mathbb{R}^N$. We assume in addition the exponent $p > 1$ subcritical : if $N \geq 3$ then $1 < p < (N+2)/(N-2)$. Moreover, we assume that

$$u_0 \geq 0 \text{ or } (3N-4)p < 3N+8. \tag{2}$$

Local Cauchy problem for equation (1) can be solved in $L^\infty(\mathbb{R}^N)$. One can show that either the solution $u(t)$ exists on $[0,+\infty)$, or on $[0,T)$ with $T < +\infty$. In this former case, u blows-up in finite time in the sense that $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \to +\infty$ when $t \to T$.

Let us consider $u(t)$ a solution to (1) which blows up in finite time T at only one blow-up point a. The study of the blow-up behavior of $u(t)$ has been done through the introduction of the following similarity variables :

$$y = \frac{x-a}{\sqrt{T-t}},\ s = -\log(T-t),\ w_{a,T}(y,s) = (T-t)^{\frac{1}{p-1}}u(x,t). \tag{3}$$

The study of the profile of u as $t \to T$ is then equivalent to the study of the asymptotic behavior of $w_{a,T}$ (or w for simplicity), as $s \to \infty$, and each result for u has an equivalent formulation in terms of w. The equation satisfied by w is the following :

$$w_s = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w. \tag{4}$$

Giga and Kohn showed first in [GK85], [GK87] and [GK89] that for each $C > 0$,

$$\lim_{s \to +\infty} \sup_{|y| \le C} |\ w(y,s) - \kappa| = 0,$$

with $\kappa = (p-1)^{-\frac{1}{p-1}}$, which gives if stated for u :

$$\lim_{t \to T} \sup_{|y| \le C} |\ (T-t)^{1/(p-1)} u(a + y\sqrt{T-t}, t) - \kappa \ | = 0.$$

This result was specified by Filippas and Liu [FL93] (see also Filippas and Kohn [FK92] and Herrero and Velázquez [HV93], [Vel92]) who established that in the (supposed to be) generic case,

$$\left\| w_{a,T}(y,s) - \left[\kappa + \frac{\kappa}{2ps} (N - \frac{1}{2}|y|^2) \right] \right\| = O(1/s^{1+\delta}) \tag{5}$$

for some $\delta > 0$, where the norm is either the L^2 norm with respect to the following Gaussian measure

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}, \tag{6}$$

or the $C^{k,\alpha}_{loc}$ norm, for all $k \in \mathbb{N}$ and $\alpha \in (0,1)$.
Let us note that Herrero and Velázquez [HV92b] (see also [HV92a]) prove that the behavior (5) is generic in the case $N = 1$ with $u_0 \ge 0$. The question remains opened in the higher dimensional case with no positivity condition. Merle and Zaag [MZ], [MZ98b], [MZ98a] and [MZ97a] (with no sign condition), and Herrero and Velázquez [HV93], [Vel92] (in the positive case) show that $w_{a,T}$ has a limiting profile in the variable $z = \frac{y}{\sqrt{s}}$ in the sense that $\forall K_0 > 0$,

$$\sup_{|y| \le K_0\sqrt{s}} \left| w_{a,T}(y,s) - f\left(\frac{y}{\sqrt{s}}\right) \right| \to 0 \text{ as } s \to +\infty \tag{7}$$

where $$f(z) = \left(p-1+\frac{(p-1)^2}{4p}|z|^2\right)^{-\frac{1}{p-1}}. \quad (8)$$

In [MZ], [MZ98b] and [Zaa98] (see also [Vel92]), the authors derive the limiting profile of u in the variables (x,t) : there exists $u^* \in C(\mathbb{R}^N \backslash \{a\})$ such that for all $x \in \mathbb{R}^N \backslash \{a\}$, $u(x,t) \to u^*(x)$ and

$$u^*(x) \sim \left[\frac{8p}{(p-1)^2}\frac{|\log|x-a||}{|x-a|^2}\right]^{\frac{1}{p-1}} \text{ as } x \to a. \quad (9)$$

Let us note that it is shown in [MZ] and [MZ98b] that all the behaviors (5), (7) and (9) are in fact equivalent (with no sign condition).

In this paper, we aim at studying some properties of the limiting behavior described by (5), (7) or (9). We first introduce the following definition :

Definition 1.1 *For all $(a,T) \in \mathbb{R}^N \times \mathbb{R}$, $S_{a,T}$ stands for the set of all solutions of (1) which blow-up with the behavior (5) (or (7) or (9)).*

Let us remark that the following invariances of equation (1) are one to one mappings between sets $S_{a,T}$:

$$\begin{array}{rcccl} \mathcal{T}_{\xi,\tau} & : & S_{a,T} & \to & S_{a+\xi,T+\tau} \\ & & u & \mapsto & (\mathcal{T}_{\xi,\tau}u : (x,t) \mapsto u(x-\xi,t-\tau)) \\ \mathcal{D}_\lambda & : & S_{a,T} & \to & S_{a\lambda^{-1},T\lambda^{-2}} \\ & & u & \mapsto & \left(\mathcal{D}_\lambda u : (x,t) \mapsto \lambda^{\frac{2}{p-1}}u(\lambda x, \lambda^2 t)\right). \end{array}$$

Given $u_1 \in S_{a_1,T_1}$ and $u_2 \in S_{a_2,T_2}$, we see from (9) that up to invariances of (1), their main singular parts at blow-up are the same. One can ask the same question about lower order terms of the expansion of their singularity. We have the following result :

Theorem 1 *Assume $N = 1$ and $p \geq 3$, and consider $u_i \in S_{a_i,T_i}$ for $i = 1, 2$. Then, there exist $\lambda > 0$, $t_0 \in [0, T_1)$ and $C_0 > 0$ such that $\tilde{u}_2 \in S_{a_1,T_1}$ where*

$$\tilde{u}_2 = \mathcal{T}_{a_1,T_1}\mathcal{D}_\lambda\mathcal{T}_{-a_2,-T_2}, \quad (10)$$

$$\forall (x,t) \in \mathbb{R}^N \times [t_0, T_1), \ |u_1(x,t) - \tilde{u}_2(x,t)| \leq C_0,$$

and $$u_1(x,t) - \tilde{u}_2(x,t) \to 0 \text{ as } (x,t) \to (a_1, T_1).$$

With this Theorem, we see that for $N = 1$ and $p \geq 3$, up to the invariances of (1), u_1 and u_2 have the same terms in the expansion of their singularity, until the order of functions with limit zero at the singularity. Moreover, their difference remains bounded, uniformly in space, until blow-up (still up to an invariance of (1)). Unfortunately, we are not able to have such a striking result in higher dimensions. However, we can prove that u_1 and u_2 have the same terms in the expansion of their singularity, until some singular order. More precisely, we have the following result which in the case $N = 1$ and $p \geq 3$ gives a sharp estimate that directly implies Theorem 1 :

Theorem 2 (Smallness of the difference of two given solutions of (1) with the behaviors (5)) *Assume $N \geq 1$ and consider $u_i \in S_{a_i,T_i}$ for $i = 1,\ 2$. Then, $\mathcal{T}_{a_1-a_2,T_1-T_2} \in S_{a_1,T_1}$ and $\forall (x,t) \in \mathbb{R}^N \times [0,T_1)$,*
$|u_1(x,t) - \mathcal{T}_{a_1-a_2,T_1-T_2} u_2(x,t)|$

$$\leq C \min\left\{ \frac{(T_1-t)^{-\frac{1}{p-1}}}{|\log(T_1-t)|}, \frac{|x-a_1|^{-\frac{2}{p-1}}}{|\log|x-a_1||^{1-\frac{1}{p-1}}} \right\}.$$

Moreover, if $N = 1$, then there exists $\lambda > 0$ such that $\tilde{u}_2$ defined in (10) belongs to S_{a_1,T_1} and :
- If $1 < p < 3$, then $\forall (x,t) \in \mathbb{R}^N \times [0,T_1)$,

$$|u_1(x,t) - \tilde{u}_2(x,t)| \leq C \min\left\{ \frac{(T_1-t)^{\frac{1}{2}-\frac{1}{p-1}}}{|\log(T_1-t)|^{\frac{3}{2}}}, \frac{|x-a_1|^{1-\frac{2}{p-1}}}{|\log|x-a_1||^{2-\frac{1}{p-1}}} \right\}.$$

- If $p \geq 3$, then $\forall (x,t) \in \mathbb{R}^N \times [0,T_1)$,

$$|u_1(x,t) - \tilde{u}_2(x,t)| \leq C \left\{ \frac{(T_1-t)^{\frac{1}{2}-\frac{1}{p-1}}}{|\log(T_1-t)|^{\frac{3}{2}}} + \frac{|x-a_1|^{1-\frac{2}{p-1}}}{|\log|x-a_1||^{2-\frac{1}{p-1}}} \right\}.$$

Even though Theorem 2 does not give in higher dimensions a result analogous to Theorem 1, it has an application for the stability of the behavior (5) (or (7) or (9)) with respect to initial data for general N. Recently, we proved this stability result with Merle in [FKMZ99] for all subcritical $p < (N+2)/(N-2)$ if $N \geq 3$, without the restriction (2). The method of [FKMZ99] relies on a dynamical system approach applied to solutions of (4). Here, we adopt a completely different point of view, and present a different proof based on a former stability result by Merle and Zaag in [MZ97b]. In that paper (see also [MZ96]), the authors construct a solution $\hat{u}(t)$ to (1) which blows-up at time $\hat{T}$ at only one point $\hat{a} \in \mathbb{R}^N$ with the behavior (7). More precisely, they obtain $\forall s \geq -\log \hat{T}$,

$$\sup_{y\in\mathbb{R}^N} \left| \hat{w}_{\hat{a},\hat{T}}(y,s) - f\left(\frac{y}{\sqrt{s}}\right) \right| \leq \frac{C}{\sqrt{s}}, \tag{11}$$

where f is defined in (8). Bricmont and Kupiainen obtain the same result in [BK94]. However, the method used by Merle and Zaag allows them to show that the behavior (11) of the constructed solution $\hat{u}(t)$ is stable with respect to initial data. As a matter of fact, their stability result is stronger : they prove that each solution $u(t)$ to (1) with the behavior (5) (or (7) or (9)) is stable provided that the function

$$q_{a,T}(y,s) = w_{a,T}(y,s) - \left\{ f\left(\frac{y}{\sqrt{s}}\right) + \frac{N\kappa}{2ps} \right\} \tag{12}$$

satisfies additional smallness conditions.

In this paper, we use Theorem 2 to estimate $\mathcal{T}_{a-\hat{a},T-\hat{T}}u(t) - \hat{u}(t)$ for each solution $u(t) \in S_{a,T}$ and then show that $u(t)$ do satisfy the same smallness conditions as $\hat{u}(t)$. Therefore, $u(t)$ is stable with respect to initial data. More precisely :

Theorem 3 (Stability of the behavior (5) with respect to initial data) *Let $\tilde{u}(t)$ be a solution to (1) such that $\tilde{u}(t)$ blows up at time $\tilde{T}$ at only one blow-up point $\tilde{a}$ and $\tilde{w}_{\tilde{a},\tilde{T}}$ satisfies the behavior (5). Then, $\forall \epsilon > 0$, there exists a neighborhood $\mathcal{V}_\epsilon$ of $\tilde{u}(0)$ in $L^\infty(\mathbb{R}^N)$ such that $\forall u_0 \in \mathcal{V}_\epsilon$, the solution $u(t)$ to (1) with initial data u_0 blows up at time T at only one blow-up point a such that*

$$|T - \tilde{T}| + |a - \tilde{a}| \leq \epsilon$$

and $w_{a,T}$ satisfies (5).

Theorem 1 follows directly from Theorem 2. We prove Theorems 2 and 3 respectively in sections 2 and 3.

The authors want to thank Frank Merle for the very interesting discussions.

