O.D.E. type behavior of blow-up solutions of nonlinear heat equations

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1 The results

We are concerned in this paper with blow-up solutions of

$$\left\{\begin{array}{rcll} \frac{\partial u}{\partial t} & = & \Delta u+|u|^{p-1}u & \text{in} \quad \mathbb{R}^N\times[0,T) \\ u(.,0) & = & u_0 & \text{in} \quad \mathbb{R}^N \end{array}\right. \tag{1}$$

where $u:(x,t)\in\mathbb{R}^N\times[0,T)\to\mathbb{R}$, $u_0\in L^\infty(R^N)$, $T>0$, with

$$p>1 \text{ and } (3N-4)p<3N+8. \tag{2}$$

Some more general condition can be consider, see [MZ] for details.

The Cauchy problem for system (1) can be solved (for example) in $L^\infty(\mathbb{R}^N,\mathbb{R})$. If the maximal solution $u(t)$ is defined on $[0,T)$ with $T<+\infty$, then

$$\lim_{t\to T}\|u(t)\|_{L^\infty}=+\infty.$$

We say that $u(t)$ blows-up at time T. If $a\in\mathbb{R}^N$ satisfies $|u(x_n,t_n)|\to+\infty$ as $n\to+\infty$ for some sequence $(x_n,t_n)\to(a,T)$, then a is called a blow-up point of u. The set of all blow-up points of $u(t)$ is called the blow-up set of $u(t)$ and will be denoted by S.

The existence of blow-up solutions for systems of the type (1) has been proved by several authors (Friedman [Fri65], Fujita [Fuj66], Levine [Lev73], Ball [Bal77],..). Many authors has been concerned by the asymptotic behavior of $u(t)$ at blow-up time, near blow-up points. See reference in [MZ]. Consider $u(t)$ a solution of (1) which blows-up at time T at a point $a\in\mathbb{R}^N$.

The study of the behavior of $u(t)$ near (a,T) has been done through the introduction of the following similarity variables :

$$y = \frac{x-a}{\sqrt{T-t}},\ s = -\log(T-t),\ w_a(y,s) = (T-t)^{\frac{1}{p-1}}u(x,t). \tag{3}$$

It is readily seen from (1) that w_a (or simply w) satisfies the following equation : $\forall s \geq -\log T$, $\forall y \in \mathbb{R}^N$,

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2}y.\nabla w - \frac{w}{p-1} + |w|^{p-1}w. \tag{4}$$

The following Lyapunov functional is associated with (4) :

$$E(w) = \int_{W_{a,s}} \left(\frac{1}{2}|\nabla w|^2 + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1}\right)\rho(y)dy \tag{5}$$

where

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}. \tag{6}$$

In the case (2) (equation (1)), Giga and Kohn showed in [GK85], [GK87] and [GK89] that

$$\forall x \in \mathbb{R}^N,\ \forall t \in [0,T),\ |u(x,t)| \leq C(T-t)^{-\frac{1}{p-1}} \tag{7}$$

for some constant $C > 0$. They also showed that

$$w_a(y,s) \to \kappa \equiv (p-1)^{-\frac{1}{p-1}} \text{ as } s \to +\infty, \tag{8}$$

uniformly on compact sets. See reference in for refinement of these results.

We now claim the following theorem which classifies all connections in L^∞_{loc} between critical points of (4). This Theorem is in some sense a classification of "critical points at infinity" (in a parabolic sense) for equation (4). Note that this Theorem is valid not only for p satisfying (2) but for all subcritical p, that is under the condition

$$p > 1 \text{ and } (N-2)p < N+2. \tag{9}$$

Theorem 1 (Liouville Theorem for equation (4)) *Assume (9) and consider w a solution of (4) defined for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^M)$. Then necessarily one of the following cases occurs :*
i) $w \equiv 0$,
ii) $w \equiv \pm\kappa$,
iii) $\exists s_0 \in \mathbb{R}$, *such that* $w(y,s) = \pm\varphi(s-s_0)$ *where*

$$\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}.$$

This Theorem has an equivalent formulation for solutions of (1) via the transformation (3).

Corollary 1 (A Liouville Theorem for equation (1)) *Assume that (9) holds and that u is a solution in L^∞ of (1) defined for $(x,t) \in \mathbb{R}^N \times (-\infty, T)$. Assume in addition that $|u(x,t)| \le C(T-t)^{-\frac{1}{p-1}}$. Then $u \equiv 0$ or there exist $T_0 \ge T$ such that $\forall (x,t) \in \mathbb{R}^N \times (-\infty, T)$, $u(x,t) = \pm\kappa(T_0-t)^{-\frac{1}{p-1}}$.*

Theorem 2 (Uniform estimates with respect to u_0) *Assume condition (2) holds and consider u a solution of (1) that blows-up at time $T < T_0$ and satisfies $\|u(0)\|_{C^2(\mathbb{R}^N)} \le C_0$. Then, there exists $C(C_0, T_0)$ such that $\forall t \in [0,T)$, $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \le Cv(t)$ where $v(t) = \kappa(T-t)^{-\frac{1}{p-1}}$ is the solution of*

$$v' = v^p \text{ and } v(T) = +\infty.$$

Remark : We suspect that this result is true with no condition on T.
Let us remark that we suspect this Theorem to be valid in the case (9).

Theorems 1 and 2 have important consequences in the understanding of the blow-up behavior for equation (1) in the case (2). We have the following localization result which compares (1) with the associated ODE

$$u' = u^p.$$

Theorem 3 (Uniform ODE Behavior) *Assume that (2) holds and consider $T \le T_0$ and $\|u_0\|_{C^2(\mathbb{R}^N)} \le C_0$. Then, $\forall \epsilon > 0$, there is $C(\epsilon, C_0, T_0)$ such that $\forall x \in \mathbb{R}^N$, $\forall t \in [0,T)$,*

$$\left| \frac{\partial u}{\partial t}(x,t) - |u|^{p-1}u(x,t) \right| \le \epsilon |u(x,t)|^p + C.$$

Remark : Note that the condition $u(0) \in C^2$ in Theorems 2 and 3 is not restrictive, because of the regularizing effect of the heat equation.

We now present in section 2 the proof of the Liouville Theorem 1 in the scalar case. Section 3 is devoted to the control of $\|u(t)\|_{L^\infty}$ (Theorem 2) and the ODE behavior (Theorem 3) uniformly with respect to initial data.

2 Liouville Theorem for equation (4)

In this section, we prove Theorem 1. Let us first introduce the following functional defined for all $W \in H^1_\rho(\mathbb{R}^N)$

$$I(W) = -2E(W) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^N} |W(y)|^2 \rho(y) dy\right)^{\frac{p+1}{2}} \tag{10}$$

where E is defined in (5), and the following blow-up criterion valid for vectorial solutions of (4) :

Proposition 2.1 (Blow-up criterion for vectorial solutions of (4)) *Let w be a solution of (4) which satisfies*

$$I(w(s_0)) > 0 \tag{11}$$

for some $s_0 \in \mathbb{R}$. Then, w blows-up at some time $S > s_0$.

