ON THE DEPENDENCE OF THE BLOW-UP TIME WITH RESPECT TO THE INITIAL DATA IN A SEMILINEAR PARABOLIC PROBLEM

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ABSTRACT. We find a bound for the modulus of continuity of the blow-up time for the semilinear parabolic problem $u_t = \Delta u + |u|^{p-1}u$, with respect to the initial data.

Introduction.

In this paper we study the dependence with respect to the initial data of the maximal time of existence for solutions of the following problem

$$(1.1) \qquad \begin{cases} u_t = \Delta u + |u|^{p-1}u, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where p is superlinear and subcritical, that is, $p > 1$ and $p(N-2) < N+2$. Here Ω is a bounded convex smooth domain in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$. In case we are dealing with a bounded domain we assume Dirichlet boundary conditions, that is $u(x,t) = 0$, for $(x,t) \in \partial\Omega \times (0,T)$. Also we assume that the initial data, $u_0(x)$, is regular in order to guarantee existence, uniqueness and regularity of the solution.

A remarkable fact is that the solution of parabolic problems may develop singularities in finite time, no matter how smooth the initial data are. It is well known that for many differential equations or systems the solutions can become unbounded in finite time (a phenomena that is known as blow-up). Typical examples where this happens are problems involving nonlinear reaction terms in the equation like (1.1), see [18], [19], [21] and the references therein.

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For our problem, if the maximal solution is defined on a finite time interval, $[0,T)$ with $T < +\infty$, then

$$\lim_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty} = +\infty.$$

We say that u blows up at time T.

The existence of blowing up solutions for (1.1) has been proved by several authors, see [1], [5], [7], [14], etc. The study of the blow-up behaviour for blow-up solutions of (1.1) has attracted a considerable attention in recent years, see for example, [2], [3], [6], [11], [8], [12], [13], [15], [18], [22]. In particular, Giga and Kohn [10] and recently, Giga *et al.* [9] have proved the following upper bound for any solution u of (1.1) that blows up at time T:

$$\forall t \in [0,T), \;\; \|u(t)\|_{L^\infty} \le Cv(t-T) = C\kappa(T-t)^{-\frac{1}{p-1}} \tag{1.2}$$

for some $C > 0$, where $\kappa = (p-1)^{-\frac{1}{p-1}}$ and v is the solution of the associated ODE $v' = v^p$, $v(0) = +\infty$. Let us mention that an easy use of the maximum principle shows that, unlike the constant C in (1.2), the power of $(T-t)$ is optimal, in the sense that

$$\forall t \in [0,T), \;\; \|u(t)\|_{L^\infty} \ge v(t-T) = \kappa(T-t)^{-\frac{1}{p-1}}. \tag{1.3}$$

Later, the following Liouville Theorem has been proved by Merle and Zaag in [16] and [18]:

Consider U a solution of (1.1) *defined for all $(x,t) \in \mathbb{R}^N \times (-\infty,T)$ such that for all $(x,t) \in \mathbb{R}^N \times (-\infty,T)$, $|U(x,t)| \le C(T-t)^{-\frac{1}{p-1}}$. Then, either $U \equiv 0$ or $U(x,t) = [(p-1)(T^*-t)]^{-\frac{1}{p-1}}$ for some $T^* \ge T$.*

This result is the key to find the best constant C in (1.2), as will be presented later.

¿From now on we assume that we are dealing with an initial datum u_0 which produces a solution u that blows up at time T with

$$\|u_0\|_{L^\infty} \le M_0 \text{ and } T \le T_0.$$

For every $h(x)$ with L^∞-norm small enough, the solution u_h of problem (1.1) that has initial datum $u_0(x) + h(x)$ also blows up in finite time, that we call T_h. Indeed, it is known that the blow-up time is continuous with respect to the initial data u_0; see [20], [2], [15] if Ω is bounded; if $\Omega = \mathbb{R}^N$, this was done in [4] under (1.2). Hence, $T_h \to T$ as $\|h\|_{L^\infty} \to 0$.