2 Smallness of the difference of two given solutions of (1) with the behaviors (5)

In this section, we prove Theorem 2. Consider $u_i \in S_{a_i,T_i}$ for $i = 1, 2$ and assume $T_2 \geq T_1$. Then, u_1 and $\mathcal{T}_{a_1-a_2,T_1-T_2}u_2$ blow up in finite time T_1 at one blow-up point a_1. Our aim is to estimate $|u_1(x,t) - \mathcal{T}_{a_1-a_2,T_1-T_2}u_2(x,t)|$ for $(x,t) \in \mathbb{R}^N \times [0, T_1)$. As in (3), we introduce the similarity variables with $T = T_1$ and $a = a_1$

$$x = a_1 + ye^{-s}, \;\; t = T_1 - e^{-s/2},$$

and we note

$$\begin{aligned} w_1(y,s) &= (T_1-t)^{\frac{1}{p-1}} u_1(x,t) \\ w_2(y,s) &= (T_1-t)^{\frac{1}{p-1}} u_2(x-a_1+a_2, t-T_1+T_2) \\ g(y,s) &= w_1(y,s) - w_2(y,s) \end{aligned}$$

We prove Theorem 2 in three parts.

Part 1: Analysis in the variables (y,s). We prove an L^2_ρ estimate on g. From (5) applied to u_1 and u_2, we have

$$\|\, g(s) \,\|_{L^2_\rho} = O(\frac{1}{s^{1+\delta}}). \tag{13}$$

We prove a better estimate by expanding g on the eigenspaces of operator $\mathcal{L} = \Delta - \frac{y}{2} \cdot \nabla + 1$. This yields an estimate in the sets $\{|\, y \,| \leq R\}$, for $R > 0$.

Part 2: Analysis in the variables $(z = \frac{y}{\sqrt{s}}, s)$. By using estimates for linear equation as in [MZ98b], we derive L^∞ estimates on $g(y,s)$ for $|\, y \,| \leq K_0 s^{1/2}$ from the L^2_ρ estimates proved before.

Part 3: Analysis in the variables (x,t). The L^∞ estimate obtained on g in Part 2 yields an estimate on $|\, u_1(x,t) - u_2(x-a_2+a_1, t-T_2+T_1) \,|$ in the sets $|\, x - a_1 \,| \leq K_0(T_1-t)^{1/2} \,|\log(T_1-t)\,|^{1/2}$. We use this estimate and the uniform comparison of u_1 and u_2 with ODEs (see Proposition 2.3 below) to obtain Theorem 2.

The following subsections are devoted to each of these three parts. Let us briefly recall some general results on blow-up solutions of (1) that we shall use in the following.

Proposition 2.1 (Equivalence of different notions of blow-up profiles at a singular point) *Let $u(t)$ be a solution to (1) which blows-up at time T at only one blow-up point $a \in \mathbb{R}^N$. The blow-up behaviors of $u(t)$ described by (5), (7) and (9) are equivalent.*

Proof : See Theorem 3 and its proof in [MZ98b] and Proposition 3 in [MZ]. ■

Proposition 2.2 (Refined L^∞ estimates for solution to (1) at blow-up) *There exist positive constants C_1, C_2 and C_3 such that if u is a solution to (1) which blows up at time $T > 0$, then, for all $\epsilon > 0$ there exists $s_1(\epsilon)$ such that for all $s \geq s_1(\epsilon)$ and for all $a \in \mathbb{R}^N$,*

$$\| w_{a,T}(s) \|_{L^\infty} \leq \kappa + (\frac{N\kappa}{2p} + \epsilon)\frac{1}{s}, \text{ and } \| \nabla^i w_{a,T}(s) \|_{L^\infty} \leq \frac{C_i}{s^{i/2}} \tag{14}$$

for all $i \in \{1, 2, 3\}$, where $w_{a,T}$ is defined in (3).

Proposition 2.3 (A uniform ODE comparison) *Let u be a solution to (1) which blows up at time T. Then, $\forall \epsilon > 0$, $\exists C_\epsilon > 0$,*

$$\forall t \in [\frac{T}{2}, T), \ \ \forall x \in \mathbb{R}^N, \ \ |\frac{\partial u}{\partial t} - u^p| \leq \epsilon u^p + C_\epsilon.$$

The reader will find a proof of these Propositions in [MZ] and [MZ98b] respectively.

2.1 L^2_ρ estimates on g.

Our aim in this section is to prove the following Proposition

Proposition 2.4 (L^2_ρ estimates on g) *We have*

$$\| g(s) \|_{L^2_\rho} = O(\frac{1}{s^2}). \tag{15}$$

Moreover, if $N = 1$, then there exists $\sigma_0 \in \mathbb{R}$ such that

$$\| w_1(s) - w_2(s + \sigma_0) \|_{L^2_\rho} = O(\frac{\mathrm{e}^{-s/2}}{s^3}). \tag{16}$$

Note that the translation in time σ_0 of $w_2(y, s)$ is associated via similarity variables to a dilatation of $\lambda = \exp(\frac{\sigma_0}{2})$ of $u_2(x - a_2 + a_1, t - T_2 + T_1)$.

Let $g = w_1 - w_2$, then we see from (4) that g satisfies

$$\partial_s g = \mathcal{L} g + \alpha g, \tag{17}$$

with $\mathcal{L} = \Delta - \frac{y}{2} \cdot \nabla + 1$ and $\forall (y, s) \in \mathbb{R}^N \times \mathbb{R}$,

$$\alpha(y, s) = \frac{|w_1|^{p-1} w_1 - |w_2|^{p-1} w_2}{w_1 - w_2} - \frac{p}{p-1}, \text{ if } w_1 \neq w_2, \tag{18}$$

and in general,

$$\alpha(y,s) = p \mid w_0(y,s) \mid^{p-1} - \frac{p}{p-1} \text{ for some } w_0(y,s) \in \big(w_1(y,s)\ ,\ w_2(y,s)\big). \tag{19}$$

Operator $\mathcal{L}$ is a self-adjoint operator on $\mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N)$ where ρ is defined in (6). The spectrum of $\mathcal{L}$ consists of eigenvalues

$$\operatorname{spec} \mathcal{L} = \{1 - \frac{m}{2},\ \ m \in \mathbb{N}\}.$$

The eigenfunctions corresponding to $1 - \frac{m}{2}$ are

$$y \mapsto h_{m_1}(y_1)...h_{m_N}(y_N),\ \ m_1 + ... + m_N = m,$$

where the Hermite polynomials

$$h_m(\xi) = \sum_{j=0}^{[m/2]} \frac{m!}{j!(m-2j)!}(-1)^j \xi^{m-2j},\ \text{ for } \ m \in \mathbb{N}, \tag{20}$$

satisfy

$$\int_{\mathbb{R}} h_m(\xi) h_j(\xi) \rho(\xi) d\xi = 2^m m! \delta_{m,j}. \tag{21}$$

For all $n \in \mathbb{N}$, for all multi-index $\beta = (\beta_1, ..., \beta_N) \in \mathbb{N}^N$, $\mid \beta \mid = n$ and for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ we note

$$h_\beta(y) = h_{\beta_1}(y_1)...h_{\beta_N}(y_N), \tag{22}$$

and $k_\beta(y) = \| h_\beta \|^{-2}_{L^2_\rho} h_\beta(y)$. Then the component of $g(s)$ on h_β is

$$g_\beta(s) = \int k_\beta(y) g(y,s) \rho(y) dy. \tag{23}$$

The component of g on the eigenspace corresponding to the eigenvalue $1 - \frac{n}{2}$ is

$$P_n g(y,s) = \sum_{|\beta|=n} g_\beta(s) h_\beta(y). \tag{24}$$

Since the eigenfunctions of $\mathcal{L}$ span the whole space $L^2_\rho(\mathbb{R}^N)$, we obtain the following expansion of g

$$g(y,s) = g_0(s) + g_1(s) \cdot y + \frac{1}{2} y^T g_2(s) y - \operatorname{tr} g_2(s) + g_-(y,s) \tag{25}$$

where $g_0(s) = P_0 g(y,s)$, $g_1(s)\cdot y = P_1 g(y,s)$, $g_-(y,s) = \sum_{n\geq 3} P_n g(y,s)$ and $g_2(s)$ is the $N\times N$ matrix defined by

$$g_2(s) = \int M(y) g(y,s)\rho(y)dy \tag{26}$$

where

$$M(y) = \big(\frac{1}{4}y_i y_j - \frac{1}{2}\delta_{ij}\big)_{1\leq i,j\leq N}. \tag{27}$$

Let us note then that we have from (25) and the orthogonality relation (21)

$$\frac{1}{2}y^T g_2(s) y - \operatorname{tr} g_2(s) = P_2 g(y,s).$$

Our aim is to study as s goes to infinity

$$I(s) = \| g(s) \|_{L^2_\rho}\ .$$

Since $I(s)^2 = \sum_{n\geq 0} l_n(s)^2$ with

$$l_n(s) = \| P_n g(s) \|_{L^2_\rho},$$

we show in the first step that either $\forall n \in \mathbb{N}$, $l_n(s) = o(I(s))$, which gives the conclusion of Proposition 2.4 rather easily, or that there exists $n_0 \in \mathbb{N}$ so that $I(s) \sim l_{n_0}(s)$. Then, in this latter case, we study precisely the cases $n_0 = 2$ and $n_0 = 3$. If $N = 1$ we prove that by modifying w_2 we can reduce to the case $n_0 \geq 3$ and thus obtain estimate (16). If $N \geq 1$ we obtain (15). Of course we shall need estimates on α which are given in the following Lemma.

Lemma 2.5 (Estimates on α) *There exist some constants $C > 0$, $\delta' \in (0,1)$ and $s_1 \in \mathbb{R}$ such that for all $y \in \mathbb{R}^N$ and $s \geq s_1$,*

$$\alpha(y,s) \leq \frac{C}{s},\ \ |\alpha(y,s)| \leq \frac{C}{s}(1+|y|^2) \tag{28}$$

$$\text{and } |\alpha(y,s) - \frac{1}{2s}(N - \frac{|y|^2}{2})| \leq \frac{C}{s^{1+\delta'}}(1+|y|^3). \tag{29}$$

Proof of Lemma 2.5 : Let us study α. Since $w_0 \in (w_1, w_2)$, uniform estimates of Proposition 2.2 yield the existence of s_1 and a constant C such that

$$\forall s \geq s_1, \quad | w_0(y,s) | \leq \kappa + \frac{C}{s}.$$

Therefore (19) yields for $s \geq s_1$,

$$\alpha(y,s) \leq p(\kappa + \frac{C}{s})^{p-1} - \frac{p}{p-1} \leq \frac{C_0}{s} \text{ for some } C_0 > 0.$$

Let us perform a Taylor expansion of $w_i(y,s)$ for $i \in \{1,2\}$,

$$w_i(y,s) = w_i(0,s) + y.\nabla w_i(0,s) + \frac{1}{2}\, y^T \nabla^2 w_i(y',s) y, \quad y' \in B(0,|y|). \quad (30)$$

The $\mathcal{C}^{k,\alpha}_{loc}$ estimates of (5) yield

$$w_i(0,s) = \kappa + \frac{N\kappa}{2ps} + O(\frac{1}{s^{1+\delta}}) \text{ and } \nabla w_i(0,s) = O(\frac{1}{s^{1+\delta}}).$$

Moreover uniform estimates of Proposition 2.2 yield $| \nabla^2 w_i(y',s) | \leq \frac{C}{s}$. Therefore (30) becomes for $i \in \{1,2\}$,

$$| w_i(y,s) - \kappa | \leq \frac{C}{s}(1 + | y |^2). \quad (31)$$

Since $w_0 \in (w_1, w_2)$, (31) holds also for $i = 0$. Note that by (19) we obtain

$$|\alpha(y,s)| \leq C\, |w_0(y,s) - \kappa|\,.$$

This yields (28). A Taylor expansion until the second order and the same arguments as before yield (29) with $\delta' = \min(\frac{1}{2}, \delta)$. ■

