Stability of blow-up profile for equation of the type $u_t = \Delta u + |u|^{p-1}u$

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Abstract In this paper, we consider the following nonlinear equation

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u \\ u(.,0) &= u_0, \end{aligned}$$

(and various extensions of this equation, where the maximum principle do not apply). We first describe precisely the behavior of a blow-up solution near blow-up time and point. We then show a stability result on this behavior.

Mathematics Subject Classification: 35K, 35B35, 35B40
Key words: Blow-up, Profile, Stability

1 Introduction

In this paper, we are concerned with the following nonlinear equation:

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u \\ u(.,0) &= u_0 \in H, \end{aligned} \qquad (1)$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$, Δ stands for the Laplacian in $\mathbb{R}^N$. We note $H = W^{1,p+1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We assume in addition the exponent p subcritical: if $N \geq 3$ then $1 < p < (N+2)/(N-2)$, otherwise, $1 < p < +\infty$. Other types of equations will be also considered.
Local Cauchy problem for equation (1) can be solved in H . Moreover, one can show that either the solution $u(t)$ exists on $[0,+\infty)$, or on $[0,T)$ with

$T < +\infty$. In this former case, u blows-up in finite time in the sense that $\|u(t)\|_H \to +\infty$ when $t \to T$.
(Actually, we have both $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \to +\infty$ and $\|u(t)\|_{W^{1,p+1}(\mathbb{R}^N)} \to +\infty$ when $t \to T$).
Here, we are interested in blow-up phenomena (for such case, see for example Ball [1], Levine [14]). We now consider a blow-up solution $u(t)$ and note T its blow-up time. One can show that there is at least one blow-up point a (that is $a \in \mathbb{R}^N$ such that: $|u(a,t)| \to +\infty$ when $t \to T$). We will consider in this paper the case of a finite number of blow-up points (see [15]). More precisely, we will focus for simplicity on the case where there is only one blow-up point. We want to study the profile of the solution near blow-up, and the stability of such behavior with respect to initial data.

Standard tools such as center manifold theory have been proven non efficient in this situation (Cf [6] [3]). In order to treat this problem, we introduce *similarity variables* (as in [8]):

$$
\begin{aligned}
y &= \frac{x-a}{\sqrt{T-t}}, & (2)\\
s &= -\log(T-t), \\
w_{T,a}(y,s) &= (T-t)^{\frac{1}{p-1}} u(x,t), & (3)
\end{aligned}
$$

where a is the blow-up point and T the blow-up time of $u(t)$.
The study of the profile of u as $t \to T$ is then equivalent to the study of the asymptotic behavior of $w_{T,a}$ (or w for simplicity), as $s \to \infty$, and each result for u has an equivalent formulation in terms of w. The equation satisfied by w is the following:

$$w_s = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w. \tag{4}$$

Giga and Kohn showed first in [8] that for each $C > 0$,

$$\lim_{s \to +\infty} \sup_{|y| \le C} |\ w(y,s) - \kappa| = 0,$$

with $\kappa = (p-1)^{-\frac{1}{p-1}}$, which gives if stated for u:

$$\lim_{t \to T} \sup_{|y| \le C} |\ (T-t)^{1/(p-1)} u(a + y\sqrt{T-t}, t) - \kappa\ | = 0.$$

This result was specified by Filippas and Kohn [6] who established that in N dimension, if w doesn't approach κ exponentially fast, then for each

$C > 0$

$$\sup_{|y|\le C} \mid w(y,s) - [\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)]| = o(1/s),$$

which gives if stated for u:

$$\sup_{|y|\le C} \mid (T-t)^{\frac{1}{p-1}} u(a + y\sqrt{T-t}, t) - [\kappa + \frac{\kappa}{2p|\log(T-t)|}(N - \frac{1}{2}|y|^2)] \mid \quad (5)$$

$$= o((-\log(T-t))^{-1}).$$

Velazquez obtained in [16] a related result, using maximum principle.