Remark : This Proposition and the fact that $I(\kappa) = 0$ yield informations on the solutions of (4) close to κ in the energy space.
In the following, we will prove Proposition 2.1 and then give a sketch of the arguments of the proof of the Liouville Theorem, since they are the same as those in [MZ98a]. Only the arguments related to the new blow-up criterion will be expanded.

Proof of Proposition 2.1 : We proceed by contradiction and suppose that w is defined for all $s \in [s_0, +\infty)$. According to (4) and (5), we have $\forall s \geq s_0$,

$$\begin{aligned}
\frac{d}{ds}\int |w(y,s)|^2 \rho dy &= 2\int \left(-|\nabla w(y,s)|^2 - \frac{|w(y,s)|^2}{p-1} + |w(y,s)|^{p+1}\right)\rho dy \\
&= -4E(w(s)) + \frac{2(p-1)}{p+1}\int |w|^{p+1}\rho dy \\
&\geq -4E(w(s_0)) + \frac{2(p-1)}{p+1}\left(\int |w|^2 \rho dy\right)^{\frac{p+1}{2}}
\end{aligned}$$

where we used Jensen's inequality ($\int \rho dy = 1$) and the fact that E is decreasing in time.

If we set

$$z(s) = \int |w(y,s)|^2 \rho dy, \ \alpha = -4E(w(s_0)) \text{ and } \beta = \frac{2(p-1)}{p+1}, \tag{12}$$

then this reads :

$$\forall s \geq s_0, \ z'(s) \geq \alpha + \beta z(s)^{\frac{p+1}{2}}. \tag{13}$$

With (12) and (10), the condition (11) reads : $\alpha + \beta z(s_0)^{\frac{p+1}{2}} > 0$. By a classical argument, we have from this and from (13)

$$\forall s \geq s_0, \ z'(s) > 0 \text{ and } \alpha + \beta z(s)^{\frac{p+1}{2}} > 0.$$

Using a direct integration, we obtain :

$$\forall s \geq s_0, \ s - s_0 \leq \int_{z(s_0)}^{z(s)} \frac{dx}{\alpha + \beta x^{\frac{p+1}{2}}} \leq \int_{z(s_0)}^{+\infty} \frac{dx}{\alpha + \beta x^{\frac{p+1}{2}}} = C(z(s_0)) < +\infty$$

since $p > 1$. Thus, a contradiction follows and Proposition 2.1 is proved. ■

Proof of Theorem 1 : We assume $p > 1$ and $p < \frac{N+2}{N-2}$ if $N \geq 3$, and consider $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ a solution of (4). We proceed in two parts in order to show that w depends only on s :

- In Part I, we show from the dissipative character of the equation that w has a limit $w_{\pm\infty}$ as $s \to \pm\infty$ with $w_{\pm\infty}$ a critical point of (4), that is $w_{\pm\infty} \equiv 0$, κ or $-\kappa$. We then focus on the nontrivial case $(w_{-\infty}, w_{+\infty}) = (\kappa, 0)$ and show from a linear study of the equation around κ that w goes to κ as $s \to -\infty$ in three possible ways.
- In Part II, we show that one of these three ways corresponds to $w(y,s) = \varphi(s - s_0)$ for some $s_0 \in \mathbb{R}$ where $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$. In the two other cases, we find a contradiction from nonlinear informations :
  - the blow-up criterion of Proposition 2.1 (for w close to κ),
  - the following geometrical transformation :

$$a \in \mathbb{R}^N \to w_a \text{ defined by } w_a(y,s) = w(y + ae^{\frac{s}{2}}, s) \tag{14}$$

which keeps (4) invariant (thanks to the translation invariance of equation (1)).

Part I : Possible behaviors of w as $s \to \pm\infty$

We proceed in two steps : First, we find limits $w_{\pm\infty}$ for w as $s \to \pm\infty$. In a second step, we focus on the linear behavior of w as $s \to -\infty$, in the case $w_{-\infty} = \kappa$.

Step 1 : Limits of w as $s \to \pm\infty$

Proposition 2.2 (Limits of w as $s \to \pm\infty$) *$w_{+\infty}(y) = \lim_{s\to+\infty} w(y,s)$ exists and is a critical point of (4). The convergence holds in L^2_ρ, the L^2 space associated to the Gaussian measure $\rho(y)dy$ where ρ is defined in (6), and uniformly on each compact subset of $\mathbb{R}^N$. The same statement holds for $w_{-\infty}(y) = \lim_{s\to-\infty} w(y,s)$.*

Proof : See Step 1 in section 3 in [MZ98a]. ■

Proposition 2.3 (Stationary problem for (4)) *The only nonnegative bounded global solutions in $\mathbb{R}^N$ of*

$$0 = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1}w \tag{15}$$

are the constant ones : $w \equiv 0$, $w \equiv -\kappa$ and $w \equiv \kappa$.

Proof : One can derive the following Pohozaev identity for each bounded solution of equation (4) in $\mathbb{R}^N$ (see Proposition 2 in [GK85]) :

$$(N+2-p(N-2)) \int |\nabla w|^2 \rho dy + \frac{p-1}{2} \int |y|^2 |\nabla w|^2 \rho dy = 0. \tag{16}$$

Hence, for $(N-2)p \leq N+2$, w is constant. Thus, $w \equiv 0$ or $w \equiv \kappa$ or $w \equiv -\kappa$. ■

¿From Propositions 2.2 and 2.3, we have $w_{\pm\infty} \equiv 0$ or $w_{\pm\infty} \equiv \kappa$ or $w_{\pm\infty} \equiv -\kappa$. Since E is a Lyapunov functional for w, one gets from (5) and (4) :

$$-\int_{-\infty}^{+\infty} ds \int_{\mathbb{R}^N} \left| \frac{\partial w}{\partial s}(y,s) \right|^2 \rho dy = E(w_{+\infty}) - E(w_{-\infty}). \tag{17}$$

Therefore, since $E(\kappa) = E(-\kappa) > 0 = E(0)$, there are only two cases :
- $E(w_{-\infty}) - E(w_{+\infty}) = 0$. This implies by (17) that $\frac{\partial w}{\partial s} \equiv 0$, hence w is a stationary solution of (4) and $w \equiv 0$ or $w \equiv \kappa$ or $w \equiv -\kappa$ by Proposition 2.3.

- $E(w_{-\infty}) - E(w_{+\infty}) > 0$. This occurs only if $w_{+\infty} \equiv 0$ and $w_{-\infty} \equiv \kappa$ or $-\kappa$. It remains to treat this case. Since (4) is invariant under the transformation $w \to -w$, it is enough to focus on the case :

$$(w_{-\infty}, w_{+\infty}) \equiv (\kappa, 0). \tag{18}$$

Step 2 : Linear behavior of w near κ as $s \to -\infty$

Let us introduce $v = w - \kappa$. From (4), v satisfies the following equation : $\forall (y,s) \in \mathbb{R}^{N+1}$,

$$\frac{\partial v}{\partial s} = \mathcal{L}v + f(v) \tag{19}$$

$$\text{where } \mathcal{L}v = \Delta v - \frac{1}{2} y.\nabla v + v \text{ and } f(v) = |v+\kappa|^{p-1}(v+\kappa) - \kappa^p - p\kappa^{p-1}v. \tag{20}$$

Since w is bounded in L^∞, we assume $|v(y,s)| \leq C$ and $|f(v)| \leq C|v|^2$.