2.1.1 Step 1: Existence of a dominating component

Proposition 2.6 (Existence of a dominating component)
(i) *For* $i \in \{0,1\}$, $l_i(s) = O(\frac{I(s)}{s})$.
(ii) *Only two cases may occur :*

- *Either there exists* $n \in \mathbb{N}$, $n \notin \{0,1\}$ *so that* $I(s) \sim l_n(s)$ *and*

$$\forall m \neq n, \;\; l_m(s) = O(\frac{l_n(s)}{s}). \quad (32)$$

Moreover, there exist σ_1, $C > 0$ *and* $C' > 0$ *such that*

$$\forall s \geq \sigma_1, \;\; (s^{C'} C')^{-1} \exp\big((1 - \frac{n}{2})s\big) \leq I(s) \leq C s^C \exp\big((1 - \frac{n}{2})s\big). \quad (33)$$

- *Either for all* $n \in \mathbb{N}$, $l_n(s) = O(\frac{I(s)}{s})$ *and there exist* σ_n, $C_n > 0$ *and* $C'_n > 0$ *such that*

$$\forall s \geq \sigma_n, \;\; I(s) \leq C_n s^{C_n} \exp\big((1 - \frac{n}{2})s\big). \tag{34}$$

This Proposition comes from two Lemmas :

Lemma 2.7 (Evolution of $I(s)$ and $l_n(s)$) *There exist* $s_2 \in \mathbb{R}$ *and* $C_0 \in \mathbb{R}$ *such that for all* $n \in \mathbb{N}$ *there exists* C_n *such that for all* $s \geq s_2$,

$$\mid l'_n(s) + (\frac{n}{2} - 1) l_n(s) \mid \leq \frac{C_n}{s} I(s) \tag{35}$$

$$\text{and } I'(s) \leq (1 - \frac{n+1}{2} + \frac{C_0}{s}) I(s) + \sum_{k=0}^{n} \frac{n+1-k}{2} l_k(s). \tag{36}$$

Lemma 2.8 *If for all* $s \geq s_0 > 0$, $x(s)$ *and* $y(s)$ *satisfy*

$$0 \leq x(s) \leq y(s) \;\; \text{and} \;\; y(s) \longrightarrow 0 \;\; \text{as} \;\; s \to +\infty,$$

with moreover

$$x'(s) \geq -\frac{C}{s} y(s) \;\; \text{and} \;\; y'(s) \leq -\frac{1}{2} y(s) + \frac{C}{s} y(s) + \frac{1}{2} x(s),$$

then, either $x \sim y$, *or there exists a constant* C *so that*

$$\forall s \geq s_0, \;\; x(s) \leq \frac{C}{s} y(s).$$

Lemma 2.8 is proved in Appendix A.

Proof of Lemma 2.7 : Consider $n \in \mathbb{N}$, $\beta \in \mathbb{N}^N$ with $\mid \beta \mid = n$. Then, we have from (17) and (23),

$$\forall s \geq s_0, \;\; g'_\beta(s) = (1 - \frac{n}{2}) g_\beta(s) + \int \alpha(y,s) g(y,s) k_\beta(y) \rho(y) dy. \tag{37}$$

Therefore, from Cauchy-Schwartz's inequality we obtain

$$\forall s \geq s_0, \;\; \mid g'_\beta(s) - (1 - \frac{n}{2}) g_\beta(s) \mid \leq I(s) \mid\mid \alpha(s) k_\beta \mid\mid_{L^2_\rho} . \tag{38}$$

By Lemma 2.5, there exists $C > 0$ so that $\| \alpha(s) k_\beta \|_{L^2_\rho} \leq \frac{C(\beta)}{s}$ for $s \geq s_1$. Since $l_n^2(s) = \sum_{|\beta|=n} g_\beta(s)^2$ we obtain (35). By (17) we obtain for all $s \geq s_0$,

$$2I(s)I'(s) = 2\big(\mathcal{L}g(s) \ \mid \ g(s)\big)_{L^2_\rho} + 2\big(\alpha(s)g(s) \ \mid \ g(s)\big)_{L^2_\rho}.$$

Using (28) we obtain

$$\big(\alpha(s)g(s) \ \mid \ g(s)\big)_{L^2_\rho} \leq \frac{C_0}{s} I(s).$$

Therefore,

$$I(s)I'(s) \leq \frac{C_0}{s} I(s)^2 + (1 - \frac{n+1}{2}) I(s)^2 + \big((\mathcal{L} - 1 + \frac{n+1}{2}) g(s) \ \mid \ g(s)\big)_{L^2_\rho}.$$

Since $g = \sum_{k\geq 0} P_k g$, $(\mathcal{L} - 1 + \frac{n+1}{2}) g = \sum_{k \geq 0} \frac{n+1-k}{2} P_k g$ and

$$\begin{array}{rcl} \big((\mathcal{L} - 1 + \frac{n+1}{2}) g(s) \ \mid \ g(s)\big)_{L^2_\rho} & = & \sum_{k\geq 0} \frac{n+1-k}{2} l_k(s)^2 \\ & \leq & \sum_{k=0}^{n} \frac{n+1-k}{2} l_k(s)^2 \\ & \leq & I(s) \sum_{k=0}^{n} \frac{n+1-k}{2} l_k(s). \end{array}$$

This yields (36). ■

Proof of Proposition 2.6 : Note that because of (13),

$$I(s) = O(\frac{1}{s^{1+\delta}}) \ \text{ and } \ l_n(s) \underset{s\to+\infty}{\longrightarrow} 0.$$

(i) Lemma 2.7 yield that $x(s) = l_0(s)\mathrm{e}^{-s}$ and $y(s) = I(s)\mathrm{e}^{-s}$ satisfy the assumptions of Lemma 2.8. Therefore $l_0(s) = O\big(\frac{I(s)}{s}\big)$ or $l_0(s) \sim I(s)$. Let us examine the case $l_0(s) \sim I(s)$. Then (35) yields either $l_0 \equiv 0$ in a neighborhood of $+\infty$ (hence $l_0 = O(I(s)/s)$) or there exists $C > 0$ such that

$$(Cs^C)^{-1} \mathrm{e}^s \leq l_0(s) \text{ for large},$$

and this is not possible because $l_0(s) \underset{s\to+\infty}{\longrightarrow} 0$. Therefore $l_0(s) = O\big(I(s)/s\big)$. The same argument applied to $x(s) = l_1(s)\mathrm{e}^{-s/2}$ and $y(s) = I(s)\mathrm{e}^{-s/2}$ yields $l_1(s) = O\big(\frac{I(s)}{s}\big)$. Hence (i).

(ii) Let us examine the following proposition

$$\forall n \in \mathbb{N}, \ \ l_n(s) = O\big(\frac{I(s)}{s}\big). \qquad (39)$$

Either (39) is true or (39) is false. Let us first examine what happens if (39) is true. Then using (36) for some fixed $n \in \mathbb{N}$ we obtain that there exists $C_n \in \mathbb{R}$ such that

$$I'(s) \leq (1 - \frac{n+1}{2} + \frac{C_n}{s}) I(s).$$

Hence (34).
Let us now examine what happens if (39) is false. Then there exists $n \in \mathbb{N}$ such that $l_n(s)$ is not $O\big(\frac{I(s)}{s}\big)$ and

$$\forall k \in \{0, ..., n-1\}, \quad l_k(s) = O\big(\frac{I(s)}{s}\big).$$

One can see from (35) and (36) that $x(s) = \exp\big((\frac{n}{2} - 1)s\big) l_n(s)$ and $y(s) = \exp\big((\frac{n}{2} - 1)s\big) I(s)$ satisfy the assumptions of Lemma 2.8. Therefore either $l_n(s) \sim I(s)$, or there exists $C > 0$ so that $l_n(s) < \frac{C}{s} I(s)$, which is ruled out by the hypothesis. Therefore $l_n(s) \sim I(s)$ and (35) yields

$$\mid l_n'(s) + (\frac{n}{2} - 1) l_n(s) \mid \leq \frac{C}{s} l_n(s). \tag{40}$$

Since $l_n \not\equiv 0$ in a neighborhood of $+\infty$ (otherwise $l_n = O(I(s)/s)$), this gives (33).
Moreover for $m > n$, (35) and $I(s) \sim l_n(s)$ imply that for all $s \geq s_0$, $s_0 \in \mathbb{R}$,

$$l_m(s) \leq e^{-(\frac{m}{2}-1)(s-s_0)} l_m(s_0) + C(m,n) \int_{s_0}^{s} \frac{l_n(t)}{t} e^{-(\frac{m}{2}-1)(s-t)} dt. \tag{41}$$

Since $m > n \geq 2$, we have from (33)

$$e^{-(\frac{m}{2}-1)(s_0-s)} l_m(s_0) = O\big(\frac{l_n(s)}{s}\big). \tag{42}$$

It remains to prove that $\int_{s_0}^{s} \frac{l_n(t)}{t} e^{-(\frac{m}{2}-1)(s-t)} dt = O\left(l_n(s)/s\right)$. By integrating by parts we obtain

$$\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \frac{l_n(t)}{t} dt \quad = \tfrac{2}{m-2} \tfrac{l_n(s)}{s} - \tfrac{2}{m-2} e^{-(\frac{m}{2}-1)(s-s_0)} \tfrac{l_n(s_0)}{s_0} \tag{43}$$

$$- \tfrac{2}{m-2} \textstyle\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \big(\tfrac{l_n'(t)}{t} - \tfrac{l_n(t)}{t^2}\big) dt. \tag{44}$$

Note that as for (42), (33) yields $e^{-(\frac{m}{2}-1)(s-s_0)} \frac{l_n(s_0)}{s_0} = O\big(\frac{l_n(s)}{s}\big)$. It remains to study $\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \big(\frac{l_n'(t)}{t} - \frac{l_n(t)}{t^2}\big) dt$. Estimating l_n' by means of (40), we obtain

$$\Big| \textstyle\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \big(\tfrac{l_n'(t)}{t} - \tfrac{l_n(t)}{t^2}\big) dt - \big(\tfrac{2-n}{2}\big) \int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \tfrac{l_n(t)}{t} dt \Big|$$
$$\leq C(n) \textstyle\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \tfrac{l_n(t)}{t^2} dt.$$

Using (44) and (2.1.1) we obtain

$$\Big|\int_{s_0}^{s} \frac{l_n(t)}{t} e^{-(\frac{m}{2}-1)(s-t)} dt - \frac{n-2}{m-2}\int_{s_0}^{s} \frac{l_n(t)}{t} e^{-(\frac{m}{2}-1)(s-t)} dt\Big|$$
$$\leq C(n,m)\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \frac{l_n(t)}{t^2} dt + C(n,m)\frac{l_n(s)}{s}.$$

Since $m > n \geq 2$, $\frac{n-2}{m-2} < 1$. Moreover, we have

$$\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \frac{l_n(t)}{t^2} dt \leq \frac{1}{s_0}\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \frac{l_n(t)}{t} dt.$$

Therefore,

$$(1 - \frac{n-2}{m-2} - \frac{C(n,m)}{s_0})\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \frac{l_n(t)}{t} dt \leq C(n,m)\frac{l_n(s)}{s}.$$

By choosing s_0 large enough so that $1 - \frac{n-2}{m-2} - \frac{C(n,m)}{s_0} > 0$ we obtain

$$\int_{s_0}^{s} e^{-(\frac{m}{2}-1)(s-t)} \frac{l_n(t)}{t} dt = O\big(\frac{l_n(s)}{s}\big). \tag{45}$$

Using (42), (45) in (41), we obtain $l_m(s) = O\big(\frac{l_n(s)}{s}\big)$. This closes the proof of Proposition 2.6. ■

2.1.2 Step 2: Description of the dominating component.

Let us now examine what happens more precisely if $I(s) \sim l_n(s)$ for $n = 2$ or $n = 3$. More precisely, in the case $n = 3$, our aim is to find an equivalent for $l_3(s)$, hence for $I(s)$.

