# Optimal estimates for blow-up rate and behavior for nonlinear heat equations

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**Abstract**: We first describe all positive bounded solutions of

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + w^p.$$

where $(y, s) \in \mathbb{R}^N \times \mathbb{R}$, $1 < p$ and $(N-2)p \leq N+2$. We then obtain for blow-up solutions $u(t)$ of

$$\frac{\partial u}{\partial t} = \Delta u + u^p$$

uniform estimates at the blow-up time and uniform space-time comparison with solutions of $u' = u^p$.

## 1 Introduction

We consider the following nonlinear heat equation:

$$\begin{array}{rcll} \frac{\partial u}{\partial t} & = & \Delta u + |u|^{p-1}u & \text{in} \quad \Omega \times [0, T) \\ u & = & 0 & \text{on} \quad \partial\Omega \times [0, T) \end{array} \tag{1}$$

where $u(t) \in H^1(\Omega)$ and $\Omega = \mathbb{R}^N$ (or $\Omega$ is a convex domain).

We assume in addition that

$$1 < p, \ (N-2)p < N+2 \text{ and } u(0) \geq 0.$$

In this paper, we are interested in blow-up solutions $u(t)$ of equation (1): $u(t)$ blows-up in finite time $T$ if $u$ exists for $t \in [0, T)$ and $\lim_{t \to T} \|u(t)\|_{H^1} = +\infty$. In this case, one can show that $u$ has at least one blow-up point, that is $a \in \Omega$ such that there exists $(a_n, t_n)_{n \in \mathbb{N}}$ satisfying $(a_n, t_n) \to (a, T)$ and

$|u(a_n,t_n)| \to +\infty$. We aim in this work at studying the blow-up behavior of $u(t)$. In particular, we are interested in obtaining uniform estimates on $u(t)$ at or near the singularity, that is estimates "basically" independent of initial data.

We will give two types of uniform estimates: the first one holds especially at the singular set (Theorem 1) and the other one consists in surprising global estimates in space and time (Theorem 3). It will be deduced from the former by some strong control of the interaction between regular and singular parts of the solution. Various applications of this type of estimates will be given in [12].

For the first type of estimates, we introduce for each $a \in \Omega$ ($a$ may be a blow-up point of $u$ or not) the following similarity variables:

$$\begin{array}{rcl} y & = & \frac{x-a}{\sqrt{T-t}} \\ s & = & -\log(T-t) \\ w_a(y,s) & = & (T-t)^{\frac{1}{p-1}}u(x,t). \end{array} \tag{2}$$

$w_a$ ($= w$) satisfies $\forall s \geq -\log T$, $\forall y \in D_{a,s}$:

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2}y.\nabla w - \frac{w}{p-1} + |w|^{p-1}w \tag{3}$$

where

$$D_{a,s} = \{y \in \mathbb{R}^N \mid a + ye^{-s/2} \in \Omega\}. \tag{4}$$

We introduce also the following Lyapunov functional:

$$E(w) = \frac{1}{2}\int |\nabla w|^2 \rho dy + \frac{1}{2(p-1)}\int |w|^2 \rho dy - \frac{1}{p+1}\int |w|^{p+1}\rho dy \tag{5}$$

$$\text{where } \rho(y) = \frac{e^{-|y|^2/4}}{(4\pi)^{N/2}} \tag{6}$$

and the integration is done over the definition set of $w$.

The study of $u(t)$ near $(a,T)$ where $a$ is a blow-up point is equivalent to the study of the long time behavior of $w_a$. Note that $D_{a,s} \neq \mathbb{R}^N$ in the case $\Omega \neq \mathbb{R}^N$. This in fact is not a problem since we know from [8] that $a \notin \partial\Omega$ in the case $\Omega$ is $C^{2,\alpha}$, and therefore, for a given $a \in \Omega$, $D_{a,s} \to \mathbb{R}^N$ as $s \to +\infty$. Let $a \in \Omega$ be a blow-up point of $u$.

If $\Omega$ is a bounded convex domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$, then Giga and Kohn prove in [7] that:

$$\begin{array}{rrcl} \forall s \geq -\log T, & \|w_a(y,s)\|_{L^\infty(D_{a,s})} & \leq & C \text{ or equivalently} \\ \forall t \in [0,T), & \|u(x,t)\|_{L^\infty(\Omega)} & \leq & C(T-t)^{-\frac{1}{p-1}}. \end{array} \tag{7}$$

They also prove in [7] and [8] (see also [6]) that for a given blow-up point $a \in \Omega$,

$$\lim_{s \to +\infty} w_a(y,s) = \lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + y\sqrt{T-t}, t) = \kappa$$

where $\kappa = (p-1)^{-\frac{1}{p-1}}$, uniformly on compact subsets of $\mathbb{R}^N$. The result is pointwise in $a$. Besides, for a.e $y$, $\lim\limits_{s \to +\infty} \nabla w_a(y,s) = 0$.

Let us denote $L^\infty(D_{a,s})$ by $L^\infty$.

In this paper, we first obtain uniform (on $a$ and in some sense on $u(0)$) sharp estimates on $w_a$, and we find a precise long time behavior for $\|w_a(s)\|_{L^\infty}$, $\|\nabla w_a(s)\|_{L^\infty}$ and $\|\Delta w_a(s)\|_{L^\infty}$ (global estimates).

**Theorem 1 (Optimal bound on $u(t)$ at blow-up time)** *Assume that $\Omega$ is a convex bounded $C^{2,\alpha}$ domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Consider $u(t)$ a blow-up solution of equation (1) which blows-up at time $T$. Assume in addition $u(0) \geq 0$ and $u(0) \in H^1(\Omega)$. Then*

$$(T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} \to \kappa = (p-1)^{-\frac{1}{p-1}} \text{ as } t \to T$$

*and*

$$(T-t)^{\frac{1}{p-1}+1} \|\Delta u(t)\|_{L^\infty(\Omega)} + (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to T,$$

*or equivalently for any $a \in \Omega$,*

$$\|w_a(s)\|_{L^\infty} \to \kappa \text{ as } s \to +\infty$$

*and*

$$\|\Delta w_a(s)\|_{L^\infty} + \|\nabla w_a(s)\|_{L^\infty} \to 0 \text{ as } s \to +\infty.$$

**Remark**: We can point out that we do not consider local norm in $w$ variable such as $L^2(d\mu)$ with $d\mu = e^{-|y|^2/4} dy$ as a center manifold theory for equation (3) would suggest. Instead, we use $L^\infty$ norm which yields results uniform with respect to $\in \Omega$. Indeed, we have from (2) that $\forall a, b \in \Omega$, $\forall (y,s) \in D_{b,s}$,

$$w_b(y,s) = w_a(y + (b-a)e^{\frac{s}{2}}, s),$$

which yields $\|w_a\|_{L^\infty} = \|w_b\|_{L^\infty}$, $\|\nabla w_a\|_{L^\infty} = \|\nabla w_b\|_{L^\infty}$ and $\|\Delta w_a\|_{L^\infty} = \|\Delta w_b\|_{L^\infty}$).

One interest of Theorem 1 is that in fact, its proof yields the following compactness result:

**Theorem 1' (Compactness of blow-up solutions of (1))** *Assume that $\Omega$ is a convex bounded $C^{2,\alpha}$ domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Consider $(u_n)_{n\in\mathbb{N}}$ a sequence of nonnegative solutions of equation (1) such that for some $T > 0$ and for all $n \in \mathbb{N}$, $u_n$ is defined on $[0,T)$ and blows-up at time $T$. Assume also that $\|u_n(0)\|_{H^2(\Omega)}$ is bounded uniformly in $n$. Then*

$$\sup_{n\in\mathbb{N}} (T-t)^{\frac{1}{p-1}} \|u_n(t)\|_{L^\infty(\Omega)} \to \kappa \text{ as } t \to T$$

*and*

$$\sup_{n\in\mathbb{N}} \left( (T-t)^{\frac{1}{p-1}+1} \|\Delta u_n(t)\|_{L^\infty(\Omega)} + (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla u_n(t)\|_{L^\infty(\Omega)} \right) \to 0$$

*as $t \to T$.*

**Remark**: The same results can be proved for the following heat equation:

$$\frac{\partial u}{\partial t} = \nabla.(a(x)\nabla u) + b(x)f(u), \;\; u(0) \geq 0$$

where $f(u) \sim u^p$ as $u \to +\infty$, $(a(x))$ is a symmetric, bounded and uniformly elliptic matrix, $b(x)$ is bounded, and $a(x)$ and $b(x)$ are $C^1$.

Let us point out that this result is optimal. One way to see it is by the following Corollary which improves the local lower bound on the blow-up solution given in [8] by Giga and Kohn.

**Corollary 1 (Lower bound on the blow-up behavior for equation (1))** *Assume that $\Omega$ is a convex bounded $C^{2,\alpha}$ domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Then for all nonnegative solution $u(t)$ of (1) such that $u(0) \in H^1(\Omega)$ and $u(t)$ blows-up at time $T$, and for all $\epsilon_0 \in (0,1)$, there exists $t_0 = t_0(\epsilon_0, u_0) < T$ such that if for some $a \in \Omega$ and some $t \in [t_0, T)$ we have*

$$0 \leq u(a,t) \leq (1-\epsilon_0)\kappa(T-t)^{-\frac{1}{p-1}}, \tag{8}$$

*then $a$ is not a blow-up point of $u(t)$.*

**Remark**: The result is still true for a sequence of nonnegative solutions $u_n$ blowing-up at $T > 0$ and satisfying the assumptions of Theorem 1', with a $t_0$ independent of $n$.

**Remark**: $\kappa$ is the optimal constant giving such a result. The result of [8] was the same except that $(1-\epsilon_0)\kappa$ was replaced by $\epsilon_0$ small and it was required that (8) is true for all $(x,t) \in B(a,r) \times [T-r^2, T)$ for some $r > 0$ (no sign condition was required there).

The proof of Theorem 1 relies strongly on the caracterization of all connections between two critical points of equation (3) in $L^\infty_{loc}$. Due to [6], the only bounded global nonnegative solutions of the stationary problem associated to (3) in $\mathbb{R}^N$ are 0 and $\kappa$, provided that $(N-2)p \leq N+2$. Here we classify the solutions $w(y,s)$ of (3) defined on $\mathbb{R}^N \times \mathbb{R}$ and connecting two of the cited critical points between them, and we obtain the surprising result:

**Theorem 2 (Classification of connections between critical points of (3))** *Assume that* $1 < p$ *and* $(N-2)p < N+2$ *and that* $w$ *is a global nonnegative solution of (3) defined for* $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ *bounded in* $L^\infty$*. Then necessarily one of the following cases occurs:*

*i)* $w \equiv 0$ *or* $w \equiv \kappa$,

*or ii) there exists* $s_0 \in \mathbb{R}$ *such that* $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$, $w(y,s) = \varphi(s - s_0)$ *where*

$$\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}. \tag{9}$$

*Note that* $\varphi$ *is the unique global solution (up to a translation) of*

$$\varphi_s = -\frac{\varphi}{p-1} + \varphi^p$$

*satisfying* $\varphi \to \kappa$ *as* $s \to -\infty$ *and* $\varphi \to 0$ *as* $s \to +\infty$.

**Remark**: This result is in the same spirit as the result of Berestycki and Nirenberg [1], and Gidas, Ni and Nirenberg [5]. Here, the moving plane technique is not used, even though the proof uses some elementary geometrical transformations. It is unclear whether the result holds without a sign condition or not. The assumption $w$ is bounded in $L^\infty$ and is defined for $s$ up to $+\infty$ is not really needed, in the following sense:

**Corollary 2** *Assume that* $1 < p$ *and* $(N-2)p < N+2$ *and that* $w$ *a nonnegative solution of (3) defined for* $(y,s) \in \mathbb{R}^N \times (-\infty, s^*)$ *where* $s^*$ *is finite or* $s^* = +\infty$*. Assume in addition that there is a constant* $C_0$ *such that* $\forall a \in \mathbb{R}^N$, $\forall s \leq s^*$, $E_a(w(s)) \leq C_0$, *where*

$$E_a(w(s)) = E(w(. + ae^{\frac{s}{2}}, s)) \tag{10}$$

*and* $E$ *is defined in (5). Then, one of the following cases occurs:*

*i)* $w \equiv 0$ *or* $w \equiv \kappa$,

*or ii)* $\exists s_0 \in \mathbb{R}$ *such that* $\forall (y,s) \in \mathbb{R}^N \times (-\infty, s^*)$, $w(y,s) = \varphi(s - s_0)$ *where*

$$\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}},$$

*or iii)* $\exists s_0 \geq s^*$ *such that* $\forall (y,s) \in \mathbb{R}^N \times (-\infty, s^*)$, $w(y,s) = \psi(s-s_0)$ *where*

$$\psi(s) = \kappa(1-e^s)^{-\frac{1}{p-1}}.$$

Theorem 2 has an equivalent formulation for solutions of (1):

**Corollary 3 (A Liouville theorem for equation (1))** *Assume that* $1 < p$ *and* $(N-2)p < N+2$ *and that* $u$ *is a nonnegative solution in* $L^\infty$ *of (1) defined for* $(x,t) \in \mathbb{R}^N \times (-\infty, T)$. *Assume in addition that* $0 \leq u(x,t) \leq C(T-t)^{-\frac{1}{p-1}}$. *Then* $u \equiv 0$ *or there exist* $T_0 \geq T$ *such that* $\forall (x,t) \in \mathbb{R}^N \times (-\infty, T)$, $u(x,t) = \kappa(T_0 - t)^{-\frac{1}{p-1}}$.

**Remark**: $u \equiv 0$ or $u$ blows-up in finite time $T_0 \geq T$.

The third main result of the paper shows that near blow-up time, the solutions of equation (1) behave globally in space like the solutions of the associated ODE:

**Theorem 3** *Assume that* $\Omega$ *is a convex bounded* $C^{2,\alpha}$ *domain in* $\mathbb{R}^N$ *or* $\Omega = \mathbb{R}^N$. *Consider* $u(t)$ *a nonnegative solution of equation (1) which blows-up at time* $T > 0$. *Assume in addition that* $u(0) \in H^1(\mathbb{R}^N)$ *if* $\Omega = \mathbb{R}^N$. *Then* $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ *such that* $\forall t \in [\frac{T}{2}, T)$, $\forall x \in \Omega$,

$$|\frac{\partial u}{\partial t} - |u|^{p-1}u| \leq \epsilon |u|^p + C_\epsilon. \tag{11}$$

**Remark**: (11) is true until the blow-up time. Let us point out that the result is global in time and in space. The same result holds for a sequence $u_n$ as before (Theorem 1'). For clear reasons, the result is optimal.

**Remark**: Let us note that the result is still true for equation

$$\frac{\partial u}{\partial t} = \nabla.(a(x)\nabla u) + b(x)f(u)$$

where $f(u) \sim u^p$ as $u \to +\infty$, $(a(x))$ is a symmetric, bounded and uniformly elliptic matrix, $b(x)$ is bounded, and $a(x)$ and $b(x)$ are $C^1$.

The conclusion in this case is

$$|\frac{\partial u}{\partial t} - b(x)f(u)| \leq \epsilon |f(u)| + C_\epsilon.$$

It is unclear whether Theorems 1, 2 and 3 hold without a sign condition.

**Remark**: $u' = u^p$ is a reversible equation. Therefore the non reversible equation behaves like a reversible equation near and at the blow-up time.

Theorem 3 localizes the equation. In particular, it shows that the interactions between two singularities or one singularity and the "regular" region are bounded up to the blow-up time.

Note that Theorem 3 has obvious corollaries. For example:
If $x_0$ is a blow-up point, then

- $u(x,t) \to +\infty$ as $(x,t) \to (x_0, T)$ (In other words, $u$ is a continuous function in $\bar{\mathbb{R}}$ of $(x,t) \in \Omega \times (0,T)$).

- $\exists \epsilon_0 > 0$ such that for all $x \in B(x_0, \epsilon_0)$ and $t \in (T - \epsilon_0, T)$, we have $\frac{\partial u}{\partial t}(x,t) > 0$.

Let us notice that theorems 1 and 3 have interesting applications in the understanding of the asymptotic behavior of blow-up solutions $u(t)$ of (1) near a given blow-up point $x_0$. Various points of view has been adopted in the literature ([8], [2], [9], [14]) to describe this behavior. In [12], we sharpen these estimates and put them in a relation.

In the second section, we see how Theorems 1 and 3 are proved using Theorem 2. The third section is devoted to the proof of Theorem 2.

## 2 Optimal blow-up estimates for equation (1)

In this section, we assume that Theorem 2 holds and prove Theorems 1 and 1', Corollary 1 and Theorem 3. The first three are mainly a consequence of compactness procedure and Theorem 2. Theorem 3 follows from Theorem 1 and scaling properties of equation (1) used in a suitable way.

### 2.1 $L^\infty$ estimates for the solution of (1)

We prove Theorems 1 and 1' and Corollary 1 in this subsection.

*Proof of Theorem 1*: Let $u(t)$ be a nonnegative solution of equation (1) defined on $[0,T)$, which blows-up at time $T$ and satisfies $u(0) \in H^1(\Omega)$. It is clear that the estimates on $w_a$ for all $a \in \Omega$ follow from the estimates on $u$ by (2). In addition, the estimates on $u$ follow from the estimates on $w_a$ for a particular $a \in \Omega$ still by (2). Hence, we consider $a \in \Omega$ a blow-up point of $u$ and prove the estimates on this particular $w_a$ defined by

$$w_a(y,s) = e^{-\frac{s}{p-1}} u(a + ye^{-\frac{s}{2}}, T - e^{-s}).$$

Note that we have $\forall a, b \in \Omega$, $\forall (y,s) \in D_{b,s}$,

$$w_b(y,s) = w_a(y + (b-a)e^{\frac{s}{2}}, s).$$

We proceed in three steps: in a first step, we show that $w_a$, $\nabla w_a$ and $\Delta w_a$ are uniformly bounded (without any precision on the bounds). Then, we show in Step 2 that blow-up for equation (1) must occur inside a compact set $K \subset \Omega$ and that $u$, $\nabla u$ and $\Delta u$ are bounded in $\Omega \backslash K$. We finally find the optimal bounds on $w_a$ through a contradiction argument.

Let us recall the expression of the energy $E(w)$ introduced in (5), since it will be useful for further estimates:

$$E(w_a) = \frac{1}{2}\int |\nabla w_a|^2 \rho dy + \frac{1}{2(p-1)}\int |w_a|^2 \rho dy - \frac{1}{p+1}\int |w_a|^{p+1}\rho dy \quad (12)$$

where $\rho$ is defined in (6) and integration is done over the definition set of $w$. By means of the transformation (2), (12) yields a local energy for equation (1):

$$\begin{aligned} \mathcal{E}_{a,t}(u) &= t^{\frac{2}{p-1}-\frac{N}{2}+1}\int \left[\frac{1}{2}|\nabla u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right]\rho(\frac{x-a}{\sqrt{t}})dx \\ &+ \frac{1}{2(p-1)}t^{\frac{2}{p-1}-\frac{N}{2}}\int |u(x)|^2 \rho(\frac{x-a}{\sqrt{t}})dx. \end{aligned} \quad (13)$$

Without loss of generality, we can suppose $a = 0$. We recall that the notation $L^\infty$ stands for $L^\infty(D_{0,s})$.