Relaying on a numerical study, Berger and Kohn [2] conjectured that in the case of a non exponential decay, the solution u of (1) would approach an explicit universal profile $f(z)$ depending only on p and independent from initial data as follows:

$$(T-t)^{\frac{1}{p-1}} u(a + \sqrt{(T-t)|\log(T-t)|}z, t) = f(z) + O((-\log(T-t))^{-1}) \quad (6)$$

in L^∞_{loc}, with

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}. \quad (7)$$

This behavior shows that in the case of one isolated blow-up point, there would be a free-boundary moving in (x,t) coordinates at the rate

$$\sqrt{(T-t)|\log(T-t)|}.$$

This free-boundary roughly separates the space into two regions:

1) the singular one, at the interior of the free-boundary, where Δu can be neglected with respect to $|u|^{p-1}u$, so equation (1) behaves like an ordinary differential equation, and blows-up.

2) the regular one, after the free-boundary, where Δu and $|u|^{p-1}u$ are of the same order.

Herrero and Velazquez in [12] and [13] showed in the case of dimension one ($N = 1$) using maximum principle that u behaves in three manners, one of them is the one suggested by Berger and Kohn, and they proved that estimate (6) is true uniformly on z belonging to compact subsets of $\mathbb{R}$ (without estimating the error).

Going further in this direction, Bricmont and Kupiainen construct a solution for (1) satisfying (6) in a global sense. For that, they used on one hand ideas close to the renormalization theory, and on the other hand hard

analysis on equation (4).
In this paper, we shall give a more elementary proof of their result, based on a more geometrical approach and on techniques of a priori estimates:

Theorem 1 Existence of a blow-up solution with a free-boundary behavior of the type (6)
There exists $T_0 > 0$ such that for each $T \in (0, T_0]$, $\forall g \in H$ with $\|g\|_{L^\infty} \leq (\log T)^{-2}$, one can find $d_0 \in I\!R$ and $d_1 \in I\!R^N$ such that for each $a \in I\!R^N$, the equation (1) with initial data

$$u_0(x) = T^{-\frac{1}{p-1}}\Big\{f(z)(1 + \frac{d_0 + d_1 z}{p - 1 + \frac{(p-1)^2}{4p}|z|^2}) + g(z)\Big\},$$

$$z = (x-a)(|\log T|T)^{-\frac{1}{2}},$$

has a unique classical solution $u(x,t)$ on $I\!R^N \times [0,T)$ and
i) u has one and only one blow-up point: a
ii) a free-boundary analogous to (6) moves through u such that

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}}z, t) = f(z) \qquad (8)$$

uniformly in $z \in I\!R^N$, with

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}.$$

Remark: We took d_0 and d_1 respectively in the direction of $h_0(y) = 1$ and $h_1(y) = y$, the two first eigenfunctions of $\mathcal{L}$ (Cf section 2), but we could have chosen other directions $D_0(y)$ and $D_1(y)$ (see Theorem 2). We can notice that we have a result in $H = W^{1,p+1}(\mathrm{I\!R}^N) \cap L^\infty(\mathrm{I\!R}^N)$. We can also obtain blow-up results in $H^1(\mathrm{I\!R}^N) \cap L^\infty(\mathrm{I\!R}^N)$. If $p < 1 + \frac{4}{N}$, then $f(z) \in H^1$, and we use the same arguments to solve the problem in $H^1(\mathrm{I\!R}^N) \cap L^\infty(\mathrm{I\!R}^N)$. If $p \geq 1 + \frac{4}{N}$, the result in H^1 follows directly from the stability result (see Theorem 2 below).
Remark: Such behavior is suspected to be generic.

Remark 1.1
One can ask the following questions:
a) Why does the free-boundary move at such a speed?
b) Why is the profile precisely the function f?
As in various physical situations, we suspect that the asymptotic behavior

of $w \to \kappa$ is described by self-similar solutions of equation (4).
Since we are dealing with equation of the heat type (Cf (4)), the natural scaling is $\frac{y}{\sqrt{s}}$. Let us hence try to find a solution of the form $v(\frac{y}{\sqrt{s}})$, with

$$v(0) = \kappa, \lim_{|z|\to\infty} |v(z)| = 0. \tag{9}$$

A direct computation shows that v must satisfy the following equation, for each $s > 0$ and each $z \in \mathbb{R}^N$:

$$-\frac{1}{2s} z.\nabla v(z) = \frac{1}{s}\Delta v(z) - \frac{1}{2} z.\nabla v(z) - \frac{1}{p-1} v(z) + |v(z)|^{p-1} v(z) \tag{10}$$