Proposition 2.9 (Description of the dominating component)

1) *If $I(s) \sim l_2(s)$ then $l_2'(s) = -\frac{2}{s} l_2(s) + O(\frac{l_2(s)}{s^{1+\delta'}})$, and there exists $C_2 > 0$, $l_2(s) = \frac{C_2}{s^2} + o(\frac{1}{s^2})$*

2) *If $I(s) \sim l_3(s)$ then $l_3'(s) = -(\frac{1}{2} + \frac{3}{s}) l_3(s) + O(\frac{l_3(s)}{s^{1+\delta'}})$, and there exists $C_3 > 0$, $l_3(s) = \frac{C_3}{s^3} e^{-s/2} + o(\frac{e^{-s/2}}{s^3})$.*

Proof of Proposition 2.9 : We shall prove 2) and the proof of 1) is quite similar. We want to calculate $l_3(s)$, therefore we study $g_\beta(s)$ for $\mid \beta \mid = 3$. Because of (37) we have to study

$$M(s) = \int \alpha(y,s) g(y,s) k_\beta(y) \rho(y) dy. \tag{46}$$

By (20) and (29) we obtain

$$M(s) = -\frac{1}{4s}\sum_{k=1}^{N}\int g(y,s)h_2(y_k)k_\beta(y)\rho(y)dy + \int \gamma(y,s)g(y,s)k_\beta(y)\rho(y)dy.$$

with

$$\mid \gamma(y,s) \mid \leq \frac{C}{s^{1+\delta'}}(1+\mid y\mid^3). \tag{47}$$

Note that by Cauchy-Schwartz's inequality we have

$$\mid \int \gamma(y,s)g(y,s)k_\beta(y)\rho(y)dy \mid \leq I(s) \mid\mid \gamma(s)k_\beta \mid\mid_{L^2_\rho}.$$

By using $I(s) \sim l_3(s)$ and (47) we obtain for

$$\mid\mid \gamma(s)k_\beta \mid\mid_{L^2_\rho} \leq \frac{C}{s^{1+\delta'}} \mid\mid (1+\mid y\mid^2)k_\beta(y) \mid\mid_{L^2_\rho},$$

therefore,

$$\mid \int \gamma(y,s)g(y,s)k_\beta(y)\rho(y)dy \mid = O\big(\frac{l_3(s)}{s^{1+\delta'}}\big).$$

We obtain

$$M(s) = -\frac{1}{4s}\sum_{k=1}^{N}\int g(y,s)h_2(y_k)k_\beta(y)\rho(y)dy + O\big(\frac{l_3(s)}{s^{1+\delta'}}\big).$$

Let us expand now g, we obtain

$$M(s) = -\frac{1}{4s}\sum_{k=1}^{N}\sum_{j\geq 0}\int P_jg(y,s)h_2(y_k)k_\beta(y)\rho(y)dy + O\big(\frac{l_3(s)}{s^{1+\delta'}}\big).$$

Note that for $j \geq 6$, $\int P_jg(y,s)h_2(y_k)k_\beta(y)\rho(y)dy = 0$ because of the orthogonality relation (21). Therefore

$$M(s) = -\frac{1}{4s}\sum_{k=1}^{N}\sum_{j=0}^{5}\int P_jg(y,s)h_2(y_k)k_\beta(y)\rho(y)dy + O\big(\frac{l_3(s)}{s^{1+\delta'}}\big).$$

Since for $j \neq 3$, $\mid\mid P_jg(s) \mid\mid_{L^2_\rho} = l_j(s) \leq \frac{C_j}{s}l_3(s)$ by (32), we have by Cauchy-Schwartz's inequality

$$\mid \frac{1}{4s}\sum_{k=1}^{N}\sum_{j=0,\, j\neq 3}^{5}\int P_jg(y,s)h_2(y_k)k_\beta(y)\rho(y)dy \mid = O\big(\frac{l_3(s)}{s^2}\big).$$

Therefore we only compute the term for $j=3$. For $k\in\{1,...,N\}$ we have from (24),

$$\int P_3 g(y,s)h_2(y_k)k_\beta(y)\rho(y)dy=\sum_{|\gamma|=3} g_\gamma(s)\int h_\gamma(y)h_2(y_k)k_\beta(y)\rho(y)dy$$

As $k_\beta(y)$ is of one of the following forms (see (22)),

$$\begin{array}{rcl} k_\beta(y) & = & k_3(y_{k'}), \\ k_\beta(y) & = & k_2(y_{k'})k_1(y_l) \text{ with } k'\neq l, \\ k_\beta(y) & = & k_1(y_{k'})k_1(y_l)k_1(y_m) \text{ with } k'\neq l\neq m, \end{array}$$

long but straightforward computations yield

$$\forall\beta\in\mathbb{N}^N,\ \ |\beta|=3,\ \sum_{k=1}^N\int P_3 g(y,s)h_2(y_k)k_\beta(y)\rho(y)dy=12g_\beta(s).$$

We obtain $M(s)=-\frac{3}{s}g_\beta(s)+O\big(\frac{l_3(s)}{s^{1+\delta'}}\big)$. Together with (46) and (37), this gives $l_3'(s)=-\left(\frac{1}{2}+\frac{3}{s}\right)l_3(s)+O\left(\frac{l_3(s)}{s^{1+\delta'}}\right)$. Therefore, there exists $C_3\geq 0$ such that

$$l_3(s)=C_3\frac{e^{-s/2}}{s^3}\exp\Big(\int_{s_0}^s\frac{\phi(u)}{u^{1+\delta'}}du\Big)$$

with ϕ bounded. Note that $\int_{s_0}^{+\infty}\frac{\phi(u)}{u^{1+\delta'}}du<+\infty$. From (33) applied with $n=3$, we have $C_3>0$ and **2)** of Proposition 2.9 is true. ■

2.1.3 Conclusion : Proof of Proposition 2.4

Propositions 2.6 and 2.9 directly yield the following Corollary.

Corollary 2.10 *As s goes to $+\infty$, $I(s)$ behaves in two ways.*

1. *Case 1 : $I(s)\sim l_2(s)$ and $l_2(s)=\frac{C_2}{s^2}+o(\frac{1}{s^2})$ for some $C_2>0$.*
2. *Case 2 : $I(s)=O\big(\frac{e^{-s/2}}{s^3}\big)$.*

(15) is an immediate consequence of this Corollary.
We concentrate now on the case $N=1$ in order to prove (16). If Case 2 of Corollary 2.10 holds, then (16) holds with $\sigma_0=0$. We are reduced to Case 1. Note that since $N=1$ and for all $s\geq s_0$, $l_2(s)\neq 0$, g_2 is a scalar and

we have either for all $s \geq s_0$, $g_2(s) = l_2(s)$, or for all $s \geq s_0$, $g_2(s) = -l_2(s)$. Therefore, from Corollary 2.10, we have for all $s \geq s_0$,

$$g_2(s) = \frac{C_2'}{s^2} + o(\frac{1}{s^2}) \tag{48}$$

for some $C_2' \in \mathbb{R}$. Roughly speaking we shall replace $w_2(y,s)$ by

$$\tilde{w}_2(y,s) = w_2(y, s+\sigma_0), \tag{49}$$

where σ_0 is to be fixed in terms of C_2' and we shall compare w_1 to $\tilde{w}_2$ rather than to w_2. Obviously all the estimates satisfied by $g = w_1 - w_2$ hold also for

$$\tilde{g} = w_1 - \tilde{w}_2, \tag{50}$$

since $\tilde{w}_2$ is also a solution to (4), in particular the alternative of Corollary 2.10 holds for $\tilde{w}_2$. In the following we denote with a $\tilde{}$ any the items associated with $\tilde{g}$. We claim that we can select a particular $\sigma_0(C_2')$ such that Case 1 of Corollary 2.10 does not hold for

$$\tilde{I}(s) = ||\tilde{g}(s)||_{L^2_\rho} = ||w_1(s) - \tilde{w}_2(s)||_{L^2_\rho} = ||w_1(s) - w_2(s+\sigma_0)||_{L^2_\rho}. \tag{51}$$

More precisely,

Lemma 2.11 *There exists $\sigma_0 \in \mathbb{R}$ such that $\tilde{l}_2(s) = o(\frac{1}{s^2})$ as s goes to $+\infty$.*

This Lemma yields that only Case 2 of Corollary 2.10 may occur for $\tilde{w}_2$ and

$$||w_1(s) - w_2(s+\sigma_0)||_{L^2_\rho(\mathbb{R}^N)} = O\big(\frac{\mathrm{e}^{-s/2}}{s^3}\big),$$

which gives (16). It remains for us to prove Lemma 2.11.
Proof of Lemma 2.11 : Let us expand w_2 as in (25)

$$w_2(y,s) = w_{2,0}(s) + w_{2,1}(s)y + \frac{1}{2}w_{2,2}(s)(y^2-2) + w_{2,-}(y,s).$$

Then there exists $\theta \in [0,1]$, $\theta = \theta(s, \sigma_0)$ such that

$$w_{2,2}(s+\sigma_0) = w_{2,2}(s) + \sigma_0 \partial_s w_{2,2}(s+\theta\sigma_0). \tag{52}$$

We estimate $\partial_s w_{2,2}$ in the following Lemma by Filippas and Liu [FL93] (see also [FK92]).

Lemma 2.12 (Filippas-Liu) *There exists a constant $C(p) > 0$ and $\sigma_3 > 0$ such that for $s \geq \sigma_3$,*

$$w_{2,2}(s) \sim -\frac{C(p)}{s} \ \textit{and}\ \partial_s w_{2,2}(s) = \frac{1}{C(p)} w_{2,2}(s)^2 + O\big(w_{2,2}(s)^2\big).$$

Proof of Lemma 2.12 : See Lemma 2.2. and Lemma 2.3. in [FL93]. ■
Because of (50), we have $\tilde{g}(y,s) = g(y,s) + w_2(y,s) - w_2(y, s+\sigma_0)$. Therefore, $\tilde{g}_2(s) = g_2(s) + w_{2,2}(s) - w_{2,2}(s+\sigma_0)$. By using (52), we obtain $\tilde{g}_2(s) = g_2(s) - \sigma_0 \partial_s w_{2,2}(s + \theta\sigma_0)$. From Lemma 2.12 and (48), we obtain

$$\tilde{g}_2(s) = \frac{C_2'}{s^2} - \frac{\sigma_0}{C(p)(s+\theta\sigma_0)^2} + o(\frac{1}{s^2}) = (C_2' - \sigma_0 C(p))\frac{1}{s^2} + o(\frac{1}{s^2})$$

since $\theta \in [0,1]$. If $\sigma_0 = -\frac{C_2'}{C(p)}$, then $\tilde{l}_2(s) = o(\frac{1}{s^2})$. This finishes the proof of Lemma 2.11 and Proposition 2.4. ■

2.2 L^∞ estimates for g in sets $\{| y | \leq K_0 s^{1/2}\}$

This section is devoted to the proof of the following :

Proposition 2.13 $\forall K_0 > 0,\ \exists s_0 \in \mathbb{R},\ \exists C(K_0) > 0,$

$$\forall s \geq s_0,\ \forall \mid y \mid \leq K_0\sqrt{s},\ \mid g(y,s) \mid \leq \frac{C(K_0)}{s}.$$

Moreover if $N = 1$, then there exists $\sigma_0 \in \mathbb{R}$ such that
$\forall K_0 \in \mathbb{R}^{*+},\ \exists s_0 \in \mathbb{R},\ \exists C(K_0) \in \mathbb{R}^{*+}, \forall s \geq s_0,\ \forall \mid y \mid \leq K_0\sqrt{s},$

$$\mid w_1(y,s) - w_2(y, s+\sigma_0) \mid \leq C(K_0)\exp(-s/2)s^{-3/2}.$$

Proof of Proposition 2.13 : We study the general case, the same arguments will lead to the particular case $N = 1$. Let us define $Z(y,s) = \mid g(y,s) \mid s^{-C_0}$ where C_0 is defined in Lemma 2.5 such that

$$\forall (y,s) \in \mathbb{R}^N \times [s_1, +\infty),\ \alpha(y,s) \leq \frac{C_0}{s}.$$

Then, because of Proposition 2.4 and equation (17), Z satisfies

$$\frac{\partial Z}{\partial s} \leq \mathcal{L} Z \quad (53)$$

$$\| Z(s) \|_{L^2_\rho} = O(s^{-2-C_0}). \quad (54)$$

We want to estimate $| Z(y,s) |$ for $| y | \leq K_0\sqrt{s}$ and for large enough s. We use the norm $N_r(\Psi)$ for $\Psi \in L^2_\rho(\mathbb{R}^N)$ defined for all $r > 0$ by

$$N_r(\Psi) = \sup_{|\xi|\leq r} \Big[\int \Psi(y)^2 \rho(y+\xi) dy\Big]^{1/2}. \tag{55}$$

We denote by $S(\tau)$ the semi-group associated with the operator $\mathcal{L}$ defined on $L^2_\rho(\mathbb{R}^N)$ with domain $H^2_\rho(\mathbb{R}^N)$. The kernel of the semi-group $S(\tau)$ is

$$S(\tau, y, z) = \frac{\mathrm{e}^\tau}{\big(4\pi(1-\mathrm{e}^{-\tau})\big)^{N/2}} \exp\Big[-\frac{| y\mathrm{e}^{-\tau/2} - z |^2}{4(1-\mathrm{e}^{-\tau})}\Big]. \tag{56}$$

We shall use the following results stated in [Vel92].