**Step 1: Preliminary estimates on $w$**

**Lemma 2.1 (Giga-Kohn, Uniform estimates on $w$)** *There exists a positive constant $M$ such that $\forall s \geq -\log T + 1$, $\forall y \in D_{0,s}$,*

$$|w_0(y,s)| + |\nabla w_0(y,s)| + |\Delta w_0(y,s)| + |\nabla \Delta w_0(t,s)| \leq M$$

$$\text{and } |\frac{\partial w}{\partial s}(y,s)| \leq M(1+|y|).$$

Let us recall the main steps of the proof:
Since $u(0) \geq 0$, we know from Giga and Kohn [8] that there exists $B > 0$ such that

$$\forall t \in [0,T),\ \forall x \in \Omega,\ |u(x,t)| \leq B(T-t)^{-\frac{1}{p-1}}. \quad (14)$$

In order to prove this, they argue by contradiction and construct by scaling properties of equation (3) a solution of

$$\begin{cases} 0 &= \Delta v + v^p \text{ in } \mathbb{R}^N \\ v &\geq 0 \\ v(0) &\geq \frac{1}{2} \end{cases}$$

which does not exist if $(N-2)p < N+2$ and $p > 1$.

The estimate on $w_0$ is equivalent to (14).

For $s_0 \geq -\log T + 1$ and $y_0 \in D_{0,s_0}$, consider $W(y', s') = w_0(y' + y_0 e^{\frac{s}{2}}, s_0 + s')$. Then $W(0,0) = w_0(y_0, s_0)$ and $W$ satisfies also (3). If $y_0 e^{-\frac{s_0}{2}}$ (which is in $\Omega$) is not near the boundary, then we have $|W(y', s')| \leq M$ for all $(y', s') \in B(0,1) \times [-1, 1]$. By parabolic regularity (see lemma 3.3 in [7] for a statement), we obtain $|\nabla W(0,0)| + |\Delta W(0,0)| + |\nabla \Delta W(0,0)| \leq M' = M'(M)$. If $y_0 e^{-\frac{s_0}{2}}$ is near the boundary, then lemma 3.4 in [7] allows to get the same conclusion. Since this is true for all $(y_0, s_0)$, we have the bound for $\nabla w_0$, $\Delta w_0$ and $\nabla \Delta w_0$.

The estimate on $\frac{\partial w_0}{\partial s}$ follows then by equation (3).

**Step 2: No blow-up for $u$ outside a compact**

**Proposition 2.1 (Uniform boundedness of $u(x,t)$ outside a compact)** *Assume that $\Omega = \mathbb{R}^N$ and $u(0) \in H^1(\mathbb{R}^N)$, or that $\Omega$ is a convex bounded $C^{2,\alpha}$ domain. Then there exist $C > 0$, $t_1 < T$ and $K$ a compact set of $\Omega$ such that $\forall t \in [t_1, T)$, $\forall x \in \Omega \backslash K$, $|u(x,t)| + |\nabla u(x,t)| + |\Delta u(x,t)| \leq C$.*

*Proof*: *Case $\Omega = \mathbb{R}^N$ and $u(0) \in H^1(\mathbb{R}^N)$*: Giga and Kohn prove in [8] that uniform estimates on $\mathcal{E}_{a,t}$ (13) give uniform estimates in $L^\infty_{loc}$ on the solution of (1). More precisely,

**Proposition 2.2 (Giga-Kohn)** *Let $u$ be a solution of equation (1).*

*i) If for all $x \in B(x_0, \delta)$, $\mathcal{E}_{x,T-t_0}(u(t_0)) \leq \sigma$, then $\forall x \in B(x_0, \frac{\delta}{2})$, $\forall t \in (\frac{t_0+T}{2}, T)$, $|u(t,x)| \leq \eta(\sigma)(T-t)^{-\frac{1}{p-1}}$ where $\eta(\sigma) \leq c\sigma^\theta$, $\theta > 0$, and $c$ and $\theta$ depend only on $p$.*

*ii) Assume in addition that $\forall x \in B(x_0, \delta)$, $|u(\frac{t_0+T}{2}, x)| \leq M$. There exists $\sigma_0 = \sigma_0(p) > 0$ such that if $\sigma \leq \sigma_0$, then $\forall x \in B(x_0, \frac{\delta}{4})$, $\forall t \in (\frac{t_0+T}{2}, T)$, $|u(t,x)| \leq M^*$ where $M^*$ depends only on $M$, $\delta$, $T$ and $t_0$.*

*Proof*: see Proposition 3.5 and Theorem 2.1 in [8]. ■

Now, since $u(0) \in H^1(\mathbb{R}^N)$, we have $u(t) \in H^1(\mathbb{R}^N)$ for all $t \in [0, T)$. Therefore, for fixed $t_0$ and $\sigma \leq \sigma_0$, (13), (6) and the dominated convergence theorem yield the existence of a compact $K_0 \subset \mathbb{R}^N$ such that $\forall x \in \mathbb{R}^N \backslash K_0$, $\mathcal{E}_{x,T-t_0}(u(t_0)) \leq \sigma$.

Hence, *ii)* of Proposition 2.2 applied to $u(.+x_1, .)$ for $x_1 \in K_0$ and with $\delta = 1$, asserts the existence of a compact $K_1 \subset \mathbb{R}^N$ such that $\forall x \in \mathbb{R}^N \backslash K_1$, $\forall t \in (\frac{t_0+T}{2}, T)$, $|u(x,t)| \leq M^*$.

Parabolic regularity (see lemma 3.3 in [7] for a statement) implies the estimates on $\nabla u$ and $\Delta u$ on $\Omega \backslash K$ with a compact $K$ containing $K_1$.

*Case $\Omega$ is a bounded convex $C^{2,\alpha}$ domain*: The main feature in the proof of the estimate on $|u(x,t)|$ is the result of Giga and Kohn which asserts that blow-up can not occur at the boundary (Theorem 5.3 in [8]). The bounds on $\nabla u$ and $\Delta u$ follow from a similar argument as before (see lemma 3.4 in [7]).

**Step 3: Conclusion of the proof**

The result has been proved pointwise. Therefore, the question is in some sense to prove it uniformly.

We want to prove that $\|w_0(s)\|_{L^\infty} \to \kappa$ as $s \to +\infty$.

From [7] and [8], we know that $|w_b(0,s)| \to \kappa$ as $s \to +\infty$ if $b$ is a blow-up point. Since $\|w_0(s)\|_{L^\infty} \geq |w_0(ae^{\frac{s}{2}}, s)| = |w_a(0,s)|$, this implies that

$$\begin{aligned} &\liminf_{s\to+\infty} \|w_0(s)\|_{L^\infty} \geq \kappa \\ \text{and } &\liminf_{s\to+\infty} \|w_0(s)\|_{L^\infty} + \|\nabla w_0(s)\|_{L^\infty} + \|\Delta w_0(s)\|_{L^\infty} \geq \kappa. \end{aligned} \tag{15}$$

The conclusion will follow if we show that

$$\limsup_{s\to+\infty} \|w_0(s)\|_{L^\infty} + \|\nabla w_0(s)\|_{L^\infty} + \|\Delta w_0(s)\|_{L^\infty} \leq \kappa. \tag{16}$$

Let us argue by contradiction and suppose that there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to +\infty$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} \|w_0(s_n)\|_{L^\infty} + \|\nabla w_0(s_n)\|_{L^\infty} + \|\Delta w_0(s_n)\|_{L^\infty} = \kappa + 3\epsilon_0 \text{ where } \epsilon_0 > 0.$$

We claim that (up to extracting a subsequence), we have

$$\begin{array}{lllll} \text{either} & \lim_{n\to+\infty} \|w_0(s_n)\|_{L^\infty} & = & \kappa + \epsilon_0 \\ \text{or} & \lim_{n\to+\infty} \|\nabla w_0(s_n)\|_{L^\infty} & = & \epsilon_0 \\ \text{or} & \lim_{n\to+\infty} \|\Delta w_0(s_n)\|_{L^\infty} & = & \epsilon_0. \end{array} \tag{17}$$

From Proposition 2.1 and the scaling (2), we deduce for $n$ large enough the existence of $y_n^{(0)}$, $y_n^{(1)}$ and $y_n^{(2)}$ in $D_{0,s_n}$ such that

$$\begin{array}{rcl} \|w_0(s_n)\|_{L^\infty} & = & |w_0(y_n^{(0)}, s_n)|, \\ \text{or } \|\nabla w_0(s_n)\|_{L^\infty} & = & |\nabla w_0(y_n^{(1)}, s_n)|, \\ \text{or } \|\Delta w_0(s_n)\|_{L^\infty} & = & |\Delta w_0(y_n^{(2)}, s_n)|. \end{array} \tag{18}$$

Let $y_n = y_n^{(i)}$ where $i$ is the number of the case which occurs. Since $y_n \in D_{0,s_n}$, (4) implies that $y_n e^{-s_n/2} \in \Omega$. Therefore, we can use (2) and define for each $n \in \mathbb{N}$

$$\begin{aligned} v_n(y,s) &= w_{y_n e^{-s_n/2}}(y, s+s_n) \\ &= e^{-\frac{s+s_n}{p-1}} u(ye^{-\frac{s+s_n}{2}} + y_n e^{-s_n/2}, T - e^{-(s+s_n)}) \\ &= w_0(y + y_n e^{s/2}, s+s_n) \end{aligned} \quad (19)$$

We claim that $(v_n)$ is a sequence of solutions of (3) which is compact in $C^3_{loc}(\mathbb{R}^N \times \mathbb{R})$. More precisely,

**Lemma 2.2** $(v_n)_{n\in\mathbb{N}}$ *is a sequence of solutions of (3) with the following properties:*

*i)* $\lim_{n\to+\infty} |v_n(0,0)| = \kappa + \epsilon_0$ *or* $\lim_{n\to+\infty} |\nabla v_n(0,0)| = \epsilon_0$ *or* $\lim_{n\to+\infty} |\Delta v_n(0,0)| = \epsilon_0$.

*ii)* $\forall R > 0$, $\exists n_0 \in \mathbb{N}$ *such that* $\forall n \geq n_0$,
- $v_n(y,s)$ *is defined for* $(y,s) \in \bar{B}(0,R) \times [-R,R]$,
- $v_n \geq 0$ *and* $\|v_n\|_{L^\infty(\bar{B}(0,R)\times[-R,R])} \leq B$ *where* $B$ *is defined in (14).*
- $\exists m(R) > 0$ *such that* $\|v_n\|_{C^3(\bar{B}(0,R)\times[-R,R])} \leq m(R)$.

*Proof*: *i)* $v_n$ satisfies (3) since $w_{y_n e^{-s_n/2}}$ does the same. From (19), (17) and (18), we obtain *i)*: $\lim_{n\to+\infty} |v_n(0,0)| = \kappa + \epsilon_0$ or $\lim_{n\to+\infty} |\nabla v_n(0,0)| = \epsilon_0$ or $\lim_{n\to+\infty} |\Delta v_n(0,0)| = \epsilon_0$.

*ii)* Let $R > 0$.

If $\Omega = \mathbb{R}^N$, then it is obvious form (19) that $v_n$ is defined for $(y,s) \in \bar{B}(0,R) \times [-R,R]$ for large $n$.

If $\Omega$ is bounded, then we can suppose that up to extracting a subsequence, $y_n e^{-s_n/2}$ converges to $y_\infty \in \bar{\Omega}$ as $n \to +\infty$. In fact $y_\infty \in \Omega$. Indeed, since $u(y_n^{(0)} e^{-s_n/2}, T - e^{-s_n}) = e^{\frac{s_n}{p-1}} v_n(0,0) \to +\infty$ as $n \to +\infty$ (or $|\nabla u(y_n^{(1)} e^{-s_n/2}, T - e^{-s_n})| = e^{s_n(\frac{1}{p-1}+\frac{1}{2})} |\nabla v_n(0,0)| \to +\infty$, or $|\Delta u(y_n^{(2)} e^{-s_n/2}, T - e^{-s_n})| = e^{s_n(\frac{1}{p-1}+1)} |\Delta v_n(0,0)| \to +\infty$), in all cases, $y_\infty$ is a blow-up point of $u$. Therefore, Step 2 implies that $y_\infty \in K$ and that $B(y_\infty, \delta_0) \subset \Omega$ for some $\delta_0 > 0$. Together with (19), this implies that $v_n$ is defined for $(y,s) \in \bar{B}(0,R) \times [-R,R]$ for large $n$.

From (19), (14) and the fact that $u \geq 0$, it directly follows that $v_n(y,s) \geq 0$ and $\|v_n\|_{L^\infty(\bar{B}(0,R)\times[-R,R])} \leq B$.

From lemma 2.1 and (19), it directly follows that $\forall (y,s) \in \bar{B}(0,R) \times [-R,R]$, $|v_n(y,s)| + |\nabla v_n(y,s)| + |\Delta v_n(y,s)| + |\nabla \Delta v_n(y,s)| \leq M$ and $|\frac{\partial v_n}{\partial s}| \leq$

$M\times(1+R)$. Since $w \geq 0$, parabolic estimates and strong maximum principle imply that $\|v_n\|_{C^3(\bar{B}(o,R)\times[-R,R])} \leq m(R)$ for some $m(R) > 0$. Just take $m(R) = M \times (1+R)$.

Now, using the compactness property of $(v_n)$ shown in lemma 2.2, we find $v \in C^2(\mathbb{R}^N \times \mathbb{R})$ such that up to extracting a subsequence, $v_n \to v$ as $n \to +\infty$ in $C^2_{loc}(\mathbb{R}^N \times \mathbb{R})$. From lemma 2.2, it directly follows that

i) $v$ satisfies equation (3) for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$

ii) $v \geq 0$ and $\|v\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \leq B$

iii) $|v(0,0)| = \kappa + \epsilon_0$ or $|\nabla v(0,0)| = \epsilon_0$ or $|\Delta v(0,0)| = \epsilon_0$ with $\epsilon_0 > 0$.

By Theorem 2, *i*) and *ii*) imply $v \equiv 0$ or $v \equiv \kappa$ or $v = \varphi(s-s_0)$ where $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$. In all cases, this contradicts *iii*). Thus, Theorem 1 is proved. ■

*Proof of Theorem 1'*: The proof of Theorem 1' is similar to the proof of Theorem 1. Let us sketch the main differences.

**Step 1**: One can remark that a uniform estimate on $E(w_{n,a}(s_0))$ where $s_0 = -\log T$ is needed. Since $\|u_0\|_{H^2(\Omega)}$ is uniformly bounded, we have the conclusion.

**Step 2**: One can use a uniform version of Giga and Kohn's estimates, as they are stated (for example) in [11].

**Step 3**: Same proof.

*Proof of Corollary 1*: Let us prove Corollary 1 now. We argue by contradiction and assume that for some $\epsilon_0 > 0$, there is $t_n \to T$ and $(a_n)_n$ a sequence of blow-up points of $u$ in $\Omega$ such that

$$\forall n \in \mathbb{N},\ 0 \leq u(a_n,t_n) \leq (1-\epsilon_0)\kappa(T-t_n)^{-\frac{1}{p-1}}.$$

Let us give two different proofs:

**Proof 1**: Consider the following solution of equation (3):

$$v_n(y,s) = w_{a_n}(y, s - \log(T-t_n)).$$

From Proposition 2.1, $a_n \in K$, since it is a blow-up point of $u$. As before, we can use a compactness procedure on $v_n$ to get a nonnegative bounded solution $v$ of (3) defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that $|v(0,0)| \leq (1-\epsilon_0)\kappa$ and $v_n \to v$ in $C^2_{loc}$. Therefore, Theorem 2 implies that $v \equiv 0$ or $v = \varphi(s-s_0)$ for some $s_0 \in \mathbb{R}$. In particular, $E(v(0)) < E(\kappa)$. Since $E(v_n(0)) \to E(v(0))$ as $n \to +\infty$, we have for $n$ large $E(w_{a_n}(-\log(T-t_n))) = E(v_n(0)) < E(\kappa)$, and in particular $a_n$ can not be a blow-up point of $u$ (we have from [6], for any blow-up point $a$ of $u$, $E(w_a(s)) \geq E(\kappa)$ for all $s \geq -\log T$). From this fact, a contradiction follows.

**Proof 2**: It is a more elementary proof based on Theorem 3. Since $a_n$ is a blow-up point and that the blow-up set is closed and bounded (see Proposition 2.1), we can assume that $a_n \to a_\infty$ where $a_\infty$ is a blow-up point.

We know from Theorem 3 that for some $C_{\frac{\epsilon_0^2}{2}}$, we have $\forall x \in \Omega$, $\forall t \in [\frac{T}{2}, T)$,

$$\left|\frac{\partial u}{\partial t}(x,t) - u^p(x,t)\right| \leq \frac{\epsilon_0^2}{2}|u(x,t)|^p + C_{\frac{\epsilon_0^2}{2}}. \tag{20}$$

In particular, $u(x,t) \to +\infty$ as $(x,t) \to (a_\infty, T)$ (see next subsection for a proof of Theorem 3 and this fact(22)-(23)). Let $\eta > 0$ such that

$$\forall (x,t) \in B(0,\eta) \times (T-\eta, T),\ C_{\frac{\epsilon_0^2}{2}} < \frac{\epsilon_0^2}{2} u^p(x,t). \tag{21}$$

For large $n$, $a_n \in B(a_\infty, \eta)$ and $t_n \in [T-\eta, T)$. Therefore (20) and (21) yield

$$\forall t \in [t_n, T),\ \frac{\partial u}{\partial t}(a_n, t) \leq (1+\epsilon_0^2) u^p(a_n, t).$$

Since $0 < u(a_n, t_n) \leq \kappa(1-\epsilon_0)(T-t_n)^{-\frac{1}{p-1}}$, we get by direct integration: $\forall t \in [t_n, \min(T, T^*(\epsilon_0)))$,

$$0 \leq u(a_n, t) \leq \kappa \left\{\frac{T-t_n}{(1-\epsilon_0)^{p-1}} - (1+\epsilon_0^2)(t-t_n)\right\}^{-\frac{1}{p-1}}$$

with $T^*(\epsilon_0) = t_n + \frac{T-t_n}{(1+\epsilon_0^2)(1-\epsilon_0)^{p-1}} > T$ if $\epsilon_0 < \epsilon_1(p)$ for some positive $\epsilon_1(p)$. Thus, $a_n$ is not a blow-up point and a contradiction follows.

## 2.2 Global approximated behavior like an ODE

We prove Theorem 3 in this subsection. It follows from Theorem 1 and propagation of flatness (through scaling arguments) observed in [14].