According to Giga and Kohn [10], the only solutions of (10) are the constant ones: $0, \kappa, -\kappa$, which are ruled out by (9). We can then try to search formally regular solutions of (4) of the form

$$V(y,s) = \sum_{j=0}^{\infty} \frac{1}{s^j} v_j(\frac{y}{\sqrt{s}})$$

and compare elements of order $\frac{1}{s^j}$ (in one dimension, in the positive case for simplicity). We obtain for $j = 0$:

$$0 = -\frac{1}{2} z v_0'(z) - \frac{1}{p-1} v_0(z) + v_0(z)^p,$$

and for $j = 1$ ($z \neq 0$)

$$v_1'(z) + a(z) v_1(z) = b(z)$$

with $a(z) = \frac{2}{z}(\frac{1}{p-1} - p v_0(z)^{p-1})$ and $b(z) = v_0'(z) + \frac{2}{z} v_0''(z)$. The solution for v_0 is given by

$$v_0(z) = (p - 1 + c_0 z^2)^{-\frac{1}{p-1}}$$

for an integration constant $c_0 > 0$. Using this to solve the equation on v_1 yields

$$v_1(z) = v_0(z)^p z^2 [c_1 + \int_1^z \zeta^{-2} v_0(\zeta)^{-p} b(\zeta) d\zeta],$$

for another integration constant c_1. Since we want V to be regular, it is natural to require that v_1 is analytic at $z = 0$. v_1 is regular if and only if the coefficient of ζ in the Taylor expansion of $v_0(\zeta)^{-p} b(\zeta)$ near $\zeta = 0$ is zero which turns to be equivalent to $c_0 = \frac{(p-1)^2}{4p}$ after simple calculation.

Therefore, $v_0(z) = (p-1+\frac{(p-1)^2}{4p}z^2)^{-\frac{1}{p-1}}$. Hence, the first term in the expansion of V is precisely the profile function f. Carrying on calculus yields:

$$v_1(z) = \frac{p-1}{2p}f(z)^p + \frac{(p-1)^2}{4p}z^2 f(z)^p \log f(z) + c_1 z^2 f(z)^p. \qquad (11)$$

We note that $v_1(0) = \frac{\kappa}{2p}$.
Unfortunately, we are not able to calculate every v_j. In conclusion, we take an other approach to obtain approximate self-similar solutions (see the proof of Theorem 1).

As in the paper of Bricmont and Kupiainen [3], we won't use maximum principle in the proof. The technique used here will allow us using geometrical interpretation of quantities of the type of d_0 and d_1 to derive stability results concerning this type of behavior for the free-boundary, with respect to perturbations of initial data and the equation.

Theorem 2 Stability with respect to initial data of the free boundary behavior
Let $\hat{u}_0$ be initial data constructed in Theorem 1. Let $\hat{u}(t)$ be the solution of equation (1) with initial data $\hat{u}_0$, $\hat{T}$ its blow-up time and $\hat{a}$ its blow-up point. Then there exists a neighborhood $\mathcal{V}_\prime$ of $\hat{u}_0$ in H which has the following property:
For each u_0 in $\mathcal{V}_\prime$, $u(t)$ blows-up in finite time $T = T(u_0)$ at only one blow-up point $a = a(u_0)$, where $u(t)$ is the solution of equation (1) with initial data u_0. Moreover, $u(t)$ behaves near $T(u_0)$ and $a(u_0)$ in an analogous way as $\hat{u}(t)$:

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(a+((T-t)|\log(T-t)|)^{\frac{1}{2}}z,t) = f(z)$$

uniformly in $z \in \mathbb{R}^N$.

Remark: Theorem 2 yields the fact that the blow-up profile $f(z)$ is stable with respect to perturbations in initial data.
Remark: From [15], we have $T(u_0) \to \hat{T}$, $a(u_0) \to \hat{a}$, as $u_0 \to \hat{u}_0$ in H.
Remark: For this theorem, we strongly use a finite dimension reduction of the problem in $\mathbb{R}^{1+N}$, which is the space of liberty degrees of the stability Theorem: (T, a).