Lemma 2.14 (Velázquez, A linear regularizing effect) *Consider $(r, r') \in (\mathbb{R}^+)^2$, then for any $\tau > 0$ and any ψ such that $N_{r'}(\psi) < +\infty$ we have*

$$N_r\big(S(\tau)\psi\big) \leq \frac{\exp(\tau)}{[4\pi(1-\mathrm{e}^{-\tau})]^{N/2}} \exp\Big(\frac{\exp(-\tau)(r - r'\mathrm{e}^{\tau/2})^2_+}{4(1-\mathrm{e}^{-\tau})}\Big) N_{r'}(\psi) \tag{57}$$

where $s_+ = \max(s, 0)$.

Proof of Lemma 2.14 : See Proposition 2.1. in [Vel92]. ■

Our first aim is to estimate $N_{K_0\sqrt{s}}\big(Z(s)\big)$ for large s. We claim that there exists $s_1(K_0)$ such that

$$\text{for all } s \geq s_1(K_0), \;\; N_{K_0\sqrt{s}}(Z(s)) \leq C K_0^2 s^{-1-C_0}. \tag{58}$$

Let us prove (58). Consider $K_0 > 0$ and define for each $s \in \mathbb{R}$, $\sigma = \sigma(s) < s$ such that

$$e^{\frac{s-\sigma}{2}} = K_0\sqrt{s}. \tag{59}$$

For all $s' \in [\sigma, s]$, we note $r(s', \sigma) = e^{\frac{s'-\sigma}{2}}$. From (53), we have for s large,

$$Z(s) \leq S(s-\sigma)Z(\sigma) = S(s-\sigma-1)S(1)Z(\sigma).$$

Therefore, in $N_{r(s,\sigma)}$ norm, we obtain,

$$N_{r(s,\sigma)}\big(Z(s)\big) \leq N_{r(s,\sigma)}\big(S(s-\sigma-1)S(1)Z(\sigma)\big).$$

Using (57) and noticing that $r(\sigma+1,\sigma)\mathrm{e}^{\frac{s-\sigma-1}{2}} = \mathrm{e}^{\frac{s-\sigma}{2}} = r(s,\sigma)$, which yields

$$\big(r(s,\sigma) - r(\sigma+1,\sigma)\mathrm{e}^{\frac{s-\sigma-1}{2}}\big)_+ = 0,$$

we obtain that there exists a constant C such that

$$N_{r(s,\sigma)}\big(Z(s)\big) \le C\mathrm{e}^{s-\sigma-1} N_{r(\sigma+1,\sigma)}\big(S(1)Z(\sigma)\big).$$

Finally, using (57) with $r' = 0$ and $r = r(\sigma+1,\sigma)$, we obtain an estimate on $N_{r(\sigma+1,\sigma)}\big(S(1)Z(\sigma)\big)$, hence a constant $\tilde{C}$ such that

$$N_{r(s,\sigma)}\big(Z(s)\big) \le \tilde{C}\mathrm{e}^{s-\sigma-1} \,||\, Z(\sigma) \,||_{L^2_\rho} . \tag{60}$$

From (54) and since $r(s,\sigma) = K_0\sqrt{s}$, (60) becomes

$$N_{K_0\sqrt{s}}\big(Z(s)\big) \le \tilde{C}\mathrm{e}^{s-\sigma-1} \,||\, Z(\sigma) \,||_{L^2_\rho} \le CK_0^2 s\sigma^{-2-C_0}.$$

Using (59), we see that $\sigma(s) \sim s$ as $s \to +\infty$. Thus, (58) follows.

It remains for us to estimate $Z(y,s)$ for $|\, y \,| \le \frac{K_0}{4}\sqrt{s}$ by means of $N_{K_0\sqrt{s'}}(Z(s'))$ for some $s' < s$, in order to finish the proof of Proposition 2.13. We do as in [Vel92]. We have from (53)

$$\begin{aligned} 0 \le Z(y,s) &\le S(K_0)Z(y,s-K_0) \\ &\le C(K_0)\int \exp\left[-\frac{|\, y\mathrm{e}^{-K_0/2} - \lambda \,|^2}{4(1-\mathrm{e}^{-K_0})}\right] Z(\lambda, s-K_0)d\lambda. \end{aligned}$$

Let us introduce $N_{K_0\sqrt{s-K_0}}\big(Z(s-K_0)\big)$. We have for all $|\, \xi \,| \le K_0\sqrt{s-K_0}$,

$$Z(y,s) \le C(K_0)\int e^{-\frac{|y\mathrm{e}^{-K_0/2}-\lambda|^2}{4(1-\mathrm{e}^{-K_0})} + \frac{|\lambda+\xi|^2}{8}} e^{-\frac{|\lambda+\xi|^2}{8}} Z(\lambda, s-K_0)d\lambda.$$

By Cauchy-Schwartz's inequality, we obtain for all $|\, \xi \,| \le K_0\sqrt{s-K_0}$,

$$Z(y,s) \le C(K_0)\left(\int \,|\, Z(\lambda, s-K_0) \,|^2 \exp\big[-\frac{|\,\lambda+\xi\,|^2}{4}\big]d\lambda\right)^{1/2} I(K_0,\xi,y)^{1/2},$$

with $I(K_0,\xi,y) = \int \exp\left[-\frac{|y\mathrm{e}^{-K_0/2}-\lambda|^2}{2(1-\mathrm{e}^{-K_0})} + \frac{|\lambda+\xi|^2}{4}\right]d\lambda$.

Therefore, for all $|\, \xi \,| \le K_0\sqrt{s-K_0}$,

$$0 \le Z(y,s) \le C(K_0)N_{K_0\sqrt{s-K_0}}\big(Z(s-K_0)\big)I(K_0,\xi,y)^{1/2}.$$

Taking the infimum in ξ with $|\xi| \leq K_0\sqrt{s-K_0}$ and then the supremum in y with $|y| \leq \frac{K_0\sqrt{s}}{4}$, we obtain for all $|y| \leq \frac{K_0\sqrt{s}}{4}$,

$$Z(y,s) \leq C(K_0) N_{K_0\sqrt{s-K_0}}\big(Z(s-K_0)\big)\Big(\sup_{|y|\leq\frac{K_0\sqrt{s}}{4}} \inf_{|\xi|\leq K_0\sqrt{s-K_0}} I(K_0,\xi,y)\Big)^{\frac{1}{2}}.$$

Note that simple computations yield

$$I(K_0,\xi,y) = \Big(\frac{4\pi(1-\mathrm{e}^{-K_0})}{1+\mathrm{e}^{-K_0}}\Big)^{N/2} \exp\Big(\frac{|\xi + y\mathrm{e}^{-K_0/2}|^2}{2(1+\mathrm{e}^{-K_0})}\Big).$$

Therefore, $\sup_{|y|\leq\frac{K_0\sqrt{s}}{4}} \inf_{|\xi|\leq K_0\sqrt{s-K_0}} |\xi + y\mathrm{e}^{-K_0/2}|^2 = 0$ for all $s \geq s_2(K_0)$ for some $s_2(K_0) \in \mathbb{R}$. Hence, we obtain that there exists $\tilde{C}(K_0)$ such that for all $|y| \leq \frac{K_0\sqrt{s}}{4}$,

$$0 \leq Z(y,s) \leq \tilde{C}(K_0) N_{K_0\sqrt{s-K_0}}\big(Z(s-K_0)\big).$$

Using (58), we obtain for all $s \geq s_3(K_0)$ and $|y| \leq \frac{K_0\sqrt{s}}{4}$, $0 \leq Z(y,s) \leq C_1(K_0)s^{-1-C_0}$. Therefore, for all $s \geq s_3(K_0)$,

$$\sup_{|y|\leq\frac{K_0\sqrt{s}}{4}} |g(y,s)| \leq \frac{C'(K_0)}{s}.$$

Hence Proposition 2.13. ■

2.3 Estimates in the variables (x,t) near the blow-up point

In this subsection we use Proposition 2.13 to prove Theorem 2. As a Corollary of Proposition 2.13 and the estimate (7) which is satisfied by w_{i,a_i,T_i}, we obtain with the notations of Theorem 1 the following :

Corollary 2.15 **(L^∞ estimates in the variables (x,t))** *For all $K_0 > 0$, there exist $\delta_0 \in (0,T_1)$ and $C(K_0) > 0$ such that for all $t \in (T_1-\delta_0, T_1)$ and $x \in B(a_1, K_0\sqrt{(T_1-t)|\log(T_1-t)|})$,*

$$\left|(T_1-t)^{\frac{1}{p-1}} u_1(x,t) - f\left(\frac{x-a_1}{\sqrt{(T_1-t)|\log(T_1-t)|}}\right)\right| \leq \epsilon(K_0,t), \qquad (61)$$

$$\left|(T_1-t)^{\frac{1}{p-1}} \bar{u}_2(x,t) - f\left(\frac{x-a_1}{\sqrt{(T_1-t)|\log(T_1-t)|}}\right)\right| \leq \epsilon(K_0,t), \qquad (62)$$

$$| u_1(x,t) - \bar{u}_2(x,t) | \leq C(K_0)(T_1-t)^{-\frac{1}{p-1}} | \log(T_1-t) |^{-1} \tag{63}$$

where $\bar{u}_2 = \mathcal{T}_{a_1-a_2,T_1-T_2}u_2$ *and* $\epsilon(K_0,t) \to 0$ *as* $t \to T_1$.
Moreover, if $N=1$*, then there exists* $\lambda > 0$ *such that*

$$| u_1(x,t) - \mathcal{T}_{a_1,T_1}\mathcal{D}_\lambda\mathcal{T}_{-a_2,-T_2}u_2(x,t) | \leq C(K_0)\frac{(T_1-t)^{\frac{1}{2}-\frac{1}{p-1}}}{| \log(T_1-t) |^{\frac{3}{2}}}. \tag{64}$$

Proof of Corollary 2.15 : The first part of Corollary 2.15 is straightforward. The second part requires to note the following fact. Consider u and $\underline{u}$ two solutions of (1) which blow-up at time $T=0$ and point $a=0$, consider w and $\underline{w}$ associated respectively to u and $\underline{u}$ by (4). We suppose that there exists σ_0 such that $\underline{w}(y,s) = w(y,s+\sigma_0)$. Then for $t \leq 0$

$$\underline{u}(x,t) = (-t)^{-\frac{1}{p-1}}\underline{w}(y,s)$$

with $y = \frac{x}{\sqrt{-t}}$ and $s = -\log(-t)$. Therefore,
$\underline{u}(x,t) = (-t)^{-\frac{1}{p-1}}w\big(\frac{x}{\sqrt{-t}}, -\log(-t)+\sigma_0\big) = (-t)^{-\frac{1}{p-1}}w\big(\frac{x}{\sqrt{-t}}, -\log(-\frac{t}{e^{-\sigma_0}})\big)$
$= e^{\frac{\sigma_0}{p-1}}u(xe^{\frac{\sigma_0}{2}}, te^{\sigma_0})$.
If we note $\lambda = e^{\sigma_0}$ we have $\underline{u}(t,x) = \mathcal{D}_\lambda u(t,x)$. This closes the proof of Corollary 2.15. ■