Let us first show how to derive the consequences of Theorem 3 announced in the introduction:
If $x_0$ is a blow-up point of $u(t)$, then

$$u(x,t) \to +\infty \text{ as } (x,t) \to (x_0, T) \tag{22}$$

and $\exists \epsilon_0 > 0$ such that $\forall (x,t) \in B(x_0,\epsilon_0) \times (T-\epsilon_0, T)$, $\dfrac{\partial u}{\partial t}(x,t) > 0$. (23)

*Proof of (22) and (23)*:
From Theorem 3 applied with $\epsilon > 0$, there exists $C_\epsilon$ such that $\forall (x,t) \in \Omega \times [\frac{T}{2}, T)$

$$\frac{\partial u}{\partial t}(x,t) \geq (1-\epsilon)u^p(x,t) - C_\epsilon. \tag{24}$$

Let $A$ be an arbitrary large positive number satisfying

$$(1-\epsilon)A^p - C_\epsilon > 0. \tag{25}$$

From the continuity of $u(x,t)$, there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\forall x \in B(x_0, \epsilon_1)$,

$$u(x, T-\epsilon_2) > A. \tag{26}$$

From (24) and (25), we have $\forall x \in B(x_0,\epsilon_1)$, $\frac{\partial u}{\partial t}(x, T-\epsilon_2) > 0$. Now we claim that $\forall (x,t) \in B(x_0,\epsilon_1) \times (T-\epsilon_2, T)$, $u(x,t) > A$ (which yields (22) and (23) also, by (24) and (25)). Indeed, if not, then there exists $(x_1,t_1) \in B(x_0,\epsilon_1) \times (T-\epsilon_2, T)$ such that $u(x_1,t_1) \leq A$. From the continuity of $u$, we get $t_2 \in (T-\epsilon_2, t_1]$ such that $\forall t \in (T-\epsilon_2, t_2)$, $u(x_1,t) > A$ and $u(x_1,t_2) = A$. From (24) and (25), we have $\forall t \in (T-\epsilon_2, t_2)$, $\frac{\partial u}{\partial t}(x_1,t) > 0$, therefore, $u(x_1,t_2) > u(x_1, T-\epsilon_2) > A$ by (26). Thus, a contradiction follows, and (22) and (23) are proved.

We now prove Theorem 3.

*Proof of Theorem 3*: Let us argue by contradiction and suppose that for some $\epsilon_0 > 0$, there exist $(x_n,t_n)_{n\in\mathbb{N}}$ a sequence of elements of $\Omega \times [\frac{T}{2}, T)$ such that $\forall n \in \mathbb{N}$,

$$|\Delta u(x_n,t_n)| \geq \epsilon_0 |u(x_n,t_n)|^p + n. \tag{27}$$

Since $\|\Delta u(t)\|_{L^\infty(\Omega)}$ is bounded on compact sets of $[\frac{T}{2}, T)$, we have that $t_n \to T$ as $n \to +\infty$. We can also assume the existence of $x_\infty \in \Omega$ such that $x_n \to x_\infty$ as $n \to +\infty$. Indeed, if not, then either $d(x_n, \partial\Omega) \to 0$ (if $\Omega$ is bounded) or $|x_n| \to +\infty$ (if $\Omega = \mathbb{R}^N$) as $n \to +\infty$, and in both cases, (27) is no longer satisfied for large $n$, thanks to Proposition 2.1.

We claim that $x_\infty$ is a blow-up point of $u$. Indeed, if not, then parabolic regularity implies the existence of a positive $\delta$ such that
$\|u(.,t)\|_{W^{2,\infty}(B(x_\infty,\delta))} \leq C$ for some positive $C$, which is a contradiction by (27).

Theorem 1 implies that $u(x_n,t_n)(T-t_n)^{\frac{1}{p-1}}$ is uniformly bounded, therefore, we can assume that it converges as $n \to +\infty$.

Let us consider two cases:
- Case 1: $u(x_n,t_n)(T-t_n)^{\frac{1}{p-1}} \to \kappa' > 0$ ($(x_n,t_n)$ is in some sense in the singular region "near" $(x_\infty,T)$). From (27), it follows that $\|\Delta u(t_n)\|_{L^\infty} \geq |\Delta u(x_n,t_n)| \geq \epsilon_0 \left(\frac{\kappa'}{2}\right)^p (T-t_n)^{-\frac{p}{p-1}}$ with $t_n \to T$, which contradicts Theorem 1.
- Case 2: $u(x_n,t_n)(T-t_n)^{\frac{1}{p-1}} \to 0$ ($(x_n,t_n)$ is in the transitory region between the singular and the regular sets).
Let us first define $(t(x_n))_n$ such that $t(x_n) \leq t_n$, $t(x_n) \to T$ and

$$u(x_n,t(x_n))(T-t(x_n))^{\frac{1}{p-1}} = \kappa_0 \tag{28}$$

where $\kappa_0 \in (0,\kappa)$ satisfies $\forall t>0$, $\forall a \in \Omega$, $\mathcal{E}_{a,t}(\kappa_0 t^{-\frac{1}{p-1}}) \leq \frac{\kappa_0^2}{2(p-1)} - \frac{\kappa_0^{p+1}}{p+1} \leq \frac{\sigma_0}{2}$ and $\sigma_0$ is defined in Proposition 2.2.

**Step 1: Existence of $t(x_n)$**

Since $x_\infty$ is a blow-up point of $u$, $\lim_{t\to T} u(x_\infty,t)(T-t)^{\frac{1}{p-1}} = \kappa$. It follows that for any $\delta > 0$ small enough, there exists a ball $B(x_\infty,\delta')$ such that $\forall x \in B(x_\infty,\delta')$, $\delta^{\frac{1}{p-1}} u(x,T-\delta) \geq \frac{3\kappa+\kappa_0}{4}$. Since $x_n \to x_\infty$ as $n \to +\infty$, this implies that

$$\forall n \geq n_1,\ \delta^{\frac{1}{p-1}} u(x_n,T-\delta) \geq \frac{\kappa+\kappa_0}{2} \tag{29}$$

for some $n_1 = n_1(\delta) \in \mathbb{N}$. Since $u(x_n,t_n)(T-t_n)^{\frac{1}{p-1}} \to 0$, we have the existence of $t_\delta(x_n) \in [T-\delta,t_n] \subset [T-\delta,T)$ such that $u(x_n,t_\delta(x_n))(T-t_\delta(x_n))^{\frac{1}{p-1}} = \kappa_0$, for all $n \geq n_2(\delta)$, where $n_2(\delta) \in \mathbb{N}$. Since $\delta$ was arbitrarily small, it follows from a diagonal extraction argument that there exists a subsequence $t(x_n) \to T$ as $n \to +\infty$ such that $t(x_n) \leq t_n$ and

$$u(x_n,t(x_n))(T-t(x_n))^{\frac{1}{p-1}} = \kappa_0.$$

Now, we claim that a contradiction follows if we prove the following Proposition:

**Proposition 2.3** *Let*

$$v_n(\xi,\tau) = (T-t(x_n))^{\frac{1}{p-1}} u(x_n + \xi\sqrt{T-t(x_n)}, t(x_n) + \tau(T-t(x_n))). \tag{30}$$

*Then, $v_n$ is a solution of (1) for $\tau \in [0,1)$, and there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$,*

$$\forall \tau \in [0,1),\ |\Delta v_n(0,\tau)| \leq \frac{\epsilon_0}{2}|v_n(0,\tau)|^p. \tag{31}$$

Indeed, from (31) and (30), we obtain: $\forall n \geq n_0$, $\forall t \in [t(x_n), T)$,
$|\Delta u(x_n,t)| = (T - t(x_n))^{-(\frac{1}{p-1}+1)}|\Delta_\xi v_n(0, \tau(t,n))|$
$\leq \frac{\epsilon_0}{2}(T - t(x_n))^{-\frac{p}{p-1}}|v_n(0,\tau(t,n))|^p = \frac{\epsilon_0}{2}|u(x_n,t)|^p$ with $\tau(t,n) = \frac{t-t(x_n)}{T-t(x_n)}$, which contradicts (27), since $t_n \geq t(x_n)$. Thus, Theorem 3 is proved.

**Step 2: Flatness of $v_n$**

In this Step we prove Proposition 2.3.

We claim that the following lemma concludes the proof of Proposition 2.3:

**Lemma 2.3** *i)* $\forall \delta_0 > 0$, $\forall A > 0$, $\exists n_3(\delta_0, A) \in \mathbb{N}$ *such that* $\forall n \geq n_3(\delta_0, A)$, *for all* $|\xi| \leq A$ *and* $\tau \in [0, \frac{3}{4}]$, $|v_n(\xi, 0) - \kappa_0| \leq \delta_0$, $|\nabla_\xi v_n(\xi,\tau)| \leq \delta_0$ *and* $|\Delta_\xi v_n(\xi,\tau)| \leq \delta_0$.

*ii)* $\forall \epsilon > 0$, $\forall A > 0$, $\exists n_4(\epsilon, A) \in \mathbb{N}$ *such that* $\forall n \geq n_4$, $\forall \tau \in [0,1)$, *for* $|\xi| \leq \frac{A}{4}$, $|v_n(\xi,\tau) - \hat{v}(\tau)| \leq \epsilon$, $|\nabla v_n(\xi,\tau)| \leq \epsilon$ *and* $|\Delta v_n(\xi,\tau)| \leq \epsilon$ *where* $\hat{v}(\tau) = \kappa\left(\left(\frac{\kappa}{\kappa_0}\right)^{p-1} - \tau\right)^{-\frac{1}{p-1}}$ *is a solution of* $\frac{d\hat{v}}{d\tau} = \hat{v}^p$ *with* $\hat{v}(0) = \kappa_0$.

Indeed, if $\epsilon$ is small enough and $n$ is large enough, then $\forall \tau \in [0,1)$, $v_n(0,\tau) \geq \frac{1}{2}\hat{v}(0) = \frac{\kappa_0}{2}$ and $|\Delta v_n(0,\tau)| \leq (\frac{\kappa_0}{2})^p \frac{\epsilon_0}{2} \leq \frac{\epsilon_0}{2}|v_n(0,\tau)|^p$.

*Proof of lemma 2.3*: *i)* Let $\delta_0 > 0$ and $A > 0$. From (28) and (30), we have: for all $|\xi| \leq A$ and $\tau \in [0, \frac{3}{4}]$:
$v_n(0,0) = \kappa_0$,
$|v_n(\xi,0) - v_n(0,0)| \leq (T - t(x_n))^{\frac{1}{p-1}+\frac{1}{2}} A \|\nabla u(t(x_n))\|_{L^\infty(\Omega)}$,
$\nabla v_n(\xi,\tau) = (T - t(x_n))^{\frac{1}{p-1}+\frac{1}{2}} \nabla u\left(x_n + \xi\sqrt{T - t(x_n)}, t(x_n) + \tau(T - t(x_n))\right)$
$= \left(\frac{1}{1-\tau}\right)^{\frac{1}{p-1}+\frac{1}{2}} (T - (t(x_n) + \tau(T - t(x_n))))^{\frac{1}{p-1}+\frac{1}{2}} \times$
$\nabla u\left(x_n + \xi\sqrt{T - t(x_n)}, t(x_n) + \tau(T - t(x_n))\right)$ and
$\Delta v_n(\xi,\tau) = (T - t(x_n))^{\frac{1}{p-1}+1} \Delta u\left(x_n + \xi\sqrt{T - t(x_n)}, t(x_n) + \tau(T - t(x_n))\right)$
$= \left(\frac{1}{1-\tau}\right)^{\frac{1}{p-1}+1} (T - (t(x_n) + \tau(T - t(x_n))))^{\frac{1}{p-1}+1} \times$
$\Delta u\left(x_n + \xi\sqrt{T - t(x_n)}, t(x_n) + \tau(T - t(x_n))\right)$.

Since $\tau \leq \frac{3}{4}$, $t(x_n) \to T$ as $n \to +\infty$, and $(T-t)^{\frac{1}{p-1}+\frac{1}{2}}\|\nabla u(t)\|_{L^\infty(\Omega)} + (T-t)^{\frac{1}{p-1}+1}\|\Delta u(t)\|_{L^\infty(\Omega)} \to 0$ as $t \to T$ (Theorem 1), *i)* is proved.

*ii)* From *i)* and continuity arguments, it follows that for all $|\xi| \leq A$, $\mathcal{E}_{\xi,1}(v_n(0)) \leq 2\mathcal{E}_{\xi,1}(\kappa_0) \leq \sigma_0$ for $n$ large enough, by definition of $\kappa_0$. Therefore, from Proposition 2.2 (applied with $\delta = 1$ and using translation invariance), we have $\forall \tau \in [\frac{1}{2}, 1)$, $\forall |\xi| \leq \frac{A}{2}$, $|v_n(\xi,\tau)| \leq M(p)$.

By classical parabolic arguments, we get

$$\forall \tau \in [\frac{3}{4}, 1),\ \forall |\xi| \leq \frac{A}{3},\ |v_n| + |\nabla v_n| + |\Delta v_n| \leq M(p). \tag{32}$$

Now, using $i$), (32) and classical estimates for the heat flow, we get for all $\epsilon > 0$: $\forall |\xi| \leq \frac{A}{4}$, $\forall \tau \in [0, 1)$, $|\nabla v_n(\xi, \tau)| \leq \epsilon$ and $|\Delta v_n(\xi, \tau)| \leq \epsilon$ if $n \geq n_5(\epsilon, A)$.

Since $v_n$ is a solution of equation (1), combining this with $i$) and ODE estimates yields for all $\epsilon > 0$: $\forall |\xi| \leq \frac{A}{4}$, $\forall \tau \in [0, 1)$, $|v_n(\xi, \tau) - \hat{v}(\tau)| \leq \epsilon$ if $n \geq n_6(\epsilon, A)$. This concludes the proof of $ii$). ■

# 3 Classification of connections between critical points of equation (3) in $L^\infty_{loc}$

We prove Theorem 2 and Corollaries 2 and 3 in this section.

We first prove Theorem 2, and then we show how Corollaries 2 and 3 can be deduced from Theorem 2.

*Proof of Theorem 2*: We assume that $1 < p$ and $(N - 2)p < N + 2$, and consider $w(y, s)$ a nonnegative global bounded solution of (3) defined for $(y, s) \in \mathbb{R}^N \times \mathbb{R}$. Our goal is to show that $w$ depends only on time $s$.

We proceed in 5 steps.

In Step 1, we show that $w$ has a limit $w_{\pm\infty}$ as $s \to \pm\infty$, where $w_{\pm\infty}$ is a critical point of (3), that is $w_{\pm\infty} \equiv 0$ or $w_{\pm\infty} \equiv \kappa$. We focus then on the non trivial case, that is $w_{-\infty} \equiv \kappa$ and $w_{+\infty} \equiv 0$.

In Step 2, we investigate the linear problem around $\kappa$, as $s \to -\infty$, and show that $w$ would behave at most in three ways.

In Step 3, we show that among these three ways we have the situation $w(y, s) = \varphi(s - s_0)$ with $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$. We then show (respectively in Step 4 and in Step 5) that the two other ways actually can not occur, we find in fact a contradiction through a blow-up argument for $w(s)$ using the geometrical transformation:

$$a \to w_a \text{ defined by } w_a(y, s) = w(y + ae^{\frac{s}{2}}, s) \tag{33}$$

($w_a$ is also a solution of (3)) and a blow-up criterion for equation (3).

**Step 1: Behavior of $w$ as $s \to \pm\infty$**

This step can be found in Giga and Kohn [6]. The results are mainly consequences of parabolic estimates and the gradient structure of equation (3). Let us recall them briefly. We first restate lemma 2.1 of section 2:

**Lemma 3.1 (Parabolic estimates)** *There is a positive constant $M$ such that $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$,*

$$|w(y,s)| + |\nabla w(y,s)| + |\Delta w(y,s)| \leq M \ \text{and} \ \left|\frac{\partial w}{\partial s}(y,s)\right| \leq M(1+|y|).$$

**Lemma 3.2 (Stationary solutions)** *Assume $p \leq (N+2)/(N-2)$ or $N \leq 2$. Then the only nonnegative bounded global solutions in $\mathbb{R}^N$ of*

$$0 = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w \tag{34}$$

*are the trivial ones: $w \equiv 0$ and $w \equiv \kappa$.*

*Proof*: The following Pohozaev identity can be derived for each bounded solutions of equation (3) in $\mathbb{R}^N$ (see Proposition 2 in [6]):

$$(N+2-p(N-2)) \int |\nabla w|^2 \rho dy + \frac{p-1}{2} \int |y|^2 |\nabla w|^2 \rho dy = 0.$$

Hence, for $(N-2)p \leq N+2$, $w$ is constant. Thus, $w \equiv 0$ or $w \equiv \kappa$. ■

**Lemma 3.3 (Gradient structure)** *Assume $p < (N+2)/(N-2)$ or $N \leq 2$. We define for each $w$ solution of (3)*

$$E(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 \rho dy + \frac{1}{2(p-1)} \int_{\mathbb{R}^N} |w|^2 \rho dy - \frac{1}{p+1} \int_{\mathbb{R}^N} |w|^{p+1} \rho dy \tag{35}$$

$$\text{where } \rho(y) = \frac{e^{-|y|^2/4}}{(4\pi)^{N/2}}. \tag{36}$$

*Then, $\forall s_1, s_2 \in \mathbb{R}$,*

$$\int_{s_1}^{s_2} \int_{\mathbb{R}^N} \left|\frac{\partial w}{\partial s}\right|^2 \rho dy ds = E(w(s_1)) - E(w(s_2)) \tag{37}$$

*Outline of the proof*: (*see* Proposition 3 in [6] for more details).

One may multiply equation (3) by $\frac{\partial w}{\partial s}\rho$ and integrate over the ball $B(0,R)$ with $R > 0$. Then, using lemma 3.1 and the dominated convergence theorem yields the result. ■

**Proposition 3.1 (Limit of $w$ as $s \to \pm\infty$)** *Assume $p < (N+2)/(N-2)$ or $N \leq 2$. Let $w$ be a bounded nonnegative global solution of (3) in $\mathbb{R}^{N+1}$. Then $w_{+\infty}(y) = \lim_{s \to +\infty} w(y,s)$ exists and equals 0 or $\kappa$. The convergence is uniform on every compact subset of $\mathbb{R}^N$. The corresponding statements hold also for the limit $w_{-\infty}(y) = \lim_{s \to -\infty} w(y,s)$.*

*Outline of the proof*: (*see* Propositions 4 and 5 in [6] for more details).

Let $(s_j)$ be a sequence tending to $+\infty$, and let $w_j(y,s) = w(y, s + s_j)$. From lemma 3.1, $(w_j)$ converges uniformly on compact sets to some $w_{+\infty}(y,s)$ and $\nabla w_j \to \nabla w_{+\infty}$ a.e. Assuming that $s_{j+1} - s_j \to +\infty$, one can use lemma 3.3 to show that $w_j$ does not depend on $s$. Therefore, $w_{+\infty} \equiv 0$ or $w_{+\infty} = \kappa$ by lemma 3.2. The continuity of $w$ then asserts that $w_{+\infty}$ does not depend on the choice of the subsequence $(s_j)$. The analysis in $-\infty$ is completely parallel. ∎

According to (37) (with $s_1 \to -\infty$ and $s_2 \to +\infty$), there are only two cases:

- $E(w_{-\infty}) - E(w_{+\infty}) = 0$: hence, $\frac{\partial w}{\partial s} \equiv 0$. Therefore, $w$ is a bounded global solution of (34). Thus, $w \equiv 0$ or $w \equiv \kappa$ according to lemma 3.2. This case has been treated by Giga and Kohn in [6].
- $E(w_{-\infty}) - E(w_{+\infty}) > 0$: since $E(\kappa) = (\frac{1}{2} - \frac{1}{p+1})\kappa^{p+1} \int \rho dy > 0 = E(0)$, we have $(w_{-\infty}, w_{+\infty}) = (\kappa, 0)$. It remains to treat this case in order to finish the proof of Theorem 2.