Remark 1.2

Theorem 2 is true for a more general $\hat{u}_0$: It is enough that $\hat{u}(t)$ satisfies the key estimate of the proof of Theorem 1.
Remark: Since we do not use the maximum principle, we suspect that such analysis can be carried on for other type of equations, for example:

$$u_t = -\Delta^2 u + |u|^2 u,$$

and

$$u_t = \Delta u + |u|^{p-1}u + i|u|^{r-1}u, \tag{12}$$

where $1 < r < p$ ($p < \frac{N+2}{N-2}$ if $N \geq 3$).
See also for other applications [18].

According to a result of Merle [15], we obtain the following corollary for Theorem 2:

Corollary 1.1 *Let D be a convex set in $I\!R^N$, or $D = I\!R^N$. For arbitrary given set of k points x_1,..., x_k in D, there exist initial data u_0 such that the solution u of (1) with initial data u_0 (with Dirichlet boundary conditions in the case $D \neq I\!R^N$) blows-up exactly at x_1,..., x_k.*

Remark: The local behavior at each blow-up point x_i ($|x - x_i| \leq \rho_i$) is also given by (8).

2 Formulation of the problem

We omit the (T, a) or (d_0, d_1) dependence in what follows to simplify the notation.

2.1 Choice of variables

As indicated before, we use *similarity variables*:

$$y = \frac{x-a}{\sqrt{T-t}},$$

$$s = -\log(T-t),$$

$$w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t).$$

We want to prove for suitable initial data that:

$$\lim_{t \to T} \|(T-t)^{\frac{1}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z, t) - f(z)\|_{L^\infty} = 0,$$

or stated in terms of w:

$$\lim_{s\to\infty} \|w(y,s) - f(\frac{y}{\sqrt{s}})\|_{L^\infty} = 0,$$

where

$$f(z) = (p-1+\frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}.$$

We will not study as usually done, this limit difference as $s \to +\infty$

$$w(.,s) - f(\frac{\cdot}{\sqrt{s}}),$$

but we introduce instead:

$$q(y,s) = w(y,s) - [\frac{N\kappa}{2ps} + (p-1+\frac{(p-1)^2}{4ps}y^2)^{-\frac{1}{p-1}}]. \tag{13}$$

The added term in (13) can be understood from Remark 1.1. There, we tried to obtain for w an expansion of the form $\sum_{j=0}^{+\infty} \frac{1}{s^j} v_j(\frac{y}{\sqrt{s}})$. We got $v_0 = f$ and for v_1 the expression (11). Hence, it is natural to study the difference $w(y,s) - (v_0(\frac{y}{\sqrt{s}}) + \frac{1}{s} v_1(\frac{y}{\sqrt{s}}))$. Since the expression of v_1 is a bit complicated (see (11)), we study instead $w(y,s) - (v_0(\frac{y}{\sqrt{s}}) + \frac{1}{s} v_1(0))$, which is (13) for $N = 1$.
Now, if we introduce

$$\varphi(y,s) = \frac{N\kappa}{2ps} + f(\frac{y}{\sqrt{s}}) = \frac{N\kappa}{2ps} + (p-1+\frac{(p-1)^2}{4ps}|y|^2)^{-\frac{1}{p-1}}, \tag{14}$$

we have

$$q(y,s) = w(y,s) - \varphi(y,s).$$

Thus, the problem in Theorem 1 is to construct a function q satisfying

$$\lim_{s\to+\infty} \|q(.,s)\|_{L^\infty} = 0.$$

From (4) and (13), the equation satisfied by q is the following:
for $s > 0$,

$$\frac{\partial q}{\partial s}(y,s) = \mathcal{L}_V(q)(y,s) + B(q(y,s)) + R(y,s), \tag{15}$$

where

- the linear term is

$$\mathcal{L}_V(q) = \mathcal{L}(q) + V(y,s)q \tag{16}$$

with

$$\mathcal{L}(q) = \Delta q - \tfrac{1}{2}y.\nabla q + q \text{ and } V(y,s) = p(\varphi^{p-1} - \tfrac{1}{p-1}),$$

- the nonlinear term (quadratic in q for p large) is

$$B(q) = |\varphi + q|^{p-1}(\varphi + q) - \varphi^p - p\varphi^{p-1}q, \tag{17}$$

- and the rest term involving φ is

$$R(y,s) = \Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{1}{p-1}\varphi + \varphi^p - \frac{\partial\varphi}{\partial s}. \tag{18}$$