By translation invariance of equation (1), we take $a_1 = 0$. We choose $K_0 > 2^7$ and $\beta = 2^6 \in (0, \frac{K_0}{2})$, hence δ_0 and $C(K_0)$ associated with K_0 by Corollary 2.15. Consider the set

$$\Omega = \{(x,t),\ t \in (T_1-\delta_0, T_1),\ | x | \leq K_0(T_1-t)^{\frac{1}{2}} | \log(T_1-t) |^{\frac{1}{2}}\} \tag{65}$$

and define for each $| x | \leq \frac{K_0}{2}\sqrt{T_1 | \log T_1 |}$, $t(x) \in [0,T_1)$ such that

$$| x | = \frac{K_0}{2}\big(T_1 - t(x)\big)^{\frac{1}{2}} | \log\big(T_1-t(x)\big) |^{\frac{1}{2}} . \tag{66}$$

Our aim is to estimate $|u_1(x,t) - \bar{u}_2(x,t)|$ or if $N=1$,
$|u_1(x,t) - \mathcal{T}_{a_1,T_1}\mathcal{D}_\lambda\mathcal{T}_{-a_2,-T_2}u_2(x,t)|$ in a set

$$\{t \in [T_1-\delta_1, T_1),\ \ | x | \leq \epsilon_0\}$$

for some $\delta_1 \in (0,\delta_0)$ and $\epsilon_0 \in (0, \frac{K_0}{2}\sqrt{T_1|\log T_1|})$. By Corollary 2.15 we already have an estimate for $(x,t) \in \Omega$. We need an estimate for (x,t)

outside Ω, namely when $|x| \geq K_0\sqrt{(T-t)|\log(T-t)|}$. Since $\big(x, t(x)\big) \in \Omega$, we will use the ODE comparison of Proposition 2.3 and deduce information for such a (x,t) from information near $\big(x, t(x)\big)$ for which either (63) or (64) holds. We explain the general case $N \geq 1$ and the special case $N = 1$ can be similarly studied. From now on, we restrict to the general case where estimate (63) holds.

Consider (x,t) with $t > t(x)$ and let us note $\tau_1(x,t) = \frac{t-t(x)}{T_1-t(x)}$, $\tau_1 \in (0,1)$. Then for all $\tau \in [0,\tau_1]$ and $|\xi| \leq \beta\sqrt{|\log(T_1-t(x))|}$, we define

$$\begin{aligned} v_1(\xi,\tau) &= \big(T_1-t(x)\big)^{\frac{1}{p-1}} u_1\Big(x+\xi\sqrt{T_1-t(x)}, t(x)+\tau\big(T_1-t(x)\big)\Big) \\ v_2(\xi,\tau) &= \big(T_1-t(x)\big)^{\frac{1}{p-1}} \bar{u}_2\Big(x+\xi\sqrt{T_1-t(x)}, t(x)+\tau\big(T_1-t(x)\big)\Big) \\ \eta(\xi,\tau) &= v_1(\xi,\tau)-v_2(\xi,\tau). \end{aligned} \tag{67}$$

From the scaling property of equation (1) and from Proposition 2.3, we have for $i \in \{1,2\}$ and for all $\tau \in [0,\tau_1]$, $|\xi| \leq \beta\sqrt{|\log(T_1-t(x))|}$ and $\epsilon > 0$,

$$\frac{\partial v_i}{\partial \tau} = \Delta_\xi v_i \quad + \quad | v_i |^{p-1} v_i, \tag{68}$$

$$\frac{\partial \eta}{\partial \tau} = \Delta_\xi \eta \quad + \quad p\, | v_0 |^{p-1} \eta, \ \text{ with } \ v_0 \in (v_1, v_2). \tag{69}$$

$$\Big| \frac{\partial v_i}{\partial \tau} - | v_i |^{p-1} v_i \Big| \quad \leq \quad \epsilon\, | v_i |^p + C(\epsilon)\big(T-t(x)\big)^{\frac{p}{p-1}} \tag{70}$$

where $C(\epsilon) > 0$.
Since $\beta < \frac{K_0}{2}$, we have for all $|\xi| \leq \beta\sqrt{|\log(T_1-t(x))|}$,
$|x+\xi|\log(T_1-t(x))|| \leq K_0\sqrt{(T_1-t)|\log(T_1-t(x))|}$. Therefore, we have from (61), (62) and (63) : for all $|\xi| \leq \beta\sqrt{|\log(T_1-t(x))|}$

$$\left| v_i(\xi,0) - f\left(\frac{K_0}{2} + \frac{\xi}{\sqrt{|\log(T_1-t(x))|}}\right)\right| \quad \leq \quad \epsilon_1(K_0,x) \tag{71}$$

$$|\eta(\xi,\tau)| \quad \leq \quad C|\log(T_1-t(x))|^{-1} \tag{72}$$

where $\epsilon_1(K_0,x) \to 0$ as $x \to 0$.
Since we have from (8), $f\left(\frac{K_0}{2} + \frac{\xi}{\sqrt{|\log(T_1-t(x))|}}\right) \leq f\left(\frac{K_0}{2}-\beta\right) < \kappa$ for all $|\xi| \leq \beta\sqrt{|\log(T_1-t(x))|}$, and since the solution of

$$v' = |v|^{p-1}v, \ \ v(0) = f\left(\frac{K_0}{2}-\beta\right)$$

is well defined and bounded for all $\tau \in [0,1]$, (70) and (71) imply from a priori estimates that for some $\epsilon_0(K_0) > 0$, for all $|x| \le \epsilon_0$, $|\xi| \le \beta\sqrt{|\log(T_1 - t(x))|}$ and $\tau \in [0,1)$,

$$|v_i(\xi,\tau)| \le M(\frac{K_0}{2} - \beta). \tag{73}$$

Now, we apply to η the following parabolic regularity result :

Lemma 2.16 (Parabolic regularity for a linear heat inequality) *Assume that $z(\xi,\tau)$ satisfies for all $|\ \xi\ | \le 4B_1$ and $\tau \in [0,\tau^*]$,*

$$\partial_\tau z \le \Delta z + \lambda z, \ \ z(\xi,0) \le z_0 \text{ and } z(\xi,\tau) \le B_2 \tag{74}$$

where $\tau^* \le 1$. Then, there exists $C > 0$ such that for all $|\ \xi\ | \le B_1$ and for all $\tau \in [0,\tau^*]$,

$$z(\xi,\tau) \le \mathrm{e}^{\lambda\tau}(z_0 + CB_2\mathrm{e}^{-\frac{B_1^2}{4}}).$$

Proof of Lemma 2.16 : See Appendix C in [MZ98b]. ■

In view of (69), (72) and (73), η satisfies the assumptions of Lemma 2.16 with $\tau^* = \tau_1$, $\lambda = p\left(M(\frac{K_0}{2} - \beta)\right)^{p-1}$, $B_1 = \frac{\beta}{4}\sqrt{|\log(T_1 - t(x))|}$, $z_0 = C|\log(T_1 - t(x))|^{-1}$ and $B_2 = 2M(\frac{K_0}{2} - \beta)$.

Therefore, there exists a constant $C_1(\frac{K_0}{2} - \beta) > 0$ such that for all $(\xi,\tau) \in B\big(0, \frac{\beta}{4}\sqrt{|\log(T_1 - t(x))|}\big) \times [0,\tau_1]$,

$$|\eta(\xi,\tau)| \le C_1\left(|\log(T_1 - t(x))|^{-1} + \exp\big(-\frac{\beta^2}{4.4^2}|\log(T_1 - t(x))|\big)\right).$$

Now, since $\beta = 2^6$, we end up with $|\eta(0,\tau_1(x,t))| \le C(K_0)|\log(T_1 - t(x))|^{-1}$ which gives from (67) : If $\epsilon_0(K_0) \ge |x| \ge \frac{K_0}{2}\sqrt{(T_1 - t)|\log(T_1 - t)}$, then

$$|u_1(x,t) - \bar{u}_2(x,t)| \le C(K_0)(T_1 - t(x))^{-\frac{1}{p-1}}|\log(T_1 - t(x))|^{-1}. \tag{75}$$

Let us conclude the proof of Theorem 2. Proposition 2.15 and (75) yield :

- if $|x| \le K_0\sqrt{(T_1 - t)|\log(T_1 - t)|}$ and $t \in (T_1 - \delta_0, T_1)$, then

$|u_1(x,t) - \bar{u}_2(x,t)| \le C(K_0)(T_1 - t)^{-\frac{1}{p-1}}|\log(T_1 - t)|^{-1}$,

- if $\epsilon_0(K_0) \ge |x| \ge K_0\sqrt{(T_1 - t)|\log(T_1 - t)|}$, then

$|u_1(x,t) - \bar{u}_2(x,t)| \le C'(K_0)(T_1 - t(x))^{-\frac{1}{p-1}}|\log(T_1 - t(x))|^{-1}$.

It is easy to see from (66) that

$$\log(T_1 - t(x)) \sim 2\log|x| \text{ and } T_1 - t(x) \sim \frac{2|x|^2}{K_0^2\,|\log|x||} \text{ as } x \to 0. \tag{76}$$

Thus, Theorem 2 falls in the case $N \geq 1$. If $N = 1$, then we have by the same techniques

- if $|x| \leq K_0\sqrt{(T_1 - t)|\log(T_1 - t)|}$ and $t \in (T_1 - \delta_0, T_1)$, then
$|u_1(x,t) - \tilde{u}_2(x,t)| \leq C(K_0)(T_1 - t)^{\frac{1}{2} - \frac{1}{p-1}}|\log(T_1 - t)|^{-\frac{3}{2}}$,
- if $\epsilon_0(K_0) \geq |x| \geq K_0\sqrt{(T_1 - t)|\log(T_1 - t)|}$, then
$|u_1(x,t) - \tilde{u}_2(x,t)| \leq C'(K_0)(T_1 - t(x))^{\frac{1}{2} - \frac{1}{p-1}}|\log(T_1 - t(x))|^{-\frac{3}{2}}$.

Using (76) gives the conclusion for Theorem 2. ■

3 Stability with respect to initial data of the behavior (5)

In this section, we prove Theorem 3. Theorem 3 is in fact a consequence of Theorem 2 and results from [MZ97b], [MZ98b] and [MZ].

We first recall geometric considerations from [MZ97b]. For this, we introduce some useful notations and definitions. Let $\chi_0 \in C_0^\infty([0, +\infty))$ with $\chi_0 \equiv 1$ on $[0, 1]$ and $\chi_0 \equiv 0$ on $[2, +\infty)$. We then fix $K_0 > 0$ and define

$$\chi(y, s) = \chi_0\left(\frac{|y|}{K_0\sqrt{s}}\right). \tag{77}$$

For each $r \in L^\infty(\mathbb{R}^N)$ and $s > 0$, we write $r(y) = r_b(y,s) + r_e(y,s)$ where $r_b(y,s) = r(y)\chi(y,s)$ and $r_e(y,s) = r(y)(1 - \chi(y,s))$. Then, we expand r_b as in (25) and write

$$r(y,s) = r_{b,0}(s) + r_{b,1}(s).y + \frac{1}{2}y^T r_{b,2}(s)y - \text{tr } r_{b,2}(s) + r_{b,-}(y,s) + r_e(y,s). \tag{78}$$

For simplicity in the notations, we drop down the subscript b in $r_{b,m}$ and $r_{b,-}$. However, one should keep in mind that r_m are the components of $r_b = r\chi$ and not those of r.

