Openness of the set of non characteristic points and regularity of the blow-up curve for the 1 D semilinear wave equation*

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February 26, 2008

Abstract

We consider here the 1 D semilinear wave equation with a power nonlinearity and with no restriction on initial data. We first prove a Liouville Theorem for that equation. Then, we consider a blow-up solution, its blow-up curve $x \mapsto T(x)$ and $I_0 \subset \mathbb{R}$ the set of non characteristic points. We show that I_0 is open and that $T(x)$ is C^1 on I_0. All these results fundamentally use our previous result in [19] showing the convergence in selfsimilar variables for $x \in I_0$.

AMS Classification: 35L05, 35L67

Keywords: Wave equation, Liouville Theorem, regularity, blow-up set.

1 Introduction

1.1 The problem and known results

We consider the one dimensional semilinear wave equation

$$\begin{cases} \partial^2_{tt} u = \partial^2_{xx} u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{1}$$

where $p > 1$, $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}$ where $\|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} = \sup_{a\in\mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$ and $\|v\|^2_{\mathrm{H}^1_{\mathrm{loc,u}}} = \|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\nabla v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}}$.
The Cauchy problem for equation (1) in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$ follows from the finite speed of propagation and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$ (see Ginibre, Soffer and Velo [6]). The existence of blow-up solutions for equation (1) follows from energy techniques (see

*This work was supported by a grant from the french Agence Nationale de la Recherche, project ONDENONLIN, reference ANR-06-BLAN-0185.

Levine [9]). More blow-up results can be found in Caffarelli and Friedman [5], [4], Alinhac [1], [2] and Kichenassamy and Litman [7], [8].

Most of the previous literature considered blow-up for the wave equation from the point of view of prediction. Indeed, most of the papers gave sufficient conditions to have blow-up or constructed special solutions with a prescribed behavior (see [7] and [8] for example). As we did in our earlier work ([19], [17], [16] and [18]), we adopt in this paper a different point of view and aim at describing the blow-up behavior for *any* blow-up solution. More precisely, this paper is dedicated to the regularity of the blow-up curve.

If u is a blow-up solution of (1), we define (see for example Alinhac [1]) a continuous curve Γ as the graph of a function $x \mapsto T(x)$ such that u cannot be extended beyond the set

$$D_u = \{(x,t) \mid t < T(x)\}. \tag{2}$$

The set D_u is called the maximal influence domain of u. From the finite speed of propagation, T is a 1-Lipschitz function. Let $\bar{T}$ be the infimum of $T(x)$ for all $x \in \mathbb{R}$. The time $\bar{T}$ and the surface Γ are called (respectively) the blow-up time and the blow-up surface of u. A point $x_0 \in \mathbb{R}$ is called a non characteristic point if

$$\exists \delta_0 = \delta_0(x_0) \in (0,1) \text{ and } t_0(x_0) < T(x_0) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0,T(x_0),\delta_0} \cap \{t \geq t_0\} \tag{3}$$

where

$$\mathcal{C}_{\bar{x},\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x-\bar{x}|\}. \tag{4}$$

We denote by I_0 the set of non characteristic points. So far, it was commonly thought that $I_0 = \mathbb{R}$, for any blow-up solution. In a forthcoming paper [20], we prove that this is not the case.

Given some (x_0, T_0) such that $0 < T_0 \leq T(x_0)$, we introduce the following self-similar change of variables:

$$w_{x_0,T_0}(y,s) = (T_0-t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x-x_0}{T_0-t}, \quad s = -\log(T_0-t). \tag{5}$$

If $T_0 = T(x_0)$, then we simply write w_{x_0} instead of $w_{x_0,T(x_0)}$. This change of variables transforms the backward light cone with vertex (x_0, T_0) into the infinite cylinder $(y,s) \in B \times [-\log T_0, +\infty)$ where $B = B(0,1)$. The function w_{x_0,T_0} (we write w for simplicity) satisfies the following equation for all $y \in B$ and $s \geq -\log T_0$:

$$\partial^2_{ss} w = \mathcal{L} w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial^2_{y,s} w \tag{6}$$

$$\text{where } \mathcal{L} w = \frac{1}{\rho} \partial_y \left(\rho (1-y^2) \partial_y w\right) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \tag{7}$$

The Lyapunov functional for equation (6)

$$E(w(s)) = \int_{-1}^{1} \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy \tag{8}$$

is defined in

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \;\mid\; \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + \left(q_1'\right)^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \quad (9)$$

In [19], we find the behavior of $w_{x_0}(y,s)$ defined in (5) as $s \to \infty$ where x_0 is a non characteristic or a characteristic point. More precisely, we proved this result (see Corollary 4 and Theorem 2 in [19]):

Blow-up profile near a non characteristic point *There exist positive μ_0 and C_0 such that if u is a solution of (1) with blow-up curve $\Gamma : \{x \mapsto T(x)\}$ and $x_0 \in \mathbb{R}$ is non characteristic (in the sense (3)), then there exist $d(x_0) \in (-1,1)$, $\theta(x_0) = \pm 1$, $s_0(x_0) \geq -\log T(x_0)$ such that for all $s \geq s_0(x_0)$:*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0), .) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s - s_0(x_0))}, \quad (10)$$

where $\mathcal{H}$ and its norm are defined in (9), $\kappa(d,y)$ is defined for all $|d| < 1$ and $|y| \leq 1$ by

$$\kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}}. \quad (11)$$

Moreover, we have

$$E(w_{x_0}(s)) \to E(\kappa_0) \text{ as } s \to \infty \quad (12)$$

and

$$\begin{aligned} &\|w_{x_0}(s) - \theta\kappa(d(x_0), y)\|_{L^\infty(-1,1)} + \|\partial_y w_{x_0}(s) - \theta \partial_y \kappa(d(x_0), y)\|_{L^2(-1,1)} \\ + \;\; &\|\partial_s w_{x_0}(s)\|_{L^2(-1,1)} \to 0 \text{ as } s \to \infty. \end{aligned} \quad (13)$$

Blow-up behavior near a characteristic point *If $x_0 \in \mathbb{R}$ is characteristic, then, there exist $k(x_0) \in \mathbb{N}$, $\theta_i(x_0) = \pm 1$ and continuous $d_i(s) = \tanh \zeta_i(s) \in (-1,1)$ for $i = 1, ..., k$ such that*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k(x_0)} \theta_i(x_0) \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0, \quad (14)$$

$$|\zeta_i(s) - \zeta_j(s)| \to \infty \text{ for } i \neq j \text{ and } E(w_{x_0}(s)) \to k(x_0) E(\kappa_0) \text{ as } s \to \infty.$$

Remark: When $k(x_0) = 0$, the sum in (14) has to be understood as 0. In [20], we prove the existence of solutions with characteristic points. Furthermore, we greatly improve estimate (14) and give a precise description of the set of characteristic points.

1.2 Statement of the results

With the result of [19], we are in a position to address the question of the regularity of the blow-up set Γ for equation (1) and the notion of non characteristic points. Note that by definition (see Alinhac [1]), Γ is the graph of a 1-Lipschitz function $T(x)$. In [5], Caffarelli and Friedman proved that it is a C^1 function for $N \le 3$ under restrictive conditions on initial data that ensure that

$$\text{for all } x \in \mathbb{R}^N \text{ and } t \ge 0, \quad u \ge 0 \text{ and } \partial_t u \ge (1+\delta_0)|\nabla u| \text{ for some } \delta_0 > 0.$$

Later on, they derived in [4] the same result in one dimension for $p \ge 3$ and initial data in $C^4 \times C^3(\mathbb{R})$. The techniques to prove these results are of elliptic type and based on the use of the maximum principle. There is no hope to generalize these techniques to the present situation. Furthermore, no results are available on the set of non characteristic points.
In this paper, we prove the following with no restrictions on the initial data:

Theorem 1 (C^1 regularity of the blow-up set and continuity of the blow-up profile on I_0) *Consider u a solution of (1) with blow-up curve $\Gamma : \{t = T(x)\}$.*
Then, the set of non characteristic points I_0 is open and $T(x)$ is C^1 on that set.
Moreover,

$$\forall x \in I_0, \;\; T'(x) = d(x) \in (-1, 1) \text{ and } \theta(x) \text{ is constant on connected components of } I_0,$$

where $d(x)$ and $\theta(x)$ are such that (10) holds.

Remark: From the remark after Proposition 3.5 (with $\delta_1 = \frac{1}{2}$), we get the existence of a non characteristic point. Thus, I_0 is never empty and Theorem 1 is always meaningful. Unlike what was commonly thought until now, we prove in a forthcoming paper [20] the existence of blow-up solutions to (1) with $\mathbb{R}\backslash I_0 \neq \emptyset$.
Remark: From this theorem, the parameter $d(x)$ related to the blow-up profile in selfsimilar variables (10), has a geometrical interpretation as the slope of $T(x)$. In [21], Nouaili uses the results and techniques of [19] and this paper with a geometrical approach to get more regularity, namely C^{1,μ_0} regularity where the universal constant μ_0 is introduced before (10).
Remark: The techniques are based on a very good understanding of the behavior of the solution in selfsimilar variables in the energy space related to the selfsimilar variable, together with a Liouville Theorem (see Theorem 2). For a similar approach of the blow-up problem, see Martel and Merle [11], [10] for the critical KdV equation; see also Merle and Raphaël [12], [13] for the NLS equation. Note that the main obstruction to extend the result in higher dimensions is the result of classification of all stationary solutions to (6).

The proof of this Theorem relies fundamentally on the convergence result (10) already obtained in [19] and on this Liouville Theorem in the u or w variable:

Theorem 2 (A Liouville Theorem for equation (1)) *Consider $u(x,t)$ a solution to equation (1) defined in the cone $\mathcal{C}_{x^*,T^*,\delta^*}$ (4) such that for all $t < T^*$,*

$$\begin{aligned} &(T^*-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(0,\frac{T^*-t}{\delta_*}))}}{(T^*-t)^{1/2}} \\ &+(T^*-t)^{\frac{2}{p-1}+1} \left(\frac{\|u_t(t)\|_{L^2(B(0,\frac{T^*-t}{\delta_*}))}}{(T^*-t)^{1/2}} + \frac{\|\nabla u(t)\|_{L^2(B(0,\frac{T^*-t}{\delta_*}))}}{(T^*-t)^{1/2}} \right) \le C^* \end{aligned} \tag{15}$$

for some $(x_*, T_*) \in \mathbb{R}^2$, $\delta_* \in (0,1)$ *and* $C^* > 0$.
Then, either $u \equiv 0$ *or* u *can be extended to a function (still denoted by* u*) defined in*

$$\{(x,t) \mid t < T_0 + d_0(x - x^*)\} \supset \mathcal{C}_{T^*,x^*,\delta_*} \text{ by } u(x,t) = \theta_0 \kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T_0 - t + d_0(x-x^*))^{\frac{2}{p-1}}}, \quad (16)$$

for some $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ *and* $\theta_0 = \pm 1$, *where* κ_0 *is defined in (11).*

Using the selfsimilar transformation (6), we have this equivalent formulation for Theorem 2:

Theorem 2' (A Liouville Theorem for equation (6)) *Consider* $w(y,s)$ *a solution to equation (6) defined for all* $(y,s) \in (-\frac{1}{\delta_*}, \frac{1}{\delta_*}) \times \mathbb{R}$ *such that for all* $s \in \mathbb{R}$,

$$\|w(s)\|_{H^1(-\frac{1}{\delta_*},\frac{1}{\delta_*})} + \|\partial_s w(s)\|_{L^2(-\frac{1}{\delta_*},\frac{1}{\delta_*})} \leq C^* \quad (17)$$

for some $\delta_* \in (0,1)$ *and* $C^* > 0$. *Then, either* $w \equiv 0$ *or* w *can be extended to a function (still denoted by* w*) defined in*

$$\{(y,s) \mid -1 - T_0 e^s < d_0 y\} \supset \left(-\frac{1}{\delta_*}, \frac{1}{\delta_*}\right) \times \mathbb{R} \text{ by } w(y,s) = \theta_0 \kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(1 + T_0 e^s + d_0 y)^{\frac{2}{p-1}}}, \quad (18)$$

for some $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ *and* $\theta_0 = \pm 1$, *where* κ_0 *defined in (11).*

Remark: The limiting case $\delta^* = 1$ is still open. The case $\delta^* = 0$ trivially follows from the case $\delta^* > 0$. We expect the result to be valid in higher dimension using the same techniques. The main obstruction comes from the classification of stationary solutions to equation (6), which is only proved in one dimension (see [19]) and remains open in higher dimension.
Remark: From (16), we see that u is a particular solution to (1) which is defined in a half-space of equation $t < T_0 + d_0(x - x_*)$ (and not just the cone $\mathcal{C}_{x^*,T^*,\delta^*}$) and blows-up on a straight line of slope d_0. Note that $u \equiv 0$ corresponds to the limiting case $T_0 = +\infty$. Similarly, w given in (18) is a particular solution to (6). In particular, when $T_0 = 0$, we recover the stationary solution $\theta_0 \kappa(d_0, y)$ defined in (11). Note also that up to a Lorentz transform, the solution given in (16) is space independent and given by

$$u(x,t) = \theta_0 \frac{\kappa_0}{(T_0 - t)^{\frac{2}{p-1}}}.$$

Remark: Note that deriving blow-up estimates through the proof of Liouville Theorems has been successful for different problems. For the case of the heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u \quad (19)$$

where $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $p > 1$ and $(N-2)p < N+2$, the blow-up time T is unique for equation (19). The blow-up set is the subset of $\mathbb{R}^N$ such that $u(x,t)$ does not remain bounded as (x,t) approaches (x_0, T). In [25], [22] and [23] (see also the note [24]), the second author proved the C^2 regularity of the blow-up set under a non degeneracy condition. A Liouville Theorem proved in [15] and [14] was crucially needed for the proof

of the regularity result in the heat equation. In the present work, we will see that the Liouville Theorem (Theorem 2) is crucial for the regularity of the blow-up set for the wave equation (Theorem 1).
Remark: The proof has a completely different structure from the proof for the heat equation (19). It is based on various energy arguments in selfsimilar variables (some of them holding even in the characteristic situation) and on a dynamical result again in selfsimilar variables obtained in [19] in the non characteristic case.