It will be useful to write equation (15) in its integral form: for each $s_0 > 0$, for each $s_1 \geq s_0$,

$$q(s_1) = K(s_1,s_0)q(s_0) + \int_{s_0}^{s_1} d\tau K(s_1,\tau)B(q(\tau)) + \int_{s_0}^{s_1} d\tau K(s_1,\tau)R(\tau), \tag{19}$$

where K is the fundamental solution of the linear operator $\mathcal{L}_V$ defined for each $s_0 > 0$ and for each $s_1 \geq s_0$ by,

$$\partial_{s_1} K(s_1,s_0) = \mathcal{L}_V K(s_1,s_0) \tag{20}$$

$$K(s_0,s_0) = Identity.$$

2.2 Decomposition of q

Since $\mathcal{L}_V$ will play an important role in our analysis, let us point some facts on it.

i) The operator $\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2(\mathbb{R}^N, d\mu)$ with

$$d\mu(y) = \frac{e^{-\frac{|y|^2}{4}}dy}{(4\pi)^{N/2}}. \tag{21}$$

Note here that there is a weight decaying at infinity. The spectrum of $\mathcal{L}$ is explicit. More precisely,

$$spec(\mathcal{L}) = \{1 - \frac{m}{2} | m \in \mathbb{N}\},$$

and it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are derived from Hermite polynomials:

- $N=1$:
All the eigenvalues of $\mathcal{L}$ are simple. For $1-\frac{m}{2}$ corresponds the eigenfunction
$$h_m(y)=\sum_{n=0}^{[\frac{m}{2}]}\frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}. \quad (22)$$
h_m satisfies
$$\int h_n h_m d\mu = 2^n n! \delta_{nm}.$$
(We will note also $k_m = h_m/\|h_m\|^2_{L^2_\mu}$.)

- $N\geq 2$:
We write the spectrum of $\mathcal{L}$ as
$$spec(\mathcal{L})=\{1-\frac{m_1+...+m_N}{2}|m_1,...,m_N\in\mathbb{N}\}.$$
For $(m_1,...,m_N)\in\mathbb{N}$, the eigenfunction corresponding to $1-\frac{m_1+...+m_N}{2}$ is
$$y\longrightarrow h_{m_1}(y_1)...h_{m_N}(y_N),$$
where h_m is defined in (22). In particular,
*1 is an eigenvalue of multiplicity 1, and the corresponding eigenfunction is
$$H_0(y)=1, \quad (23)$$
*$\frac{1}{2}$ is of multiplicity N, and its eigenspace is generated by the orthogonal basis $\{H_{1,i}(y)|i=1,...,N\}$, with $H_{1,i}(y)=h_1(y_i)$; we note
$$H_1(y)=(H_{1,1}(y),...,H_{1,N}(y)), \quad (24)$$
*0 is of multiplicity $\frac{N(N+1)}{2}$, and its eigenspace is generated by the orthogonal basis $\{H_{2,ij}(y)|i,j=1,...,N,i\leq j\}$, with $H_{2,ii}(y)=h_2(y_i)$, and for $i<j$, $H_{2,ij}(y)=h_1(y_i)h_1(y_j)$; we note
$$H_2(y)=(H_{2,ij}(y),i\leq j). \quad (25)$$

ii) The potential $V(y,s)$ has two fundamental properties that will influence strongly our analysis.

a) We have $V(.,s)\to 0$ in the $L^2(\mathbb{R},d\mu)$ when $s\to+\infty$. In particular, the effect of V on the bounded sets or in the "blow-up" region

($|x| \le C\sqrt{s}$) inside the free boundary will be a "perturbation" of the effect of $\mathcal{L}$.

b) Outside the free boundary, we have the following property: $\forall \epsilon > 0$, $\exists C_\epsilon > 0$, $\exists s_\epsilon$ such that

$$\sup_{s \ge s_\epsilon, \frac{|y|}{\sqrt{s}} \ge C_\epsilon} |V(y,s) - (-\frac{p}{p-1})| \le \epsilon$$

with $-\frac{p}{p-1} < -1$.
Since 1 is the biggest eigenvalue of $\mathcal{L}$, we can consider that outside the free boundary, the operator $\mathcal{L}_V$ will behave as one with fully negative spectrum, which simplifies greatly the analysis in this region.