Definition 3.1 *For each $A > 0$, for each $s > 0$, we define $V_A(s)$ as being the set of all functions r in $L^\infty(\mathbb{R}^N)$ such that*

$$\begin{array}{rclrcl} |r_m(s)| & \leq & As^{-2}, \; m = 0, 1, & |r_2(s)| & \leq & A^2 s^{-2}\log s, \\ |r_-(y,s)| & \leq & A(1 + |y|^3)s^{-2}, & |r_e(y,s)| & \leq & A^2 s^{-\frac{1}{2}}, \end{array} \tag{79}$$

where r is expanded in (78).

We recall in the following Proposition the results of [MZ97b].

Proposition 3.2 (Merle-Zaag, Existence and stability of a solution of (1) with the behavior (5) under the geometric condition of $V_A(s)$) *i) There exists $\hat{u}(t)$ a solution to (1) which blows-up at time $\hat{T}$ at only one blow-up point $\hat{a}$. Moreover, $\hat{w}_{\hat{a},\hat{T}}$ satisfies (11) and $\forall s \geq -\log \hat{T}$, $\hat{q}_{\hat{a},\hat{T}}(s) \in V_{\hat{A}}(s)$ for some $\hat{A} > 0$, where $\hat{w}_{\hat{a},\hat{T}}$, $\hat{q}_{\hat{a},\hat{T}}$ and $V_{\hat{A}}(s)$ are defined respectively in (3), (12) and (79).*

ii) Assume that $\tilde{u}(t)$ is a solution of (1) such that $\tilde{u}(t)$ blows-up at time $\tilde{T}$ at only one point $\tilde{a} \in \mathbb{R}^N$ and $\forall s \geq -\log \tilde{T}$, $\tilde{q}_{\tilde{a},\tilde{T}}(s) \in V_{\tilde{A}}(s)$ for some $\tilde{A} > 0$. Then, there exists $A > \tilde{A}$ such that for all $\epsilon > 0$, there exists a neighborhood $\mathcal{V}_\epsilon$ of $\tilde{u}(0)$ in $L^\infty(\mathbb{R}^N)$ such that $\forall u_0 \in \mathcal{V}_\epsilon$, the solution $u(t)$ to (1) with initial data u_0 blows-up at time T at only one blow-up point a such that

$$|T - \tilde{T}| + |a - \tilde{a}| \leq \epsilon$$

and $w_{a,T}$ satisfies (11) and $\forall s \geq -\log T$, $q_{a,T}(s) \in V_A(s)$.

Proof :
i) See Theorem 1 and subsection 3.2 in [MZ97b].
ii) See in [MZ97b] Theorem 2 and the second and the fourth remark after it, and the first remark in Section 4. Let us emphasize that the results of [MZ97b] actually hold with as Cauchy space $L^\infty(\mathbb{R}^N)$ and not $L^\infty \cap W^{1,p+1}(\mathbb{R}^N)$ as stated there. ■

In the following Proposition, we consider $u(t)$ a solution to (1) which blows-up at time T at only one blow-up point $a \in \mathbb{R}^N$. We use Theorem 2 to estimate the difference $\hat{u}(t) - \mathcal{T}_{a-\hat{a},T-\hat{T}}u(t)$ where $\hat{u}$ is the solution to (1) constructed in i) of Proposition 3.2, which allows us to show that $q_{a,T}$ satisfies the same smallness conditions as $\hat{q}_{\hat{a},\hat{T}}$. More precisely,

Proposition 3.3 (Smallness condition for solutions of (1) with the behavior (5)) *Let $u(t)$ be a solution to (1) which blows-up at time T at only one blow-up point $a \in \mathbb{R}^N$. $u(t)$ satisfies (5) if and only if*

$$\exists A > 0 \text{ such that } \forall s \geq -\log T,\ q_{a,T}(s) \in V_A(s) \tag{80}$$

where $q_{a,T}$ is defined in (12).

It is obvious that Theorem 3 is a direct consequence of ii) in Proposition 3.2 and Proposition 3.3. Thus, we focus on the the proof of Proposition 3.3. ■

Proof of Proposition 3.3 : For simplicity, we write q for $q_{a,T}$.

We first assume (80) and prove that it yields (5). From (78), we write :

$$q(y,s) = q_b(y,s) + q_e(y,s) = q_b(y,s).1_{\{|y|\leq 2K_0\sqrt{s}\}} + q_e(y,s)$$

$= \left(q_0(s) + q_1(s).y + \frac{1}{2}y^T q_2(s)y - \mathrm{tr}q_2(s) + q_-(y,s)\right) 1_{\{|y|\leq 2K_0\sqrt{s}\}} + q_e(y,s)$.
From (79) and the definition of h_m, we get

$$\sup_{y\in\mathbb{R}^N} |q(y,s)| \leq \frac{C(A)}{\sqrt{s}},$$

which implies (7) by (12). Using Proposition 2.1, we get the conclusion.

We now assume (5) and prove (80). If $\hat{u}$ is the solution constructed in i) of Proposition 3.2, then, we have $\forall s \geq -\log \hat{T}$, $\hat{q}_{\hat{a},\hat{T}}(s) \in V_{\hat{A}}(s)$ for some $\hat{A} > 0$. If we define

$$g(y,s) = q_{a,T}(y,s) - \hat{q}_{\hat{a},\hat{T}}(y,s),$$

then, it is enough to prove that for some s_0 and $B > 0$

$$g(s) \in V_B(s) \text{ for all } s \geq s_0. \tag{81}$$

From (12), we have

$$g(y,s) = w_{a,T}(y,s) - \hat{w}_{\hat{a},\hat{T}}(y,s). \tag{82}$$

Applying Theorem 2 (Case $N \geq 1$) and Proposition 2.4 to u and $\hat{u}$, we get through the transformation (3) and straightforward calculations :

$$\forall s \geq s_1,\ \|g(s)\|_{L^2_\rho} \leq \frac{C_0}{s^2},\ \|g(s)\|_{L^\infty} \leq \frac{C_0}{s}, \tag{83}$$

where $s_1 \geq 1$.
We now use (83) in order to estimate g_m, g_- and g_e, and prove (81).

Estimate on g_m, $m \in \{0,1,2\}$: For all $m \in \mathbb{N}$, $|g_m(s)| \leq \|\chi g(s)\|_{L^2_\rho} \leq \|g(s)\|_{L^2_\rho} \leq C_0 s^{-2} \leq C s^{-2}\log s$ by (83).

Estimate on g_e : $|g_e(y,s)| = |(1-\chi)g| \leq |g| \leq C_0 s^{-1}$ by (83).

Estimate on g_- : From (82) and (17), we see that g satisfies for all $s \geq 2s_1 \geq 2$ and $\sigma \leq s-1$,

$$g(s) = S(s-\sigma)g(\sigma) + \int_\sigma^s S(s-\tau)\alpha(\tau)g(\tau)d\tau$$

where $\alpha(y,s)=\frac{|w|^{p-1}w-|\hat{w}|^{p-1}\hat{w}}{w-\hat{w}}-\frac{p}{p-1}$ if $w\neq\hat{w}$ and $\alpha(y,s)=p|w|^{p-1}-\frac{p}{p-1}$ if $w=\hat{w}$, and $S(\tau)$ is the semigroup of $\mathcal{L}$ introduced in (56).
Therefore, $g_-(y,s)=-P_-\left((1-\chi)g(s)\right)+P_-(g(s))=E_1+E_2+E_3$ where

$$\begin{aligned} E_1 &= -P_-\left((1-\chi)g(s)\right), \\ E_2 &= P_-\left(S(s-\sigma)g(\sigma)\right) \text{ and } E_3=P_-\left[\int_\sigma^s S(s-\tau)\alpha(\tau)g(\tau)d\tau\right]. \end{aligned} \tag{84}$$

We now estimate E_1, E_2 and E_3.

If $G=(1-\chi)g$, then we can write as in (25)

$$G(y,s)=G_0(s)+G_1(s).y+\frac{1}{2}y^TG_2(s)y-\text{tr }G_2(s)+G_-(y,s). \tag{85}$$

From (77) and (83), we have

$$|G(y,s)|\leq 1_{\{|y|\geq K_0\sqrt{s}\}}|g(y,s)|\leq\frac{|y|^2}{K_0^2s}\frac{C_0}{s}. \tag{86}$$

From (6), (77), (83) and (26), we write for $m=2$:
$|G_2(s)|=\left|\int M(y)(1-\chi)g\rho dy\right|$ where M is defined in (27). Therefore,
$|G_2(s)|\leq C\int_{\{|y|\geq K_0\sqrt{s}\}}(1+|y|^2)|g(y,s)|e^{-\frac{|y|^2}{8}}e^{-\frac{|y|^2}{8}}dy$
$\leq e^{-\frac{K_0^2s}{8}}\frac{C_0}{s}\int_{\mathbb{R}^N}(1+|y|^2)e^{-\frac{|y|^2}{8}}dy\leq\frac{C}{s}e^{-\frac{K_0^2s}{8}}$. Doing analogous calculations for $m=0$ or 1, we get for all $m\in\{0,1,2\}$,

$$|G_m(s)|\leq\frac{C}{s}e^{-\frac{K_0^2s}{8}}. \tag{87}$$

Combining (85), (86) and (87), we end up with

$$|E_1|=|P_-\left((1-\chi)g\right)(y,s)|=|G_-(y,s)|\leq\frac{C}{s^2}(1+|y|^3). \tag{88}$$

It remains to estimate E_2 and E_3. We use the following estimate on the operator $S(\theta)$:

Lemma 3.4 (Linear estimate for $S(\theta)$) *Let $\theta\geq 1$ and consider $h:\mathbb{R}^N\to\mathbb{R}$ such that*

$$\forall x\in\mathbb{R}^N,\ |h(x)|\leq\mu(1+|x|^3). \tag{89}$$

Then, for all $y\in\mathbb{R}^N$:
i) $|S(\theta)h(y)|\leq C\mu e^\theta(1+|y|^3)$,
ii) $|P_-(S(\theta)h)(y)|=|S(\theta)P_-h(y)|\leq C\mu e^{-\frac{\theta}{2}}(1+|y|^3)$

Proof : see Appendix B. ■

If we set $\theta = s - \sigma$, $h(x) = g(x,\sigma)$, then, using ii) of this Lemma and (83), we obtain

$$|E_2| \leq \frac{C}{\sigma} e^{-\frac{(s-\sigma)}{2}} (1 + |y|^3). \tag{90}$$

We now estimate E_3. Since P_- is the L^2_ρ projector on the negative eigenspace of $\mathcal{L}$, we have

$$E_3 = \int_\sigma^s S(s-\tau) P_- \left(\alpha(\tau) g(\tau)\right) d\tau. \tag{91}$$

From (83) and (28), we have $|\alpha(x,\tau)g(x,\tau)| \leq \frac{C}{\tau^2}(1+|x|^3)$. Therefore, if $\tau \leq s-1$, then we have from ii) of Lemma 3.4

$$|S(s-\tau)P_- \left(\alpha(\tau)g(\tau)\right)(y)| \leq \frac{C}{\tau^2} e^{-\frac{(s-\tau)}{2}} (1+|y|^3). \tag{92}$$

If $\tau \geq s-1$, we can do as we did for G in the proof of the estimate (88) on E_1 to get $|P_-(\alpha(\tau)g(\tau))(x)| \leq \frac{C}{\tau^2}(1+|x|^3)$.

Applying i) of Lemma 3.4, we obtain

$$|S(s-\tau)P_-(\alpha(\tau)g(\tau))(y)| \leq \frac{C}{\tau^2} e^{s-\tau}(1+|y|^3) \leq \frac{C}{\tau^2} e^{-\frac{(s-\tau)}{2}} (1+|y|^3) \tag{93}$$

since $0 \leq s-\tau \leq 1$.