The paper is organized as follows:
- In section 2, we assume the Liouville Theorem and prove Theorem 1.
- In section 3, we prove the Liouville Theorem (Theorems 2 and 2').

The authors wish to thank the referee for his meticulous reading and valuable suggestions which undoubtedly improved the paper.

2 C^1 regularity of the blow-up curve at non characteristic points

This section is devoted to the proof of Theorem 1 assuming Theorem 2.

Proof of Theorem 1: We consider a solution u of (1) with blow-up curve $\Gamma : \{t = T(x)\}$.
We proceed in two parts, each making a separate subsection:
- In subsection 2.1, we consider a non characteristic point x_0 and show that $T(x)$ is differentiable at $x = x_0$ with $T'(x_0) = d(x_0)$ where $d(x_0)$ is such that (10) holds.
- In subsection 2.2, we conclude the proof of Theorem 1.

2.1 Differentiability of the blow-up curve at a given point

In this subsection, we prove the proposition:

Proposition 2.1 (Differentiability of the blow-up curve at a given non characteristic point) *If x_0 is a non characteristic point, then $T(x)$ is differentiable at x_0 and $T'(x_0) = d(x_0)$ where $d(x_0)$ is such that (10) holds.*

Proof: We consider x_0 a non characteristic point. From (3), we have

$$\mathcal{C}_{x_0,T(x_0),\delta_0} \cap \{t \geq t_0\} \subset D_u \tag{20}$$

for some $\delta_0 \in (0, 1)$ and $t_0 < T(x_0)$.
From translation invariance, we can assume that

$$x_0 = T(x_0) = 0.$$

Using the convergence result of [19], we see that (10) holds for some $d(0) \in (-1, 1)$ and $\theta(0) = \pm 1$. Up to replacing $u(x, t)$ by $-u(x, t)$ (also solution to equation (1)), we can assume that $\theta(0) = 1$.

We proceed in 2 steps.

- In Step 1, we use the Liouville Theorem 2 to show that the convergence of $w_0(s)$ in (10) holds also in $H^1 \times L^2 \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)$ where

$$\delta_0' \in (\delta_0, 1) \text{ is fixed.} \tag{21}$$

- In Step 2, we use energy and continuation arguments in selfsimilar variables to conclude the proof by contradiction.

Step 1: Convergence of w_0 to $\kappa(d(0), .)$ on larger sets
We claim:

Lemma 2.2 (Convergence in selfsimilar variables on larger sets) *It holds that*

$$\left\| \begin{pmatrix} w_0(s) \\ \partial_s w_0(s) \end{pmatrix} - \begin{pmatrix} \kappa(d(0), .) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2 \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)} \to 0 \text{ as } s \to \infty.$$

Proof: For simplicity, we denote w_0 by w. Using the uniform bound on the solution at blow-up (Theorem 2' in [17]) and the covering technique in that paper (Proposition 3.3 in [17]), we get for all $s \geq -\log T(0) + 1$,

$$\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} \right\|_{H^1 \times L^2 (-\frac{1}{\delta_0'}, \frac{1}{\delta_0'})} \leq K \tag{22}$$

for some constant K.
We proceed by contradiction and assume that for some $\epsilon_0 > 0$ and some sequence $s_n \to \infty$, we have

$$\forall n \in \mathbb{N}, \left\| \begin{pmatrix} w(s_n) \\ \partial_s w(s_n) \end{pmatrix} - \begin{pmatrix} \kappa(d(0), .) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2 \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)} \geq \epsilon_0 > 0. \tag{23}$$

Let us introduce the sequence

$$w_n(y, s) = w(y, s + s_n). \tag{24}$$

Using the uniform bound stated in (22), we can assume that

$$w_n(0) \rightharpoonup z_0 \text{ in } H^1 \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right) \text{ and } \partial_s w_n(0) \rightharpoonup v_0 \text{ in } L^2 \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right) \tag{25}$$

as $n \to \infty$ for some $(z_0, v_0) \in H^1 \times L^2 \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)$. Since we have from the convergence result (10), the definitions (9) and (24) of the norm in $\mathcal{H}$ and w_n,

$$\left\| \begin{pmatrix} w_n(0) \\ \partial_s w_n(0) \end{pmatrix} - \begin{pmatrix} \kappa(d(0), .) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-1+\epsilon, 1-\epsilon)} \to 0 \text{ as } n \to \infty$$

for any $\epsilon \in (0, 1)$, we deduce from (25) that

$$\forall y \in (-1, 1), z_0(y) = \kappa(d(0), y) \text{ and } v_0(y) = 0 \tag{26}$$

(note that we still need to determine (z_0, v_0) for $1 < |y| < \frac{1}{\delta_0'}$). The following claim allows us to conclude, thanks to the Liouville Theorem 2:

Claim 2.3 (Existence of a limiting object) *There exists $W(y,s)$ a solution to (6) defined for all $(y,s) \in \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right) \times \mathbb{R}$ such that:*
(i) $W(0) = z_0$ and $\partial_s W(0) = v_0$ and the convergence is strong in (25).
(ii) For all $s \in \mathbb{R}$,

$$\left\| \begin{pmatrix} W(s) \\ \partial_s W(s) \end{pmatrix} \right\|_{H^1 \times L^2\left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)} \leq K \tag{27}$$

where K is defined in (22).

Proof: See Appendix A. ■

Indeed, from this claim, we see that $W(y,s)$ satisfies the hypothesis of Theorem 2'. Therefore, either $W \equiv 0$ or there exists $T_0 \geq 0$, $d_0 \in [-\delta_0', \delta_0']$ and $\theta_0 = \pm 1$ such that:

$$\forall (y,s) \in \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right) \times \mathbb{R}, \quad W(y,s) = \theta_0 \kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(1+T_0 e^s + d_0 y)^{\frac{2}{p-1}}} \tag{28}$$

on the one hand, where κ_0 is defined in (11). On the other hand, using (26), (i) of Claim 2.3, and the definition (11) of $\kappa(d,y)$, we see that

$$\forall y \in (-1,1), \quad W(y,0) = z_0(y) = \kappa(d(0), y) = \kappa_0 \frac{(1-d(0)^2)^{\frac{1}{p-1}}}{(1+d(0)y)^{\frac{2}{p-1}}}. \tag{29}$$

Comparing (28) and (29) when $y \in (-1,1)$ and $s = 0$, we see that $\theta_0 = 1$, $d_0 = d(0)$ and $T_0 = 0$, hence, from (28),

$$\forall (y,s) \in \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right) \times \mathbb{R}, \quad W(y,s) = \kappa(d(0), y).$$

In particular, from (24), (25) and (i) of Claim 2.3, this implies that

$$\left\| \begin{pmatrix} w(s_n) \\ \partial_s w(s_n) \end{pmatrix} - \begin{pmatrix} \kappa(d(0), .) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2\left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)} \to 0 \text{ as } n \to \infty,$$

which contradicts (23). Thus, Lemma 2.2 holds. ■

Step 2: Conclusion of the proof of Proposition 2.1

Our goal is to prove that $T(x)$ is differentiable when $x = 0$ and that $T'(0) = d(0)$. We proceed by contradiction. From the fact that $T(x)$ is 1-Lipschitz, we assume that there is a sequence x_n such that

$$x_n \to 0 \text{ and } \frac{T(x_n)}{x_n} \to d(0) + \lambda \text{ with } \lambda \neq 0 \text{ as } n \to \infty. \tag{30}$$

Up to extracting a subsequence and to considering $u(-x,t)$ (also solution to (1)), we can assume that $x_n > 0$. Since 0 is non characteristic, we see from (20) and (21) that

$$\lambda + d(0) \geq -\delta_0 > -\delta_0'. \tag{31}$$

The following corollary transposing the convergence of $w_0(s)$ to $w_{x_n}(s)$ follows from Lemma 2.2:

Corollary 2.4 *Let* $\delta_1 = \frac{1+\delta_0'}{2}$*. For* $\sigma_n' = -\log\left(\frac{\delta_1(T(x_n)+\delta_0' x_n)}{\delta_1-\delta_0'}\right)$*, we have*

$$\left\| \begin{pmatrix} w_{x_n}(\sigma_n') \\ \partial_s w_{x_n}(\sigma_n') \end{pmatrix} - \begin{pmatrix} w_{\pm}(\sigma^*) \\ \partial_s w_{\pm}'(\sigma^*) \end{pmatrix} \right\|_{H^1\times L^2\left(-\frac{1}{\delta_1},\frac{1}{\delta_1}\right)} \to 0 \text{ as } n\to\infty$$

where $\pm = -\operatorname{sgn}\lambda$,

$$\sigma^* = \log\left(\frac{|\lambda|(\delta_1-\delta_0')}{\delta_1(\lambda+d(0)+\delta_0')}\right) \text{ and } w_{\pm}(y,s) = \kappa_0 \frac{(1-d(0)^2)^{\frac{1}{p-1}}}{(1\pm e^s + d(0)y)^{\frac{2}{p-1}}} \tag{32}$$

is a solution to (6).

Proof:
We define for n large enough a time τ_n as the largest t such that the section of the cone $\mathcal{C}_{x_n,T(x_n),\delta_1}$ at time t is included in the cone $\mathcal{C}_{0,0,\delta_0'}$. Note by definition of τ_n that we have

$$B\left(x_n, \frac{T(x_n)-\tau_n}{\delta_1}\right) \subset B\left(0, \frac{-\tau_n}{\delta_0'}\right) \tag{33}$$

and $\tau_n = T(x_n) - \delta_1(\xi_n' - x_n) = -\delta_0'\xi_n'$ for some $\xi_n \in \mathbb{R}$, hence

$$\tau_n = -\frac{\delta_0'}{\delta_1-\delta_0'}(T(x_n)+\delta_1 x_n). \tag{34}$$

From (30), (31) and the fact that $0<\delta_0'<\delta_1<1$, note that $\tau_n \le 0$.
Using the selfsimilar transformation (5), we write

$$\begin{aligned} u(x,t) &= (-t)^{-\frac{2}{p-1}} w_0(y,s) \text{ where } y = \frac{x}{-t},\quad s = -\log(-t), \qquad (35)\\ u(x,t) &= (T(x_n)-t)^{-\frac{2}{p-1}} w_{x_n}(z,\sigma) \text{ where } z = \frac{x-x_n}{T(x_n)-t},\ \sigma = -\log(T(x_n)-t). \end{aligned}$$

Therefore, we have

$$\begin{aligned} w_{x_n}(z,\sigma) &= (1+e^sT(x_n))^{\frac{2}{p-1}} w_0(y,s), & (36)\\ \partial_y w_{x_n}(z,\sigma) &= (1+e^sT(x_n))^{\frac{2}{p-1}+1} \partial_y w_0(y,s), & (37)\\ \partial_s w_{x_n}(z,\sigma) &= (1+e^sT(x_n))^{\frac{2}{p-1}} \Big\{ (1+e^sT(x_n))\partial_s w_0(y,s) \\ &+ \frac{2e^sT(x_n)}{p-1} w_0(y,s) + e^s(yT(x_n)+x_n)\partial_y w_0(y,s) \Big\} & (38) \end{aligned}$$

where

$$y = \frac{z+x_ne^\sigma}{1-e^\sigma T(x_n)} \text{ and } s = \sigma - \log(1-e^\sigma T(x_n)). \tag{39}$$

Using the same process as in (35) with $\kappa(d(0),.)$ instead of w_0 (note that $\kappa(d(0),.)$ is also solution to (6)), we define this solution to (6)

$$\bar{w}_n(z,\sigma) = (1+e^sT(x_n))^{\frac{2}{p-1}}\kappa(d(0),y) = \frac{\kappa_0(1-d(0)^2)^{\frac{1}{p-1}}}{(1+e^\sigma(d(0)x_n - T(x_n)) + d(0)z)^{\frac{2}{p-1}}}. \tag{40}$$