Since the behavior of V inside and outside the free boundary is different, let us decompose q as the following:
Let $\chi_0 \in C_0^\infty([0,+\infty))$, with $\mathrm{supp}(\chi_0) \subset [0,2]$ and $\chi_0 \equiv 1$ on $[0,1]$. We define then

$$\chi(y,s) = \chi_0(\frac{|\ y\ |}{K_0 s^{\frac{1}{2}}}), \tag{26}$$

where $K_0 > 0$ is chosen large enough so that various technical estimates hold.
We write $q = q_b + q_e$ where

$$q_b = q\chi \text{ and } q_e = q(1-\chi).$$

Let us remark that

$$\mathrm{supp}\ q_b(s) \subset B(0, 2K_0\sqrt{s}) \text{ and } \mathrm{supp}\ q_e(s) \subset \mathbb{R}\backslash B(0, K_0\sqrt{s}).$$

Then we study q_b using the structure of $\mathcal{L}$. Since $\mathcal{L}$ has $1+N$ expanding directions (corresponding to eigenvalues 1 and $\frac{1}{2}$) and $\frac{N(N+1)}{2}$ neutral ones, we write q_b with respect to the eigenspaces of $\mathcal{L}$ as follows:

$$q_b(y,s) = \sum_{m=0}^{2} q_m(s).H_m(y) + q_-(y,s) \tag{27}$$

where
$q_0(s)$ is the projection of q_b on H_0,
$q_{1,i}(s)$ is the projection of q_b on $H_{1,i}$, $q_1(s) = (q_{1,i}(s), ..., q_{1,N}(s))$, $H_1(y)$ is given by (24),
$q_{2,ij}(s)$ is the projection of q_b on $H_{2,ij}$, $i \le j$, $q_2(s) = (q_{2,ij}(s), i \le j)$, $H_2(y)$ is given by (25),
$q_-(y,s) = P_-(q_b)$ and P_- the projector on the negative subspace of $\mathcal{L}$.

In conclusion, we write q into 5 "components" as follows:

$$q(y,s)=\sum_{m=0}^{2} q_m(s).H_m(y)+q_-(y,s)+q_e(y,s). \tag{28}$$

(Note here that q_m are coordinates of q_b and not of q).
In particular, if $N=1$ and $m=0,1,2$, $q_m(s)$ and $H_m(y)$ are scalar functions, and $H_m(y)=h_m(y)$. We write in this case:

$$q(y,s)=\sum_{m=0}^{2} q_m(s)h_m(y)+q_-(y,s)+q_e(y,s). \tag{29}$$

Let us now prove Theorem 1.

3 Existence of a blow-up solution with the given free-boundary profile

This section is devoted to the proof of Theorem 1.

3.1 Transformation of the problem

As in [3], we give the proof in one dimension (same proof holds in higher dimension). We also assume a to be zero, without loss of generality.
Let us consider initial data:

$$u_{0,d_0,d_1}(x)=T^{-\frac{1}{p-1}}\Big\{f(z)(1+\frac{d_0+d_1z}{p-1+\frac{(p-1)^2}{4p}z^2})+g(z)\Big\},$$

where

$$z=x(|\log T|T)^{-\frac{1}{2}}.$$

We want to prove first that there exists $T_0>0$ such that for each $T\in(0,T_0]$, for every $g\in H$ with $\|g\|_{L^\infty}\le(\log T)^{-2}$, we can find $(d_0,d_1)\in\mathbb{R}^2$ such that

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u_{d_0,d_1}(((T-t)|\log(T-t)|)^{\frac{1}{2}}z,t)=f(z) \tag{30}$$

uniformly in $z\in\mathbb{R}$, where u_{d_0,d_1} is the solution of (1) with initial data u_{0,d_0,d_1}, and

$$f(z)=(p-1+\frac{(p-1)^2}{4p}z^2)^{-\frac{1}{p-1}}. \tag{31}$$

This property will imply that u_{d_0,d_1} blows-up at time T at one single point: $x=0$. Indeed,

Proposition 3.1 Single blow-up point properties of solutions
Let $u(t)$ be a solution of equation (1). If u satisfies the following property

$$\lim_{t\to T}\|(T-t)^{\frac{1}{p-1}}u(\sqrt{(T-t)|\log(T-t)|}z,t)-f(z)\|_{L^\infty}=0 \tag{32}$$

then $u(t)$ blows-up at time T at one single point: $x=0$.