Combining (91), (92) and (93), we get

$$|E_3| \leq \frac{C}{\sigma^2}(1+|y|^3) \int_\sigma^s e^{-\frac{(s-\tau)}{2}} d\tau \leq \frac{C}{\sigma^2}(1+|y|^3). \tag{94}$$

Using (84), (88), (90) and (94), and taking $\sigma = s - 2\log s$, we end up with

$$|g_-(y,s)| \leq \frac{C}{s^2}(1+|y|^3).$$

This concludes the proof of (81) and the proof of Proposition 2.1. This concludes also the proof of Theorem 3. ■

A Proof of Lemma 2.8.

Either $x \sim y$ as $s \to +\infty$, or there exists $\epsilon > 0$ and $(s_n) \in \mathbb{R}$ with $s_n \longrightarrow +\infty$ as $n \to +\infty$ so that $x(s_n) < (1-\epsilon)y(s_n)$. Our first claim is that

$$\exists n_0 \in \mathbb{N}, \ \ \forall s > s_{n_0}, \ \ x(s) < (1-\epsilon)y(s). \tag{95}$$

If (95) is false we can find σ_1 and σ_2 as big as we want such that

$$x(\sigma_2) = (1-\epsilon)y(\sigma_2) \text{ and } \forall \sigma \in [\sigma_1, \sigma_2[,\ \ x(\sigma) \geq (1-\epsilon)y(\sigma).$$

Then necessarily $x'(\sigma_2) \leq (1-\epsilon)y'(\sigma_2)$. But

$$\begin{array}{rl} (x' - (1-\epsilon)y')(\sigma_2) & \geq -\frac{C}{\sigma_2}y(\sigma_2) + (1-\epsilon)\big(\frac{1}{2}y(\sigma_2) - \frac{C}{\sigma_2}y(\sigma_2) - \frac{1}{2}x(\sigma_2)\big) \\ & \geq \frac{\epsilon}{2}(1-\epsilon)y(\sigma_2) - \frac{C}{\sigma_2}(2-\epsilon)y(\sigma_2). \end{array}$$

Therefore, if σ_2 is big enough $x'(\sigma_2) > (1-\epsilon)y'(\sigma_2)$. Hence a contradiction and (95) is true.
From (95), we can define $z(s) = \frac{x(s)}{y(s)} < 1-\epsilon$ for $s > s_{n_0}$. Therefore, from the inequalities satisfied by x and y we have

$$z'(s) \geq -\frac{C}{s} + \frac{1}{2}z(s) - \frac{C}{s}z(s) - \frac{1}{2}z(s)^2 \geq -\frac{C}{s} - \frac{C}{s}z(s) + \frac{\epsilon}{2}z(s).$$

Hence, $\frac{d}{ds}\big(\exp(-\frac{\epsilon}{2}s)s^C z(s)\big) \geq -\frac{C}{s}\exp(-\frac{\epsilon}{2}s)s^C$. This gives
$z(s) \leq C\exp(\frac{\epsilon}{2}s)s^{-C}\int_s^{+\infty}\exp(-\frac{\epsilon}{2}\sigma)\sigma^{C-1}d\sigma$.

$$\begin{array}{rcl} \text{Since } \gamma(s) & = & \int_s^{+\infty}\exp(-\frac{\epsilon}{2}\sigma)\sigma^{C-1}d\sigma \\ & = & \frac{2}{\epsilon}\exp(-\frac{\epsilon}{2}s)s^{C-1} + \frac{2}{\epsilon}(C-1)\int_s^{+\infty}\exp(-\frac{\epsilon}{2}\sigma)\sigma^{C-2}d\sigma \\ & \leq & \frac{2}{\epsilon}\exp(-\frac{\epsilon}{2}s)s^{C-1} + \frac{2}{\epsilon s}(C-1)\gamma(s), \end{array}$$

there exist $C' > 0$ and $s_4 \in \mathbb{R}$ such that for all $s > s_4$,
$\gamma(s) \leq C'\exp(-\frac{\epsilon}{2}s)s^{C-1}$. This yields $z(s) = O(\frac{1}{s})$ as $s \to +\infty$, which closes the proof. ∎

B Linear estimates for the fundamental solution of $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$

We prove Lemma 3.4 in this appendix.

Proof of i) : This follows easily from (56) by using a simple change of variables.

Proof of ii) : From linearity, we can assume $\mu = 1$. Since P_- is the L^2_ρ projector on the negative eigenspace of $\mathcal{L}$ and the eigenfunctions of $\mathcal{L}$ are given by (20), we have $P_-(S(\theta)h) = S(\theta)P_-(h)$ and

$$\forall \beta \in \mathbb{N}^N \text{ with } |\beta| \leq 2,\ \ \int_{\mathbb{R}^N} P_- h(x)x^\beta\rho(x)dx = 0. \tag{96}$$

From (56), we write

$$S(\theta)P_-(h) = \int dxK(y,x)\psi(x) \tag{97}$$

where $\psi(x) = \rho(x)P_-(h(x))$ and

$$K(y,x) = S(\theta,y,x)\rho(x)^{-1} = \frac{e^\theta}{[4\pi(1-e^{-\theta})]^{N/2}}\exp[-\frac{|ye^{-\theta/2}-x|^2}{4(1-e^{-\theta})}]\rho(x)^{-1}. \tag{98}$$

Using (89) and techniques used for G in the proof of (88), we obtain

$$|\psi(x)| \le C\rho(x)(1+|x|^3). \tag{99}$$

Note that (96) is equivalent to the fact that

$$\forall \beta \in \mathbb{N}^N \text{ with } |\beta| \le 2, \ \int_{\mathbb{R}^N} \psi(x)x^\beta dx = 0. \tag{100}$$

We have the following Lemma :

Lemma B.1 (Existence of a fast decaying flux for a fast decaying L^1 function with a null integral)
Assume that $F \in L^1(\mathbb{R}^N,\mathbb{R})$ satisfies $\int_{\mathbb{R}^N} F = 0$ and

$$\textit{if } |y| \le 1, \ |F(y)| \le \frac{C}{|y|^{N-1}}; \ \textit{if } |y| \ge 1, \ |F(y)| \le C\rho(y)|y|^m \tag{101}$$

for some $m \in \mathbb{N}^*$. Then, there exists $U \in L^1(\mathbb{R}^N,\mathbb{R}^N)$ such that $\operatorname{div} U = F$ *in the distribution sense, and*

$$\textit{if } |y| \le 1, \ |U(y)| \le \frac{C}{|y|^{N-1}}; \ \textit{if } |y| \ge 1, \ |U(y)| \le C\rho(y)|y|^{m-1}. \tag{102}$$

We let the proof of this Lemma to the end and use it now to finish the proof of ii) of Lemma 3.4.

From (99), (100), Lemma B.1 and a very simple finite induction, we obtain the existence of $\phi_{ijk} \in W^{3,1}(\mathbb{R}^N,\mathbb{R})$ such that $\sum_{ijk}\partial^3_{ijk}\phi_{ijk} = \psi$ and $\forall i,j,k \in \{1,...,N\}$,

$$\text{if } |y| \le 1, \ |\phi_{ijk}| \le \frac{C}{|y|^{N-1}}; \ \text{if } |y| \ge 1, |\phi_{ijk}| \le C\rho(y) \tag{103}$$

where ∂^3_{ijk} stands for $\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}$.
Therefore, using (97), we have

$$S(\theta)P_-h = \sum_{ijk} \int dx K(y,x)\partial^3_{ijk}\phi_{ijk}(x) = -\sum_{ijk} \int dx \partial^3_{ijk} K(y,x)\phi_{ijk}(x) \tag{104}$$

by integration by parts (this is possible, since $\phi_{ijk} \in W^{3,1}(\mathbb{R}^N)$ and $K(y,.) \in W^{3,\infty}(\mathbb{R}^N)$).
By simple calculations, we can derive from (98) the fact that for $\theta \geq 1$,

$$|\partial^3_{ijk} K(y,x)| \leq Ce^{-\frac{3}{2}\theta}(1 + |x|^3 + |y|^3)K(y,x). \tag{105}$$

Using (104), (105) and (103), we write : if $\theta \geq 1$, then

$$|S(\theta)P_-h| \leq \sum_{ijk} \int |\partial^3_{ijk} K(y,x)||\phi_{ijk}(x)|dx \leq Ce^{-\frac{3}{2}\theta}(I_1 + I_2) \tag{106}$$

where $I_1 = \int_{B(0,1)} K(y,x)(1 + |x|^3 + |y|^3)\frac{dx}{|x|^{N-1}}$ and
$I_2 = \int_{\mathbb{R}^N \setminus B(0,1)} K(y,x)(1 + |x|^3 + |y|^3)\rho(x)dx$.
From (98), $I_1 = \int_{B(0,1)} S(\theta, y, x)\rho(x)^{-1}(1 + |x|^3 + |y|^3)|x|^{1-N}dx$

$$\leq \rho(1)^{-1}(2 + |y|^3)\int_{B(0,1)} |x|^{1-N}dx \leq C(1 + |y|^3). \tag{107}$$

Moreover,

$$I_2 \leq \int_{\mathbb{R}^N} S(\theta, y, x)(1 + |x|^3 + |y|^3)dx \leq Ce^{\theta}(1 + |y|^3) \tag{108}$$

by i) of Lemma 3.4.
Gathering (106), (107) and (108), we get $|S(\theta)P_-h(y)| \leq Ce^{-\frac{\theta}{2}}(1 + |y|^3)$ which is the conclusion of ii) of Lemma 3.4.

Now we prove Lemma B.1.
Proof of Lemma B.1 : We aim at finding U such that for all $q \in C_0^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} F(x)q(x)dx = -\int_{\mathbb{R}^N} U(x).\nabla q(x)dx$.
We follow here a duality method introduced by Bouchut and Perthame (see Lemma A1.1 in [BP98]).
Since $\int F = 0$, we write
$\int_{\mathbb{R}^N} F(x)q(x)dx = \int_{\mathbb{R}^N} F(x)\left(q(x) - q(0)\right)dx = \int_{\mathbb{R}^N} F(x)\int_0^1 x.\nabla q(tx)dt =$

$\int_0^1 dt \int_{\mathbb{R}^N} F(x)x.\nabla q(tx)dx = \int_0^1 t^{-(N+1)}dt \int_{\mathbb{R}^N} F(t^{-1}y)y.\nabla q(y)dy$
$= -\int_{\mathbb{R}^N} U(y).\nabla q(y)dy$ where

$$U(y) = -y\int_0^1 t^{-(N+1)}F(t^{-1}y)dt. \tag{109}$$

The argument will be completed if we show that $U(y)$ satisfies (102) (which implies that $U(y)$ is well defined and is in $L^1(\mathbb{R}^N)$).
From (109) and (101), we have :
- If $|y| \geq 1$, then $|U(y)| \leq |y| \int_0^1 t^{-(N+1)}|F(t^{-1}y)|dt$
$\leq C|y| \int_0^1 t^{-(N+1)} e^{-\frac{|y|^2}{4t^2}} t^{-m}|y|^m dt$
$= C|y|^{1-N} \int_{\frac{|y|^2}{4}}^{+\infty} e^{-\theta}\theta^{\frac{N-2+m}{2}} d\theta \leq C|y|^{1-N}e^{-\frac{|y|^2}{4}}|y|^{N-2+m} = C\rho(y)|y|^{m-1}$.
- If $|y| \leq 1$, then, we have from (101)
$|U(y)| \leq C|y| \left(\int_0^{|y|} t^{-(N+1)}e^{-\frac{|y|^2}{4t^2}}t^{-m}|y|^m dt + \int_{|y|}^1 t^{-(N+1)}|y|^{1-N}t^{N-1}dt\right)$
$\leq C|y|^{2-N}\left(\int_{\frac{1}{4}}^{+\infty} e^{-\theta}\theta^{\frac{N-2+m}{2}}d\theta + \frac{1}{|y|} - 1\right) \leq C|y|^{1-N}$.

This finishes the proof of Lemma B.1, and also the proof of Lemma 3.4.
■

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