Our interest here is to provide a bound for $|T - T_h|$ in terms of $\|h\|_{L^\infty}$. To this end, we obviously need estimates on u_h, uniform with respect to h. It is to be noticed that before the proof of the Liouville Theorem of [18], no such estimates were available in the literature. We should emphasize that the only exception to this statement is the fact that the argument of [10] and [9] in the proof of (1.2) *does* hold uniformly with respect to initial data. Indeed, a simple use of the continuity of $u_h(t_0)$ for a fixed t_0, with respect to initial data, shows that Giga and Kohn actually proved that

There exists $C = C(M_0, T_0) > 0$ such that for all h small enough, $\forall (x,t) \in \Omega \times [0, T_h)$,

$$|u_h(x,t)| \le Cv(t - T_h) = C\kappa(T_h - t)^{-\frac{1}{p-1}}. \tag{1.4}$$

See Theorem 2 in [18] and its proof for a simple sketch of this argument.

Using the Liouville Theorem, the estimate (1.4) has been refined in [17] and [18] uniformly with respect to h:

Given $\varepsilon > 0$, there exists $\tau = \tau(M_0, T_0, \varepsilon)$ such that for every h with $\|h\|_{L^\infty}$ small, the solution of problem (1.1) *verifies*

$$|u_h(x,t)| \le \kappa(T_h - t)^{-\frac{1}{p-1}} + \left(\frac{N\kappa}{2p} + \varepsilon\right) \frac{(T_h - t)^{-\frac{1}{p-1}}}{|\ln(T_h - t)|}, \tag{1.5}$$

for every $(x,t) \in \Omega \times [T_h - \tau, T_h)$.

We will call this fact hypothesis (H). Note that all the results of [18] hold under (1.2) which is true for all subcritical p with no sign condition. Hence (H) is valid for any subcritical p.

Now we state our result,

Theorem 1.1. *Assume that hypothesis* (H) *holds. Let T and T_h be the blow-up times for u and u_h, two solutions of problem* (1.1) *with initial data u_0 and $u_0 + h$ respectively, then for every $\varepsilon > 0$ there exist a positive constant $C = C(M_0, T_0, \varepsilon)$ such that for every h smaller than some $\eta(M_0, T_0, \varepsilon) > 0$,*

$$|T - T_h| \le C\|h\|_{L^\infty} |\ln(\|h\|_{L^\infty})|^{\frac{N+2}{2} + \varepsilon}. \tag{1.6}$$

Remark: We suspect that an estimate of the form $|T - T_h| \le C\|h\|_{L^\infty}$ is valid, as has been proved by Herrero and Velázquez [13] in one dimension. However, we cannot get rid of the logarithmic term in (1.6).

Remark: We want to remark that the restriction on p, namely $1 < p < (N+2)/(N-2)$ if $N \ge 3$, is not technical. Indeed, if it does not hold, then the blow-up time is not even continuous as a function of the initial data, see [8].

Proof of Theorem 1.1

First, let us give an idea of the proof. Let u and u_h be the solutions of (1.1) with initial data u_0 and u_0+h respectively. When $T\neq T_h$, the function $\|e(\cdot,t)\|_{L^\infty}=\|(u_h-u)(\cdot,t)\|_{L^\infty}$ will grow from its initial value $\|h\|_{L^\infty}$ to $+\infty$, as $t\to\min(T,T_h)$. If t_0 is the first time where $\|e(t)\|_{L^\infty}$ reaches a given size, we get a bound on $\|e(t)\|_{L^\infty}$ that allows us to control $T-t_0$ and T_h-t_0 in terms of $\|h\|_{L^\infty}$. We then use a triangular inequality to conclude.