$\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2_\rho$. Its spectrum is

$$\text{spec}(\mathcal{L}) = \{1 - \frac{m}{2} \mid m \in \mathbb{N}\}, \tag{21}$$

and it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are derived from Hermite polynomials :

- $N = 1$:
  All the eigenvalues of $\mathcal{L}$ are simple. For $1 - \frac{m}{2}$ corresponds the eigenfunction

$$h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!} (-1)^n y^{m-2n}. \tag{22}$$

- $N \geq 2$:
  We write the spectrum of $\mathcal{L}$ as

$$\text{spec}(\mathcal{L}) = \{1 - \frac{m_1 + ... + m_N}{2} | m_1, ..., m_N \in \mathbb{N}\}.$$

  For $(m_1, ..., m_N) \in \mathbb{N}^N$, the eigenfunction corresponding to $1 - \frac{m_1+...+m_N}{2}$ is

$$h_{(m_1,...,m_N)} : y \longrightarrow h_{m_1}(y_1)...h_{m_N}(y_N), \tag{23}$$

where h_m is defined in (22). In particular,

*1 is an eigenvalue of multiplicity 1, and the corresponding eigenfunction is

$$H_0(y) = 1, \tag{24}$$

*$\frac{1}{2}$ is of multiplicity N, and its eigenspace is generated by the orthogonal basis $\{H_{1,i}(y) | i = 1, ..., N\}$, with $H_{1,i}(y) = h_1(y_i)$; we note

$$H_1(y) = (H_{1,1}(y), ..., H_{1,N}(y)), \tag{25}$$

*0 is of multiplicity $\frac{N(N+1)}{2}$, and its eigenspace is generated by the orthogonal basis $\{H_{2,ij}(y) | i, j = 1, ..., N, i \leq j\}$, with $H_{2,ii}(y) = h_2(y_i)$, and for $i < j$, $H_{2,ij}(y) = h_1(y_i)h_1(y_j)$; we note

$$H_2(y) = (H_{2,ij}(y), i \leq j). \tag{26}$$

Since the eigenfunctions of $\mathcal{L}$ constitute a total orthonormal family of L^2_ρ, we expand v as follows :

$$v(y,s) = \sum_{m=0}^{2} v_m(s).H_m(y) + v_-(y,s) \tag{27}$$

where
$v_0(s)$ is the projection of v on H_0,
$v_{1,i}(s)$ is the projection of v on $H_{1,i}$, $v_1(s) = (v_{1,i}(s), ..., v_{1,N}(s))$, $H_1(y)$ is given by (25),
$v_{2,ij}(s)$ is the projection of v on $H_{2,ij}$, $i \leq j$, $v_2(s) = (v_{2,ij}(s), i \leq j)$, $H_2(y)$ is given by (26),
$v_-(y,s) = P_-(v)$ and P_- is the projector on the negative subspace of $\mathcal{L}$.

With respect to the positive, null and negative subspaces of $\mathcal{L}$, we write

$$v(y,s) = v_+(y,s) + v_{null}(y,s) + v_-(y,s) \tag{28}$$

where $v_+(y,s) = P_+(v) = \sum_{m=0}^{1} v_m(s).H_m(y)$,
$v_{null}(y,s) = P_{null}(v) = v_2(s).H_2(y)$, P_+ and P_{null} are the L^2_ρ projectors respectively on the positive subspace and the null subspace of $\mathcal{L}$.

Now, we show that as $s \to -\infty$, either $v_0(s)$, $v_1(s)$ or $v_2(s)$ is predominant with respect to the expansion (27) of v in L^2_ρ. At this level, we are not able to use a center manifold theory to get the result (see [FK92] page 834-835 for more details). In some sense, we are not able to say that the nonlinear terms in the function of space are small enough. However, using similar techniques as in [FK92], we are able to prove the result. We have the following :

Proposition 2.4 (Linear classification of the behaviors of w as $s \to -\infty$) *As $s \to -\infty$, one of the following cases occurs :*
i) $|v_1(s)| + \|v_{null}(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(v_0(s))$,

$$\forall s \le s_0,\ v_0'(s) = v_0(s) + O\left(v_0(s)^2\right) \tag{29}$$

and there exists $C_0 \in \mathbb{R}$ such that

$$\|v(y,s) - C_0 e^s\|_{H^1_\rho} = o(e^s), \tag{30}$$

and $\forall \epsilon > 0$,

$$v_0(s) = C_0 e^s + O(e^{(2-\epsilon)s}) \text{ and } v_1(s) = O(e^{(2-\epsilon)s}). \tag{31}$$

ii) $|v_0(s)| + \|v_{null}(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(v_1(s))$ *and* $\exists C_1 \in \mathbb{R}^N \backslash \{0\}$ *such that* $\|v(y,s) - e^{\frac{s}{2}} C_1.y\|_{H^1_\rho} = o(e^{\frac{s}{2}})$, $v_1(s) \sim C_1 e^{s/2}$ *and* $v_0(s) \sim \frac{p}{\kappa}|C_1|^2 e^s$,
iii) $\|v_+(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(\|v_{null}(y,s)\|_{L^2_\rho})$ *and there exists $l \in \{1,...,N\}$ and Q an orthonormal $N \times N$ matrix such that*
$$\left\| v(Qy,s) - \frac{\kappa}{4ps}\left(2l - \sum_{i=1}^{l} y_i^2\right)\right\|_{H^1_\rho} = o(\tfrac{1}{s}),$$
$$v_{null}(Qy,s) = \frac{\kappa}{4ps}\left(2l - \sum_{i=1}^{l} y_i^2\right) + O\left(\frac{1}{s^{1+\delta}}\right),\ v_1(s) = O\left(\frac{1}{s^2}\right) \text{ and } v_0(s) = O\left(\frac{1}{s^2}\right) \text{ for some } \delta > 0.$$

Proof: See Propositions 3.5, 3.6, 3.9 and 3.10 in [MZ98a]. Although only L^2_ρ norms appear in those Propositions, one can see that the proof of Proposition 3.5 in [MZ98a] can be adapted without difficulties to yield H^1_ρ estimates (see section 6 in [FK92] for a similar adaptation). ■

Part II : Conclusion of the proof

The crucial point is to note that $I(\kappa) = 0$ where I is defined in (10). Thus, the use of the geometrical transformation $w \to w_a$ (see (14)) and the blow-up argument of Proposition 2.1 applied to $w_a(s)$ will introduce some rigidity on the behavior of $w(s)$ as $s \to -\infty$.