In the following steps, we consider the case

$$(w_{-\infty}, w_{+\infty}) = (\kappa, 0).$$

**Step 2: Classification of the behavior of $w$ as $s \to -\infty$:**

Since $w$ is globally bounded in $L^\infty$ and $w \to \kappa$ as $s \to -\infty$, uniformly on compact subsets of $\mathbb{R}^N$, we have $\lim_{s \to -\infty} \|w - \kappa\|_{L^2_\rho} = 0$ where $L^2_\rho$ is the $L^2$-space associated to the Gaussian measure $\rho(y)dy$ and $\rho$ is defined in (36).

In this part, we classify the $L^2_\rho$ behavior of $w - \kappa$ as $s \to -\infty$. Let us introduce $v = w - \kappa$. From (3), $v$ satisfies the following equation: $\forall (y,s) \in \mathbb{R}^{N+1}$,

$$\frac{\partial v}{\partial s} = \mathcal{L}v + f(v) \tag{38}$$

$$\text{where } \mathcal{L}v = \Delta v - \frac{1}{2} y.\nabla v + v \text{ and } f(v) = |v+\kappa|^{p-1}(v+\kappa) - \kappa^p - p\kappa^{p-1} v. \tag{39}$$

Since $w$ is bounded in $L^\infty$, we can assume $|v(y,s)| \le M$, and then $|f(v)| \le Cv^2$ with $C = C(M)$.

$\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2_\rho$. Its spectrum is

$$spec(\mathcal{L}) = \{1 - \frac{m}{2} | m \in \mathbb{N}\},$$

and it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are derived from Hermite polynomials:

- $N = 1$:
  All the eigenvalues of $\mathcal{L}$ are simple. For $1 - \frac{m}{2}$ corresponds the eigenfunction
$$h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}. \tag{40}$$
  $h_m$ satisfies $\int h_n h_m \rho dy = 2^n n! \delta_{nm}$. Let us introduce
$$k_m = h_m / \|h_m\|^2_{L^2_\rho}. \tag{41}$$

- $N \geq 2$:
  We write the spectrum of $\mathcal{L}$ as
$$spec(\mathcal{L}) = \{1 - \frac{m_1 + ... + m_N}{2} | m_1, ..., m_N \in \mathbb{N}\}.$$
  For $(m_1, ..., m_N) \in \mathbb{N}^N$, the eigenfunction corresponding to $1 - \frac{m_1 + ... + m_N}{2}$ is
$$y \longrightarrow h_{m_1}(y_1)...h_{m_N}(y_N),$$
  where $h_m$ is defined in (40). In particular,

  *1 is an eigenvalue of multiplicity 1, and the corresponding eigenfunction is
$$H_0(y) = 1, \tag{42}$$
  *$\frac{1}{2}$ is of multiplicity $N$, and its eigenspace is generated by the orthogonal basis $\{H_{1,i}(y) | i = 1, ..., N\}$, with $H_{1,i}(y) = h_1(y_i)$; we note
$$H_1(y) = (H_{1,1}(y), ..., H_{1,N}(y)), \tag{43}$$
  *0 is of multiplicity $\frac{N(N+1)}{2}$, and its eigenspace is generated by the orthogonal basis $\{H_{2,ij}(y) | i, j = 1, ..., N, i \leq j\}$, with $H_{2,ii}(y) = h_2(y_i)$, and for $i < j$, $H_{2,ij}(y) = h_1(y_i)h_1(y_j)$; we note
$$H_2(y) = (H_{2,ij}(y), i \leq j). \tag{44}$$

Since the eigenfunctions of $\mathcal{L}$ constitute a total orthonormal family of $L^2_\rho$, we expand $v$ as follows:
$$v(y, s) = \sum_{m=0}^{2} v_m(s).H_m(y) + v_-(y, s) \tag{45}$$

where
$v_0(s)$ is the projection of $v$ on $H_0$,
$v_{1,i}(s)$ is the projection of $v$ on $H_{1,i}$, $v_1(s) = (v_{1,i}(s), ..., v_{1,N}(s))$, $H_1(y)$ is given by (43),
$v_{2,ij}(s)$ is the projection of $v$ on $H_{2,ij}$, $i \leq j$, $v_2(s) = (v_{2,ij}(s), i \leq j)$, $H_2(y)$ is given by (44),
$v_-(y, s) = P_-(v)$ and $P_-$ the projector on the negative subspace of $\mathcal{L}$.

With respect to the positive, null and negative subspaces of $\mathcal{L}$, we write

$$v(y, s) = v_+(y, s) + v_{null}(y, s) + v_-(y, s) \tag{46}$$

where $v_+(y, s) = P_+(v) = \sum_{m=0}^{1} v_m(s).H_m(y)$,
$v_{null}(y, s) = P_{null}(v) = v_2(s).H_2(y)$, $P_+$ and $P_{null}$ are the $L^2_\rho$ projectors respectively on the positive subspace and the null subspace of $\mathcal{L}$.

Now, we show that as $s \to -\infty$, either $v_0(s)$, $v_1(s)$ or $v_2(s)$ is predominant with respect to the expansion (45) of $v$ in $L^2_\rho$. At this level, we are not able to use a center manifold theory to get the result (see [3] page 834-835 for more details). In some sense, we are not able to say that the nonlinear terms in the function of space are small enough. However, using similar techniques as in [3], we are able to prove the result. We have the following:

**Proposition 3.2 (Classification of the behavior of $v(y, s)$ as $s \to -\infty$ )** *As $s \to -\infty$, one of the following situations occurs:*
*i)* $|v_1(s)| + \|v_{null}(y, s)\|_{L^2_\rho} + \|v_-(y, s)\|_{L^2_\rho} = o(v_0(s))$,
*ii)* $|v_0(s)| + \|v_{null}(y, s)\|_{L^2_\rho} + \|v_-(y, s)\|_{L^2_\rho} = o(|v_1(s)|)$,
*iii)* $\|v_+(y, s)\|_{L^2_\rho} + \|v_-(y, s)\|_{L^2_\rho} = o(\|v_{null}(y, s)\|_{L^2_\rho})$.

*Proof*: See Appendix A.

Now we handle successively the three cases suggested by proposition 3.2 to show that only case $i$) occurs.

In case $i$), we end up to show that $w(y, s) = \varphi(s - s_0)$ for some $s_0 \in \mathbb{R}$, where $\varphi$ is defined in (9). In cases $ii$) and $iii$), we show that the solutions satisfy through an elementary geometrical transformation a blow-up condition for equation (3) considered for increasing $s$, which contradicts their boundedness, and concludes the proof of Theorem 2.

**Step 3: Case $i$) of Proposition 3.2: $\exists s_0 \in \mathbb{R}$ such that $w(y, s) = \varphi(s - s_0)$**

**Proposition 3.3** *Suppose that $|v_1(s)| + \|v_{null}(y, s)\|_{L^2_\rho} + \|v_-(y, s)\|_{L^2_\rho} = o(v_0(s))$ as $s \to -\infty$, then there exists $s_0 \in \mathbb{R}$ such that:*

*i)* $\forall \epsilon > 0,\ v_0(s) = -\frac{\kappa}{p-1}e^{s-s_0} + O(e^{(2-\epsilon)s})$ *as* $s \to -\infty$,
*ii)* $\forall (y,s) \in \mathbb{R}^{N+1}\ w(y,s) = \varphi(s-s_0)$ *where* $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$.

**Remark**: This proposition asserts that if a solution of (38) behaves like a constant independent of $y$ (that is like $v_0(s)$), then it is exactly a constant.

*Proof*: *i)* See Step 3 of Appendix A and take $s_0 = -\log(-\frac{(p-1)C_0}{\kappa})$.

We remark that we already know a solution of equation (38) which behaves like *i*). Indeed, $\varphi(s-s_0) - \kappa = (\varphi(s-s_0)-\kappa)h_0$ is a solution of (38) which satisfies

$$\varphi(s-s_0) - \kappa = -\frac{\kappa}{p-1}e^{s-s_0} + O(e^{(2-\epsilon)s}) \text{ as } s \to -\infty.$$

From a dimension argument, we expect that for each parameter, there is at most one solution such that:

$$v_0(s) \sim -\frac{\kappa}{p-1}e^{s-s_0} \text{ as } s \to -\infty.$$

(if for example, center manifold analysis applies). We propose to prove this fact.

In other words, our goal is to show that

$$\forall (y,s) \in \mathbb{R}^{N+1}, v(y,s) = \varphi(s-s_0) - \kappa.$$

Since (38) is invariant under translations in time, we can assume $s_0 = 0$ without loss of generality.

For this purpose, we introduce

$$V(y,s) = v(y,s) - (\varphi(s) - \kappa) = w(y,s) - \varphi(s). \tag{47}$$

From (3), $V$ satisfies the following equation:

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V + F(V)$$

where $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$, $l(s) = -\frac{pe^s}{(p-1)(1+e^s)}$ and
$F(V) = |\varphi+V|^{p-1}(\varphi+V) - \varphi^p - p\varphi^{p-1}V$. Note that $\forall s \le 0$, $|F(V)| \le C|V|^2$ where $C = C(M)$ and $M \ge \|v\|_{L^\infty}$.

We know from Step 3 in Appendix A that

$$|V_0(s)| + |V_1(s)| = O(e^{(2-\epsilon)s}),\ \|V_{null}(s)\|_{L^2_\rho} = o(e^s) \text{ as } s \to -\infty.$$

The following Proposition asserts that $V \equiv 0$, which concludes the proof of Proposition 3.3:

**Proposition 3.4** *Let $V$ be an $L^\infty$ solution of*

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V + F(V)$$

*defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that $V \to 0$ as $s \to \pm\infty$ uniformly on compact sets of $\mathbb{R}^N$,*

$$|V_0(s)| + |V_1(s)| = O(e^{(2-\epsilon)s}) \text{ and } \|V_{null}(s)\|_{L^2_\rho} = o(e^s) \text{ as } s \to -\infty.$$

*Then $V \equiv 0$.*

*Proof*: see Appendix B.

**Step 4: Irrelevance of the case where $v_1(s)$ is preponderant**
In this case $ii)$ of Proposition 3.3, we use the main term in the expansion of $v(s)$ as $s \to -\infty$ to find $a_0$ and $s_0$ such that

$$\int w_{a_0}(y, s_0)\rho(y)dy > \kappa \tag{48}$$

where $w_{a_0}$ is defined in (33). Since $w \geq 0$, we find that (48) implies that $w_{a_0}$ (which is also a solution of (3)) blows-up in finite time $S > s_0$ (and so does $w$), which contradicts the fact that $w$ is globally bounded. It is in fact mainly the only place where the hypothesis

$$w \geq 0$$

is used. More precisely, let us state the following Proposition:

**Proposition 3.5 (A blow-up criterion for equation (3))** *Consider $W \geq 0$ a solution of (3) and suppose that for some $s_0 \in \mathbb{R}$, $\int W(y, s_0)\rho(y)dy > \int \kappa\rho dy = \kappa$. Then $W$ blows-up in finite time $S > s_0$.*

*Proof*: We argue by contradiction and suppose that $W$ is defined for all $s \in [s_0, +\infty)$. If $V = W - \kappa$, then $V$ satisfies equation (38). Let us define

$$z_0(s) = \int V(y,s)\rho(y)dy.$$

Integrating (38) with respect to $\rho dy$, we obtain

$$z_0'(s) = z_0(s) + \int f(V(y,s))\rho dy$$

$$\text{where} \qquad f(x) = (\kappa + x)^p - \kappa^p - p\kappa^{p-1}x \text{ for } \kappa + x \geq 0.$$

It is obvious that $f$ is nonnegative and convex on $[-\kappa, +\infty)$. Since $W = \kappa + V \geq 0$, $\rho \geq 0$ and $\int \rho dy = 1$, we have the following Jensen's inequality:

$$\int f(V(y,s))\rho dy \geq f(\int V(y,s)\rho dy) = f(z_0(s)).$$

Therefore,

$$z_0'(s) \geq z_0(s) + f(z_0(s)). \tag{49}$$

Since $f(x) > 0$ for $x > 0$ ($f$ is strictly convex and $f(0) = f'(0) = 0$) and $z_0(s_0) > 0$ by the hypothesis, by classical arguments, we have $\forall s \geq s_0$, $z_0'(s) \geq 0$, therefore, $\forall s \geq s_0$, $z_0(s) > 0$. By direct integration, we have $\forall s \geq s_0$,

$$s - s_0 \leq \int_{z_0(s_0)}^{z_0(s)} \frac{dx}{f(x)} \leq \int_{z_0(s_0)}^{+\infty} \frac{dx}{f(x)}.$$

Since $\frac{1}{f(x)} \sim \frac{1}{|x|^p}$ as $s \to +\infty$, a contradiction follows and Proposition 3.5 is proved. ∎

**Proposition 3.6 (Case where $v_1(s)$ is preponderant)** *Suppose that* $|v_0(s)| + \|v_{null}(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(|v_1(s)|)$, *then:*

*i)* $\exists C_1 \in \mathbb{R}^N \backslash \{0\}$ *such that* $v_0(s) \sim \frac{p}{\kappa}|C_1|^2 s e^s$ *and* $v_1(s) \sim C_1 e^{s/2}$ *as* $s \to -\infty$.

*ii)* $\exists a_0 \in \mathbb{R}^N$, $\exists s_0 \in \mathbb{R}$ *such that* $\int w_{a_0}(y, s_0)\rho(y)dy > \kappa$ *where* $w_{a_0}$ *introduced in (33) is a solution of equation (3) defined for* $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ *satisfying*
$\|w_{a_0}\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq B$.

From Proposition 3.5, *ii)* is a contradiction.

**Remark**: $w_a$ has a geometrical interpretation in terms of $w(y,s)$. Indeed, from $w(y,s)$, we introduce $u(x,t)$ (as in (2)) defined for $(x,t) \in \mathbb{R}^N \times (-\infty, 0)$ by:

$$x = \frac{y}{\sqrt{-t}},\ s = -\log(-t),\ u(x,t) = (-t)^{-\frac{1}{p-1}} w(y,s).$$

Now, if we define $\hat{w}_a(y,s)$ from $u(x,t)$ by (2) as

$$x = \frac{y-a}{\sqrt{-t}},\ s = -\log(-t),\ \hat{w}_a(y,s) = (-t)^{\frac{1}{p-1}} u(x,t),$$

then, $\hat{w}_a \equiv w_a$.

*Proof of Proposition 3.6*: *i*) follows from Step 3 in Appendix A.

Therefore, we prove *ii*). It is easy to check that $w_a$ satisfies (3). Moreover, from (33) we get $\|w_a\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} = \|w\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \le B$. We want to show that there exist $a\in\mathbb{R}^N$ and $s_0\in\mathbb{R}$ such that $\int w_a(y,s_0)\rho(y)dy > \kappa$. From (33), we have:
$\int w_a(y,s)\rho dy = \int w(y+ae^{s/2},s)\rho dy$.
Let us note $\alpha = ae^{s/2}$. The conclusion follows if we show that there exist $s_0\in\mathbb{R}$ and $\alpha(s_0)\in\mathbb{R}^N$ such that $\int w(y+\alpha(s_0),s_0)\rho dy > \kappa$.

For this purpose, we search an expansion for $\int w(y+\alpha,s)\rho dy$ as $s\to-\infty$ and $\alpha\to 0$.

$$\int w(y+\alpha,s)\rho dy = \int w(y,s)\rho(y-\alpha)dy = \kappa + \int v(y,s)\frac{e^{-\frac{|y-\alpha|^2}{4}}}{(4\pi)^{N/2}}dy$$
$$= \kappa + e^{-\frac{|\alpha|^2}{4}}\int v(y,s)\rho(y)e^{\frac{\alpha.y}{2}}dy$$
$$= \kappa + e^{-\frac{|\alpha|^2}{4}}\int v(y,s)\rho(y)\left(1+\frac{\alpha.y}{2}+\frac{(\alpha.y)^2}{8}\int_0^1(1-\xi)e^{\xi\frac{\alpha.y}{2}}d\xi\right)dy$$
$$= \kappa + (1+O(|\alpha|^2))\left(v_0(s)+\alpha.v_1(s)+(I)\right)$$

where $(I) = e^{-\frac{|\alpha|^2}{4}}\int dy v(y,s)\rho(y)\frac{(\alpha.y)^2}{8}\int_0^1 d\xi(1-\xi)e^{\xi\frac{\alpha.y}{2}}$.
Using Schwartz's inequality, we have

$$|(I)| \le \left(\int v(y,s)\rho(y)dy\right)^{1/2}\left(\int dy\frac{(\alpha.y)^4}{64}\rho(y)\left(\int_0^1 d\xi(1-\xi)e^{\xi\frac{\alpha.y}{2}}\right)^2\right)^{1/2}$$
$$\le \|v(s)\|_{L^2_\rho}\times\frac{|\alpha|^2}{8}\left(\int dy|y|^4\rho(y)\left(\int_0^1 d\xi(1-\xi)e^{\frac{|y|}{2}}\right)^2\right)^{1/2}$$
$$\le \|v(s)\|_{L^2_\rho}\times\frac{|\alpha|^2}{16}\left(\int dy|y|^4\rho(y)e^{|y|}\right)^{1/2} = C|\alpha|^2\|v(s)\|_{L^2_\rho}.$$

Therefore, using the fact that $\|v(s)\|_{L^2_\rho}\sim 2|v_1(s)| = O(e^{s/2})$ and *i*), we get:

$$\int w(y+\alpha,s)\rho dy = \kappa + v_0(s)+\alpha.v_1(s)+O(|\alpha|^2e^{s/2})$$
$$= \kappa + \frac{p}{\kappa}|C_1|^2se^s + o(se^s) + \alpha.C_1e^{s/2} + o(|\alpha|e^{s/2}).$$

Now, if we make $\alpha = \alpha(s) = -\frac{1}{s}\frac{C_1}{|C_1|}$ and take $-s$ large enough, then $\int w(y+\alpha(s),s) - \kappa \ge \frac{1}{2}\alpha(s).C_1e^{s/2} = -\frac{e^{s/2}}{2s}|C_1| > 0$, and the existence of $a_0$ and $s_0$ is proved.

This concludes the proof of Proposition 3.6. ■

**Step 5: Irrelevance of the case where $v_2(s)$ is preponderant**

As in the previous part, we use the information given by the linear theory at $-\infty$ to find a contradiction in the case where *iii*) holds in Proposition 3.2.