Hence, estimates (36)-(38) hold when (w_{x_n}, w_0) is replaced by $(\bar w_n, \kappa_0)$. If we take now $\sigma = \sigma_n' \equiv -\log(T(x_n) - \tau_n)$, then we see from (39) that $s = \sigma_n \equiv -\log(-\tau_n)$ and from (34) and (30) that as $n \to \infty$,

$$\begin{aligned} &e^{\sigma_n'}(d(0)x_n - T(x_n)) \to -\frac{\lambda(\delta_1 - \delta_0')}{\delta_1(\lambda + d(0) + \delta_0')}, \\ &e^{\sigma_n} x_n \to \frac{\delta_1 - \delta_0'}{\delta_0'(\delta_1 + d(0) + \lambda)} \text{ and } 1 + e^{\sigma_n} T(x_n) \to \frac{\delta_1(\delta_0' + d(0) + \lambda)}{\delta_0'(\delta_1 + d(0) + \lambda)} > 0 \end{aligned} \tag{41}$$

because of (31). Therefore, if n is large enough, we get from (36)-(38)

$$\begin{aligned} &\left\| \begin{pmatrix} w_{x_n}(\sigma_n') \\ \partial_s w_{x_n}(\sigma_n') \end{pmatrix} - \begin{pmatrix} \bar w_n(\sigma_n') \\ \partial_s \bar w_n(\sigma_n') \end{pmatrix} \right\|_{H^1 \times L^2\left(-\frac{1}{\delta_1}, \frac{1}{\delta_1}\right)} \\ \le\ & C \left\| \begin{pmatrix} w_0(\sigma_n) \\ \partial_s w_0(\sigma_n) \end{pmatrix} - \begin{pmatrix} \kappa(d(0), .) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(J_n)} \end{aligned} \tag{42}$$

where the interval $J_n \subset \left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)$ by (33). Since we have from (32), (40) and (41)

$$\left\| \begin{pmatrix} \bar w_n(\sigma_n') \\ \partial_s \bar w_n(\sigma_n') \end{pmatrix} - \begin{pmatrix} w_\pm(\sigma^*) \\ \partial_s w_\pm(\sigma^*) \end{pmatrix} \right\|_{H^1 \times L^2\left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)} \to 0 \text{ as } n \to \infty$$

where $w_\pm(\sigma^*)$ is defined in (32), we get the conclusion of (i) in Lemma 2.4 from Lemma 2.2 and (42). ■

Let us conclude the proof of Lemma 2.2. We consider two cases depending on the sign of λ and reach a contradiction in both cases. Using the selfsimilar transformation (5), we introduce the following solutions to equation (1):

$$v_n(\xi, \tau) = (1-\tau)^{-\frac{2}{p-1}} w_{x_n}(y, s) \text{ with } y = \frac{\xi}{1-\tau}, \quad s = \sigma_n' - \log(1-\tau), \tag{43}$$

and

$$\bar v_\pm(\xi, \tau) = (1-\tau)^{-\frac{2}{p-1}} w_\pm\left(\frac{\xi}{1-\tau}, \sigma^* - \log(1-\tau)\right) = \kappa_0 \frac{(1 - d(0)^2)^{\frac{1}{p-1}}}{((1-\tau) \pm e^{\sigma^*} + d(0)\xi)^{\frac{2}{p-1}}}. \tag{44}$$

From Corollary 2.4, we have

$$\left\| \begin{pmatrix} v_n(0) \\ \partial_\tau v_n(0) \end{pmatrix} - \begin{pmatrix} \bar v_\pm(0) \\ \partial_\tau \bar v_\pm(0) \end{pmatrix} \right\|_{H^1 \times L^2\left(-\frac{1}{\delta_1}, \frac{1}{\delta_1}\right)} \to 0 \text{ as } n \to \infty. \tag{45}$$

Case where $\lambda < 0$. Here, we will reach a contradiction using Corollary 2.4 and the fact that $u(x,t)$ cannot be extended beyond its maximal influence domain D_u defined by (2).

In this case, (45) holds with $\pm = +$. Since v_n and $\bar{v}_+$ are solutions to (1) and $\bar{v}_+$ is defined on $\{(\xi,\tau), \ | \ \tau < 1 + e^{\sigma^*} + d(0)\xi\}$ which contains the closure of $D_+(\tau_0) \equiv \{(\xi,\tau), \ | \ 0 \le \tau < \tau_0 - |\xi|\}$ where τ_0 is fixed in $(1, \min\left(\frac{1}{\delta_1}, 1 + e^{\sigma^*}\right))$, using (45) and the solution of the Cauchy problem for (1), we see that for some $n_0 \in \mathbb{N}$ and for all $n \ge n_0$, $v_n(\xi,\tau)$ is well defined in $D_+(\tau_0)$. Since we have from the selfsimilar transformation (5) and (43)

$$v_n(\xi,\tau) = e^{-\frac{2\sigma_n'}{p-1}} u(x,t) \text{ with } x = x_n + \xi e^{-\sigma_n'}, \ \ t = T(x_n) - e^{-\sigma_n'}(1-\tau),$$

this means that for all $n \ge n_0$, $u(x,t)$ is well defined in the set

$$\{(x,t) \ | \ T(x_n) - e^{-\sigma_n'} \le t \le T(x_n) + (\tau_0 - 1)e^{-\sigma_n'} - |x - x_n|\},$$

which contains $(x_n, T(x_n))$. This is a contradiction by definition (2) of the maximal influence domain.

Case where $\lambda > 0$. Here, a contradiction follows from the fact that $w_{x_n}(y,s)$ exists for all $(y,s) \in (-1,1) \times [-\log T(x_n), +\infty)$ and satisfies a blow-up criterion at the same time.

In this case, (45) holds with $\pm = -$. Since $\bar{v}_-$ is defined on $\{(\xi,\tau), \ | \ \tau < 1 - e^{\sigma^*} + d(0)\xi\}$ which contains the closure of $D_-(\tau_0) \equiv \{(\xi,\tau), \ | \ 0 \le \tau < \min(\tau_0, 1 - |\xi|)\}$ for any $\tau_0 \in \left[0, 1 - \frac{e^{\sigma^*}}{1-|d(0)|}\right)$, using (45) and the solution of the Cauchy problem for (1), we see that for some $n_0(\tau_0) \in \mathbb{N}$ and for all $n \ge n_0$, $v_n(\xi,\tau)$ is well defined in $D_+(\tau_0)$. Moreover,

$$\left\| \begin{pmatrix} v_n(\tau_0) \\ \partial_\tau v_n(\tau_0) \end{pmatrix} - \begin{pmatrix} \bar{v}_-(\tau_0) \\ \partial_\tau \bar{v}_-(\tau_0) \end{pmatrix} \right\|_{H^1 \times L^2(-1+\tau_0, 1-\tau_0)} \to 0 \text{ as } n \to \infty.$$

Using back the transformation (43) and taking $s = \sigma^* - \log(1 - \tau_0)$, we get for all $s \in [\sigma^*, \log(1 - |d(0)|))$,

$$\left\| \begin{pmatrix} w_{x_n}(\sigma_n' - \sigma^* + s) \\ \partial_s w_{x_n}(\sigma_n' - \sigma^* + s) \end{pmatrix} - \begin{pmatrix} w_-(s) \\ \partial_s w_-(s) \end{pmatrix} \right\|_{H^1 \times L^2(-1,1)} \to 0 \text{ as } n \to \infty. \tag{46}$$

Since we have from (46)

$$E\left(w_{x_n}(\sigma_n' - \sigma^* + s)\right) \to E\left(w_-(s)\right) \text{ as } n \to \infty$$

and

$$\forall s \in [s_-, \log(1 - |d(0)|)), \ \ E(w_-(s)) < 0 \text{ for some } s_- < \log(1 - |d(0)|) \tag{47}$$

(see Appendix B for the proof), we see that for all n large enough, we have

$$E(w_{x_n}(\sigma_n' - \sigma^* + s_-)) \le \frac{1}{2} E(w_-(s_-)) < 0. \tag{48}$$

Since by the definition (5), w_{x_n} is defined for all $(y,s) \in (-1,1) \times [-\log T(x_n), +\infty)$, a contradiction follows from the following claim:

Claim 2.5 (Blow-up criterion for equation (6), see Antonini and Merle[3]) *Consider $W(y,s)$ a solution to equation (6) such that $W(y,s_0)$ is defined for all $|y|<1$ and $E(W(s_0))<0$ for some s_0. Then, $W(y,s)$ cannot exist for all $(y,s)\in(-1,1)\times[s_0,\infty)$.*

Proof: See Theorem 2 page 1147 in [3]. ■

Thus, (30) does not hold and $T(x)$ is differentiable at $x=0$ with $T'(0)=d(0)$. This concludes the proof of Proposition 2.1. ■

2.2 Proof of Theorem 1

Let x_0 be a non characteristic point. From (3), we know that (20) holds. One can assume that $x_0=T(x_0)=0$ from translation invariance. From [19] and Proposition 2.1, we know (up to replacing $u(x,t)$ by $-u(x,t)$) that (10) holds with some $d(0)\in(-1,1)$ and $\theta(0)=1$, and that $T(x)$ is differentiable at 0 with

$$T'(0)=d(0). \tag{49}$$

We proceed in 3 steps.

- In Step 1, we use [19] and the fact that 0 is non characteristic to derive that (10) holds in a small neighborhood of 0 for some $d(x)\in(-1,1)$ and $\theta(x)=1$ with $d(x)\to d(0)$ as $x\to 0$.

- In Step 2, using a geometrical construction and the previous step, we show that in a small open interval containing 0, the Lipschitz constant of $T(x)$ is less than $\frac{1+d(0)}{2}$.

- In Step 3, using Steps 1 and 2, we conclude the proof of Theorem 1.

Step 1: Openness of the set of x such that (10) holds

We have from the dynamical study in selfsimilar variables (5) in [19]:

Lemma 2.6 (Convergence in selfsimilar variables for x close to 0) *For all $\epsilon>0$, there exists η such that if $|x|<\eta$ and x is non characteristic, then, (10) holds for w_x with $|d(x)-d(0)|\le\epsilon$ and $\theta(x)=1$.*

Remark: Here, we don't assume that all the points in some neighborhood of 0 are uniformly non characteristic (that is, $\delta_0(x)$ defined in (3) may have no positive lower bound in any neighborhood of 0). We use instead the fact that in [19], we have completely understood the dynamical structure of equation (6) in $\mathcal{H}$ close to the stationary solution $\kappa(d(0),y)$.

Proof:

- Since 0 is non characteristic, we have from (22), for all $s\ge s_1$ for some $s_1\in\mathbb{R}$, $\|(w_0(s),\partial_s w_0(s))\|_{H^1\times L^2\left(-\frac{1}{\delta_0'},\frac{1}{\delta_0'}\right)}\le K$ for some constant K, where $\delta_0'\in(\delta_0,1)$ is fixed. Again from the fact that 0 is non characteristic, note that (10) holds, hence $(w_0(s),\partial_s w_0(s))$ converges to $(\kappa(d(0),.),0)$ as $s\to\infty$ in the norm of $\mathcal{H}$ defined by (9).

- Since for fixed s, we have $(w_x(y,s),\partial_s w_x(y,s))\to(w_0(y,s),\partial_s w_0(y,s))$ in $\mathcal{H}$ from the continuity of solutions to equation (6) with respect to initial data, we know that for all $\epsilon>0$, there exists $s_0(\epsilon)\ge s_1$ and $\eta(\epsilon)>0$ such that for all $x\in(-\eta(\epsilon),\eta(\epsilon))$,

$$\left\|\begin{pmatrix} w_x(\cdot,s_0(\epsilon))\\ \partial_s w_x(\cdot,s_0(\epsilon))\end{pmatrix}-\begin{pmatrix}\kappa(d(0),.)\\ 0\end{pmatrix}\right\|_{\mathcal{H}}\le\epsilon.$$

- From Theorem 3 in [19] (use in particular the first remark following Theorem 3), for a small enough fixed $\epsilon > 0$, we have that for all $x \in (-\eta(\epsilon), \eta(\epsilon))$, there exists $d(x)$ such that

$$\left\| \begin{pmatrix} w_x(y,s) \\ \partial_s w_x(y,s) \end{pmatrix} - \begin{pmatrix} \kappa(d(x),y) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty$$

and

$$|d(x) - d(0)| \le C\epsilon.$$

This conclude the proof of Lemma 2.6. ■

Step 2: The Lipschitz constant of $T(x)$ around 0 is less than $(1+|d(0)|)/2$
Fix ϵ_0 small enough such that

$$\epsilon_0 > 0 \text{ and } d(0) + 2\epsilon_0 < 1. \tag{50}$$

Using (49) and Lemma 2.6, we see that there exists $\eta_0 > 0$ such that

$$\forall |x| \le \eta_0, \quad |T(x) - T(0) - d(0)x| \le \epsilon_0 |x|, \tag{51}$$

and if in addition, x is non characteristic in the sense (3), then, (10) holds for w_x with

$$|d(x) - d(0)| \le \epsilon_0. \tag{52}$$

We now claim:

Lemma 2.7 (The slope of $T(x)$ around 0 is less than $(1+|d(0)|)/2$) *It holds that*

$$\forall x,\ y \in [-\frac{\eta_0}{10}, \frac{\eta_0}{10}], \quad |T(x) - T(y)| \le \frac{1+|d(0)|}{2}|x-y|.$$

Proof: We proceed by contradiction and assume that

$$\text{for some } x_0 \text{ and } y_0 \in [-\frac{\eta_0}{10}, \frac{\eta_0}{10}], \quad |T(x_0) - T(y_0)| > \frac{1+|d(0)|}{2}|x_0 - y_0|. \tag{53}$$

Up to considering $u(-x,t)$ (also a solution to equation (1)) and to renaming x_0 and y_0, we can assume that

$$y_0 < x_0, T(y_0) \le T(x_0) \text{ and } \frac{1+|d(0)|}{2}(x_0 - y_0) \le T(x_0) - T(y_0). \tag{54}$$

We can also assume that y_0 is the minimum in $[-\eta_0, x_0]$ satisfying (54). Therefore, we have

$$\forall x \in [-\eta_0, y_0), \quad \frac{1+|d(0)|}{2}(x_0 - x) \ge T(x_0) - T(x). \tag{55}$$

Let us define

$$x^* \in [y_0, x_0] \text{ such that } T(x^*) - d(0)x^* = \min_{y_0 \le x \le x_0} T(x) - d(0)x. \tag{56}$$

Note from (54) that $x^* < x_0$. Indeed, if not, then $T(x_0) - d(0)x_0 \le T(y_0) - d(0)y_0$, hence, $T(x_0) - T(y_0) \le d(0)(x_0 - y_0)$ which contradicts (54).