We proceed in two steps :
- In Step 1, we show that if the case $i)$ of Proposition 2.4 occurs, then $w(y,s) = \varphi(s - s_0)$ for some $s_0 \in \mathbb{R}$.
- In Step 2, we show by means of Proposition 2.1 and the transformation (14) that cases $ii)$ and $iii)$ of Proposition 2.4 yield a contradiction.

Step 1 : Case $i)$ of Proposition 2.4 : the relevant case

Proposition 2.5 *Assume that case $i)$ of Proposition 2.4 occurs, then :*
i) $C_0 < 0$,
ii) $\forall y \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$, $w(y,s) = \varphi(s - s_0)$ *where* $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$ *and* $s_0 = -\log\left(-\frac{(p-1)C_0}{\kappa}\right)$.

Proof :
i) We proceed by contradiction in order to eliminate successively the cases $C_0 = 0$ and $C_0 > 0$.
- Suppose $C_0 = 0$, then one can see from (29) and (31) that $\forall s \le s_1$, $v_0(s) = 0$ for some $s_1 \in \mathbb{R}$. Since $\|v(s)\|_{L^2_\rho} \sim v_0(s)$ as $s \to -\infty$, we have $\forall s \le s_2$, $\forall y \in \mathbb{R}^N$, $v(y,s) = 0$ and $w(y,s) = \kappa$ for some $s_2 \in \mathbb{R}$. From the uniqueness of the solution of the Cauchy problem for equation (4), we have $w \equiv \kappa$ in all $\mathbb{R}^N \times \mathbb{R}$, which contradicts the fact that $w \to 0$ as $s \to +\infty$ (see (18)). Hence, $C_0 \ne 0$.
- Suppose now that $C_0 > 0$. We will prove that

$$I(w(s)) = -2E(w(s)) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^N} |w(y,s)|^2 \rho(y)dy\right)^{\frac{p+1}{2}} > 0 \qquad (32)$$

for some $s \in \mathbb{R}$, which is the blow-up condition of Proposition 2.1, in contradiction with the global boundedness of w.

Since $w = \kappa + v$ and κ is a critical point of $E : H^1_\rho(\mathbb{R}^N) \to \mathbb{R}$ (see Proposition 2.3), we have

$$E(w(s)) = E(\kappa) + O\left(\|v(s)\|^2_{H^1_\rho}\right) = \frac{\kappa^2}{2(p+1)} + O\left(\|v(s)\|^2_{H^1_\rho}\right). \qquad (33)$$

For the second term in (32), we use $w = \kappa + v$ and write
$\int |w(y,s)|^2 \rho dy = \kappa^2 + 2\kappa \int v(y,s)\rho dy + \int |v(y,s)|^2 \rho dy$
$= \kappa^2 + 2\kappa v_0(s) + \int |v(y,s)|^2 \rho dy$. Therefore,
$\frac{p-1}{p+1}\left(\int |w(y,s)|^2 \rho dy\right)^{\frac{p+1}{2}} = \frac{\kappa^2}{p+1} + \kappa v_0(s) + O(\|v(s)\|^2_{L^2_\rho})$. Combining this with (33) and using (31) and (30), we end up with

$$I(w(s)) \sim \kappa v_0(s) \sim \kappa C_0 e^s > 0 \text{ as } s \to -\infty$$

which is the blow-up condition of Proposition 2.1. Contradiction. Thus, $C_0 < 0$.

ii) Let us introduce $V(y,s) = w(y,s) - \varphi(s-s_0)$ where $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$ and $s_0 = -\log\left(-\frac{(p-1)C_0}{\kappa}\right)$. Since φ is a solution of

$$\varphi'(s) = -\frac{\varphi(s)}{p-1} + \varphi(s)^p,$$

we see from (4) that V satisfies the following equation :

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V + F(V) \tag{34}$$

where $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$, $l(s) = -\frac{pe^{s-s_0}}{(p-1)(1+e^{s-s_0})}$ and $F(V) = |\varphi+V|^{p-1}(\varphi+V) - \varphi^p - p\varphi^{p-1}V$. Note that $\forall s \leq 0$, $|F(V)| \leq C|V|^2$.

Besides, we have from $i)$ of Proposition 2.4 and the choice of s_0 that

$$|V_0(s)| + |V_1(s)| = O(e^{(2-\epsilon)s}) \text{ and } \|V_{null}(s)\|_{L^2_\rho} + \|V_-(s)\|_{L^2_\rho} = o(e^s) \tag{35}$$

as $s \to -\infty$. Using the linear classification at infinity of solutions of equation (34) under the conditions (35) (see Proposition 3.7 in [MZ98a]), we get $V \equiv 0$ on $\mathbb{R}^N \times \mathbb{R}$. Thus, $\forall y \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$,

$$w(y,s) = \varphi(s-s_0).$$

■

Step 2 : Cases $ii)$ and $iii)$ of Proposition 2.4 : blow-up cases In both cases $ii)$ and $iii)$ of Proposition 2.4, we will find $s_0 \in \mathbb{R}$ and $|a_0| \leq e^{-\frac{s_0}{2}}$ such that $I(w_{a_0}(s_0)) > 0$ where I is defined in (10), which implies by Proposition 2.1 that w_{a_0} blows-up in finite time $S > s_0$, in contradiction with $\|w_{a_0}\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} = \|w\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} < +\infty$. We give in the following lemma an expansion of $I(w_a(s))$ as $s \to -\infty$ and $ae^{s/2} \to 0$, which will allow us to conclude :

Lemma 2.6
a - Assume that case $ii)$ or $iii)$ of Proposition 2.4 holds, then

$$I(w_a(s)) = \kappa \int v(y,s)\rho(y - ae^{s/2})dy + O\left(\|v(s)\|^2_{H^1_\rho}\right)$$

as $s \to -\infty$ and $ae^{s/2} \to 0$. Moreover,
b - In case $ii)$: $\int v(y,s)\rho(y-ae^{s/2})dy = a.C_1e^s + o(|a|e^s) + O(se^s)$,

c - In case iii) : $\int v(y,s)\rho(y-ae^{s/2})dy =$
$\frac{\kappa}{4p|s|}\sum_{i=1}^{l}\int(z_i^2-2)(Qae^{s/2}.z)^2\rho(z)dz+O\left(\frac{1}{s^2}\right)+O\left(\frac{|a|^2e^s}{|s|^{1+\delta}}\right)+O\left(\frac{|a|^3e^{\frac{3s}{2}}}{|s|}\right).$

Proof : see **. ■

This lemma allows us to conclude. Indeed,
- if case *ii)* of Proposition 2.4 holds, then
$I(w_a(s)) = \kappa a.C_1e^s + o(|a|e^s) + O(se^s)$. We fix s_0 negative enough and $a_0 = \frac{1}{|s_0|}\frac{C_1}{|C_1|}e^{-s_0/2}$ to get

$$I(w_{a_0}(s_0)) \geq \frac{1}{2}\kappa a_0.C_1e^{s_0} = \kappa\frac{e^{s_0/2}}{2|s_0|}|C_1| > 0.$$