**Proposition 3.7 (Case where $v_2(s)$ is preponderant)** *Assume that* $\|v_+(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(\|v_{null}(y,s)\|_{L^2_\rho})$, *then:*

*i) there exists $\delta \geq 0$, $k \in \{0, 1..., N-1\}$ and $Q$ an orthonormal $N \times N$ matrix such that*

$$\begin{aligned} v_{null}(y,s) &= y^T A(s) y - 2trA(s) \\ \text{where } A(s) &= -\frac{\kappa}{4ps} A_0 + O(\frac{1}{s^{1+\delta}}) \text{ as } s \to -\infty, \end{aligned}$$

$$A_0 = Q \begin{pmatrix} I_{N-k} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}$$

*and $I_{N-k}$ is the $(N-k) \times (N-k)$ identity matrix. Moreover,*

$$\|v(s)\|_{L^2_\rho} = -\frac{\kappa}{ps}\sqrt{\frac{N-k}{2}} + O\left(\frac{1}{|s|^{1+\delta}}\right), \ v_0(s) = O(\frac{1}{s^2}) \text{ and } v_1(s) = O(\frac{1}{s^2}).$$

*ii) $\exists a_0 \in \mathbb{R}^N$, $\exists s_0 \in \mathbb{R}$ such that $\int w_{a_0}(y, s_0)\rho(y)dy > \kappa$ where $w_{a_0}$ defined in (33) is a solution of equation (3) satisfying $\|w_{a_0}\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq B$.*

From *ii)* and Proposition 3.5, a contradiction follows.

*Proof of i) of Proposition 3.7*:

The first part of the proof follows as before the ideas of Filippas and Kohn in [3]. Then, we carry on the proof similarly as Filippas and Liu did in [4] for the same equation when the null mode dominates as $s \to +\infty$. Since the used techniques are the same than in [3] and [4], we leave the proof in Appendix C.

*Proof of ii) of Proposition 3.7*:

We proceed exactly in the same way as for the proof of *ii)* of Proposition 3.6. $w_a$ satisfies equation (3), and the $L^\infty$ bound on $w_a$ follows as before.

By setting $\alpha = ae^{s/2}$, the proof reduces then to find $s_0$ and $\alpha = \alpha(s_0)$ such that $\int w(y + \alpha(s_0), s_0)\rho dy > \kappa$.

For this purpose, we search an expansion for $\int w(y+\alpha, s)\rho dy$ as $s \to -\infty$ and $\alpha \to 0$.

$\int w(y+\alpha, s)\rho dy = \int w(y,s)\rho(y-\alpha)dy = \kappa + \int v(y,s)\frac{e^{-\frac{|y-\alpha|^2}{4}}}{(4\pi)^{N/2}}dy$
$= \kappa + e^{-\frac{|\alpha|^2}{4}} \int v(y,s)\rho(y)e^{\frac{\alpha.y}{2}}dy$
$= \kappa + e^{-\frac{|\alpha|^2}{4}} \int v(y,s)\rho(y)\left(1 + \frac{\alpha.y}{2} + \frac{(\alpha.y)^2}{8} + \frac{(\alpha.y)^3}{16}\int_0^1 (1-\xi)^2 e^{\xi\frac{\alpha.y}{2}}d\xi\right)dy.$

We write

$$\int w(y+\alpha, s)\rho dy = \kappa + (I) + (II), \tag{50}$$

where

$(I) = e^{-\frac{|\alpha|^2}{4}}(v_0(s) + \alpha.v_1(s)) + e^{-\frac{|\alpha|^2}{4}} \int dy v(y,s)\rho(y)\frac{(\alpha.y)^3}{16}\int_0^1 d\xi(1-\xi)^2 e^{\xi\frac{\alpha.y}{2}}$
and $(II) = \frac{1}{8}e^{-\frac{|\alpha|^2}{4}}\int v(y,s)(\alpha.y)^2\rho(y)dy.$

From $i)$ of Proposition 3.7 and Schwartz's inequality, we have

$$|(I)| \le \frac{C}{s^2} + C\frac{|\alpha|^3}{|s|}. \tag{51}$$

Since $v = v_- + v_{null} + v_+ = v_- + v_{null} + v_1.y + v_0$, we have from the orthogonality of $v_-$ and $v_{null} + v_+$:

$(II) = \frac{e^{-\frac{|\alpha|^2}{4}}}{8} \int v(y,s)(\alpha.y)^2 \rho dy$

$= \frac{e^{-\frac{|\alpha|^2}{4}}}{8}(v_0(s)\int(\alpha.y)^2\rho dy + v_1(s).\int y(\alpha.y)^2\rho dy) + \frac{e^{-\frac{|\alpha|^2}{4}}}{8}\int v_{null}(\alpha.y)^2\rho dy$

$= v_0(s)O(|\alpha|^2) + \frac{e^{-\frac{|\alpha|^2}{4}}}{8}\int(y^TA(s)y - 2trA(s))(\alpha.y)^2\rho dy$

$= O\left(\frac{|\alpha|^2}{|s|^{1+\delta}}\right) + \frac{e^{-\frac{|\alpha|^2}{4}}}{8}\frac{\kappa}{4p|s|}\int(y^TA_0y - 2trA_0)(\alpha.y)^2\rho dy$

for some $\delta > 0$, according to $i)$ of Proposition 3.7, with

$$A_0 = Q\begin{pmatrix} I_{N-k} & 0 \\ 0 & 0 \end{pmatrix}Q^{-1}.$$

With the change of variable $y = Q^{-1}z$ ($Q$ is an orthonormal matrix) we write:

$(II) = O\left(\frac{|\alpha|^2}{|s|^{1+\delta}}\right) + \frac{e^{-\frac{|\alpha|^2}{4}}}{8}\frac{\kappa}{4p|s|}\int\sum_{i=1}^{N-k}(z_i^2 - 2) \times (Q\alpha.z)^2\rho(z)dz,$

therefore,

$$(II) = \frac{\kappa}{4p|s|}\sum_{i=1}^{N-k}\int(z_i^2-2)(Q\alpha.z)^2\rho dz + O\left(\frac{|\alpha|^2}{|s|^{1+\delta}}\right) + O\left(\frac{|\alpha|^4}{|s|}\right). \tag{52}$$

Gathering (50), (51) and (52), we write:

$\int w(y+\alpha,s)\rho dy$

$$= \kappa + \frac{\kappa}{4p|s|}\sum_{i=1}^{N-k}\int(z_i^2-2)(Q\alpha.z)^2\rho dz + O(\frac{1}{s^2}) + O\left(\frac{|\alpha|^2}{|s|^{1+\delta}}\right) + O\left(\frac{|\alpha|^3}{|s|}\right).$$

Now, if we take $\alpha = \alpha(s) = \frac{1}{|s|^{1/4}}Q^{-1}e^1$ where $e^1 = (1,0,...,0)$, then

$$\int w(y+\alpha(s),s)\rho dy = \kappa + \frac{\kappa}{4p|s|^{3/2}} \times 8 + O\left(\frac{1}{|s|^{3/2+\delta}}\right).$$

If we take $-s$ large enough, and $a(s) = e^{-s/2}\alpha(s)$, then

$$\int w(y+\alpha(s)e^{s/2},s) > \kappa.$$

This concludes the proof of $ii)$ of Proposition 3.7 and the proof of Theorem 2.

We now prove Corollaries 1 and 2:

*Proof of Corollary 2*:

We consider $w$ a nonnegative solution of (3) defined for $(y,s) \in \mathbb{R}^N \times (-\infty, s^*)$ where $s^* \in \mathbb{R} \cup \{+\infty\}$. We assume that there is a constant $C_0$ such that

$$\forall a \in \mathbb{R}^N,\ \forall s \le s^*,\ E_a(w(s)) \le C_0 \tag{53}$$

where $E_a$ is defined in (10).

Through some geometrical transformations, we define below $\hat{w}$, a solution of (3) defined on $\mathbb{R}^N \times \mathbb{R}$, which satisfies the hypotheses of Theorem 2. Then, we deduce the characterization of $w$ from the one given in Theorem 2 for $\hat{w}$.

Let us define $u(t)$ a solution of (1) by:

$$y = \frac{x}{\sqrt{-t}},\ s = -\log(-t),\ u(x,t) = (-t)^{-\frac{1}{p-1}} w(y,s) \tag{54}$$

where $(x,t) \in \mathbb{R}^N \times (-\infty, T^*)$ with $T^* = -e^{-s^*}$ if $s^*$ is finite and $T^* = 0$ if $s^* = +\infty$. Then we introduce $\hat{w}$ a solution of (3):

$$y = \frac{x}{\sqrt{T^* - t}},\ s = -\log(T^* - t),\ \hat{w}(y,s) = (T^* - t)^{\frac{1}{p-1}} u(x,t) \tag{55}$$

defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$. We have then $\forall (y,s) \in \mathbb{R}^N \times (-\infty, s^*)$,

$$w(y,s) = (1 + T^* e^s)^{-\frac{1}{p-1}} \hat{w}\Big(\frac{y}{\sqrt{1 + T^* e^s}}, s - \log(1 + T^* e^s)\Big). \tag{56}$$

We claim that $\hat{w} \in L^\infty(\mathbb{R}^N \times \mathbb{R})$. Indeed, from (53), (54) and $i)$ of Proposition 2.2, we have $\forall (x,t) \in \mathbb{R}^N \times (-\infty, T^*)$, $|u(x,t)| \le M(C_0)(T^* - t)^{-\frac{1}{p-1}}$. Hence, (55) implies that $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$, $|w(y,s)| \le M(C_0)$.

Since $w$ is nonnegative, $\hat{w}$ is also nonnegative, and then, by Theorem 2 we have:

either $\hat{w} \equiv 0$, or $\hat{w} \equiv \kappa$

or $\hat{w}(y,s) = \varphi(s - s_0)$ for some $s_0 \in \mathbb{R}$, where $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$.

Therefore, by (56), we have:

either $w \equiv 0$, or $w(y,s) = \kappa(1 - e^{s - s^*})^{-\frac{1}{p-1}}$

or $w(y,s) = (1 - e^{s-s^*})^{-\frac{1}{p-1}} \kappa \left(1 + \exp(s - \log(1 - e^{s-s^*} - s_0))\right)^{-\frac{1}{p-1}}$

$= \kappa \left(1 + e^s(e^{-s_0} - e^{-s^*})\right)^{-\frac{1}{p-1}}$.

Since $s_0$ is arbitrary in $\mathbb{R}$, this concludes the proof of Corollary 2.

*Proof of Corollary 3*:

Let $u(x,t)$ be a nonnegative solution of (1) defined for $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ which satisfies $|u(x,t)| \le C(T-t)^{-\frac{1}{p-1}}$. We introduce $w(y,s) = w_0(y,s)$ where $w_0$ is defined in (2). Then, it is easy to see that $w$ satisfies all the hypotheses of Theorem 2. Therefore, either $w \equiv 0$ of there exists $t_0 \ge 0$ such that $\forall (y,s) \in \mathbb{R}^{N+1}$, $w(y,s) = \kappa(1+t_0e^s)^{-\frac{1}{p-1}}$. Thus, either $u \equiv 0$ or $u(x,t) = \kappa(T+t_0-t)^{-\frac{1}{p-1}}$. This concludes the proof of Corollary 3.

# A Proof of Proposition 3.2

We proceed in 3 steps: In Step 1, we give a new version of an ODE lemma by Filippas and Kohn [3] which will be applied in Step 2 in order to show that either $v_{null}$ or $v_+$ is predominant in $L^2_\rho$ as $s \to -\infty$. In Step 3, we show that in the case where $v_+$ is predominant, then either $v_0(s)$ or $v_1(s)$ predominates the other.

**Step 1: An ODE lemma**

**Lemma A.1** *Let $x(s)$, $y(s)$ and $z(s)$ be absolutely continuous, real valued functions which are non negative and satisfy*

*i) $(x,y,z)(s) \to 0$ as $s \to -\infty$, and $\forall s \le s_*$, $x(s)+y(s)+z(s) \ne 0$,*

*ii) $\forall \epsilon > 0$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \le s_0$*

$$\begin{cases} \dot{z} \ge c_0 z - \epsilon(x+y) \\ |\dot{x}| \le \epsilon(x+y+z) \\ \dot{y} \le -c_0 y + \epsilon(x+z). \end{cases} \tag{57}$$

*Then, either $x+y = o(z)$ or $y+z = o(x)$ as $s \to -\infty$.*

*Proof*: Filippas and Kohn showed in [3] a slightly weaker version of this lemma (with in the conclusion $x, y, z \to 0$ exponentially fast instead of $x+y = o(z)$). We adapt here their proof to get the proof of lemma A.1.

By rescaling in time, we may assume $c_0 = 1$.

**Part 1**: Let $\epsilon > 0$. We show in this part that either:

$$\exists s_2(\epsilon) \text{ such that } \forall s \le s_2,\ z(s)+y(s) \le C\epsilon x(s), \tag{58}$$

$$\text{or } \exists s_2(\epsilon) \text{ such that } \forall s \le s_2,\ x(s)+y(s) \le C\epsilon z(s). \tag{59}$$

We first show that $\forall s \le s_0(\epsilon)$, $\beta(s) \le 0$ where $\beta = y - 2\epsilon(x+z)$.

We argue by contradiction and suppose that there exists $s_* \leq s_0(\epsilon)$ such that $\beta(s_*) > 0$. Then, if $s \leq s_*$ and $\beta(s) > 0$, we have form (57)
$\dot{\beta}(s) = \dot{y} - 2\epsilon(\dot{x} + \dot{z}) \leq -y + \epsilon(x+z) + 2\epsilon^2(x+y+z) - 2\epsilon(z - \epsilon(x+y)) \leq -\epsilon(1 - 4\epsilon - 8\epsilon^2)x - \epsilon(3 - 2\epsilon - 8\epsilon^2)z \leq 0$.
Therefore, $\forall s \leq s_*$, $\beta(s) \geq \beta(s_*) > 0$, which contradicts $\beta(s) \to 0$ as $s \to -\infty$ . Thus

$$\forall s \leq s_0(\epsilon),\ y \leq 2\epsilon(x+z). \tag{60}$$

Therefore, (57) yields

$$\begin{array}{rcl} \dot{z} & \geq & \frac{1}{2}z - 2\epsilon x \\ |\dot{x}| & \leq & 2\epsilon(x+z) \end{array} \tag{61}$$

Let $\gamma(s) = 8\epsilon x(s) - z(s)$. Two cases arise then:

*Case 1*: $\exists s_2 \leq s_0(\epsilon)$ such that $\gamma(s_2) > 0$.

Suppose then $\gamma(s) = 0$ and compute $\dot{\gamma}(s)$.
$\dot{\gamma}(s) = 8\epsilon\dot{x} - \dot{z} \leq 16\epsilon^2(x+z) - \frac{1}{2}z + 2\epsilon x = -z(s)(\frac{1}{4} - 2\epsilon - 16\epsilon^2)$.
Since $z(s) > 0$ (otherwise $z(s) = 0$, $x(s) = 0$ and then $y(s) = 0$ by (60), which is excluded by the hypothesis), we have

$$\gamma(s) = 0 \Longrightarrow \dot{\gamma}(s) < 0.$$

Since $\gamma(s_2) > 0$, this implies $\forall s \leq s_2$, $\gamma(s) > 0$, i.e. $8\epsilon x(s) > z(s)$. Together with (60), this yields (58).

*Case 2*: $\forall s \leq s_0(\epsilon)$, $\gamma(s) \leq 0$ i.e. $8\epsilon x \leq z(s)$.

In this case, (61) yields

$$\forall s \leq s_0(\epsilon),\ \dot{z} \geq \frac{1}{4}z,\ \text{and}\ \dot{x} \leq (2\epsilon + \frac{1}{4})z.$$

Therefore, we get by integration:

$$z(s) \geq \frac{1}{4}\int_{-\infty}^{s} z(t)dt \text{ and } x(s) \leq (2\epsilon + \frac{1}{4})\int_{-\infty}^{s} z(t)dt,$$

which yields $x(s) \leq (8\epsilon + 1)z(s)$. We inject this in (61) and get
$\dot{x}(s) \leq 2\epsilon(x+z) \leq 2\epsilon z(2 + 8\epsilon)$. Again, by integration:
$x(s) \leq 2\epsilon(2+8\epsilon)\int_{-\infty}^{s} z(t)dt \leq 8\epsilon(2+8\epsilon)z(s)$. Together with (60), this yields (59).

**Part 2**: Let $\epsilon < \frac{1}{C}$. Then either (58) or (59) occurs.

For example, (58) occurs, that is $\exists s_2(\epsilon) \leq s_0$ such that $\forall s \leq s_2$, $z + y \leq C\epsilon x$. Let $\epsilon' \leq \epsilon$ be an arbitrary positive number. Then, according to Part 1, either $\forall s \leq s_2'$, $z + y \leq C\epsilon' x$ for some $s_2'(\epsilon')$,
or $\forall s \leq s_2'$, $y + x \leq C\epsilon' z$ for some $s_2'(\epsilon')$.

Only the first case occurs. Indeed, if not, then for $s \le \min(s_2, s_2')$, $x \le C\epsilon' z \le C\epsilon' C\epsilon x \le C^2\epsilon^2 x$ since $\epsilon' \le \epsilon$. Since $(C\epsilon)^2 < 1$, we have $x \equiv 0$ and $z \equiv y \equiv 0$ for $s \le \min(s_2, s_2')$, which is excluded by the hypotheses.

Do the same if (59) occurs.

This concludes the proof of lemma A.1. ∎

**Step 2: Competition between $v_+$, $v_{null}$ and $v_-$**

In this step we show that either $\|v_-(s)\|_{L^2_\rho} + \|v_+(s)\|_{L^2_\rho} = o(\|v_{null}(s)\|_{L^2_\rho})$ (which is case *iii*) of Proposition 3.2) or
$\|v_-(s)\|_{L^2_\rho} + \|v_{null}(s)\|_{L^2_\rho} = o(\|v_+(s)\|_{L^2_\rho})$ (which yields case *i*) or *ii*) of Proposition 3.2) in Step 3).

This situation is exactly symmetric to the one in section 4 in Filippas and Kohn's paper [3]. Indeed, we are treating the same equation (38), but we have $\|v(s)\|_{L^\infty_{loc}} \to 0$ as $s \to -\infty$ whereas in [3], $\|v(s)\|_{L^\infty_{loc}} \to 0$ as $s \to +\infty$. Nevertheless, the derivation of the differential inequalities satisfied by $v_-$, $v_{null}$ and $v_+$ in [3] is still valid here with the changes: "$s \to +\infty$" becomes $s \to -\infty$ and "$s$ large enough" becomes "$-s$ large enough". Therefore, we claim that [3] implies:

**Lemma A.2** *$\forall \epsilon > 0$, $\exists s_0 \in \mathbb{R}$ such that for a.e. $s \le s_0$:*

$$\begin{aligned} \dot{z} &\ge (\tfrac{1}{2} - \epsilon) z - \epsilon(x + y) \\ |\dot{x}| &\le \epsilon(x + y + z) \\ \dot{y} &\le -(\tfrac{1}{2} - \epsilon) y + \epsilon(x + z) \end{aligned}$$

*where $z(s) = \|v_+(s)\|_{L^2_\rho}$, $x(s) = \|v_{null}(s)\|_{L^2_\rho}$ and $y(s) = \|v_-(s)\|_{L^2_\rho} + \||y|^{\frac{k}{2}} v^2(s)\|_{L^2_\rho}$ for a fixed integer $k$.*

Now, since $\|v(s)\|_{L^\infty_{loc}} \to 0$ as $s \to -\infty$, we have $(x, y, z)(s) \to 0$ as $s \to -\infty$. We can not have $x(s_1) + y(s_1) + z(s_1) = 0$ for some $s_1 \in \mathbb{R}$, because this implies that $\forall y \in \mathbb{R}^N$, $v(y, s_1) = 0$, and from the uniqueness of the solution to the Cauchy problem of equation (38) and $v(s_1) = 0$, we have $\forall (y, s) \in \mathbb{R}^N \times \mathbb{R}$, $v(y, s) = 0$, which contradicts $\kappa + v \to 0$ as $s \to +\infty$. Applying lemma A.1 with $c_0 = \frac{1}{4}$, we get:
either $\|v_-(s)\|_{L^2_\rho} + \|v_+(s)\|_{L^2_\rho} = o(\|v_{null}(s)\|_{L^2_\rho})$
or $\|v_-(s)\|_{L^2_\rho} + \|v_{null}(s)\|_{L^2_\rho} = o(\|v_+(s)\|_{L^2_\rho})$.