Considering a family of straight lines of slope $d(0)+2\epsilon_0$, there exists one which is "tangent" to the blow-up curve on $[x^*, x_0]$ at some point $(m^*, T(m^*))$ with

$$-\eta_0 \le y_0 \le x^* \le m^* \le x_0, \tag{57}$$

in the sense that

$$\forall x \in [x^*, x_0], \;\; T(m^*) + (d(0)+2\epsilon_0)(x-m^*) \le T(x). \tag{58}$$

We have the following:

Claim 2.8 (m^* is a non characteristic point) *There exists $\eta_1 > 0$ such that*

$$\forall x \in [m^*-\eta_1, m^*+\eta_1], \;\; T(m^*) - \frac{1+|d(0)|}{2}|x-m^*| \le T(x). \tag{59}$$

Proof: We claim first that

$$-\frac{\eta_0}{2} < y_0 < x_0, \tag{60}$$

which yields by minimality in (55)

$$\frac{(1+|d(0)|)}{2}(x_0-y_0) = T(x_0) - T(y_0) \tag{61}$$

Note first from (54) and (51) that

$$\begin{aligned} x_0 - y_0 &\le \frac{2}{1+|d(0)|}(T(x_0)-T(y_0)) \\ &\le \frac{2}{1+|d(0)|}\left[(T(x_0)-T(0)) - (T(y_0)-T(0))\right] \\ &\le \frac{2d(0)}{1+|d(0)|}(x_0-y_0) + \frac{2\epsilon_0}{1+|d(0)|}\left(|x_0|+|y_0|\right). \end{aligned} \tag{62}$$

Since $\frac{2|d(0)|}{1+|d(0)|} < 1$, this yields

$$x_0 - y_0 \le C(d(0))\epsilon_0\left(|y_0|+|x_0|\right) \le 2C(d(0))\epsilon_0\eta_0 \le \frac{\eta_0}{10}$$

for ϵ_0 small enough. Since $|x_0| \le \frac{\eta_0}{10}$ by(53), (60) follows.

Let us now prove (59). First note that $m^* < x_0$. Indeed, from (56), (61), (57) and (54), we have $T(x^*) \le T(y_0) + d(0)(x^*-y_0) = T(x_0) + d(0)(x^*-x_0) - \left(\frac{(1+|d(0)|)}{2} - d(0)\right)(x_0 - y_0) < T(x_0) + (d(0)+2\epsilon_0)(x^*-x_0)$ since $|d(0)| < 1$, so (58) cannot hold with $m^* = x_0$ (it fails with $x = x^*$).

- If $m^* \in (y_0, x_0)$, then $T(m^*)$ is a local minimum for $T(x) - (d(0)+2\epsilon_0)(x-m^*)$, hence, since $d(0)+2\epsilon_0 < 1$ by (50), m^* is non characteristic.
- If $m^* = y_0$, then $y_0 = x^* = m^*$ by (57). From (55) and the fact that $T(y_0) - d(0)y_0$ is the minimum of $T(x) - d(0)x_0$ for $x \in [y_0, x_0]$, we see that m^* is non characteristic, which concludes the proof of Claim 2.8. ■

Thus, m^* is non characteristic in the sense (3). Using (52) and Proposition 2.1, we see that $T(x)$ is differentiable at $x = m^*$ and

$$|T'(m^*) - d(0)| = |d(m^*) - d(0)| \le \epsilon_0$$

on the one hand. On the other hand, from (57), (58) and the fact that $m^* < x_0$, we have $T'(m^*) \ge d(0) + 2\epsilon_0$, which leads to a contradiction. This concludes the proof of Lemma 2.7. ■

Step 3: Conclusion of the proof
Using Lemma 2.7, we see that for all $x \in [-\frac{\eta_0}{20}, \frac{\eta_0}{20}]$, x is non characteristic in the sense (3). Using Proposition 2.1, we see that T is differentiable at x and $T'(x) = d(x)$ where $d(x)$ is such that (10) holds for w_x. Using Lemma 2.6, we see from (49) that $T'(x) = d(x) \to d(0) = T'(0)$ as $x \to 0$ and $\theta(x) = 1$. This concludes the proof of Theorem 1. ■

3 Proof of the Liouville Theorem

Remark first that Theorem 2' follows from Theorem 2.

Proof of Theorem 2' assuming Theorem 2 : Consider $w(y, s)$ a solution to equation (6) defined for all $(y, s) \in (-\frac{1}{\delta_*}, \frac{1}{\delta_*}) \times \mathbb{R}$ for some $\delta^* \in (0, 1)$ such that for all $s \in \mathbb{R}$, (17) holds. If we introduce the function $u(x, t)$ defined by

$$u(x,t) = (-t)^{-\frac{2}{p-1}} w(y, s) \text{ where } y = \frac{x}{-t} \text{ and } s = -\log(-t), \tag{63}$$

then we see that $u(x, t)$ satisfies the hypotheses of Theorem 2 with $T_* = x_* = 0$, in particular (15) holds. Therefore, the conclusion of Theorem 2 holds for u. Using back (63), we directly get the conclusion of Theorem 2'. ■

The rest of the section is now devoted to the proof of Theorem 2.

Proof of Theorem 2: Consider a solution $u(x, t)$ to equation (1) defined in the backward cone $\mathcal{C}_{x_*,T_*,\delta^*}$ (see (4)) such that (15) holds, for some $(x_*, T_*) \in \mathbb{R}^2$ and $\delta_* \in (0, 1)$. From the bound (15) and the resolution of the Cauchy problem of equation (1), we can extend the solution by a function still denoted by $u(x, t)$ and defined in some influence domain D_u of the form

$$D_u = \{(x, t) \in \mathbb{R}^2 \mid t < T(x)\} \tag{64}$$

for some 1-Lipschitz function $T(x)$ where one of the following cases occurs:
- Case 1: For all $x \in \mathbb{R}$, $T(x) \equiv +\infty$.
- Case 2: For all $x \in \mathbb{R}$, $T(x) < +\infty$. In this case, since $u(x, t)$ is known to be defined on C_{x_*,T_*,δ_*} (4), we have $\mathcal{C}_{x_*,T_*,\delta_*} \subset D_u$, hence from (4) and (64)

$$\forall x \in \mathbb{R}, \;\; T(x) \ge T_* - \delta_*|x - x_*|. \tag{65}$$

In this case, we will denote the set of non characteristic points by I_0.

We proceed in 4 steps:

- In Step 1, we consider $w_{\bar{x},\bar{T}}$ defined in (5) where $(\bar{x},\bar{T})$ is arbitrary in $\bar{D}_u$ and show it converges to a stationary solution of (6) as $s\to-\infty$. Using some energy estimates from [16], we conclude the proof of Theorem 2 when Case 1 holds.

- From this step on, we focus on Case 2. In Step 2, we show that there exists a non characteristic point (in the sense (3)) with a given location.

- In Step 3, we first use the convergence result (10) to show that when x_0 is a non characteristic point, w_{x_0} is a stationary solution to (6). It follows then that I_0 is an interval. We also reduce the proof of Theorem 2 to the fact that $I_0=\mathbb{R}$.

- In Step 4, we proceed by contradiction and assume that $I_0\neq\mathbb{R}$. We first show that the blow-up set is a straight line of slope ±1 outside I_0 (that is, on the set of characteristic points). Then, we conclude the proof thanks to a space-time energy argument applied at a characteristic point.

Step 1: Behavior for $s\to-\infty$ of $w_{\bar{x},\bar{T}}(s)$ and conclusion when Case 1 holds
We first recall some dispersion estimates from [3] and [16]:

Lemma 3.1 (A Lyapunov functional for equation (6)) *Consider $w(y,s)$ a solution to (6) defined for all $(y,s)\in(-1,1)\times[s_0,+\infty)$ for some $s_0\in\mathbb{R}$. Then:*
(i) For all $s_2\geq s_1\geq s_0$, we have

$$E(w(s_2))-E(w(s_1))=-\frac{4}{p-1}\int_{s_1}^{s_2}\int_{-1}^{1}(\partial_s w(y,s))^2\frac{\rho(y)}{1-y^2}dyds$$

where E is defined in (8).
(ii) For all $s\geq s_0+1$, $\displaystyle\int_{-\frac12}^{\frac12}|w(y,s)|^{p+1}dy\leq C(E(w(s_0))+1)^p$.

Proof: See [3] for (i). For (ii), see Proposition 2.2 in [17] for a statement and the proof of Proposition 3.1 page 1156 in [16] for the proof. ■

Using energy arguments together with the finite speed of propagation similar to what we did in the blow-up situation (see [19]), we claim:

Proposition 3.2 (Behavior of $w_{\bar{x},\bar{T}}(s)$ as $s\to-\infty$)
For any $(\bar{x},\bar{T})\in\bar{D}_u$, it holds that as $s\to-\infty$,
either

$$\|w_{\bar{x},\bar{T}}(s)\|_{H^1(-1,1)}+\|\partial_s w_{\bar{x},\bar{T}}(s)\|_{L^2(-1,1)}\to0 \text{ in } H^1\times L^2(-1,1),$$

or for some $\theta(\bar{x},\bar{T})=\pm1$,

$$\inf_{|d|<1}\left\|w_{\bar{x},\bar{T}}(s)-\theta(\bar{x},\bar{T})\kappa(d,.)\right\|_{H^1(-1,1)}+\left\|\partial_s w_{\bar{x},\bar{T}}(s)\right\|_{L^2(-1,1)}\to0$$

where $\kappa(d,y)$ is defined in (11).

Remark: Here, from the fact that $\delta^*<1$, we have

$$\{t\leq\bar{\tau}\}\cap\mathcal{C}_{\bar{x},\bar{T},\frac{1+\delta^*}{2}}\subset\{t\leq\bar{\tau}\}\cap\mathcal{C}_{x^*,T^*,\delta^*}\subset D_u$$

for some $\bar{\tau}$, hence all the point $(\bar{x}, \bar{T})$ are "non characteristic" (in a sense adapted to $s \to -\infty$).

Proof: The proof is similar to what we did as $s \to \infty$ in Proposition 3 in [19] (see [19] for details). We claim first that for some $\bar{s} \in \mathbb{R}$, we have

$$\forall s \le \bar{s}, \quad \|w_{\bar{x},\bar{T}}(s)\|_{H^1(-1,1)} + \|\partial_s w_{\bar{x},\bar{T}}(s)\|_{L^2(-1,1)} \le C \tag{66}$$

where C is related to the bound given in the hypotheses of Theorem 2. Let us prove the estimate for $\|w_{\bar{x},\bar{T}}(s)\|_{L^2(-1,1)}$ first. The estimate for $\partial_s w_{\bar{x},\bar{T}}$ and $\partial_y w_{\bar{x},\bar{T}}$ follows in a similar way. From the selfsimilar transformation (5), we have

$$w_{\bar{x},\bar{T}}(y,s) = e^{-\frac{2s}{p-1}} u\left(\bar{x} + ye^{-s}, \bar{T} - e^{-s}\right).$$

Therefore,

$$\int_{-1}^{1} w_{\bar{x},\bar{T}}(y,s)^2 dy = (\bar{T} - t)^{\frac{4}{p-1}-1} \int_{B(\bar{x},\bar{T}-t)} u(x,t)^2 dx \text{ where } t = \bar{T} - e^{-s}.$$

Since $\delta^* < 1$, there exists $\bar{s}(\bar{x}, \bar{T}) \in \mathbb{R}$ such that for all $s \le \bar{s}(\bar{x}, \bar{T})$ (or $t \le \bar{t}(\bar{x}, \bar{T}) = \bar{T} - e^{-\bar{s}}$)

$$B(\bar{x}, \bar{T} - t) \subset B\left(x^*, \frac{T^* - t}{\delta^*}\right),$$

we see that the bound on $\|w_{\bar{x},\bar{T}}(s)\|_{L^2(-1,1)}$ follows from (15). Using Lemma 3.1, we see from (66) that

$$\forall s \le \bar{s}, \quad |E(w_{\bar{x},\bar{T}}(s))| \le C \text{ and } \int_{-\infty}^{\bar{s}} \int_{-1}^{1} |\partial_s w_{\bar{x},\bar{T}}(s)|^2 \frac{\rho(y)}{1-y^2} dy ds \le C,$$

and from the monotonicity of $E(w_{\bar{x},\bar{T}}(s))$, we see that $E(w_{\bar{x},\bar{T}}(s)) \to E_-$ as $s \to -\infty$.

We now follow exactly the steps of the proof of Proposition 3 in Section 6 of [19]. See [19] for more details.