This implies by Proposition 2.1 that w_{a_0} blows-up at time $S > s_0$. Contradiction.
- If case *iii)* of Proposition 2.4 holds, then
$I(w_a(s)) = \frac{\kappa^2}{4p|s|}\sum_{i=1}^{l}\int(z_i^2-2)(Qae^{s/2}.z)^2\rho(z)dz + O\left(\frac{1}{s^2}\right) + O\left(\frac{|a|^2e^s}{|s|^{1+\delta}}\right) + O\left(\frac{|a|^3e^{\frac{3s}{2}}}{|s|}\right)$. We fix s_0 negative enough and $a_0 = \frac{e^{-s_0/2}}{|s_0|^{1/4}}Q^{-1}e_1$ where $e_1 = (1,0,...,0)$ so that we get

$$I(w_{a_0}(s_0)) \geq \frac{1}{2}\frac{\kappa^2}{4p|s_0|}\sum_{i=1}^{l}\int(z_i^2-2)(\frac{e_1}{|s_0|^{1/4}}.z)^2\rho(z)dz = \frac{\kappa^2}{p|s_0|^{3/2}} > 0$$

by (6). This implies by Proposition 2.1 that w_{a_0} blows-up at time $S > s_0$. Contradiction.

This concludes the proof of Theorem 1. ■

3 Uniform estimates for nonlinear heat equations

In this section, we prove uniform bounds and the ODE like behavior of the solution.

Proof of Theorem 2 : Uniform L^∞ bounds on the solution

Consider $u_0 \in C^2$ such that $\|u_0\|_{C^2} \leq C_0$ and $u(t)$ solution of (1) with initial data u_0 blows-up at T with $T < T_0$. We claim that there is $C = C(C_0, T_0)$ such that $\|u(t)\|_{L^\infty}$ is controlled by $Cv(t)$ where v is the solution of the ODE $v' = v^p$ which blows-up at the same time T as $u(t)$. The result mainly follows from blow-up argument giving local energy estimates and the fact that these estimates yield L^∞ estimates (from Giga-Kohn [GK87]).

Step 1 : Estimates on $u(t)$ for small time

Lemma 3.1 (C^2 bounds for small time) *There is $t_0 = t_0(C_0) > 0$ such that :*
i) for all $t \in [0, t_0]$, $\|u(t)\|_{L^\infty} \leq 2C_0$,
ii) for all $t \in [0, t_0]$, $\|u(t)\|_{C^2} \leq 2C_0$,
iii) for all $\alpha \in (0,1)$, $\|\Delta u\|_{C^\alpha(D)} \leq C_1(\alpha, C_0)$where

$$\|a\|_\alpha = \sup_{(x,t)\neq(x',t')\in D} \frac{|a(x,t) - a(x',t')|}{\left(|x - x'| + |t - t'|^{1/2}\right)^\alpha}$$

where $D = \mathbb{R}^N \times [\frac{t_0}{2}, t_0]$.

Proof : We start with i) and ii). Since u satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-s)|u(s)|^{p-1}u(s)ds,$$

we have

$$\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \int_0^t \|u(s)\|_{L^\infty}^p ds.$$

Thus, by a priori estimates, we have $\forall t \in [0, t_0]$, $\|u(t)\|_{L^\infty} \leq 2C_0$ where $t_0 = 2^{-p}C_0^{1-p}$.
Similarly, we obtain $\forall t \in [0, t_0]$, $\|u(t)\|_{C^2} \leq 2C_0$ where $t_0 = t_0(C_0)$.

iii) We use the following lemma :

Lemma 3.2 *Assume that h solves*

$$\frac{\partial h}{\partial \tau} = \Delta h + a(\xi, \tau)h$$

for $(\xi, \tau) \in D$ where $D = B(0,3) \times [0, t_0]$ and $t_0 \leq T_0$. Assume in addition that $\|a\|_{L^\infty} + |a|_{\alpha,D}$ is finite, where

$$|a|_{\alpha,D} = \sup_{(\xi,\tau),(\xi',\tau')\in D} \frac{|a(\xi,\tau) - a(\xi',\tau')|}{\left(|\xi - \xi'| + |\tau - \tau'|^{1/2}\right)^\alpha} \quad (36)$$

and $\alpha \in (0,1)$. Then,

$$\|h\|_{C^2(D')} + |\nabla^2 h|_{\alpha,D'} \leq K\|h\|_{L^\infty(D)}$$

where $K = K\left(\|a\|_{L^\infty(D)} + |a|_{\alpha,D}\right)$ and $D' = B(0,1) \times [\frac{t_0}{2}, t_0]$.

Proof : see Lemma 2.10 in [MZ98b]. ■

Step 2 : Energy bounds in similarity variables

¿From the blow-up argument for equation (4) (Proposition 2.1) and the monotonicity of the energy E, we have :

Lemma 3.3 *There is $C_1 = C_1(C_0, T_0)$ such that $\forall s \geq s_0 = -\log T$, $\forall a \in \mathbb{R}^N$,*
i) $|E(w_a(s))| \leq C_1$ and $\int |w_a(y,s)|^2 \rho(y)dy \leq C_1$,
ii) $\int_s^{s+1} \int \left(|w_a(y,s)|^{p+1} + |\nabla w_a(y,s)|^2 + \left|\frac{\partial w_a}{\partial s}(y,s)\right|^2 \right) \rho(y)dyds \leq C_1$,
iii) $\int_s^{s+1} \left(\int |w_a(y,s)|^{p+1}\rho(y)dy\right)^2 ds \leq C_1$ where w_a and E are defined respectively in (3) and (5).

Proof : Following [GK87], we note $w = w_a$.

i) First we have that $\forall s \in [s_0, +\infty)$, $\frac{d}{ds}E(w_a(s)) \leq 0$, $E(w_a(s)) \leq E(w_a(s_0)) \leq C(C_0, T_0)$. Let us note from the blow-up result of Proposition 2.1 that $\forall s \in [s_0, +\infty)$,

$$I(w(s)) = -2E(w(s)) + \frac{p-1}{p+1}\left(\int |w(y,s)|^2\rho(y)dy\right)^{\frac{p+1}{2}} \leq 0.$$

Thus, $\left(\int |w(y,s)|^2\rho(y)dy\right)^{\frac{p+1}{2}} \leq \frac{2(p+1)}{p-1}E(w(s)) \leq C(C_0,T_0)$ and we have *i*).

ii) We have

$$\frac{d}{ds}\int |w(y,s)|^2\rho(y)dy = -2E(w(s)) + \frac{p-1}{p+1}\int |w(y,s)|^{p+1}\rho(y)dy.$$

Therefore, by integration and *i*), $\int_s^{s+1}\int |w(y,s)|^{p+1}\rho(y)dyds \leq C_1$.
¿From the bound on $\int |w(y,s)|^2\rho(y)dy$, $E(w(s))$ and $\int_s^{s+1}\int |w(y,s)|^{p+1}\rho(y)dyds$, we obtain the bound on $\int_s^{s+1}\int |\nabla w(y,s)|^2\rho(y)dyds$, and from the variation of the energy, $\left|\int_s^{s+1}\int \left|\frac{\partial w}{\partial s}(y,s)\right|^2\rho(y)dyds\right| \leq |E(w(s))| + |E(w(s+1))| \leq 2C_1$.