**Step 3: Competition between $v_0$ and $v_1$**

In this step, we focus on the case where $\|v_-(s)\|_{L^2_\rho} + \|v_{null}(s)\|_{L^2_\rho} = o(\|v_+(s)\|_{L^2_\rho})$. We will show that it leads either to case *i*) or case *ii*) of Proposition 3.2.

Let us first remark that lemma A.1 implies in this case that

$$\forall \epsilon > 0,\ z(s) = \|v_+(s)\|_{L^2_\rho} = O(e^{(\frac{1}{2}-\epsilon)}) \text{ as } s \to -\infty. \tag{62}$$

Now, we want to derive from (38) the equations satisfied by $v_0$ and $v_1$. We must estimate $\int f(v(y,s))k_m(y_i)\rho(y)dy$ for $m = 0, 1$ and $i = 1, ..N$ (see (41) for $k_m$). Let us give this crucial estimate:

**Lemma A.3** *There exists $\delta_0 > 0$ and an integer $k' > 4$ such that for all $\delta \in (0, \delta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \le s_0$, $\int v^2 |y|^{k'} \rho dy \le c_0(k')\delta^{4-k'} z(s)^2$.*

*Proof*: Let $I(s) = \left(\int v^2|y|^{k'}\rho dy\right)^{1/2}$. We first derive a differential inequality satisfied by $I(s)$. If we multiply (38) by $v|y|^{k'}\rho$ and integrate over $\mathbb{R}^N$, we obtain:

$$\frac{1}{2}\frac{d}{ds}(I(s)^2) = \int v.\mathcal{L}v|y|^{k'}\rho dy + \int vf(v)|y|^{k'}\rho dy.$$

Since $v$ is bounded by $M$, we get $\int vf(v)|y|^{k'}\rho dy \le MC\int v^2|y|^{k'}\rho dy$.

After some calculations, we show that
$\int v.\mathcal{L}v|y|^{k'}\rho dy \le \frac{k}{2}(k+N-2)\int |y|^{k'-2}v^2\rho dy + (1-\frac{k}{4})I(s)^2$.
Using Schwartz's inequality, we find:
$\int v^2|y|^{k'-2}\rho dy \le I(s)\left(\int v^2|y|^{k'-4}\rho dy\right)^{1/2}$.
Let us bound $\left(\int v^2|y|^{k'-4}\rho dy\right)^{1/2}$. If $k' > 4$ and $\delta > 0$, then
$\left(\int v^2|y|^{k'-4}\rho dy\right)^{1/2} \le \left(\int_{|y|\le\delta^{-1}} v^2|y|^{k'-4}\rho dy\right)^{1/2} + \left(\int_{|y|\ge\delta^{-1}} v^2|y|^{k'-4}\rho dy\right)^{1/2}$
$\le \delta^{2-k'/2}\left(\int v^2\rho dy\right)^{1/2} + \delta^2 I$
$\le 2\delta^{2-k'/2}z(s) + \delta^2 I$ since $\left(\int v^2\rho dy\right)^{1/2} \sim \left(\int v_+^2\rho dy\right)^{1/2} = z(s)$ as $s \to -\infty$.

Combining all the previous bounds, we obtain:
$I'(s) \le -\theta I + d\delta^{2-k'/2}z$ with $\theta = \frac{k'}{4} - 1 - MC - \frac{k'}{2}(k'+N-2)\delta^2$ and $d = k'(k'+N-2)$.

We claim that there exist an integer $k' > 4$ and $\delta_0 > 0$ such that $\forall\delta \in (0,\delta_0)$, $\theta \ge 1$. Hence,

$$I'(s) \le -I(s) + d\delta^{2-k'/2}z(s). \tag{63}$$

Now, we will derive a differential inequality satisfied by $z$ in order to couple it with (63), and then prove lemma A.3.

We project (38) onto the positive subspace of $\mathcal{L}$, we multiply the result by $v_+\rho$ and then, we integrate over $\mathbb{R}^N$ to get:

$$\frac{1}{2}\frac{d}{ds}(z(s)^2) = \int \mathcal{L}v_+.v_+\rho dy + \int P_+(f(v))v_+\rho dy.$$

Since $(\mathrm{Spec}\ \mathcal{L}) \cap \mathbb{R}_+^* = \{1, \frac{1}{2}\}$, we have $\int \mathcal{L} v_+.v_+ \rho dy \geq \frac{1}{2} z(s)^2$.
Using Schwartz's inequality, we obtain:
$|\int P_+(f(v)) v_+ \rho dy| \leq \left(\int P_+(f(v))^2 \rho dy\right)^{1/2} \left(\int v_+^2 \rho dy\right)^{1/2}$
$\leq \left(\int f(v)^2 \rho dy\right)^{1/2} z(s)$.
Since $v \to 0$ as $s \to -\infty$ uniformly on compact sets, we have:
$\int f(v)^2 \rho dy \leq C^2 \int v^4 \rho dy = C^2 \int_{|y| \leq \delta^{-1}} v^4 \rho dy + C^2 \int_{|y| \geq \delta^{-1}} v^4 \rho dy$
$\leq \epsilon^2 \int v^2 \rho dy + C^2 M^2 \delta^{k'} \int v^2 |y|^{k'} \rho \leq 4\epsilon^2 z^2 + C^2 M^2 \delta^{k'} I^2$ for all $\epsilon > 0$, provided that $s \leq s_0(\epsilon, \delta)$.
Thus, $\left(\int f(v)^2 \rho dy\right)^{1/2} \leq 2\epsilon z + CM\delta^{k'/2} I$.

Combining all the previous estimates, we obtain:

$$z'(s) \geq \frac{1}{2} z(s) - 2\epsilon z - CM\delta^{k'/2} I(s). \tag{64}$$

With $\epsilon = 1/8$, (63) and (64) yield:

$$\forall s \leq s_0 \begin{cases} z'(s) & \geq & \frac{1}{4} z(s) - CM\delta^{k'/2} I(s) \\ I'(s) & \leq & -I(s) + d\delta^{2-k'/2} z(s). \end{cases}$$

Now, we are ready to conclude the proof of lemma A.3:

Let $\gamma(s) = I(s) - 2d\delta^{2-k'/2} z(s)$. Let us assume $\gamma(s) > 0$ and show that $\gamma'(s) < 0$.

$\gamma'(s) = I' - 2d\delta^{2-k'/2} z' \leq (-I + d\delta^{2-k'/2} z) - 2d\delta^{2-k'/2}(\frac{1}{4} z - CM\delta^{k'/2} I)$
$\leq I(-1 + \frac{1}{4} + 2CMd\delta^2) = I(-\frac{3}{4} + 2CM\delta^2 d)$

If we choose $\delta_0$ such that $\forall \delta \in (0, \delta_0)$, $-\frac{3}{4} + 2CM\delta^2 d < 0$, then $\gamma(s) > 0$ implies $I(s) > 0$ and $\gamma'(s) < 0$. Since $\gamma(s) \to 0$ as $s \to -\infty$ (because $v \to 0$ uniformly on compact sets), we conclude that for some $s_1 \in \mathbb{R}$, $\forall s \leq s_1$, $\gamma(s) \leq 0$. Since $d = k'(k' + N - 2)$, lemma A.3 is proved. ■

Using lemma A.3, we try to estimate $\int f(v) k_m(y_i) \rho dy$.
Since $|f(v) - \frac{p}{2\kappa} v^2| \leq C(M) v^3$, we write:

$$\int f(v) \rho dy = \frac{p}{2\kappa} \int v^2 \rho dy + O\left(\int v^3 \rho dy\right). \tag{65}$$

For all $\epsilon > 0$, $\delta > 0$ and $s \leq s_0$, we write:
$|\int v^3 \rho dy| \leq |\int_{|y| \leq \delta^{-1}} v^3 \rho dy| + |\int_{|y| \geq \delta^{-1}} v^3 \rho dy|$
$\leq |\int_{|y| \leq \delta^{-1}} v^3 \rho dy| + M\delta^{k'} \int v^2 |y|^{k'} \rho dy \leq |\int_{|y| \leq \delta^{-1}} v^3 \rho dy| + Mc_0(k')\delta^4 z(s)^2$.
We fix $\delta > 0$ small enough such that $Mc_0(k')\delta^4 \leq \frac{\epsilon}{2}$. Then, we take $s \leq s_1(\epsilon)$ such that $|\int_{|y| \leq \delta^{-1}} v^3 \rho dy| \leq \frac{\epsilon}{4} \int_{|y| \leq \delta^{-1}} v^2 \rho dy \leq \frac{\epsilon}{4} \int v^2 \rho dy$ (because $v \to 0$ in $L^\infty(B(0, \delta))$).

Since $\int v^2\rho dy \sim z(s)^2$ as $s \to -\infty$, we get for $s \le s_2(\epsilon)$, $|\int v^3 \rho dy| \le \epsilon z(s)^2$. Therefore, equation (38) and (65) yield:

$$v_0'(s) = v_0(s) + \frac{p}{2\kappa} z(s)^2 (1 + \alpha(s)) \tag{66}$$

where $\alpha(s) \to 0$ as $s \to -\infty$.

Using the same type of calculations as for $\int v^3\rho dy$, we can prove that $\int v^2 k_1(y_i)\rho dy = O(z(s)^2)$. Therefore, (38) yields the following vectorial equation:

$$v_1'(s) = \frac{1}{2} v_1(s) + \beta(s) z(s)^2 \tag{67}$$

where $\beta$ is bounded.

From (62), (66), (67) and standard ODE techniques, we get:

$$\forall \epsilon > 0,\ v_0(s) = O(e^{(1-\epsilon)s}) \text{ and } v_1(s) = C_1 e^{\frac{s}{2}} + O(e^{(1-\epsilon)s}).$$

Since $z(s)^2 = v_0(s)^2 + 2|v_1(s)|^2$, we write (66) as

$$v_0'(s) = v_0(s) + \frac{p}{2\kappa}|C_1|^2 e^s (1 + \alpha(s)) + \gamma(s)$$

where $\gamma(s) = O(e^{2(1-\epsilon)s})$. Therefore,

$$\forall \epsilon > 0,\ v_0(s) = \frac{p}{\kappa}|C_1|^2 s e^s (1 + o(1)) + C_0 e^s + O(e^{2(1-\epsilon)s}) \tag{68}$$

as $s \to -\infty$.

Two cases arise:

i) If $C_1 \ne 0$, then $v_1(s) \sim C_1 e^{\frac{s}{2}} \gg \frac{p}{\kappa}|C_1|^2 s e^s \sim v_0(s)$. This is case *ii*) of Proposition 3.2.

ii) If $C_1 = 0$, then $|z(s)| \le Ce^{(2-\epsilon)s}$, and (67) yields $v_1 = O(e^{(2-\epsilon)s})$. From (68), we have $v_0(s) = C_0 e^s + O(e^{(2-\epsilon)s})$.
We claim that $C_0 < 0$ (If not, then the function $F(s) = e^{-s} v_0(s)$ goes to $C_0 \ge 0$ as $s \to -\infty$ and is increasing if $s \le s_0$. Therefore, $\forall s \le s_0$, $v_0(s) \ge C_0 e^s \ge 0$. Since $v$ is bounded and $\kappa + v \ge 0$, we have from Proposition 3.5 $\forall s \in \mathbb{R}$, $\int (\kappa + v(y,s))\rho dy \le \kappa$, that is $v_0(s) \le 0$.

Hence, $\forall s \le s_0$, $v_0(s) = 0$ and $z(s) = \sqrt{2}|v_1(s)|$. Then, (67) implies that $\forall s \le s_0$, $v_1(s) = 0$ and $z(s) = 0$. Since $\int v^2 \rho dy \sim z(s)$, we have $v \equiv 0$ and $w \equiv \kappa$ in a neighborhood of $-\infty$ and then on $\mathbb{R}^N \times \mathbb{R}$ which contradicts $w \to 0$ as $s \to +\infty$).
Thus, $v_0(s) \sim C_0 e^s \gg Ce^{(2-\epsilon)s} \ge |v_1(s)|$. This is Case *i*) of Proposition 3.2.

This concludes the proof of Proposition 3.2. ∎

# B Proof of Proposition 3.4

Let us recall Proposition 3.4:

**Proposition B.1** *Let $V$ be an $L^\infty$ solution of*

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V + F(V) \tag{69}$$

*defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$, where $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$, $l(s) = -\frac{pe^s}{(p-1)(1+e^s)}$ and $F(V) = |\varphi + V|^{p-1}(\varphi + V) - \varphi^p - p\varphi^{p-1}V$.*

*Assume that $V \to 0$ as $s \to \pm\infty$ uniformly on compact sets of $\mathbb{R}^N$,*

$$|V_0(s)| + |V_1(s)| = O(e^{(2-\epsilon)s}) \text{ and } \|V_{null}(s)\|_{L^2_\rho} = o(e^s) \text{ as } s \to -\infty. \tag{70}$$

*Then $V \equiv 0$.*

In order to show that $V \equiv 0$ in $\mathbb{R}^{N+1}$, we proceed in three steps: in Step 1, we do an $L^2_\rho$ analysis for $V$ as $s \to -\infty$ , similarly as in Part 2 of section 2 to show that either $\|V(s)\|_{L^2_\rho} \sim \|V_+(s)\|_{L^2_\rho}$ or $\|V(s)\|_{L^2_\rho} \sim \|V_{null}(s)\|_{L^2_\rho}$. Then, we treat these two cases successively in Steps 2 and 3 to show that $V \equiv 0$.

**Step 1: $L^2_\rho$ analysis for $V$ as $s \to -\infty$**

**Lemma B.1** *As $s \to -\infty$ , either*
*i) $\|V_-(s)\|_{L^2_\rho} + \|V_{null}(s)\|_{L^2_\rho} = o(\|V_+(s)\|_{L^2_\rho})$*
*or ii) $\|V_-(s)\|_{L^2_\rho} + \|V_+(s)\|_{L^2_\rho} = o(\|V_{null}(s)\|_{L^2_\rho})$.*

*Proof*: One can adapt easily the proof of Filippas and Kohn in [3] here. Indeed, $V$ satisfies almost the same type of equation (because $l(s) \to 0$ as $s \to -\infty$ , and $|F(V)| \le CV^2$), and $V \to 0$ as $s \to -\infty$ uniformly on compact sets. Therefore, we claim that up to the change of "$s \to -\infty$ " into "$s \to +\infty$", section 4 of [3] implies

**Lemma B.2** *$\forall \epsilon > 0$, $\exists s_0 \in \mathbb{R}$ such that for a.e. $s \le s_0$:*

$$\begin{array}{rcl} \dot{Z} & \ge & (\frac{1}{2} - \epsilon)Z - \epsilon(X+Y) \\ |\dot{X}| & \le & \epsilon(X+Y+Z) \\ \dot{Y} & \le & -(\frac{1}{2} - \epsilon)Y + \epsilon(X+Z) \end{array}$$

*where $Z(s) = \|V_+(s)\|_{L^2_\rho}$, $X(s) = \|V_{null}(s)\|_{L^2_\rho}$ and $Y(s) = \|V_-(s)\|_{L^2_\rho} + \||y|^{\frac{k}{2}}V^2(s)\|_{L^2_\rho}$ for a fixed integer $k$.*

Since $\|V(s)\|_{L^\infty_{loc}} \to 0$ as $s \to -\infty$ and $V$ is bounded in $L^\infty$, we have $(X,Y,Z)(s) \to 0$ as $s \to -\infty$. Similarly as in Step 2 of Appendix A, we can not have $X(s)+Y(s)+Z(s) = 0$ for some $s \in \mathbb{R}$. Therefore, the conclusion follows from lemma A.1, in the same way as in Step 2 of Appendix A.

**Step 2: Case** $\|V_-(s)\|_{L^2_\rho} + \|V_{null}(s)\|_{L^2_\rho} = o(\|V_+(s)\|_{L^2_\rho})$

Since (69) and (38) are very similar (the only real difference is the presence in (69) of $l(s)$ which goes to zero as $s \to -\infty$ ), one can adapt without difficulty all the Step 3 of Appendix A and show that $V_0$ and $V_1$ satisfy equations analogous to (66) and (67): $\forall s \le s_0$

$$\left\{ \begin{array}{rcl} V_0'(s) & = & V_0(s)(1+l(s)) + a_0(s)(V_0(s)^2 + 2|V_1(s)|^2) \\ V_1'(s) & = & V_1(s)(\frac{1}{2}+l(s)) + a_1(s)(V_0(s)^2 + 2|V_1(s)|^2) \end{array} \right. \quad (71)$$

where $a_0$ and $a_1$ are bounded.

According to (70), there exist $B > 0$ and $s_1 \le s_0$ such that $\forall s \le s_1$

$$|a_0(s)| \le B, |a_1(s)| \le B, |V_0(s)| \le e^{\frac{3s}{2}} \text{and } |V_1(s)| \le e^{\frac{3s}{2}}. \quad (72)$$

We claim then that the following lemma yields $V \equiv 0$:

**Lemma B.3** *$\forall n \in \mathbb{N}$, $\forall s \le s_1$, $|V_m(s)| \le (\frac{3}{2}e(s_1)B)^{2^n-1}e^{3\times 2^{n-1}s}$ for $m = 0$ and $m = 1$, where $e(s_1) = e^{-\int_{-\infty}^{s_1} l(t)dt}$.*

Indeed, the lemma yields that $\forall s \le s_2$ $V_0(s) = V_1(s) = 0$ for some $s_2 \le s_1$. Since $\|V(s)\|_{L^2_\rho} \sim \|V_+(s)\|_{L^2_\rho}$ as $s \to -\infty$ , we have $\forall s \le s_3$, $\forall y \in \mathbb{R}^N$, $V(y,s) = 0$ for some $s_3 \le s_2$. The uniqueness of the solution of the Cauchy problem: $\forall s \ge s_3$, $V$ satisfies equation (69) and $V(s_3) = 0$ yields $V \equiv 0$ in $\mathbb{R}^{N+1}$.

*Proof of lemma B.3*: We proceed by induction:

- $n = 0$, the hypothesis is true by (72).
- We suppose that for $n \in \mathbb{N}$, we have

$\forall s \le s_1$, $|V_m(s)| \le (\frac{3}{2}e(s_1)B)^{2^n-1}e^{3\times 2^{n-1}s}$ for $m = 0,1$. Let us prove that $\forall s \le s_1$, $|V_m(s)| \le (\frac{3}{2}e(s_1)B)^{2^{n+1}-1}e^{3\times 2^n s}$ for $m = 0,1$.