- From Proposition 6.2 in [19], we reduce to prove that

$$\inf_{\tilde{w} \in S} \left\| \begin{pmatrix} w_{\bar{x},\bar{T}}(s) \\ \partial_s w_{\bar{x},\bar{T}}(s) \end{pmatrix} - \begin{pmatrix} \tilde{w} \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-1,1)} \to 0 \text{ as } s \to -\infty, \tag{67}$$

where S is the set of $H^1(-1,1)$ stationary solutions to (6).
Recall from Proposition 2 in [19] that $S = S_1 \cup S_2 \cup \{0\}$ where

$$S_1 = \{\kappa(d,.) \mid |d| < 1\} \text{ and } S_2 = \{-\kappa(d,.) \mid |d| < 1\}.$$

- Then, we proceed by contradiction to prove (67) and assume that there exists $s_n \to -\infty$ as $n \to \infty$ and $\epsilon_0 > 0$ such that for all $n \in \mathbb{N}$,

$$\inf_{\tilde{w} \in S} \left\| \begin{pmatrix} w_{\bar{x},\bar{T}}(s_n) \\ \partial_s w_{\bar{x},\bar{T}}(s_n) \end{pmatrix} - \begin{pmatrix} \tilde{w} \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-1,1)} \ge \epsilon_0 > 0.$$

As in Lemma 6.4 in [19], we prove that there exists $\tilde{s}_n \to -\infty$ as $n \to \infty$ such that for some $w^* \in S$,

$$\begin{pmatrix} w_{\bar{x},\bar{T}}(\tilde{s}_n) \\ \partial_s w_{\bar{x},\bar{T}}(\tilde{s}_n) \end{pmatrix} \rightharpoonup \begin{pmatrix} w^* \\ 0 \end{pmatrix} \text{ weakly in } H^1 \times L^2(-1,1) \text{ as } n \to \infty.$$

By the same way as in Lemma 6.5 in [19], we obtain a contradiction. This concludes the proof of Proposition 3.2. ■

From Proposition 3.2, we derive the behavior of the Lyapunov functional $E(w(s))$ defined by (8) as $s \to -\infty$.

Corollary 3.3 (Behavior of $E(w_{\bar{x},\bar{T}}(s))$ as $s \to -\infty$)
(i) For all $d \in (-1,1)$,

$$E(\kappa(d,.)) = E(-\kappa(d,.)) = E(\kappa_0) > 0. \tag{68}$$

(ii) For any $(\bar{x},\bar{T}) \in \bar{D}_u$, either $E(w_{\bar{x},\bar{T}}(s)) \to 0$ or $E(w_{\bar{x},\bar{T}}(s)) \to E(\kappa_0) > 0$ as $s \to -\infty$. In particular,

$$\forall s \in \mathbb{R}, \quad E(w_{\bar{x},\bar{T}}(s)) \le E(\kappa_0). \tag{69}$$

Proof: (ii) is a direct consequence of Proposition 3.2, the definition (8) of $E(w(s))$ and (i). Thus we only prove (i). Since $\kappa(d,y)$ is a stationary solution to (6), we have

$$\mathcal{L}\kappa(d,y) - \frac{2(p+1)}{(p-1)^2}\kappa(d,y) + |\kappa(d,y)|^{p-1}\kappa(d,y) = 0.$$

Multiplying this equation by $\kappa(d,y)\rho(y)$ and integrating with respect to $y \in (-1,1)$, we get from the definition (7) of $\rho(y)$

$$-\int_{-1}^{1} |\partial_y \kappa(d,y)|^2(1-y^2)\rho(y) - \frac{2(p+1)}{(p-1)^2}\int_{-1}^{1} \kappa(d,y)^2\rho(y)dy + \int_{-1}^{1} \kappa(d,y)^{p+1}\rho(y)dy = 0.$$

Therefore, from the definition (8) of $E(\kappa(d,.))$, we see that

$$E(\kappa(d,.)) = \frac{p-1}{2(p+1)}\int_{-1}^{1} \kappa(d,y)^{p+1}\rho(y)dy. \tag{70}$$

Making the change of variables $Y = \frac{y+d}{1+dy}$, we see from the definitions (11) and (7) of $\kappa(d,y)$ and $\rho(y)$ that

$$\frac{p-1}{2(p+1)}\int_{-1}^{1} \kappa(d,y)^{p+1}\rho(y)dy = \frac{p-1}{2(p+1)}\kappa_0^{p+1}\int_{-1}^{1} \rho(Y)dY = E(\kappa_0) > 0.$$

Thus, (68) follows from (70). This concludes the proof of Corollary 3.3. ■

This result allows us to conclude the proof of the Liouville Theorem by energy arguments, when Case 2 holds. Indeed,

Corollary 3.4 (Conclusion of the proof when Case 2 holds) *If for all $x \in \mathbb{R}$, $T(x) \equiv +\infty$, then $u \equiv 0$.*

Proof: In this case, $u(x,t)$ is defined for all $(x,t) \in \mathbb{R}^2$. The conclusion is a consequence of the uniform bounds stated in the hypothesis of Theorem 2 and the bound for solutions of equation (6) in terms of the Lyapunov functional stated in (ii) of Lemma 3.1. Indeed, consider for arbitrary $t \in \mathbb{R}$ and $T > t$ the function $w_{0,T}$ defined from $u(x,t)$ by means of the transformation (5). Note that $w_{0,T}$ is defined for all $(y,s) \in \mathbb{R}^2$. If $s = -\log(T-t)$, then we see from (ii) in Lemma 3.1 and (69) that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |w_{0,T}(y,s)|^{p+1} dy \le C(E(w_{0,T}(s_0)) + 1)^p \le C(E(\kappa_0) + 1)^p \equiv C_1.$$

Using (5), this gives in the original variables

$$\int_{-\frac{T-t}{2}}^{\frac{T-t}{2}} |u(x,t)|^{p+1} dx \le C_1 (T-t)^{-\frac{2(p+1)}{p-1}+1}.$$

Fix t and let T go to infinity to get $u(x,t) = 0$ for all $x \in \mathbb{R}$, and then $u \equiv 0$, which concludes the proof of Corollary 3.4 and thus the proof of Theorem 2 in the case where $T(x) \equiv +\infty$. ■

Step 2: Existence of non characteristic point with a given location

From now on, we assume that Case 2 holds. We claim the following general result on the existence of a non characteristic point in a given cone with slope $\delta_1 > 1$.

Proposition 3.5 (Existence of a non characteristic point with a given location) *For all $x_1 \in \mathbb{R}$ and $\delta_1 \in (\delta_*, 1)$, there exists $x_0 = x_0(x_1, \delta_1)$ such that*

$$(x_0, T(x_0)) \in \bar{\mathcal{C}}_{x_1, T(x_1), \delta_1} \text{ and } \mathcal{C}_{x_0, T(x_0), \delta_1} \subset D_u. \tag{71}$$

In particular, x_0 is non characteristic.

Remark: This proposition remains valid for general solutions defined for all (x,t) such that $0 \le t < T(x)$ with $T(x) \ge \bar{T} > 0$.

Proof of Proposition 3.5: Consider $x_1 \in \mathbb{R}$ and $\delta_1 \in (\delta_*, 1)$. Note that it is enough to show the existence of x_0 such that (71) holds, since this implies by the definition (3) that x_0 is non characteristic. Let us introduce

$$E_1 = \{x \in \mathbb{R} \mid T(x) \le T(x_1) - \delta_1 |x - x_1|\}.$$

Since $\delta_1 > \delta_*$, we have for $|x|$ large

$$T(x_1) - \delta_1 |x - x_1| < T_* - \delta_* |x - x_*| \le T(x)$$

where we used (65) for the last inequality. Hence the boundedness of E_1. Since $E_1 \neq \emptyset$ ($x_1 \in E_1$) and E_1 is closed, there exists $x_2 \in E_1$ such that

$$|x_2 - x_1| = \max_{x \in E_1} |x - x_1|, \text{ if } |x - x_1| > |x_2 - x_1|, \text{ then } T(x) > T(x_1) - \delta_1 |x - x_1|, \tag{72}$$

$$\text{and } T(x_2) = T(x_1) - \delta_1 |x_1 - x_2| \text{ (i.e. } (x_2, T(x_2)) \in \partial \mathcal{C}_{x_1, T(x_1), \delta_1}). \tag{73}$$

By continuity of $T(x)$, there exists $x_0 \in \mathbb{R}$ such that

$$|x_0 - x_1| \le |x_2 - x_1| \text{ and } T(x_0) = \min_{|x-x_1|\le|x_2-x_1|} T(x). \tag{74}$$

We claim that x_0 satisfies (71). Indeed, note first from (74) and (73) that $T(x_0) \le T(x_2) = T(x_1) - \delta_1|x_1 - x_2| \le T(x_1) - \delta_1|x_1 - x_0|$, hence, $(x_0, T(x_0)) \in \bar{\mathcal{C}}_{x_1,T(x_1),\delta_1}$ by the definition (4). For the second inequality in (71), note from (4) and (64) that it is enough to prove that for all $x \in \mathbb{R}$,

$$T(x_0) - \delta_1|x - x_0| \le T(x). \tag{75}$$

- If $|x - x_1| \le |x_2 - x_1|$, then $T(x_0) - \delta_1|x - x_0| \le T(x_0) \le T(x)$ by (74).
- If $|x - x_1| \ge |x_2 - x_1|$, then since we have just proved the first inequality in (71), it holds that $\mathcal{C}_{x_0,T(x_0),\delta_1} \subset \mathcal{C}_{x_1,T(x_1),\delta_1}$ (use the fact that the two cones have the same slope), hence $T(x_0) - \delta_1|x - x_0| \le T(x_1) - \delta_1|x - x_1|$. Using (72), we get (75) when $|x - x_1| \ge |x_2 - x_1|$. Therefore, (75) holds for all $x \in \mathbb{R}$ and $\mathcal{C}_{x_0,T(x_0),\delta_1} \subset D_u$. Thus, the second inequality in (71) holds. This concludes the proof of Proposition 3.5. ■

Taking $x_1 = x_*$ and $\delta_1 = \frac{1+\delta_*}{2}$ in Proposition 3.5, we get:

Corollary 3.6 *There exists a non characteristic point $x_0 \in \mathbb{R}$ (in the sense (3)).*

Step 4: The set of non characteristic points is an interval

We claim the following from energy arguments (Lemma 3.1):

Proposition 3.7 (Characterization of w_{x_0} when x_0 is non characteristic) *If x_0 is non characteristic, then, there exist $d(x_0) \in (-1, 1)$ and $\theta(x_0) = \pm 1$ such that for all $(y, s) \in (-1, 1) \times \mathbb{R}$, $w_{x_0}(y, s) = \theta(x_0)\kappa(d(x_0), y)$ (11).*

Proof: From Corollary 3.3, we know that

$$E(w_{x_0}(s)) \to e_- \text{ as } s \to -\infty \text{ with } e_- = 0 \text{ or } e_- = E(\kappa_0) > 0. \tag{76}$$

Using the convergence result of [19] stated in (10), we see that there exists $d(x_0) \in (-1, 1)$ such that

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0), y) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty. \tag{77}$$

where $\mathcal{H}$ and its norm are defined in (9). Using the definition (8) of $E(w)$ and (68), we see that

$$E(w_{x_0}(s)) \to e_+ = E(\kappa(d(x_0), .) = E(\kappa_0) > 0 \text{ as } s \to \infty.$$

Using Lemma 3.1, we see that

$$e_+ - e_- = -\frac{4}{p-1} \int_{-\infty}^{\infty} \int_{-1}^{1} (\partial_s w_{x_0}(y, s))^2 \frac{\rho(y)}{1 - y^2} dyds \le 0. \tag{78}$$

Hence, from (76), $e_- = E(\kappa_0)$ and $e_+ - e_- = 0$. Therefore, from (78), we obtain $\partial_s w_{x_0}(y, s) \equiv 0$ for all $(y, s) \in (-1, 1) \times \mathbb{R}$, which means that w_{x_0} is a stationary solution to (6). Using (77), we see that $w_{x_0}(y, s) = \theta(x_0)\kappa(d(x_0), y)$ for all $(y, s) \in (-1, 1) \times \mathbb{R}$. This concludes the proof of Proposition 3.7. ■

We claim:

Corollary 3.8 *Consider $x_1 < x_2$ two non characteristic points. Then, there exists $d_0 \in (-1,1)$ and $\theta_0 = \pm 1$ such that:*
(i) for all $(y,s) \in (-1,1) \times \mathbb{R}$, $w_{x_1}(y,s) = w_{x_2}(y,s) = \theta_0 \kappa(d_0, y)$,
(ii) for all $\bar{x} \in [x_1, x_2]$, $T(\bar{x}) = T(x_1) + d_0(\bar{x} - x_1)$ and for all $(x,t) \in \mathcal{C}_{\bar{x},T(\bar{x}),1}$,

$$u(x,t) = \theta_0 \kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T(\bar{x}) - t + d_0(x - \bar{x}))^{\frac{2}{p-1}}}. \tag{79}$$

Proof : Consider $x_1 < x_2$ two non characteristic points.