The hypothesis (H) is crucial in getting a sharp estimate on $\|e(t)\|_{L^\infty}$. Given $\varepsilon>0$, it is easy to see from (1.5) that (H) holds uniformly for u (resp. for u_h), for all $t\in[T-\tau,T)$ (resp. $t\in[T-\tau,T_h)$), where $\tau=\tau(M_0,T_0,\epsilon)$, whenever $\|h\|_{L^\infty}$ is small enough. Since $T-\frac{\tau}{2}$ is independent of h, we prove in the following lemma that the size of e at $t=T-\frac{\tau}{2}$ is comparable to its size at $t=0$. Therefore, we can make a shift in time to simplify the notation and assume that (H) holds for u (resp. for u_h) for all $t\in[0,T)$ (resp. $t\in[0,T_h)$). We claim the following lemma:

Lemma 1.1. *Given $\varepsilon>0$ there exists a constant $C=C(M_0,T_0,\varepsilon)>0$ such that for every h with $\|h\|_{L^\infty}$ small enough, it holds that u_h, the solution with initial datum u_0+h, is defined on $[0,T-\frac{\tau}{2}]$ where $\tau=\tau(M_0,T_0,\epsilon)$, and moreover*

$$\|u-u_h\|_{L^\infty}(T-\frac{\tau}{2})\leq C\|h\|_{L^\infty}. \tag{1.7}$$

Proof: Accroding to (1.4), there exists a constant $M(M_0,T_0,\epsilon)$ such that

$$M\geq\|u\|_{L^\infty(\Omega\times[0,T-\frac{\tau}{2}])}.$$

Consider a uniformly Lipschitz function $f(u)$ that coincides with the power nonlinearity $|u|^{p-1}u$ for values of u with $|u|\leq M+1$. Now let v and v_h the solutions of a problem like (1.1) with the source term $|u|^{p-1}u$ replaced by $f(u)$ and initial datum u_0 and u_0+h respectively. As $f(u)$ is uniformly Lipschitz we have that both v and v_h are global and moreover there exists a constant C such that

$$\|v-v_h\|_{L^\infty}(T-\frac{\tau}{2})\leq C\|h\|_{L^\infty}.$$

Now we only have to observe that if $\|h\|_{L^\infty}$ is small enough we have that both v and v_h verify that

$$\|v\|_{L^\infty(\Omega\times[0,T-\frac{\tau}{2}])},\|v_h\|_{L^\infty(\Omega\times[0,T-\frac{\tau}{2}])}\leq M+1.$$

Hence, by uniqueness, $v(T-\frac{\tau}{2})$ and $v_h((T-\frac{\tau}{2})$ coincides with $u(T-\frac{\tau}{2})$ and $u_h((T-\frac{\tau}{2}))$ respectively and the result follows. $\square$

Now we can prove Theorem 1.1,

Proof of Theorem 1.1: First of all we observe that for u_0 fixed, the set of functions h such that u_h blows up in finite time is open. So if u blows up, the same occurs with u_h for h small enough. From Lemma 1.1, taking as initial datum $u(T-\frac{\tau}{2})$ and $u_h((T-\frac{\tau}{2}))$ if necessary, we can assume that hypothesis (H) is valid for all t in $[0,T)$ and $[0,T_h)$ respectively.

Let $e(x,t) = u_h(x,t) - u(x,t)$. We have that for every $(x,t) \in \Omega \times [0, \min(T,T_h))$,

$$\left\{ \begin{array}{l} e_t = \Delta e + |u_h|^{p-1}u_h - |u|^{p-1}u = \Delta e + p|\xi(x,t)|^{p-1}e, \\ \\ \|e(x,0)\|_{L^\infty} \le C\|h\|_{L^\infty}, \end{array} \right. \tag{1.8}$$

where $\xi(x,t)$ lies between $u(x,t)$ and $u_h(x,t)$. Given $\varepsilon > 0$, we can define for $\|h\|_{L^\infty}$ small enough, a time $t_0 \in (0,T]$ as the maximal time such that

$$\|e(\cdot,t)\|_{L^\infty} \le \varepsilon \frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|} \quad \text{for all } t \in [0,t_0). \tag{1.9}$$