Let $F_m(s) = V_m(s)e^{-(1-\frac{m}{2})s-\int_{-\infty}^s l(t)dt}$. From (71) and the induction hypothesis, we have: $\forall s \le s_1$,
$|F_m'(s)| \le e^{-(1-\frac{m}{2})s-\int_{-\infty}^{s_1} l(t)dt} B \times 3(\frac{3}{2}e(s_1)B)^{2(2^n-1)}e^{3\times 2^n s}$. By the induction hypothesis, $\lim\limits_{s\to-\infty} F_m(s) = 0$. Hence, $\forall s \le s_1$,
$|F_m(s)| = |\int_{-\infty}^s F_m'(\sigma)d\sigma| \le \int_{-\infty}^s |F_m'(\sigma)|d\sigma$

$\leq 3e(s_1)B(\frac{3}{2}e(s_1)B)^{2^{n+1}-2}\int_{-\infty}^{s} e^{(3\times 2^n-(1-\frac{m}{2}))\sigma}d\sigma$
$= \frac{2}{3\times 2^n-(1-\frac{m}{2})}(\frac{3}{2}e(s_1)B)^{2^{n+1}-1}e^{(3\times 2^n-(1-\frac{m}{2}))s}$.

Since $3\times 2^n-(1-\frac{m}{2})\geq 2$ and $l(s)\leq 0$, this yields $\forall s\leq s_1$, $|V_m(s)|\leq(\frac{3}{2}e(s_1)B)^{2^{n+1}-1}e^{3\times 2^n s}$ for $m=0,1$. This concludes the proof of lemma B.3. ∎

**Step 3: Case $\|V_-(s)\|_{L^2_\rho}+\|V_+(s)\|_{L^2_\rho}=o(\|V_{null}(s)\|_{L^2_\rho})$**

In order to show that $V\equiv 0$, it is enough to show that $V_{null}\equiv 0$ or equivalently that $\forall i,j\in\{1,..,N\}$, $V_{2,ij}\equiv 0$.

For this purpose, we derive form (69) an equation satisfied by $V_{2,ij}$ as $s\to-\infty$ :

$$V'_{2,ij}(s)=l(s)V_{2,ij}(s)+\int F(V)\frac{H_{2,ij}}{\|H_{2,ij}\|^2_{L^2_\rho}}\rho dy. \quad (73)$$

We have to estimate the last term of (73):

- if $i=j$, then $H_{2,ij}(y)=y_i^2-2$ and

$$|\int F(V)H_{2,ii}\rho dy|\leq C\int V^2\rho dy+C\int V^2|y|^2\rho dy, \quad (74)$$

- if $i\neq j$, then $H_{2,ij}(y)=y_iy_j$ and

$$|\int F(V)H_{2,ij}\rho dy|\leq C\int V^2|y|^2\rho dy. \quad (75)$$

The hypothesis of this step implies that

$$\int V^2\rho dy\leq 2\int V_{null}^2\rho dy. \quad (76)$$

It remains then to bound $\int V^2|y|^2\rho dy$. This will be done through this lemma, which is analogous to lemma A.3:

**Lemma B.4** *There exists $\delta_0>0$ and an integer $k'>5$ such that for all $\delta\in(0,\delta_0)$, $\exists s_0\in\mathbb{R}$ such that $\forall s\leq s_0$, $\int V^2|y|^{k'}\rho dy\leq c_0(k')\delta^{4-k'}\int V_{null}^2\rho dy$.*

*Proof*: We will argue similarly as in the proof of lemma A.3. Let $I(s)=\left(\int V^2|y|^{k'}\rho dy\right)^{1/2}$ and use the notation $X(s)=\left(\int V_{null}^2|y|^{k'}\rho dy\right)^{1/2}$. From (69), we derive the following equation for $I(s)$:

$$\frac{1}{2}\frac{d}{ds}(I(s)^2)=\int V.\mathcal{L}V|y|^{k'}\rho dy+l(s)I(s)^2+\int VF(V)|y|^{k'}\rho dy.$$

Since $v$ is bounded by $M$, we can assume $|V|\leq M+1=M'$ and get $\int VF(V)|y|^{k'}\rho dy\leq M'C\int V^2|y|^{k'}\rho dy$. We can also assume that $|l(s)|\leq\frac{1}{12}$.

As for lemma A.3, we can show that for all $\delta > 0$
$\int V.\mathcal{L}V|y|^{k'}\rho dy \le \frac{k'}{2}(k'+N-2)I(s)(\delta^{2-k'/2}\left(\int V^2\rho dy\right)^{1/2}+\delta^2 I)+(1-\frac{k}{4})I(s)^2$.

Combining these bounds with (76), we get:
$I'(s) \le -\theta I + d\delta^{2-k'/2}z$ with $\theta = \frac{k'}{4} - 1 - \frac{1}{12} - M'C - \frac{k'}{2}(k'+N-2)\delta^2$ and $d = k'(k'+N-2)$.

It is clear that there exist an integer $k' > 5$ and $\delta_0 > 0$ such that $\forall \delta \in (0, \delta_0)$, $\theta \ge 1$. Hence,

$$I'(s) \le -I(s) + d\delta^{2-k'/2}X(s). \tag{77}$$

Let us derive a differential equation satisfied by $X$.

From (69), we obtain:

$$\frac{1}{2}\frac{d}{ds}(X(s)^2) = l(s)X(s)^2 + \int P_{null}(F(V))V_{null}\rho dy.$$

By Schwartz's inequality, we have:
$|\int P_{null}(F(V))V_{null}\rho dy| \le \left(\int P_{null}(F(V))^2\rho dy\right)^{1/2}\left(\int V_{null}^2\rho dy\right)^{1/2}$
$\le \left(\int F(V)^2\rho dy\right)^{1/2} X(s)$.
Since $V \to 0$ as $s \to -\infty$ uniformly on compact sets, we have:
$\int F(V)^2\rho dy \le C^2\int V^4\rho dy = C^2\int_{|y|\le\delta^{-1}} V^4\rho dy + C^2\int_{|y|\ge\delta^{-1}} V^4\rho dy$
$\le \epsilon^2\int V^2\rho dy + C^2M'^2\delta^{k'}\int V^2|y|^{k'}\rho \le 4\epsilon^2X^2 + C^2M'^2\delta^{k'}I^2$ for all $\epsilon > 0$, provided that $s \le s_0(\epsilon, \delta)$.
Thus, $\left(\int F(V)^2\rho dy\right)^{1/2} \le 2\epsilon X + CM'\delta^{k'/2}I$.

Since $|l(s)| \le \frac{1}{12}$, we combine all the previous bounds to get:

$$|X'(s)| \le (2\epsilon + \frac{1}{12})X(s) + CM'\delta^{k'/2}I(s). \tag{78}$$

With $\epsilon = 1/12$, (77) and (78) yield:

$$\forall s \le s_1 \left\{ \begin{array}{rcl} |X'(s)| & \le & \frac{1}{4}X(s) + CM'\delta^{k'/2}I(s) \\ I'(s) & \le & -I(s) + d\delta^{2-k'/2}X(s). \end{array} \right.$$

Now, we conclude the proof of lemma A.3:

Let $\gamma(s) = I(s) - 2d\delta^{2-k'/2}X(s)$. Let us assume $\gamma(s) > 0$ and show that $\gamma'(s) < 0$.

$\gamma'(s) = I' - 2d\delta^{2-k'/2}X'$
$\le (-I + d\delta^{2-k'/2}X) + 2d\delta^{2-k'/2}(\frac{1}{4}X(s) + CM'\delta^{k'/2}I)$
$\le I(-1 + \frac{1}{2} + 2CM'd\delta^2 + \frac{1}{4}) = I(-\frac{1}{4} + 2CM'\delta^2 d)$

If we choose $\delta_0$ such that $\forall \delta \in (0, \delta_0)$, $-\frac{1}{4} + 2CM'\delta^2 d < 0$, then $\gamma(s) > 0$ implies $I(s) > 0$ and $\gamma'(s) < 0$. Since $\gamma(s) \to 0$ as $s \to -\infty$ (because $V \to 0$

uniformly on compact sets), we conclude that for some $s_2 \in \mathbb{R}$, $\forall s \leq s_1$, $\gamma(s) \leq 0$. Since $d = k'(k' + N - 2)$, lemma B.4 is proved. ■

Lemma B.4 allows us to bound $\int V^2 |y|^2 \rho dy$. Indeed, for fixed $\delta \in (0, \delta_0)$ and $s \leq s_0$, we have:
$\int V^2 |y|^2 \rho dy \leq \int_{|y| \leq \delta^{-1}} V^2 |y|^2 \rho dy + \int_{|y| \geq \delta^{-1}} V^2 |y|^2 \rho dy$
$\leq \delta^{-2} \int_{|y| \leq \delta^{-1}} V^2 \rho dy + \delta^{k'-2} \int_{|y| \geq \delta^{-1}} V^2 |y|^{k'} \rho dy$
$\leq \delta^{-2} \int V^2 \rho dy + c_0(k') \delta^2 \int V^2 \rho dy = C(\delta, k') \int V^2 \rho dy$.

With this bound, (74) and (75), equation (69) yields: $\forall s \leq s_0$,

$$V'_{2,ij}(s) = l(s) V_{2,ij}(s) + a_{2,ij}(s) \|V_{null}(s)\|^2_{L^2_\rho}$$

where $a_{2,ij}$ is bounded.

According to (70), there exist then $B > 0$ and $s_1 \leq s_0$ such that $\forall s \leq s_1$, $\forall i, j \in \{1, ..., N\}$,

$$|a_{2,ij}(s)| \leq B, \ |V_{2,ij}(s)| \leq e^s.$$

We claim that the following lemma yields $V \equiv 0$:

**Lemma B.5** *$\forall n \in \mathbb{N}$, $\forall s \leq s_1$, $\forall i, j \in \{1, ..., N\}$,*
*$|V_{2,ij}(s)| \leq (8N^2(N+1)^2 e(s_1) B)^{2^n - 1} e^{2^n s}$ where $e(s_1) = e^{-\int_\infty^{s_1} l(t) dt}$.*

Indeed, this lemma yields $\forall s \leq s_1$, $\forall i, j \in \{1, ..., N\}$, $V_{2,ij}(s) = 0$ for some $s_2 \leq s_1$. Hence, $\forall s \leq s_2$, $\forall y \in \mathbb{R}^N$, $V_{null}(y, s) = 0$, and by the hypothesis of this step, $\forall s \leq s_3$, $\forall y \in \mathbb{R}^N$, $V(y, s) = 0$ for some $s_3 \leq s_3$. The uniqueness of the solutions to the Cauchy problem: $\forall s \geq s_3$, $V$ satisfies equation (69) and $V(s_3) = 0$ yields $V \equiv 0$ in $\mathbb{R}^{N+1}$.

We escape the proof of lemma B.5 since it is completely analogous to the proof of lemma B.3.

# C Proof of $i)$ of Proposition 3.7

We proceed in 4 steps: in Step 1, we derive form the fact that $\|v(s)\|_{L^2_\rho} \sim \|v_{null}(s)\|_{L^2_\rho}$ an equation satisfied by $v_{null}(s)$ as $s \to -\infty$ . Then, we find in Step 2 $c > 0$, $C > 0$ and $s_0 \in \mathbb{R}$ such that $c|s|^{-1} \leq \|v(s)\|_{L^2_\rho} \leq C|s|^{-1}$ for $s \leq s_0$. In Step 3, we use this estimate to derive a more accurate equation for $v_{null}$. We use this equation in Step 4 to get the asymptotic behaviors of $v_{null}(y, s)$, $v_0(s)$ and $v_1(s)$.

**Step 1: An ODE satisfied by $v_{null}(y, s)$ as $s \to -\infty$**

This step is very similar to Step 3 in Appendix B where we handled the equation (69) instead of (38) as in the present context.

From (38) we have by projection:

$$v_{2,ij}'(s) = \int f(v)\frac{H_{2,ij}(y)}{\|H_{2,ij}\|^2_{L^2_\rho}}\rho(y)dy \qquad (79)$$

We will prove the following proposition here:

**Proposition C.1** *i)* $\forall i,j\in\{1,...,N\}$,

$$v_{2,ij}'(s) = \frac{p}{2\kappa}\int v_{null}^2(y,s)\frac{H_{2,ij}(y)}{\|H_{2,ij}\|^2_{L^2_\rho}}\rho(y)dy + o(\|v_{null}(s)\|^2_{L^2_\rho}). \qquad (80)$$

*as* $s\to-\infty$ .

*ii) There exists a symmetric* $N\times N$ *matrix* $A(s)$ *such that* $\forall s\in\mathbb{R}$,

$$v_{null}(y,s) = y^TA(s)y - 2tr(A(s)) \qquad (81)$$

$$\textit{and } c_0\|A(s)\| \le \|v_{null}(s)\|_{L^2_\rho} \le C_0\|A(s)\| \qquad (82)$$

*for some positive constants* $c_0$ *and* $C_0$. *Moreover,*

$$A'(s) = \frac{4p}{\kappa}A^2(s) + o(\|A(s)\|^2) \textit{ as } s\to-\infty. \qquad (83)$$

**Remark**: $\|A\|$ stands for any norm on the space of $N\times N$ symmetric matrices.

**Remark**: The interest of the introduction of the matrix $A(s)$ is that it generalizes to $N\ge 2$ the situation of $N=1$. Indeed, if $N=1$, then it is obvious that $v_{null}(y,s) = yv_2(s)y - 2v_2(s)$ and that (80) implies $v_2'(s) = \frac{4p}{\kappa}v_2(s)^2 + o(v_2(s)^2)$. Let us remark that in the case $N=1$, we get immediately $v_2(s)\sim -\frac{\kappa}{4ps}$ as $s\to-\infty$, which concludes the proof of Proposition 3.7. Unfortunately, we can not solve the system (83) so easily if $N\ge 2$. Nevertheless, the intuition given by the case $N=1$ will guide us in next steps in order to refine the system (83) and reach then a similar result (see Step 2).

*Proof of Proposition C.1*:

Let us remark that *ii*) follows directly form *i*). Indeed, we have by definition of $H_{2,ij}$ and $v_{2,ij}$ (see (44) and (45)):

$v_{null}(y,s) = \sum_{i\le j} v_{2,ij}(s)H_{2,ij}(y) = \sum_{i=1}^N v_{2,ii}(s)(y_i^2-2) + \sum_{i<j} v_{2,ij}(s)y_iy_j$. If we define $A(s) = (a_{ij}(s))_{i,j}$ by

$$a_{ii}(s) = v_{2,ii}(s), \text{ and for } i<j,\ a_{ij}(s) = a_{ji}(s) = \frac{1}{2}v_{2,ij}(s), \qquad (84)$$

then (81) follows. (82) follows form the equivalence of norms in finite dimension $\frac{N(N+1)}{2}$. (83) follows from (80) by simple but long calculations which we escape here.

Now, we focus on the proof of $i$). For this purpose, we try to estimate the right-hand side of equation (79).

As in Step 3 of Part 3, this will be possible thanks to the following lemma:

**Lemma C.1** *There exists $\delta_0 > 0$ and an integer $k' > 4$ such that for all $\delta \in (0, \delta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \le s_0$, $\int v^2 |y|^{k'} \rho dy \le c_0(k') \delta^{4-k'} \int v_{null}^2 \rho dy$.*

*Proof*: The proof of lemma B.4 holds for lemma C.1 with the changes $V \to v$, $F \to f$ and $l(s) \to 0$. ■

Now we estimate $\int f(v) H_{2,ij} \rho dy$:
Since $f(v) = \frac{p}{2\kappa} v^2 + g(v)$ where $|g(v)| \le C|v|^3$, we write:

$$\int f(v) H_{2,ij} \rho dy = \frac{p}{2\kappa} \int v_{null}^2 H_{2,ij} \rho dy + (I) + (II) \tag{85}$$

where

$$(I) = \frac{p}{2\kappa} \int (v^2 - v_{null}^2) H_{2,ij} \rho dy \tag{86}$$

$$\text{and } (II) = \int g(v) H_{2,ij} \rho dy. \tag{87}$$

The proof of Proposition C.1 will be complete if we show that $(I)$ and $(II)$ are $o(\|v_{null}(s)\|_{L^2_\rho})$. Since $H_{2,ij}(y) = y_i^2 - 2$ if $i = j$ and $H_{2,ij}(y) = y_i y_j$ if $i \ne j$, it is enough to show that for all $\epsilon > 0$, $I_1$, $I_2$, $II_1$ and $II_2$ are lower that $\epsilon \|v_{null}(s)\|_{L^2_\rho}$ for all $s \le s_0(\epsilon)$, where

$$I_1 = \int |v^2 - v_{null}^2| \rho dy, \quad I_2 = \int |v^2 - v_{null}^2| |y|^2 \rho dy,$$
$$II_1 = \int |g(v)| \rho dy, \quad II_2 = \int |g(v)| |y|^2 \rho dy.$$

We start with $I_1$: Since $\int v^2 \rho dy \sim \int v_{null}^2 \rho dy$,
$I_1 = \int (v_+^2 + v_-^2) \rho dy \le \epsilon \int v_{null}^2 \rho dy$ if $s \le s_1(\epsilon)$.

For $I_2$, we consider $\delta \in (0, \delta_0)$, and write:
$I_2 \le \int_{|y| \le \delta^{-1}} |v^2 - v_{null}^2| |y|^2 \rho dy + \int_{|y| \ge \delta^{-1}} |v^2 - v_{null}^2| |y|^2 \rho dy := I_{21} + I_{22}$.
We first estimate $I_{21}$:

Since $v = v_- + v_{null} + v_+$, we have $v^2 - v_{null}^2 = (v_+ + v_-)^2 + 2 v_{null}(v_+ + v_-)$.
Hence,
$I_{21} \le \int_{|y| \le \delta^{-1}} (v_+ + v_-)^2 |y|^2 \rho dy + 2 \int_{|y| \le \delta^{-1}} |v_{null}(v_+ + v_-)| |y|^2 \rho dy$

$\leq \delta^{-2}\int(v_++v_-)^2\rho dy+2\left(\int v_{null}^2|y|^4\rho dy\right)^{1/2}\left(\int(v_++v_-)^2\rho dy\right)^{1/2}$.
Since $\int v^2\rho dy\sim\int\int v_{null}^2\rho dy$, we have
$\int(v_++v_-)^2\leq\delta^3\int v_{null}^2\rho dy$ if $s\leq s_2(\delta)$.
Since the null subspace of $\mathcal{L}$ in finite dimensional, all the norms on it are equivalent, therefore, there exists $C_4(N)$ such that:
$\int v_{null}^2|y|^4\rho dy\leq C_4(N)^2\int v_{null}^2\rho dy$.
Therefore, $I_{21}\leq(\delta+2C_4(N)\delta^{3/2})\int v_{null}^2\rho dy$ if $s\leq s_2(\delta)$.