(i) Using Proposition 3.7, we see that there exist $d(x_1)$ and $d(x_2)$ in $(-1,1)$, and $\theta(x_1)$ and $\theta(x_2)$ in $\{-1,1\}$ such that for all $(y,s) \in (-1,1) \times \mathbb{R}$, $w_{x_i}(y,s) = \theta_i \kappa(d(x_i), y)$ where $i = 0$ and 1. Going back to the original variables with (5), we see that for $i = 0$ and 1,

$$\forall (x,t) \in \mathcal{C}_{x_i,T(x_i),1}, \quad u(x,t) = \theta(x_i)\kappa_0 \frac{(1-d(x_i)^2)^{\frac{1}{p-1}}}{(T(x_i) - t + d(x_i)(x - x_i))^{\frac{2}{p-1}}}. \tag{80}$$

Therefore, if $V = \mathcal{C}_{x_1,T(x_1),1} \cap \mathcal{C}_{x_2,T(x_2),1}$, we have for all $(x,t) \in V$,

$$\theta(x_1)\kappa_0 \frac{(1-d(x_1)^2)^{\frac{1}{p-1}}}{(T(x_1) - t + d(x_1)(x - x_1))^{\frac{2}{p-1}}} = \theta(x_2)\kappa_0 \frac{(1-d(x_2)^2)^{\frac{1}{p-1}}}{(T(x_2) - t + d(x_2)(x - x_2))^{\frac{2}{p-1}}}.$$

Since we know from (4) that V is a non empty open set of $\mathbb{R}^2$, this yields $\theta(x_1) = \theta(x_2)$ and $d(x_1) = d(x_2) = \frac{T(x_2) - T(x_1)}{x_2 - x_1}$. Thus, (i) holds with

$$d_0 = d(x_1) = d(x_2) = \frac{T(x_2) - T(x_1)}{x_2 - x_1} \text{ and } \theta_0 = \theta(x_1) = \theta(x_2). \tag{81}$$

(ii) Consider $\bar{x} \in [x_1, x_2]$ and define

$$\tilde{T}(\bar{x}) = T(x_1) + d_0(\bar{x} - x_1). \tag{82}$$

Since $(\bar{x}, \tilde{T}(\bar{x}))$ is on the segment connecting $(x_1, T(x_1))$ and $(x_2, T(x_2))$ (use (81)), we see from (4) that for some $t(\bar{x}) \in \mathbb{R}$, we have

$$V_{\bar{x}} \equiv \mathcal{C}_{\bar{x},\tilde{T}(\bar{x}),1} \cap \{t < t(\bar{x})\} \subset \{\mathcal{C}_{x_1,T(x_1),1} \cup \mathcal{C}_{x_2,T(x_2),1}\} \cap \{t < t(\bar{x})\}.$$

Therefore, from (80) and (81), we see that $\forall (x,t) \in V_{\bar{x}}$,

$$u(x,t) = \theta_0 \kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T(x_1) - t + d_0(x - x_1))^{\frac{2}{p-1}}}. \tag{83}$$

From the uniqueness of the solution of (1), we see that $u(x,t)$ is defined everywhere in $\mathcal{C}_{\bar{x},\tilde{T}(\bar{x}),1}$. In particular, the identity (83) holds in all $\mathcal{C}_{\bar{x},\tilde{T}(\bar{x}),1}$, which means by (82) that $T(\bar{x}) = \tilde{T}(\bar{x}) = T(x_1) + d_0(\bar{x} - x_1)$. Hence, (79) follows from (83). This concludes the proof of Corollary 3.8. ■

From Corollaries 3.6 and 3.8, we see that the set of non characteristic points is some non empty interval I_0, and that for any $\bar{x} \in I_0$ and $x_1 \in I_0$, we have

$$T(x_1) - T(\bar{x}) = d_0(x_1 - \bar{x}) \tag{84}$$

where d_0 is the slope of the blow-up curve on I_0. Thus, using (79), we get:

Corollary 3.9 *The set of non characteristic points is a non empty interval I_0 and there exist $d_0 \in (-1,1)$ and θ_0 such that on I_0, the blow-up curve is a straight line with slope d_0. Moreover, for any $x_1 \in I_0$,*

$$\forall (x,t) \in \bigcup_{\bar{x}\in I_0} \mathcal{C}_{\bar{x},T(\bar{x}),1}, \quad u(x,t) = \theta_0\kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T(x_1)-t+d_0(x-x_1))^{\frac{2}{p-1}}}. \tag{85}$$

We then have the following reduction of the proof of Theorem 2:

Corollary 3.10 *It is enough to prove that*

$$I_0 = \mathbb{R} \tag{86}$$

in order to conclude the proof of Theorem 2.

Proof: Let us conclude here the proof of Theorem 2 assuming (86). If $I_0 = \mathbb{R}$, then we see from Corollary 3.9 that the blow-up curve is a straight line of slope d_0 whose equation is

$$t = T(x) \text{ with } \forall x \in \mathbb{R}, \;\; T(x) = T(x_*) + d_0(x - x_*) \tag{87}$$

and that

$$D_u = \{(x,t) \mid t < T(x_*) + d_0(x-x_*)\}$$

which contains $\mathcal{C}_{x_*,T(x_*),1}$ by the fact that $T(x_*) \geq T_*$ (see (65)). Using (65) and (87), we see that $|d_0| \leq \delta_*$, hence $\mathcal{C}_{x_*,T(x_*),\delta^*} \subset D_u$. Moreover, since $\bigcup_{\bar{x}\in I_0} \mathcal{C}_{\bar{x},T(\bar{x}),1} = D_u$, we see that (85) implies (16) with $T_0 = T(x_*)$ and $x_0 = x_*$. This concludes the proof of Theorem 2 assuming (86).

Step 5: Conclusion of the proof

In this step, we proceed by contradiction to prove (86). Therefore, we assume that $I_0 \neq \mathbb{R}$. From Corollary 3.9, up to changing $u(x,t)$ in $u(-x,t)$ (also a solution to (1)), we can assume that

$$\bar{I}_0 = (-\infty, b] \text{ or } \bar{I}_0 = [a,b] \tag{88}$$

for some $a \leq b$.

From Corollary 3.9, we know that the blow-up set is a straight line of slope $d_0 \in (-1,1)$ on I_0.

In this step, we first show that to the right of the interval I_0, the blow-up curve is a straight line of slope 1, by the fact that I_0 is an interval. We then find a contradiction by energy arguments in space-time valid in the case of a non characteristic point. More precisely:

Lemma 3.11 *It holds that $\forall x \geq b, \;\; T(x) = T(b) + x - b.$*

Proof: Let us remark first that since the blow-up curve is 1-Lipschitz from the finite speed of propagation, we already know that for all $x \geq b$, $T(x) \leq T(b) + x - b$. We proceed by contradiction, and assume that for some $x_1 > b$, we have $T(x_1) < T(b) + x_1 - b$. Therefore, if $\lambda \in \mathbb{R}$ is the slope of the straight line connecting $(b, T(b))$ and $(x_1, T(x_1))$, we have

$$\lambda = \frac{T(x_1) - T(b)}{x_1 - b} < 1.$$

Since $\delta_* \in (0,1)$ by the hypothesis of Theorem 2 and $|d_0| < 1$ where d_0 is the slope of the blow-up set on the interval I_0, we can fix some $\delta_1 \in (0,1)$ such that $\delta_1 \in (\max(\delta_*, d_0, \lambda), 1)$. Since $\delta_1 > \lambda$, we have $(b, T(b)) \notin \bar{\mathcal{C}}_{x_1, T(x_1), \delta_1}$, therefore, since $\delta_1 > d_0$, $\bar{\mathcal{C}}_{x_1, T(x_1), \delta_1}$ does not contain any point from the left half-line finishing in $(b, T(b))$ and whose slope is d_0. Since this half-line contains the set

$$\mathcal{N} \equiv \{(\bar{x}, T(\bar{x})) \mid \bar{x} \text{ is non characteristic } \}, \tag{89}$$

this means that for any non characteristic point $\bar{x}$, $(\bar{x}, T(\bar{x})) \notin \bar{\mathcal{C}}_{x_1, T(x_1), \delta_1}$ on the one hand. On the other hand, since $\delta_1 \in (\delta_*, 1)$, Proposition 3.5 applies and we know that there exists a non characteristic point $x_0 = x_0(x_1, \delta_1)$ such that $(x_0, T(x_0)) \in \bar{\mathcal{C}}_{x_1, T(x_1), \delta_1}$, which is a contradiction. Thus, for all $x > b$, $T(x) = T(b) + x - b$. This concludes the proof of Lemma 3.11. ■

From Corollary 3.9 and Lemma 3.11, we have:

Claim 3.12

$$\int_{-1}^{0} \int_{-1}^{1} |\partial_s w_{b+1, T(b+1)}(y, s')|^2 \frac{\rho(y)}{1 - y^2} dy ds' = \infty.$$

Proof: Note first that b is non characteristic. Indeed,
- if $I_0 = [a, b]$ with $a = b$ in (88), then $I_0 = \{b\} = \{x_0\}$ by Corollary 3.6 and b is non characteristic.
- if not, then Corollary 3.9 and Lemma 3.11 imply that b is non characteristic.

We now know from Corollary 3.9 (put $x_1 = b$) that for all $(x, t) \in \mathcal{C}_{b, T(b), 1}$,

$$u(x, t) = \theta_0 \kappa_0 \frac{(1 - d_0^2)^{\frac{1}{p-1}}}{(T(b) - t + d_0(x - b)))^{\frac{2}{p-1}}}.$$

Up to changing u in $-u$, we can assume $\theta_0 = 1$. Using the selfsimilar transformation (5) and the fact that $T(b+1) = T(b) + 1$ (from Lemma 3.11), we see that when $s < 0$ and $-1 < y < 1 - 2e^s$, we have

$$\begin{aligned} w_{b+1, T(b+1)}(y, s) &= \frac{\kappa_0 (1 - d_0^2)^{\frac{1}{p-1}}}{(1 + d_0 y + e^s(d_0 - 1))^{\frac{2}{p-1}}}, \\ \partial_s w_{b+1, T(b+1)}(y, s) &= \frac{2e^s(1 - d_0)}{p - 1} \frac{\kappa_0 (1 - d_0^2)^{\frac{1}{p-1}}}{(1 + d_0 y + e^s(d_0 - 1))^{\frac{p+1}{p-1}}}. \end{aligned} \tag{90}$$

If we introduce

$$y_1(s) = -e^s \text{ and } y_2(s) = 1 - 2e^s, \tag{91}$$

then we see that $(y_1(s), y_2(s)) \subset (-1, 1 - 2e^s)$ and from (90) and the definition (7) of ρ, we have for all $s \in (-2, 0)$ and $y \in (y_1(s), y_2(s))$,

$$|\partial_s w_{b+1, T(b+1)}(y, s)| \geq C \frac{e^s}{(1 - e^s)^{\frac{p+1}{p-1}}} \geq \frac{C}{|s|^{\frac{p+1}{p-1}}} \text{ and } \frac{\rho(y)}{1 - y^2} \geq C(1 - e^s)^{\frac{2}{p-1} - 1} \geq C|s|^{\frac{2}{p-1} - 1}.$$

Since $y_2(s) - y_1(s) \geq C|s|$ by (91), this implies that

$$\int_{-1}^{0} \int_{y_1(s)}^{y_2(s)} |\partial_s w_{b+1,T(b+1)}(y,s)|^2 \frac{\rho(y)}{1-y^2} dyds$$
$$\geq \int_{-1}^{0} (y_2(s) - y_1(s)) \frac{C}{|s|^{\frac{2(p+1)}{p-1}}} |s|^{\frac{2}{p-1}-1} ds = C \int_{-1}^{0} |s|^{-\frac{2p}{p-1}} ds = +\infty.$$

Since $(y_1(s), y_2(s)) \subset (-1,1)$, this concludes the proof of Claim 3.12. ■

Using energy estimates for $w_{b+1,T(b+1)}$ still valid in the characteristic situation (Lemma 3.1), we write

$$\int_{-1}^{0} \int_{-1}^{1} \left|\partial_s w_{b+1,T(b+1)}(y,s)\right|^2 \frac{\rho(y)}{1-y^2} dyds \tag{92}$$
$$= \frac{p-1}{4} \left[E(w_{b+1,T(b+1)}(-1)) - E(w_{b+1,T(b+1)}(0))\right].$$

Since $E(w_{b+1,T(b+1)}(0)) \geq 0$ (use Claim 2.5 and the fact that $w_{b+1,T(b+1)}(y,s)$ is defined for all $(y,s) \in (-1,1) \times \mathbb{R}$) and $E(w_{b+1,T(b+1)}(0)) \leq E(\kappa_0)$ (use (68)), we get from (92)

$$\int_{-1}^{0} \int_{-1}^{1} \left|\partial_s w_{b+1,T(b+1)}(y,s)\right|^2 \frac{\rho(y)}{1-y^2} dyds < +\infty,$$

which is a contradiction with Claim 3.12. Thus $I_0 = \mathbb{R}$ and by Corollary 3.10, Theorem 2 is proved. ■

A Proof of Claim 2.3

Let us first introduce the following continuity result for solutions to equation (6):

Proposition A.1 (Weak continuity of solutions to (6) with respect to initial data in $H^1 \times L^2$) *Consider a sequence of solutions W_n to equation (6) defined for all $(y,s) \in (-A,A) \times [0,s_0]$ for some $A \geq 1$ and $s_0 \geq 0$ such that*

$$\forall s \in [0,s_0], \ \ \forall n \in \mathbb{N}, \ \ \|W_n(s), \partial_s W_n(s)\|_{H^1 \times L^2(-A,A)} \leq M \tag{93}$$

for some $M > 0$.
If $(W_n(0), \partial_s W_n(0))$ weakly converges to some (z^*, v^*) in $H^1 \times L^2(-A,A)$ as $n \to \infty$, then, there exists $\bar{W}(y,s)$ a solution to (6) with initial data (z^*, v^*) defined for all $(y,s) \in (-A,A) \times [0,s_0]$ with the following properties:
(a) For all $s \in [0,s_0]$, $(W_n(s), \partial_s W_n(s)) \rightharpoonup \left(\bar{W}(s), \partial_s \bar{W}(s)\right)$ as $n \to \infty$ in $H^1 \times L^2(-A,A)$.
There exists $n_0 = n_0(M,s_0) \in \mathbb{N}$ such that for all $n \geq n_0$ and $s \in [0,s_0]$,
(b) $\|W_n(s) - \bar{W}(s)\|_{L^\infty(-A,A)} \leq e^{-\frac{2s}{p-1}}$,
(c) $e^{\frac{s}{2}} \|\partial_y W_n(s) - \partial_y \bar{W}(s)\|_{L^2(-A,A)} + \|\partial_s W_n(s) - \partial_s \bar{W}(s)\|_{L^2(-A,A)} \leq C(A,M) e^{-\frac{2s}{p-1}}$.