For times $t \in [0,t_0)$ we have that

$$|\xi(x,t)| \le |u(x,t)| + \varepsilon \frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|}$$

and as we are assuming (H) we get that

$$|\xi(x,t)| \le \kappa (T-t)^{-\frac{1}{p-1}} + \left(\frac{N\kappa}{2p} + 2\varepsilon \right) \frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|},$$

hence,

$$p|\xi(x,t)|^{p-1} \le \frac{p}{p-1}(T-t)^{-1} + \left(\frac{N}{2} + C\varepsilon \right) \frac{(T-t)^{-1}}{|\ln(T-t)|} \equiv \alpha(t).$$

Now, let $\overline{e}(t)$ be the solution of the ODE

$$\overline{e}'(t) = \alpha(t)\overline{e}(t)$$

with initial datum $\overline{e}(0) = C\|h\|_{L^\infty}$. Integrating we obtain

$$\overline{e}(t) = C\|h\|_{L^\infty}(T-t)^{-\frac{p}{p-1}}|\ln(T-t)|^{\frac{N}{2}+C\varepsilon},$$

and, since e is a solution of (1.8), by a comparison argument we have for every $t \in [0,t_0)$,

$$\|e(\cdot,t)\|_{L^\infty} \le \overline{e}(t) = C\|h\|_{L^\infty}(T-t)^{-\frac{p}{p-1}}|\ln(T-t)|^{\frac{N}{2}+C\varepsilon}. \tag{1.10}$$

We want to use this to obtain a bound on $|T-t_0|$ in terms of $\|h\|_{L^\infty}$. To this end let us introduce for $\|h\|_{L^\infty}$ small enough, the time $t_1 = t_1(h) \le T$ as the maximal time such that

$$C\|h\|_{L^\infty}(T-t)^{-\frac{p}{p-1}}|\ln(T-t)|^{\frac{N}{2}+C\varepsilon} \le \varepsilon\frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|}$$

for all $t \in [0, t_1)$.

It is clear from (1.10) that $t_1 \le t_0$, hence $0 \le T - t_0 \le T - t_1$. Note that

$$T - t_1(h) \sim \frac{C}{\varepsilon}\|h\|_{L^\infty}|\ln\|h\|_{L^\infty}|^{\frac{N+2}{2}+C\epsilon} \quad \text{as } \|h\|_{L^\infty} \to 0.$$

Therefore,

$$T - t_0 \le T - t_1 \le \frac{C}{\varepsilon}\|h\|_{L^\infty}\left|\ln\|h\|_{L^\infty}\right|^{\frac{N+2}{2}+C\varepsilon}.$$

Now we observe that (1.4) and the definition of e yield

$$|T_h - t_0| \le C\left(\|u_h(t_0,\cdot)\|_{L^\infty(\Omega)}\right)^{1-p} \le C\left(\|u(t_0,\cdot)\|_{L^\infty} + \|e(t_0,\cdot)\|_{L^\infty}\right)^{1-p}.$$

Using (1.3) and (1.9), we get

$$|T_h - t_0| \le C\left(\|u(t_0,\cdot)\|_{L^\infty(\Omega)} - \varepsilon\frac{(T-t_0)^{-\frac{1}{p-1}}}{\ln(T-t_0)}\right)^{1-p}$$

$$\le C\left(\kappa(T-t_0)^{-\frac{1}{p-1}} - \varepsilon\frac{(T-t_0)^{-\frac{1}{p-1}}}{|\ln(T-t_0)|}\right)^{1-p} \le C(T-t_0),$$

and so we obtain

$$|T - T_h| \le |T - t_0| + |T_h - t_0| \le$$

$$|T - t_0| + C|T - t_0| \le \frac{C}{\varepsilon}\|h\|_{L^\infty}\left|\ln(\|h\|_{L^\infty})\right|^{\frac{N+2}{2}+C\varepsilon}.$$

This completes the proof. □

Remark: The above proof gives a constant $C(M_0, T_0, \varepsilon)$ for (1.6) with $C(M_0, T_0, \varepsilon) \to +\infty$ as $\varepsilon \to 0$.

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