For $I_{22}$, we write:
$I_{22}\leq\int_{|y|\geq\delta^{-1}}|v^2-v_{null}^2||y|^2\rho dy\leq\delta^{k'-2}\int v^2|y|^{k'}\rho dy+\delta^{k'-2}\int v_{null}^2|y|^{k'}\rho dy$
$\leq c_0(k')\delta^2\int v_{null}^2\rho dy+\delta^{k'-2}C_{k'}(N)^2\int v_{null}^2\rho dy$
by lemma C.1 and the equivalence of norms for $v_{null}$. Collecting all the above estimates, we get
$I_2\leq(\delta+2C_4(N)\delta^{3/2}+c_0(k')+\delta^{k'-2}C_{k'}(N)^2)\int v_{null}^2\rho dy$ for $s\leq s_2(\delta)$. If $\delta=\delta(\epsilon)$ is small enough, then
$I_2\leq\epsilon\int v_{null}^2\rho dy$ for $s\leq s_3(\epsilon)$.

Now, we handle $II_1$ and $II_2$ in the same time: we consider $\delta\in(0,\delta_0)$ and write for $m=0$ or $m=2$:
$|\int|g(v)||y|^m\rho dy\leq C\int|v|^3|y|^m\rho dy$
$\leq C\int_{|y|\leq\delta^{-1}}|v|^3|y|^m\rho dy+C\int_{|y|\geq\delta^{-1}}|v|^3|y|^m\rho dy$
$\leq C\epsilon'\delta^{-m}\int_{|y|\leq\delta^{-1}}v^2\rho dy+CM\delta^{k'-m}\int_{|y|\geq\delta^{-1}}v^2|y|^{k'}\rho dy$
$\leq(C\epsilon'\delta^{-m}+CMc_0(k')\delta^{4-m})2\int v_{null}^2\rho dy$
where we used the fact that $v\to 0$ as $s\to-\infty$ in $L^\infty(B(0,\delta^{-1}))$, $|v(y,s)|\leq M$, lemma C.1 and $\int v^2\rho dy\leq\int v_{null}^2\rho dy$.
Now, we can choose $\delta=\delta(\epsilon)$ and then $\epsilon'=\epsilon'(\epsilon)$ such that for $s\leq s_5(\epsilon)$
$\int|g(v)||y|^m\rho dy\leq\epsilon\int v_{null}^2\rho dy$.

Setting $s_0(\epsilon)=\min(s_1(\epsilon),s_3(\epsilon),s_5(\epsilon))$, we have: $\forall\epsilon>0$, $\forall s\leq s_0(\epsilon)$,
$I_1+I_2+II_1+II_2\leq 4\epsilon\int v_{null}^2\rho dy$. Therefore $(I)+(II)=o(\|v_{null}(s)\|_{L^2_\rho})$ as $s\to-\infty$ .

Thus, combining this with (79) and (85) concludes the proof of Proposition C.1. ■

**Step 2: $\|v(s)\|_{L^2_\rho}$ behaves like $\frac{1}{|s|}$ as $s\to-\infty$**

In this step, we show that although we can not derive directly from (80) the asymptotic behavior of $v_{null}(s)$ (and then the one of $v(s)$), we can use it to show that $\|v(s)\|_{L^2_\rho}$ behaves like $\frac{1}{|s|}$ as $s\to-\infty$ . More precisely, we have the following Proposition:

**Proposition C.2** *If $\|v_-(s)\|_{L^2_\rho}+\|v_+(s)\|_{L^2_\rho}=o(\|v_{null}(s)\|_{L^2_\rho})$, then for $-s$*

*large enough, we have*

$$\frac{c}{|s|} \le \|v(s)\|_{L^2_\rho} \le \frac{C}{|s|}$$

*for some positive constants $c$ and $C$.*

*Proof*: Since $\|v(s)\|_{L^2_\rho} \sim \|v_{null}(s)\|_{L^2_\rho}$, and because of (82), it is enough to show that

$$\frac{c}{|s|} \le \|A(s)\| \le \frac{C}{|s|} \tag{88}$$

for $-s$ large. The proof is completely parallel to section 3 of Filippas and Liu [4]. Therefore, we give only its main steps.

We first give a result from the perturbation theory of linear operators which asserts that $A(s)$ has continuously differentiable eigenvalues:

**Lemma C.2** *Suppose that $A(s)$ is a $N \times N$ symmetric and continuously differentiable matrix-function in some interval $I$. Then, there exist continuously differentiable functions $\lambda_1(s), ..., \lambda_N(s)$ in $I$, such that for all $j \in \{1, .., N\}$,*

$$A(s)\phi^{(j)}(s) = \lambda_j(s)\phi^{(j)}(s),$$

*for some (properly chosen) orthonormal system of vector-functions $\phi^{(1)}(s), ..., \phi^{(N)}(s)$.*

The proof of this lemma is contained (for instance) in Kato [10] or Rellich [13].

We consider then $\lambda_1(s), ..., \lambda_N(s)$ the eigenvalues of $A(s)$. It is well-known that $\sum_{i=1}^{N} |\lambda_i|$ is a norm on the space of $N \times N$ symmetric matrices. We choose this norm to prove (88). From (83), we can derive an equation satisfied by $(\lambda_i(s))_i$:

**Lemma C.3** *The eigenvalues of $A(s)$ satisfy $\forall i \in \{1, ..., N\}$*

$$\lambda_i'(s) = \frac{4p}{\kappa}\lambda_i^2(s) + o\left(\sum_{i=1}^{N} \lambda_i^2(s)\right).$$

The proof of lemma 3.3 in [4] holds here with the slight change: $s \to +\infty$ becomes $s \to -\infty$ and $s$ large enough becomes $-s$ large enough.

Now, we claim that with the introduction of $\Lambda_i(\sigma) = -\lambda_i(-\sigma)$, we have:

- $\forall i \in \{1, ..., N\}$

$$\Lambda_i'(\sigma) = \frac{4p}{\kappa}\Lambda_i^2(\sigma) + o\left(\sum_{i=1}^{N} \Lambda_i^2(\sigma)\right) \text{ as } \sigma \to +\infty,$$

-$\forall \sigma \geq \sigma_0$, $\sum_i |\Lambda_i(\sigma)| \neq 0$ (Indeed, if not, then for all $i$, $\Lambda_i \equiv 0$, $\lambda_i \equiv 0$, and then $A(s)$, $v_{null}(s)$ and $v(s)$ are identically zero.)

Section 3 of [4] yields (directly and without any adaptations) that for all $\sigma \geq \sigma_1$,

$$\frac{c}{\sigma} \leq \sum_i |\Lambda_i(\sigma)| \leq \frac{C}{\sigma}.$$

Since $\|A(s)\| = \sum_i |\lambda_i(s)| = \sum_i |\Lambda_i(-s)|$, this concludes the proof of (88) and the proof of Proposition C.2. ■

**Step 3: A new ODE satisfied by $v_{null}(y,s)$**

In this step, we show that since $\|v\|_{L^2_\rho}$ behaves like $\frac{1}{|s|}$, then all the $L^q_\rho$ norms are in some sense equivalent as $s \to -\infty$ for this particular $v$. Then, we will do a kind of center-manifold theory for this particular $v$ to show that $\|v_+(s)\|_{L^2_\rho} + \|v_-(s)\|_{L^2_\rho}$ is in fact $O(\|v_{null}(s)\|^2_{L^2_\rho})$ and not only $o(\|v_{null}(s)\|_{L^2_\rho})$. These two estimates are used then to rederive a more accurate equation satisfied by $v_{null}(y,s)$.

**Lemma C.4** *If $\|v_+(s)\|_{L^2_\rho} + \|v_-(s)\|_{L^2_\rho} = o(\|v_{null}(s)\|_{L^2_\rho})$, then*

*i) for every $r > 1$, $q > 1$, there exists $C = C(r,q)$ such that*

$$\left(\int v^r(y,s)\rho dy\right)^{1/r} \leq C \left(\int v^q(y,s)\rho dy\right)^{1/q}$$

*for $-s$ large enough.*

*ii) $\|v_+(s)\|_{L^2_\rho} + \|v_-(s)\|_{L^2_\rho} = O(\|v_{null}(s)\|^2_{L^2_\rho})$ as $s \to -\infty$ .*

*Proof of i) of lemma C.4*: The crucial estimate is an a priori estimate of solutions of (38) shown by Herrero and Velazquez in [9]. This a priori estimate is a version of $i)$ holding for all bounded (in $L^\infty$) solutions of (38), but with a delay time; although they proved their result in the case $N = 1$ for solutions defined for $s \in [0, +\infty)$, their proof holds in higher dimensions with $s \in \mathbb{R}$.

**Lemma C.5 (Herrero-Velazquez)** *Assume that $v$ solves (38) and $|v| \leq M < \infty$. Then for any $r > 1$, $q > 1$ and $L > 0$, there exist $s_0^* = s_0^*(q,r)$ and $C = C(r,q,L) > 0$ such that*

$$\left(\int v^r(y,s+s^*)\rho dy\right)^{1/r} \leq C \left(\int v^q(y,s)\rho dy\right)^{1/q}$$

*for any $s \in \mathbb{R}$ and any $s^* \in [s_0^*, s_0^* + L]$.*

Set $s_1^* = s_0^*(2,r)$ and $s_2^* = s_0^*(q,2)$. For $-s$ large enough, we write according to lemma C.5 and Proposition C.2:
$\left(\int v^r(y,s)\rho dy\right)^{1/r} \le C_1 \left(\int v^2(y,s-s_1^*)\rho dy\right)^{1/2} \le C_2/(s-s_1^*)$
$\le C_3/(s+s_2^*) \le C_4 \left(\int v^2(y,s+s_2^*)\rho dy\right)^{1/2} \le C_5 \left(\int v^q(y,s)\rho dy\right)^{1/q}$. Thus, $i)$ of lemma C.4 follows .

*Proof of ii) of lemma C.4*: We argue as in Step 2 of Appendix A, and use the same notations: $x(s) = \|v_{null}(s)\|_{L^2_\rho}$, $y(s) = \|v_-(s)\|_{L^2_\rho}$, $z(s) = \|v_+(s)\|_{L^2_\rho}$ and $N(s) = \|V^2\|_{L^2_\rho}$. We have already derived (in the proofs of lemmas A.3 and B.4) two differential inequalities satisfied by $x$ and $z$. By the same techniques (see also [3]), we can show that

$$\begin{aligned} z' &\ge \frac{1}{2}z - CN \\ |x'| &\le CN \\ y' &\le -\frac{1}{2}y + CN. \end{aligned}$$

By $i)$ of lemma C.4, we have $N(s) \le C\|v(s)\|^2_{L^2_\rho} = C(x^2(s)+y^2(s)+z^2(s))$ for large $-s$.

Since $x, y, z \to 0$ as $s \to -\infty$ , we can write for $-s$ large:

$$\begin{aligned} z' &\ge \frac{1}{3}z - C(x+y)^2 \\ |x'| &\le C(x+y+z)^2 \\ y' &\le -\frac{1}{3}y + C(x+z)^2. \end{aligned}$$

The conclusion then follows form the following ODE lemma by Filippas and Liu:

**Lemma C.6 (Filippas-Liu)** *Let $x(s)$, $y(s)$ and $z(s)$ be absolutely continuous, real valued functions which are non negative and satisfy*

*i) $(x,y,z)(s) \to 0$ as $s \to -\infty$,*

*ii) $\forall s \le s_0$*

$$\left\{ \begin{aligned} \dot z &\ge c_0 z - c_1(x+y)^2 \\ |\dot x| &\le c_1(x+y+z)^2 \\ \dot y &\le -c_0 y + c_1(x+z)^2. \end{aligned} \right.$$

*for some positive constants $c_0$ and $c_1$. Then,*
*either (i) $x, y, z \to 0$ exponentially fast as $s \to -\infty$,*
*or (ii) for all $s \le s_1$, $y + z \le b(c_0,c_1)x^2$ for some $s_1 \le s_0$.*

*Proof*: see lemma 4.1 in [4].

Now, using lemma C.4, we derive a new equation satisfied by $v_{null}$:

**Proposition C.3** $\forall i,j \in \{1,...,N\}$,

$$v'_{2,ij}(s) = \frac{p}{2\kappa}\int v_{null}^2(y,s)\frac{H_{2,ij}(y)}{\|H_{2,ij}\|^2_{L^2_\rho}}\rho(y)dy + O(\|v_{null}(s)\|^3_{L^2_\rho}).$$

*as* $s \to -\infty$ .

*Moreover,*

$$A'(s) = \frac{4p}{\kappa}A^2(s) + O(\frac{1}{s^3}) \text{ as } s \to -\infty.$$

The proof of Proposition 4.1 in [4] holds here with the usual changes: $s \to +\infty$ becomes $s \to -\infty$ .

**Step 4: Asymptotic behavior of $v_{null}(y,s)$, $v_0(s)$ and $v_1(s)$**

Setting $\mathcal{A}(\sigma) = -A(-\sigma)$, we see that

$$\mathcal{A}'(\sigma) = \frac{4p}{\kappa}\mathcal{A}^2(\sigma) + O(\frac{1}{\sigma^3}) \text{ as } \sigma \to +\infty.$$

Therefore, Proposition 5.1 in [4] yields (directly and without any adaptations) the existence of $\delta > 0$ and a $N \times N$ orthonormal matrix $Q$ such that

$$\mathcal{A}(\sigma) = -\frac{\kappa}{4p\sigma}A_0 + O(\frac{1}{\sigma^{1+\delta}})$$

where

$$A_0 = Q\begin{pmatrix} I_{N-k} & 0 \\ 0 & 0\end{pmatrix}Q^{-1}$$

for some $k \in \{0,1,...,N-1\}$. Together with (81), this yields the behavior of $v_{null}(y,s)$ announced in $i)$ of Proposition 3.7.

It also yields

$$\begin{aligned}
\|v_{null}(s)\|_{L^2_\rho} &= \left(\int v_{null}^2(y,s)\rho(y)dy\right)^{1/2} \\
&= \left(\int (y^T A(s)y - 2trA(s))^2\rho(y)dy\right)^{1/2} \\
&= -\frac{\kappa}{4ps}\left(\int (y^T A_0 y - 2trA_0)^2\rho(y)dy\right)^{1/2} + O\left(\frac{1}{|s|^{1+\delta}}\right).
\end{aligned}$$

With the change of variables, $y = Qz$, we get since $Q$ is orthonormal:

$$\begin{aligned}
\|v_{null}(s)\|_{L^2_\rho} &= -\frac{\kappa}{4ps}\left(\int\left(\sum_{i=1}^{N-k}(z_i^2-2)\right)^2\rho(z)dz\right)^{1/2} + O\left(\frac{1}{|s|^{1+\delta}}\right)\\
&= -\frac{\kappa}{4ps}\left(\sum_{i=1}^{N-k}\int(z_i^2-2)^2\rho(z)dz\right)^{1/2} + O\left(\frac{1}{|s|^{1+\delta}}\right)\\
&= -\frac{\kappa}{ps}\sqrt{\frac{N-k}{2}} + O\left(\frac{1}{|s|^{1+\delta}}\right)
\end{aligned}$$

where we used the fact that $(y_i^2-2)_i$ is an orthogonal system with respect to the measure $\rho dy$.

Since $\|v(s)\|_{L^2_\rho} = \|v_{null}(s)\|_{L^2_\rho} + O\left(\|v_{null}(s)\|^2_{L^2_\rho}\right)$ $(ii)$ of lemma C.4), we get

$$\|v(s)\|_{L^2_\rho} = -\frac{\kappa}{ps}\sqrt{\frac{N-k}{2}} + O\left(\frac{1}{|s|^{1+\delta}}\right). \tag{89}$$

Integrating (38) with respect to $\rho dy$, we find

$$v_0'(s) = v_0(s) + \int f(v)\rho dy.$$

Since $|f(v)| \le Cv^2$, we get from (89)

$$v_0'(s) = v_0(s) + O(\frac{1}{s^2}) \text{ as } s \to -\infty.$$

Therefore, it follows that

$$v_0(s) = O(\frac{1}{s^2}) \text{ as } s \to -\infty.$$

Using lemma C.1, we have: for all $\eta \in (0, \delta_0)$,

$$\int v^2|y|^{k'}\rho dy \le c_0(k')\eta^{4-k'}\int v_{null}^2\rho dy \le 2c_0(k')\eta^{4-k'}\int v^2\rho dy.$$

Therefore,
$\int v^2|y|\rho dy \le \int_{|y|\le\eta^{-1}} v^2|y|\rho dy + \int_{|y|\ge\eta^{-1}} v^2|y|\rho dy$
$\le \eta^{-1}\int v^2\rho dy + \eta^{k'-1}\int v^2|y|^{k'}\rho dy$
$\le (\eta^{-1} + 2c_0(k')\eta^3)\int v^2\rho dy.$

If we fix $\eta > 0$, then

$$\int v^2|y|\rho dy \le C(\eta, k')\int v^2\rho dy. \tag{90}$$

Integrating (38) with respect to $y_i \rho dy$, we find

$$v'_{1,i}(s) = \frac{1}{2} v_{1,i}(s) + \int f(v) \frac{y_i}{2} \rho dy.$$

Since $|f(v)| \leq C v^2$, we get from (90) and (89)

$$v_1(s) = O(\frac{1}{s^2}) \text{ as } s \to -\infty.$$

This concludes the proof of $i)$ of Proposition 3.7.

**Acknowledgment**

The second author is thankful for the invitation of the Institute for Advanced Study where part of this work was done.

# References

[1] Berestycki, H., and Nirenberg, L., *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. (N.S.) 22, 1991, pp. 1-37.

[2] Bricmont, J., and Kupiainen, A., *Universality in blow-up for nonlinear heat equations,* Nonlinearity 7, 1994, pp. 539-575.

[3] Filippas, S., and Kohn, R., *Refined asymptotics for the blowup of* $u_t - \Delta u = u^p$*,* Comm. Pure Appl. Math. 45, 1992, pp. 821-869.

[4] Filippas, S., and Liu, W., *On the blow-up of multidimensional semilinear heat equations*, Ann. Inst. Henri Poincaré Anal. Non Linéaire 10, 1993, pp. 313-344.

[5] Gidas, B., Ni, W.M., and Nirenberg, L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68, 1979, pp. 209-243.

[6] Giga, Y., and Kohn, R., *Asymptotically self-similar blowup of semilinear heat equations,* Comm. Pure Appl. Math. 38, 1985, pp. 297-319.

[7] Giga, Y., and Kohn, R., *Characterizing blowup using similarity variables,* Indiana Univ. Math. J. 36, 1987, pp. 1-40.

[8] Giga, Y., and Kohn, R., *Nondegeneracy of blow-up for semilinear heat equations,* Comm. Pure Appl. Math. 42, 1989, pp. 845-884.

[9] Herrero, M.A, et Velazquez, J.J.L., *Blow-up behavior of one-dimensional semilinear parabolic equations,* Ann. Inst. Henri Poincaré Anal. Non Linéaire 10, 1993, pp. 131-189.

[10] Kato, T., *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1980.

[11] Merle, F., *Solution of a nonlinear heat equation with arbitrary given blow-up points,* Comm. Pure Appl. Math. 45, 1992, pp. 263-300.

[12] Merle, F., and Zaag, H., *Refined uniform estimates at blow-up and applications for nonlinear heat equations*, preprint, 1997.

[13] Rellich, F., *Perturbation theory for eigenvalue problems*, Lecture Notes, New York University, 1953.

[14] Zaag, H., *Blow-up results for vector valued nonlinear heat equations with no gradient structure*, Ann. Inst. Henri Poincaré, to appear.

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