iii) We write
$-\int |\nabla w(y,s)|^2\rho(y)dy + \int |w(y,s)|^{p+1}\rho(y)dy$
$= \int \frac{\partial w}{\partial s}(y,s)w(y,s)\rho(y)dy + \frac{1}{p-1}\int |w(y,s)|^2\rho(y)dy.$
Since $\left|\int |\nabla w(y,s)|^2\rho(y)dy - \frac{2}{p+1}\int |w(y,s)|^{p+1}\rho(y)dy\right| \leq C_1$, we have
$\int |w(y,s)|^{p+1}\rho(y)dy \leq C_1\left(\int \left|\frac{\partial w}{\partial s}(y,s)\right|^2\rho dy\right)^{\frac{1}{2}}\left(\int |w(y,s)|^2\rho(y)dy\right)^{\frac{1}{2}} + C_1,$

then,
$\left(\int |w(y,s)|^{p+1}\rho(y)dy\right)^2 \leq C_1 \left(1+\int \left|\frac{\partial w}{\partial s}(y,s)\right|^2 \rho(y)dy\right)$.
Thus, by integration we have the conclusion. ■

Step 3 : L^∞ bound in similarity variables

We have the following proposition, where L^∞ bound can be derived from energy bounds :

Proposition 3.4 (Giga-Kohn, L^∞ bound on w) *Assume that we have the bounds of lemma 3.3 on w in the interval $[s, s+1]$ for a given C_1, then for all $\delta \in (0,1)$, there exists $C_2(C_1,\delta)$ such that $|w_a(0, s+\delta)| \leq C_2$.*

Proof : See lemma 3.2 in [GK87]. ■

Step 4 : Conclusion of the proof : L^∞ bounds with respect to C_0 and T_0

We can see that these arguments yield uniform bounds on the solution.
- On one hand, we have from Step 1,

$$\forall t \in [0, t_0(C_0)],\ \|u(t)\|_{L^\infty} \leq 2C_0. \tag{37}$$

- On the other hand, we have from Proposition 3.4 and Step 2, for all $\delta_0 \in (0,1)$, $\forall s \in [s_0+\delta_0, +\infty)$, $\|w(s)\|_{L^\infty} \leq C_2(C_1,\delta_0)$, therefore

$$\forall t \in [T(1-e^{-\delta_0}), T),\ \|u(t)\|_{L^\infty} \leq \frac{C_2}{(T-t)^{\frac{1}{p-1}}}. \tag{38}$$

Taking $\delta_0 = \delta_0(T_0, t_0)$ such that $T_0(1-e^{-\delta_0}) \leq \frac{t_0}{2}$, and using (37) and (38) we obtain $\forall t \in [0,T)$, $\|u(t)\|_{L^\infty} \leq \frac{C_3}{(T-t)^{\frac{1}{p-1}}}$ where
$C_3(C_0,T_0) = \max(C_2(C_1,\delta_0), 2C_0 T_0^{\frac{1}{p-1}})$.

This concludes the proof of Theorem 2. ■

Let us prove now the uniform pointwise control of the diffusion term by the nonlinear term, which asserts that the solution $u(t)$ behaves everywhere like the ODE $v' = v^p$.

Proof of Theorem 3 (Uniform ODE behavior) :

We argue by contradiction. Let us consider u_n solution of (1) with initial data u_{0n} such that $\|u_{0n}(t)\|_{C^2} \leq C_0$, $u_n(t)$ blows-up at time $T_n < T_0$ and for some $\epsilon_0 > 0$, the statement

$$|\Delta u| \leq \epsilon_0 |u|^p + n \text{ on } \mathbb{R}^N \times [0, T_n) \tag{39}$$

is not valid. Therefore, there is $(x_n, t_n) \in \mathbb{R}^N \times [0, T_n)$ such that

$$|\Delta u_n(x_n, t_n)| \geq \epsilon_0 |u_n(x_n, t_n)|^p + n. \qquad (40)$$

Considering $\tilde{u}_n(x, t) = u_n(x_n + x, t)$, we can assume

$$x_n = 0.$$

From the uniform estimates and the parabolic regularity, we have

$$T_n - t_n \to 0 \text{ as } n \to +\infty.$$

Indeed, from Theorem 2, $\exists C_2(C_0, T_0) > 0$ such that $\forall t \in [0, T_n)$,
$\|u_n(t)\|_{L^\infty} \leq \frac{C_2}{(T_n - t)^{\frac{1}{p-1}}}$.
Introducing $w_n(y, s)$ for all $y \in \mathbb{R}^N$ and $s \geq s_{0n} = -\log T_n$ by

$$y = \frac{x - a}{\sqrt{T_n - t}}, \; s = -\log(T_n - t), \; w_n(y, s) = (T_n - t)^{\frac{1}{p-1}} u_n(x, t),$$

we have $\forall s \in [s_{0n}, +\infty)$, $\|w_n(s)\|_{L^\infty} \leq C_2$, where $s_{0n} = -\log T_n$. From parabolic regularity applied to equations (1) and (4), there is C' such that $\forall s \in [s_0, +\infty)$, $\|\Delta w_n(s)\|_{L^\infty} \leq C'$.
Thus, $\forall t \in [0, T_n)$, $\|\Delta u_n(t)\|_{L^\infty} \leq \frac{C'}{(T_n - t)^{\frac{p}{p-1}}}$.
¿From (40), we have

$$\frac{C'}{(T_n - t_n)^{\frac{p}{p-1}}} \geq \|\Delta u_n(t_n)\|_{L^\infty} \geq |\Delta u_n(x_n, t_n)| \geq n$$

and $T_n - t_n \to 0$ as $n \to +\infty$.

Let us now consider two cases.
In the region where the solution $u_n(t)$ is of the same order as the solution of the ODE blowing-up at T_n (called the very singular region), the Liouville Theorem 1 in similarity variables yields a contradiction.
For the other regions, we can control the nonlinear term by using in some sense wellposedness for small data in some localized energy space (subcritical behavior). This allows us to transport the information from the very singular region everywhere.

i) Estimates in the very singular region. $|u_n(0, t_n)|(T_n - t_n)^{\frac{1}{p-1}} \to \delta_0 \neq 0$ *as* $n \to +\infty$.