Proof: If we introduce

$$
\begin{aligned}
u_n(\xi,\tau) &= (1-\tau)^{-\frac{2}{p-1}} W_n\left(\frac{\xi}{1-\tau}, -\log(1-\tau)\right), \\
z_1^*(\xi) &= v^*(\xi) - \frac{2}{p-1} z^*(\xi) - \xi {z^*}'(\xi),
\end{aligned} \tag{94}
$$

then we see that $u_n(\xi,\tau)$ is a solution of equation (1) defined in $\mathcal{C}_{\tau_0}$ where

$$
\tau_0 = 1 - e^{-s_0}, \;\; \mathcal{C}_t = \{(\xi,\tau) \;|\; 0 < \tau < t \text{ and } |\xi| < A(1-\tau)\}, \tag{95}
$$

and

$$
\forall \tau \in [0,\tau_0], \;\; \|(u_n(\tau), \partial_\tau u_n(\tau))\|_{H^1\times L^2(D(\tau))} \le M_0 (1-\tau)^{-\frac{2}{p-1}-\frac{1}{2}}, \tag{96}
$$

$$
(u_n(0), \partial_\tau u_n(0)) \rightharpoonup (z^*, z_1^*) \text{ as } n \to \infty, \quad \text{in} \quad H^1 \times L^2(-A,A) \tag{97}
$$

where $M_0 = C_0(p)M$ for some $C_0(p) > 0$ and $D(\tau) = (-A(1-\tau), A(1-\tau))$. Note from (96) and (97) that

$$
\|(z^*, z_1^*)\|_{H^1\times L^2(-A,A)} \le M_0. \tag{98}
$$

Therefore, we can define $u(\xi,\tau)$ as the maximal solution of (1) with initial data (z^*, z_1^*) defined in $\mathcal{C}_{\tau^*}$ where $\tau^* \le 1$ is maximal. Note that

$$
\text{either } \tau^* = 1 \text{ or } \tau^* < 1 \text{ and } \limsup_{\tau\to\tau^*} \|(u(\tau), \partial_\tau u(\tau))\|_{H^1\times L^2(D(\tau))} = \infty. \tag{99}
$$

Defining

$$
v_n = u_n - u, \tag{100}
$$

we see that the following Lemma allows us to conclude:

Lemma A.2 *For all $\epsilon > 0$, we have the following with $\tau_\epsilon = \min(\tau_0, \tau^* - \epsilon)$:*
(a) $\sup_{\tau\in[0,\tau_\epsilon]} \|v_n(\tau)\|_{L^\infty(D(\tau))} \to 0$ as $n \to \infty$.
(b) There exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and $\tau \in [0,\tau_\epsilon]$,
$\|\partial_\xi v_n(\tau)\|_{L^2(D(\tau))} + \|\partial_\tau v_n(\tau)\|_{L^2(D(\tau))} \le 20M_0$.
(c) For all $\tau \in [0,\tau_\epsilon]$, $(v_n(\tau), \partial_\tau v_n(\tau)) \rightharpoonup 0$ weakly in $H^1 \times L^2(D(\tau))$.

Indeed, let us first show that $\tau_\epsilon = \tau_0$ for ϵ small (in other words that $\tau^* > \tau_0$) before deriving Proposition A.1 from Lemma A.2. Assume by contradiction that $\tau^* \le \tau_0$. Then, using (a) and (b) of this lemma and (96), we see that for all $\epsilon > 0$, $\tau_\epsilon = \tau^* - \epsilon$ and for all $\tau \in [0, \tau^* - \epsilon]$,

$$
\begin{aligned}
&\|u(\tau), \partial_\tau u(\tau)\|_{H^1\times L^2(D(\tau))} \le \|u_n(\tau), \partial_\tau u_n(\tau)\|_{H^1\times L^2(D(\tau))} \\
+ \;\; &\|u_n(\tau) - u(\tau), \partial_\tau u_n(\tau) - \partial_\tau u(\tau)\|_{H^1\times L^2(D(\tau))} \le C(A, M_0, \tau^*).
\end{aligned}
$$

Letting $\epsilon \to 0$, we see that $\limsup_{\tau\to\tau^*} \|(u(\tau), \partial_\tau u(\tau))\|_{H^1\times L^2(D(\tau))} < \infty$. Since we also have $\tau^* \le \tau_0 < 1$, a contradiction follows from (99). Thus, $\tau^* > \tau_0$ and Lemma A.2 is valid for all $\tau \in [0,\tau_0]$. Let us now derive Proposition A.1 from Lemma A.2.

If we define $s^* = -\log(1-\tau^*)$ and for all $(y,s) \in (-A,A) \times [0,s^*)$, $\bar{W}(y,s)$ by

$$
u(\xi,\tau) = (1-\tau)^{-\frac{2}{p-1}} \bar{W}\left(\frac{\xi}{1-\tau}, -\log(1-\tau)\right), \tag{101}
$$

then we see from (94) that

$$\begin{aligned}\partial_\xi u_n(\xi,\tau) &= (1-\tau)^{-\frac{2}{p-1}-1}\partial_y W_n\left(\tfrac{\xi}{1-\tau}, -\log(1-\tau)\right),\\ \partial_\tau u_n(\xi,\tau) &= (1-\tau)^{-\frac{2}{p-1}-1}\left(\partial_s W_n + y.\partial_y W_n + \tfrac{2}{p-1}W_n\right)\left(\tfrac{\xi}{1-\tau}, -\log(1-\tau)\right),\end{aligned} \tag{102}$$

and the same holds between u and $\bar{W}$. Using (94), (101), (102) and (100) we obtain for all $s \in [0, s_0]$,

$$\begin{aligned}\|W_n(s) - \bar{W}(s)\|_{L^\infty(-A,A)} &\leq e^{-\frac{2s}{p-1}}\|v_n(\tau)\|_{L^\infty(D(\tau))},\\ \|\partial_y W_n(s) - \partial_y \bar{W}(s)\|_{L^2(-A,A)} &\leq e^{-\frac{2s}{p-1}-\frac{s}{2}}\|\partial_\xi v_n(\tau))\|_{L^2(D(\tau))},\\ \|\partial_s W_n(s) - \partial_s \bar{W}(s)\|_{L^2(-A,A)} &\leq e^{-\frac{2s}{p-1}-\frac{s}{2}}\|\partial_\tau v_n(\tau))\|_{L^2(D(\tau))}\\ +Ae^{-\frac{2s}{p-1}-\frac{s}{2}}\|\partial_\xi v_n(\tau))\|_{L^2(D(\tau))} &+ \frac{2}{p-1}\sqrt{2A}e^{-\frac{2s}{p-1}}\|v_n(\tau))\|_{L^\infty(D(\tau))}\end{aligned}$$

where $\tau = 1 - e^{-s} \in [0, \tau_0]$. Therefore, the conclusion of Proposition A.1 follows from Lemma A.2. Remains to prove Lemma A.2 to conclude.

Proof of Lemma A.2:
(a) From the definition (100) of v_n, we see that

$$\left(\partial^2_{\tau\tau} - \partial^2_{\xi\xi}\right) v_n = |u_n|^{p-1}u_n - |u|^{p-1}u \equiv f_n(\xi,\tau) \tag{103}$$

where

$$|f_n| \leq p\left(|u|^{p-1} + |u_n|^{p-1}\right)|v_n|.$$

Since we have from (96) and the Sobolev injection that for all $\tau \in [0, \tau_\epsilon]$, $\|u_n(\tau)\|_{L^\infty(D(\tau))} \leq C(\tau_0)\|u_n(\tau)\|_{H^1(D(\tau))} \leq C(\tau_0)M_0$, we get for all $\tau \in [0, \tau_\epsilon]$ and $\xi \in D(\tau)$,

$$|f_n(\xi,\tau)| \leq C(\tau_0, M_0)\left(1 + \|v_n(\tau)\|^{p-1}_{L^\infty(D(\tau))}\right)|v_n(\xi,\tau)|. \tag{104}$$

Translating (97), (96) and (98) for v_n, we write

$$\left(v_n, \partial_\tau v_n\right)(0) \rightharpoonup 0 \text{ weakly in } H^1 \times L^2(-A,A) \text{ and } v_n(0) \to 0 \text{ strongly in } L^q(-A,A) \tag{105}$$

for any $q \in [2, +\infty]$ as $n \to \infty$, and

$$\forall n \in \mathbb{N},\quad \|v_n(0)\|_{H^1(-A,A)} + \|\partial_\tau v_n(0)\|_{L^2(-A,A)} \leq 2M_0. \tag{106}$$

Using equation (103), we write

$$v_n(\xi,\tau) = S(\tau)v_n(0)(\xi) + S_1(\tau)\partial_\tau v_n(0)(\xi) + \int_0^\tau \left(S_1(\tau - s)f_n(s)\right)(\xi)ds \tag{107}$$

where

$$S(t)h(x) = \frac{1}{2}\left(h(x+t) + h(x-t)\right) \text{ and } S_1(t)h(x) = \frac{1}{2}\int_{x-t}^{x+t} h(x')dx'. \tag{108}$$

Using (108), we write from (107)

$$\alpha_n(\tau) \le I_n(\tau) + J_n(\tau) + \int_0^\tau K_n(\tau, s)ds \tag{109}$$

where

$$\begin{aligned} &\alpha_n(\tau) = \|v_n(\tau)\|_{L^\infty(D(\tau))}, \quad I_n(\tau) = \|S(\tau)v_n(0)\|_{L^\infty(D(\tau))}, \\ &J_n(\tau) = \|S_1(\tau)\partial_\tau v_n(0)\|_{L^\infty(D(\tau))} \text{ and } K_n(\tau, s) = \|S_1(\tau - s)f_n(s)\|_{L^\infty(D(\tau))}. \end{aligned} \tag{110}$$

In the following, we use (105) to estimate $I_n(\tau)$, $J_n(\tau)$ and $K_n(\tau, s)$.

Estimate of $I_n(\tau)$. From (110) and (108), we have

$$\sup_{\tau\in[0,\tau_\epsilon]} I_n(\tau) \le \alpha_n(0) \to 0 \text{ as } n \to \infty \tag{111}$$

by (105).

Estimate of $J_n(\tau)$. Note first from (108) and (106) that for any $\tau \in [0, \tau_\epsilon]$, we have

$$\begin{aligned} \|\partial_\xi S_1(\tau)\partial_\tau v_n(0)\|_{L^2(D(\tau))} &= \frac{1}{2}\|\partial_\tau v_n(.+\tau, 0) - \partial_\tau v_n(.-\tau, 0)\|_{L^2(D(\tau))} \\ &\le C\|\partial_\tau v_n(0)\|_{L^2(-A,A)} \le CM_0. \end{aligned}$$

Similarly, for any $\xi \in (-A, A)$, we have

$$\begin{aligned} \int_0^{1-\frac{|\xi|}{A}} |\partial_\tau S_1(\tau)\partial_\tau v_n(0)(\xi)|^2 d\tau &= \frac{1}{4}\int_0^{1-\frac{|\xi|}{A}} |\partial_\tau v_n(\xi+\tau, 0) + \partial_\tau v_n(\xi-\tau, 0)|^2 d\tau \\ &\le C\|\partial_\tau v_n(0)\|^2_{L^2(-A,A)} \le CM_0^2, \end{aligned}$$

which means that $S_1(\tau)\partial_\tau v_n(0)$ is $\frac{1}{2}$-Holder for $(\xi, \tau) \in \mathcal{C}_{\tau_\epsilon}$ defined by (95). Then, since $\partial_\tau v_n(0) \rightharpoonup 0$ as $n \to \infty$ in $L^2(-A, A)$ by (105), we use $1_{\xi-\tau<x<\xi+\tau}(x)$ as a test function and write from (108), for all $(\xi, \tau) \in \mathcal{C}_{\tau_\epsilon}$ (95),

$$S_1(\tau)\partial_\tau v_n(0)(\xi) = \frac{1}{2}\int_{-A}^{A} \partial_\tau v_n(x, 0)1_{\xi-\tau<x<\xi+\tau}(x)dx \to 0 \text{ as } n \to \infty.$$

This is the pointwise convergence in $\mathcal{C}_{\tau_\epsilon}$. Since pointwise convergence and $\frac{1}{2}$-Holder imply uniform convergence, this means that

$$\sup_{\tau\in[0,\tau_\epsilon]} J_n(\tau) \le \|S_1(\cdot)\partial_\tau v_n(0)\|_{L^\infty(\mathcal{C}_{\tau_\epsilon})} \to 0 \text{ as } n \to \infty. \tag{112}$$

Estimate of $K_n(\tau, s)$. Fix some s and τ such that $0 \le s \le \tau \le \tau_\epsilon$. Using (110), (108) and (104), we write,

$$K_n(\tau, s) = \|S_1(\tau - s)f_n(s)\|_{L^\infty(D(\tau))} \le \|f_n(s)\|_{L^\infty(D(s))} \le C(\tau_0, M_0)\left(1 + \alpha_n(s)^{p-1}\right)\alpha_n(s). \tag{113}$$

Using (109), (111), (112) and (113), we write for all $\tau \in [0, \tau_\epsilon]$,

$$\alpha_n(\tau) - C(\tau_0, M_0) \int_0^\tau \left(1 + \alpha_n(s)^{p-1}\right) \alpha_n(s) ds \leq \sup_{\tau \in [0, \tau_\epsilon]} \left(I_n(\tau) + J_n(\tau)\right) \to 0 \text{ as } n \to \infty.$$

Using Gronwall inequality with some a priori estimates, (105) and (110) , we get

$$\sup_{\tau \in [0, \tau_\epsilon]} \|v_n(\tau)\|_{L^\infty(D(\tau))} = \sup_{\tau \in [0, \tau_\epsilon]} \alpha_n(\tau) \to 0 \text{ as } n \to \infty,$$

which is precisely the conclusion of (a) of Lemma A.2.

(b) Using (103), we can write

$$v_n = v_{n,1} + v_{n,2} \tag{114}$$

where

$$\begin{aligned} \left(\partial^2_{\tau\tau} - \partial^2_{\xi\xi}\right) v_{n,1} &= 0, \ \ v_{n,1}(0) = v_n(0) \text{ and } \partial_\tau v_{n,1}(0) = \partial_\tau v_n(0) & (115) \\ \left(\partial^2_{\tau\tau} - \partial^2_{\xi\xi}\right) v_{n,2} &= f_n(\xi, \tau), \ \ v_{n,2}(0) = 0 \text{ and } \partial_\tau v_{n,1}(0) = 0. & (116) \end{aligned}$$

From (107), we have

$$\begin{aligned} v_{n,1}(\xi, \tau) &= S(\tau) v_n(0)(\xi) + S_1(\tau) \partial_\tau v_n(0)(\xi), & (117) \\ v_{n,2}(\xi, \tau) &= \frac{1}{2} \int_0^\tau \int_{\xi-\tau+s}^{\xi+\tau-s} f_n(y, s) dy ds. & (118) \end{aligned}$$

Since the energy on slices of the light cone for the linear equation (115)

$$\int_{D(\tau)} \left((\partial_\xi v_{n,1}(\xi, \tau))^2 + (\partial_\tau v_{n,1}(\xi, \tau))^2 \right) d\xi$$

is decreasing, we write from (106)

$$\begin{aligned} & \|\partial_\xi v_{n,1}(\tau)\|_{L^2(D(\tau))} + \|\partial_\tau v_{n,1}(\tau)\|_{L^2(D(\tau))} \\ \leq \ & 2 \left(\|\partial_\xi v_{n,1}(0)\|_{L^2(|\xi| < A(1-\tau)+\tau)} + \|\partial_\tau v_{n,1}(0)\|_{L^2(|\xi| < A(1-\tau)+\tau))} \right) \\ \leq \ & 2 \left(\|\partial_\xi v_n(0)\|_{L^2(-A,A)} + \|\partial_\tau v_n(0)\|_{L^2(-A,A)} \right) \leq 10 M_0. \end{aligned} \tag{119}$$

From (118), we write

$$\begin{aligned} \partial_\xi v_{n,2}(\xi, \tau) &= \frac{1}{2} \int_0^\tau \left(f_n(\xi + \tau - s, s) - f_n(\xi - \tau + s, s)\right) ds, \\ \partial_\tau v_{n,2}(\xi, \tau) &= \frac{1}{2} \int_0^\tau \left(f_n(\xi + \tau - s, s) + f_n(\xi - \tau + s, s)\right) ds. \end{aligned}$$

Using (a) and (104), we see that for all $n \in \mathbb{N}$,

$$\sup_{\tau \in [0, \tau_\epsilon]} \|f_n(\tau)\|_{L^\infty(D(\tau))} \to 0 \text{ as } n \to \infty.$$

Therefore, we write for all $\tau \in [0, \tau_\epsilon]$,

$$\begin{aligned}
&\sup_{\tau\in[0,\tau_\epsilon]} \|\partial_\xi v_{n,2}(\tau)\|_{L^2(D(\tau))} + \|\partial_\tau v_{n,2}(\tau)\|_{L^2(D(\tau))} \\
\le\ & \sqrt{2A} \sup_{\tau\in[0,\tau_\epsilon]} \|\partial_\xi v_{n,2}(\tau)\|_{L^\infty(D(\tau))} + \|\partial_\tau v_{n,2}(\tau)\|_{L^\infty(D(\tau))} \\
\le\ & 2\sqrt{2A} \sup_{\tau\in[0,\tau_\epsilon]} \|f_n(\tau)\|_{L^\infty(D(\tau))} \to 0 \qquad (120)
\end{aligned}$$

as $n \to \infty$. Thus, (b) follows from (119) and (120) for n large.

(c) Since $\forall \tau \in [0, \tau_\epsilon]$, $\|v_n(\tau)\|_{H^1(D(\tau))} \le C$ and $\sup_{\tau\in[0,\tau_\epsilon]} \|v_n(\tau)\|_{L^\infty(D(\tau))} \to 0$ by (a) and (b), this implies that $v_n \rightharpoonup 0$ in $H^1(D(\tau))$. Remains to prove the weak convergence of $\partial_\tau v_n(\tau)$ to 0 in $L^2(D(\tau))$ as $n \to \infty$ to conclude the proof of (c).
From (114), we have

$$\partial_\tau v_n(\xi,\tau) = \partial_\tau v_{n,1}(\xi,\tau) + \partial_\tau v_{n,2}(\xi,\tau),$$

Using (117) and (108), we write

$$\begin{aligned}
\partial_\tau v_{n,1}(\xi,\tau) &= \partial_\tau S(\tau) v_n(0)(\xi) + \partial_\tau S_1(\tau)\partial_\tau v_n(0)(\xi) \\
&= \frac{1}{2}\left(\partial_\xi v_n(\xi+\tau,0) - \partial_\xi v_n(\xi-\tau,0)\right) + \frac{1}{2}\left(\partial_\tau v_n(\xi+\tau,0) + \partial_\tau v_n(\xi-\tau,0)\right)
\end{aligned}$$

Since $(v_n, \partial_\tau v_n)(0) \rightharpoonup 0$ in $H^1 \times L^2(-A, A)$ by (105), we have $\partial_\tau v_{n,1}(\cdot,\tau) \rightharpoonup 0$ in $L^2(D(\tau))$ as $n \to \infty$.
Using (120), we see that

$$\sup_{\tau\in[0,\tau_\epsilon]} \|\partial_\tau v_{n,2}(\tau)\|_{L^2(D(\tau))} \to 0 \text{ as } n \to \infty.$$

Thus, $\partial_\tau v_n(\tau) = \partial_\tau v_{n,1}(\tau) + \partial_\tau v_{n,2}(\tau) \rightharpoonup 0$ in $L^2(D(\tau))$ as $n \to \infty$. This concludes the proof of (c) and Lemma A.2. ■

We now prove Claim 2.3.
Proof of Claim 2.3: Using the uniform bound stated in (22) and a diagonal process, we extract a subsequence (still denoted by w_n) such that for all $k \in \mathbb{N}$,

$$w_n(-k) \rightharpoonup z_k \text{ in } H^1\left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right) \text{ and } \partial_s w_n(-k) \rightharpoonup v_k \text{ in } L^2\left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)$$

as $n \to \infty$ for some $(z_k, v_k) \in H^1 \times L^2\left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)$. From the bound (22), we can apply Proposition A.1 on the time interval $[-k, 0]$ and obtain the existence of $W_k(y,s)$, a solution to (6) such that

$$W_k(-k) = z_k \text{ and } \partial_s W_k(-k) = v_k$$

with the following properties:
- First,

$$\|(w_n(0), \partial_s w_n(0)) - (W_k(0), \partial_s W_k(0))\|_{H^1\times L^2\left(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'}\right)} \le C(\delta_0', K) e^{-\frac{2k}{p-1}}. \qquad (121)$$

- Second, for all $s \in [-k,0]$, $(w_n(s),\partial_s w_n(s)) \rightharpoonup (W_k(s),\partial_s W_k(s))$ as $n\to\infty$ in $H^1\times L^2(-1,1)$. Therefore, from (22), we obtain for all $s\in[-k,0]$,

$$\left\|\begin{pmatrix} W_k(s)\\ \partial_s W_k(s)\end{pmatrix}\right\|_{H^1\times L^2\left(-\frac{1}{\delta_0'},\frac{1}{\delta_0'}\right)} \le \liminf_{n\to\infty}\left\|\begin{pmatrix} w_n(s)\\ \partial_s w_n(s)\end{pmatrix}\right\|_{H^1\times L^2\left(-\frac{1}{\delta_0'},\frac{1}{\delta_0'}\right)} \le K. \qquad (122)$$

Using a diagonal process, we can assume that

$$\forall k,l\in\mathbb{N},\ \ W_k(y,s)\equiv W_l(y,s) \text{ on } \left(-\frac{1}{\delta_0'},\frac{1}{\delta_0'}\right)\times[-\min(k,l),0].$$

Therefore, we can define $W(y,s)$ for all $(y,s)\in\left(-\frac{1}{\delta_0'},\frac{1}{\delta_0'}\right)\times(-\infty,0]$ by the fact that the restriction of W to $\left(-\frac{1}{\delta_0'},\frac{1}{\delta_0'}\right)\times[-k,0]$ is W_k, for any k. Hence, from (122), we see that (27) holds for all $s\le 0$. Since $W(0)=W_k(0)$ for any $k\in\mathbb{N}$, letting k go to infinity in (121) gives the strong convergence of $(w_n(0),\partial_s w_n(0))$ to $(W(0),\partial_s W(0))$ in $H^1\times L^2\left(-\frac{1}{\delta_0'},\frac{1}{\delta_0'}\right)$. From the continuity of the solution to the Cauchy problem, the same strong convergence holds for any $s\ge 0$. Thus, using (22), we see that (27) holds for any $s\ge 0$. This concludes the proof of Claim 2.3. ■

B Sign of $E(w_-(s))$ for s close to $\log(1-|d(0)|)$

We prove (47) here. Let $\sigma = s-\log(1-|d(0)|)$. In the following, we will make expansions as $\sigma\to 0^-$.
Using (8) and (32), we write

$$\begin{aligned} E(w_-(s)) &= \frac{2(1-|d(0)|)^2e^{2\sigma}}{(p-1)^2}\kappa_0^2(1-d(0)^2)^{\frac{2}{p-1}}I(\sigma)+\frac{2d(0)^2}{(p-1)^2}\kappa_0^2(1-d(0)^2)^{\frac{2}{p-1}}J(\sigma)\\ &+ \frac{(p+1)}{(p-1)^2}\kappa_0^2(1-d(0)^2)^{\frac{2}{p-1}}K(\sigma)-\frac{1}{p+1}\kappa_0^{p+1}(1-d(0)^2)^{\frac{p+1}{p-1}}I(\sigma) \end{aligned} \qquad (123)$$

where

$$I(\sigma)=\int_{-1}^1\frac{(1-y^2)^{\frac{2}{p-1}}}{g(y,\sigma)^{\frac{2(p+1)}{p-1}}},\ \ J(\sigma)=\int_{-1}^1\frac{(1-y^2)^{\frac{p+1}{p-1}}}{g(y,\sigma)^{\frac{2(p+1)}{p-1}}},\ \ K(\sigma)=\int_{-1}^1\frac{(1-y^2)^{\frac{2}{p-1}}}{g(y,\sigma)^{\frac{4}{p-1}}},$$

with $g(y,\sigma)=1-(1-|d(0)|)e^\sigma+d(0)y$. Since $d(0)J(\sigma)=d(0)o(I(\sigma))$ and $K(\sigma)=o(\sigma I(\sigma))$ as $\sigma\to 0^-$ from straightforward computations, and $\frac{2\kappa_0^2}{(p-1)^2}=\frac{\kappa_0^{p+1}}{p+1}$ from (11), we see from (123) that $E(w_-(\sigma))$ has the same sign as $e^{2\sigma}(1-|d(0)|)^2+d(0)o(1)+o(\sigma)-(1-d(0)^2)$ which is equal to $2|d(0)|(|d(0)|-1)+d(0)o(1)+2\sigma(1-|d(0)|)^2+o(\sigma)$ as $\sigma\to 0^-$. Since $|d(0)|<1$, (47) is proved.

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