A compactness procedure and the Liouville Theorem yield a contradiction. We now consider $\tilde{w}_n(y, s) = w_n(s_n + s, y)$ where $s_n = -\log(T_n - t_n) \to$

$+\infty$ as $n \to +\infty$.
$\tilde{w}_n$ is a solution of (4) for $(y,s) \in \mathbb{R}^N \times [s_{0n} - s_n, +\infty)$ such that $\forall s \geq s_{0n} - s_n + 1$, $\|\tilde{w}_n(s)\|_{L^\infty(\mathbb{R}^N)} \leq C$, $\forall R > 0$, $\|\tilde{w}_n\|_{C^{2,1}_\alpha(B(0,R)\times[-R,R])} \leq C'(R)$, and
$|\Delta \tilde{w}_n(0,0)| \geq \epsilon_0 |\tilde{w}_n(0,0)|^p \geq \epsilon_0 \frac{\delta_0^p}{2} \geq \delta_0' > 0$, where for all $D \subset \mathbb{R}^N \times \mathbb{R}$,

$$\begin{aligned} \|w\|_{C^{2,1}_\alpha(D)} &= \|w\|_{L^\infty(D)} + \|\nabla w\|_{L^\infty(D)} + \|\nabla^2 w\|_{L^\infty(D)} + \|\nabla^2 w\|_{\alpha,D} \\ &+ \|\frac{\partial w}{\partial t}\|_{L^\infty(D)} + \|\frac{\partial w}{\partial s}\|_{\frac{\alpha}{2},D} \end{aligned}$$

and $\|u\|_{\alpha,D}$ is defined in (36). Note that $s_n \to +\infty$ and $s_{0n} = -\log T_n \leq -\log t_0(C_0)$ by lemma 3.1. Therefore, $s_{0n} - s_n \to -\infty$. By compactness procedure, $\tilde{w}_n \to w$ as $n \to +\infty$ on compact sets of $\mathbb{R}^N \times \mathbb{R}$ where w is solution of (4) for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that

$$\forall s \in \mathbb{R},\ \|w(s)\|_{L^\infty} \leq C \text{ and } |\Delta w(0,0)| \geq \delta_0' > 0.$$

¿From Theorem 1, we have a contradiction, since all the globally bounded solutions w of (4) defined on $\mathbb{R}^N \times \mathbb{R}$ satisfy $w(y,s) = w(s)$ and $\Delta w(y,s) = 0$.

ii) Estimates in the singular region : $u_n(0,t_n)(T_n - t_n)^{\frac{1}{p-1}} \to 0$.
We now consider the case where

$$u(0,t_n)(T_n - t_n)^{\frac{1}{p-1}} \to 0 \text{ as } n \to +\infty. \tag{41}$$

Again, by the Liouville Theorem and the local energy estimates (which allow us to control the nonlinear term), we transport the information obtained in the very singular region to obtain a contradiction in this case.

Step 1 : Compactness procedure outside the singular region
We have from Theorem 2 and its proof

$$\forall t \in [0,T_n),\ \forall n,\ \|u_n(t)\|_{L^\infty} \leq \frac{C}{(T_n - t)^{\frac{1}{p-1}}} \text{ and } \|u_n(t)\|_{C^2} \leq \frac{C}{(T_n - t)^{\frac{p}{p-1}}}.$$

By a compactness procedure, we can assume that $T_n \to T^*$ where $t_0(C_0) < T^* \leq T_0$ and $u_n(x,t) \to u(x,t)$ in $C^{2,1}_{loc}(\mathbb{R}^N \times [0,T^*))$ where $\forall t \in [0,T^*)$, $\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u$,

$$\|u(t)\|_{L^\infty} \leq \frac{C_1}{(T^* - t)^{\frac{1}{p-1}}} \text{ and } \|u(t)\|_{C^2} \leq \frac{C_1}{(T^* - t)^{\frac{p}{p-1}}},$$

and for all $D \subset \mathbb{R}^N \times \mathbb{R}$,

$$\|u\|_{C^{2,1}(D)} = \|u\|_{L^\infty(D)} + \|\nabla u\|_{L^\infty(D)} + \|\nabla^2 u\|_{L^\infty(D)} + \|\frac{\partial u}{\partial t}\|_{L^\infty(D)}.$$

We claim :

Lemma 3.5 *$u(t)$ blows-up at T^* and 0 is a blow-up point of $u(t)$.*

Let us recall the following result which asserts that the smallness of the following weighted energy (related to the energy $E(w_a)$ defined in (5)) :

$$\begin{aligned} \mathcal{E}_{a,t}(u) &= t^{\frac{2}{p-1}-\frac{N}{2}+1} \int \left[\frac{1}{2}|\nabla u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right] \rho(\frac{x-a}{\sqrt{t}})dx \\ &+ \frac{1}{2(p-1)} t^{\frac{2}{p-1}-\frac{N}{2}} \int |u(x)|^2 \rho(\frac{x-a}{\sqrt{t}})dx \end{aligned}$$

implies an L^∞ bound on $u(x,t)$ locally in space-time.

Proposition 3.6 (Local energy smallness result) *There exists $\sigma_0 > 0$ such that for all $\delta' > 0$ and $\theta' > 0$, $\forall t' \in [0, T_n - \theta']$, if $\forall x \in B(0,\delta')$, $\mathcal{E}_{x,T_n-t'}(u_n) \le \sigma_0$, then*
- $\forall |x| \le \delta'$, $\forall t \in [\frac{t'+T_n}{2}, T_n)$, $|u_n(x,t)| \le \frac{C\sigma_0^\theta}{(T_n-t)^{\frac{1}{p-1}}}$
- Moreover, if $\forall |x| \le \delta'$, $|u_n(x, \frac{t'+T_n}{2})| \le M'$ then $\forall |x| \le \frac{\delta'}{2}$, $\forall t \in [\frac{t'+T_n}{2}, T_n)$, $|u_n(x,t)| \le M^*$ where $M^* = M^*(M', \delta', \theta')$.

Proof : See [GK89] and [Mer92] (Proposition 2.5). ■
Proof of lemma 3.5 : By contradiction, there is M, $\delta > 0$ such that

$$\forall |x| \le 4\delta,\ \forall t \in [0, T^*),\ |u(x,t)| \le M. \quad (42)$$

¿From a stability result with respect to the initial data of this property, we obtain a contradiction.
Indeed, from (42) and direct calculations, there is then t^* such that $\forall |x| \le \delta$, $\mathcal{E}_{x,T^*-t^*}(u(t^*)) \le \frac{\sigma_0}{2}$. We now fix t^*. Then, for n large, $\forall |x| \le \delta$, $\mathcal{E}_{x,T_n-t^*}(u_n)(t^*) \le \sigma_0$, and $\forall |x| \le \delta$, $\forall t \in [0, \frac{t^*+T_n}{2}]$, $|u_n(x,t)| \le 2M$. Therefore, form Proposition 3.6, $\forall |x| \le \frac{\delta}{2}$, $\forall t \in [\frac{t^*+T_n}{2}, T_n)$, $|u_n(x,t)| \le M^*$.

By a classical regularity argument, we have $\forall |x| \le \frac{\delta}{4}$, $\forall t \in [\frac{3T_n}{4}, T_n)$, $|\Delta u_n(0,t_n)| \le M^{**}(M^*, M)$ which is a contradiction with the fact that $|\Delta u_n(0,t_n)| \to +\infty$ as $n \to +\infty$ and the fact that $T_n - t_n \to 0$. This concludes the proof of lemma 3.5. ■

Step 2 : Choice of the scaling parameter

¿From the fact that 0 is a blow-up point of u, we are able to choose a suitable scaling parameter connecting $(0, t_n)$ and the "very singular region" of u_n. We are now reduced to the same proof as in [MZ98a]. Consider $\kappa_0 \in (0, \kappa)$ a constant such that $\mathcal{E}_{0,1}(\kappa_0) \le \frac{\sigma_0}{2}$ ($\mathcal{E}_{0,1}(0) = 0$ yields the existence of such a κ_0).
Since 0 is a blow-up point of u,

$$u(0,t)(T^* - t)^{\frac{1}{p-1}} \to \kappa \mathbb{R}^N.$$

where $\mathbb{R}^N \in S^{M-1}$. (Note that this follows from the results of Giga and Kohn [GK89] and Filippas and Merle [FM95]. If $M = 1$, then $\omega = \pm 1$).
In particular, there is $t_0 \ge 0$ such that $\forall t \in [t_0, T^*)$, $|u(0,t)|(T^* - t)^{\frac{1}{p-1}} \ge \frac{3\kappa + \kappa_0}{4}$.
Therefore, by continuity arguments, for all $t \in [t_0, T^*)$, there is a $n(t)$ such that

$$\forall n \ge n(t),\ |u_n(0,t)|(T_n - t)^{\frac{1}{p-1}} \ge \frac{\kappa + \kappa_0}{2}. \qquad (43)$$

¿From (41) and (43), we have the existence of $\tilde{t}_n \in [0, t_n]$ such that
$|u_n(0, \tilde{t}_n)|(T_n - \tilde{t}_n)^{\frac{1}{p-1}} = \kappa_0$ and $\forall t \in (\tilde{t}_n, t_n]$, $|u_n(0,t)|(T_n - t)^{\frac{1}{p-1}} < \kappa_0$.
We will see in Step 3 that $u(0, \tilde{t}_n) \sim \frac{C}{(T_n - \tilde{t}_n)^{\frac{1}{p-1}}}$.
We have $\tilde{t}_n \to T^*$ from (43).

Let us now consider

$$v_n(\xi, \tau) = (T_n - \tilde{t}_n)^{\frac{1}{p-1}} u_n(\xi \sqrt{T_n - \tilde{t}_n}, \tilde{t}_n + \tau (T_n - \tilde{t}_n)).$$

Step 3 : Conclusion of the proof

¿From the Liouville Theorem stated for equation (1) (Corollary 1) and energy estimates, we show that the nonlinear term is "subcritical" on compact sets of $\mathbb{R}^N \times (-\infty, 1]$. In particular, we have $v_n(\xi, \tau) \to v(\tau)\omega_0$ where $\omega_0 \in S^{M-1}$, $v' = v^p$ and $v(0) = \kappa_0$ uniformly on compact sets of $\mathbb{R}^N \times (-\infty, 1]$ (Note that $v(\tau) = \kappa \left(\left(\frac{\kappa}{\kappa_0} \right)^{p-1} - \tau \right)^{-\frac{1}{p-1}}$ and $v(1) < +\infty$).
We have from the definition of v_n that
- v_n is defined for all $\tau \in [\tau_n, 1)$ where $\tau_n \to -\infty$ (since $T_n - \tilde{t}_n \to 0$) and satisfies

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + |v_n|^{p-1} v_n.$$

- $\|v_n(\tau)\|_{L^\infty} \le C\frac{(T_n-\tilde{t}_n)^{\frac{1}{p-1}}}{[(1-\tau)(T_n-\tilde{t}_n)]^{\frac{1}{p-1}}} \le \frac{C}{(1-\tau)^{\frac{1}{p-1}}}$, $\|v_n(\tau)\|_{C^2} \le \frac{C'}{(1-\tau)^{\frac{p}{p-1}}}$ and $|v_n(0,0)| = \kappa_0$.

We can assume $v_n \to v$ in $C^{2,1}_{loc}(\mathbb{R}^N \times (-\infty, 1))$ where

$$\begin{aligned}\frac{\partial v}{\partial \tau} &= \Delta v + |v|^{p-1}v \\ |v(0,0)| &= \kappa_0 \text{ and } \|v(\tau)\|_{L^\infty} \le \frac{C'}{(1-\tau)^{\frac{1}{p-1}}}.\end{aligned}$$

¿From Corollary 1, (that is using in some sense the Liouville Theorem in the very singular region), we have $v(\xi,\tau) = v(\tau)\omega_0$ for some $\omega_0 \in S^{M-1}$. Thanks to this result, we have uniformly with respect to $|\xi| \le 2$,

$$\mathcal{E}_{\xi,1}(v_n(0)) \to \mathcal{E}_{\xi,1}(v(0)) = \mathcal{E}_{\xi,1}(\kappa_0) \le \frac{\sigma_0}{2}.$$

Thus, for n large, $\forall |\xi| \le 2$, $\mathcal{E}_{\xi,1}(v_n(0)) \le \sigma_0$, $|v_n(\xi, \frac{1}{2})| \le 2v(\frac{1}{2})$, and by Proposition 3.6, $\forall |\xi| \le \frac{1}{2}$, $\forall \tau \in [\frac{1}{2}, 1)$, $|v_n(\xi,\tau)| \le M^*$.
By lemma 3.2, there is M^* such that $\forall |\xi| \le \frac{1}{4}$, $\forall \tau \in [\frac{3}{4}, 1]$,
$\left|\frac{\partial v_n}{\partial t}\right|_{\frac{1}{2},[-\frac{1}{4},\frac{1}{4}]^N\times[\frac{3}{4},1]} + |\Delta v_n|_{\frac{1}{2},[-\frac{1}{4},\frac{1}{4}]^N\times[\frac{3}{4},1]} \le M^{**}$ where $|a|_{\alpha,D}$ is defined in (36).
In particular, $|\Delta v_n|$ and $\left|\frac{\partial v_n}{\partial t}\right|$ are uniformly continuous on $(\xi,\tau) \in B_{1/4} \times [\frac{3}{4}, 1]$ (with a constant independent from n). Thus, $v_n(0,\tau) \to v(\tau)\omega_0$ and $\Delta v_n(0,\tau) \to \Delta v(0,\tau)\omega_0 = 0$ uniformly for $\tau \in [0,1]$ as $n \to +\infty$.
For $\tau_n = \frac{t_n - \tilde{t}_n}{T_n - \tilde{t}_n} \in [0,1]$, we have from (39)
$|\Delta v_n(\tau_n, 0)| = (T_n - \tilde{t}_n)^{\frac{p}{p-1}}|\Delta u_n(0,t_n)| \ge \frac{\epsilon_0}{2}|u_n(0,t_n)|^p(T_n - \tilde{t}_n)^{\frac{p}{p-1}}$
$\ge \frac{\epsilon_0}{2}|v_n(0,\tau_n)|^p$. Let $n \to +\infty$, we obtain

$$0 \ge \frac{\epsilon_0}{2}\left(\min_{\tau\in[0,1]} v(\tau)\right)^p \ge \frac{\epsilon_0}{2}\kappa_0^p$$

which is a contradiction. This concludes the proof of Theorem 3. ■

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