Proof: For each $b\in\mathbb{R}$, we have from (32)

$$\lim_{t\to T}\{(T-t)^{\frac{1}{p-1}}u(b,t)-f(\frac{b}{\sqrt{(T-t)|\log(T-t)|}})\}=0.$$

Using (31), we obtain $\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(0,t)=\kappa$ and for $b\neq 0$, $\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(b,t)=0$. A result by Giga and Kohn in [8] shows that b is a blow-up point if and only if $\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(b,t)=\pm\kappa$. This concludes the proof of proposition 3.1.

Therefore, it remains to find $(d_0,d_1)\in\mathbb{R}^2$ so that (30) holds to conclude the proof of Theorem 1.
If we use the formulation of the problem in section 2, the problem reduces to find $S_0>0$ such that for each $s_0\geq S_0$, $g\in H$ with $\|g\|_{L^\infty}\leq\frac{1}{s_0^2}$, we can find $(d_0,d_1)\in\mathbb{R}^2$ so that the equation (15)

$$\frac{\partial q}{\partial s}(y,s)=\mathcal{L}_V(q)(y,s)+B(q(y,s))+R(y,s),$$

with initial data at $s=s_0$

$$q_{d_0,d_1}(y,s_0)=(p-1+\frac{(p-1)^2}{4ps_0}y^2)^{-\frac{p}{p-1}}(d_0+d_1y/\sqrt{s_0})-\frac{\kappa}{2ps_0}+g(y/\sqrt{s_0}), \tag{33}$$

has a solution $q(d_0,d_1)$ satisfying

$$\lim_{s\to\infty}\sup_{y\in\mathbb{R}}|q_{d_0,d_1}(y,s)|=0. \tag{34}$$

q will always depend on g, d_0 and d_1, but we will omit theses dependences in the notations (except when it is necessary).
The convergence of q to zero in $L^\infty(\mathbb{R})$ follows directly if we construct $q(s)$ solution of equation (15) satisfying a geometrical property, that is q belongs to a set $V_A\subset C([s_0,+\infty),L^2(\mathbb{R},d\mu))$, such that V_A shrinks to $q\equiv 0$ when $s\to\infty$.
More precisely we have the following definitions:

Definition 3.1 *For each $A > 0$, for each $s > 0$, we define $V_A(s)$ as being the set of all functions r in $L^2(I\!R, d\mu)$ such that*

$$\begin{aligned}|r_m(s)| &\leq As^{-2}, m = 0,1,\\ |r_2(s)| &\leq A^2(\log s)s^{-2},\\ |r_-(y,s)| &\leq A(1+|y|^3)s^{-2},\\ \|r_e(s)\|_{L^\infty} &\leq A^2 s^{-\frac{1}{2}},\end{aligned}$$

where $r(y) = \sum_{m=0}^{2} r_m(s)h_m(y) + r_-(y,s) + r_e(y,s)$ (Cf decomposition (29)).

Definition 3.2 *For each $A > 0$, we define V_A as being the set of all functions q in $C([s_0,+\infty), L^2(I\!R, d\mu))$ satisfying $q(s) \in V_A(s)$ for each $s \geq s_0$.*

Indeed, assume that $\forall s \geq s_0$ $q(s) \in V_A(s)$. Let us show that $\forall s \geq s_0$ $\sup_{y\in\mathbb{R}} |q(y,s)| \leq \frac{C(A)}{\sqrt{s}}$, which implies (34).
We have from the definitions of q_b and q_e

$$\begin{aligned}q(y,s) &= q_b(y,s) + q_e(y,s)\\ &= q_b(y,s).1_{\{|y|\leq 2K_0\sqrt{s}\}} + q_e(y,s)\\ &= \Big(\sum_{m=0}^{2} q_m(s)h_m(y) + q_-(y,s)\Big).1_{\{|y|\leq 2K_0\sqrt{s}\}}(y,s) + q_e(y,s).\end{aligned}$$

Using the definitions of h_m (Cf (22)) and V_A, the conclusion follows.

3.2 Proof of Theorem 1

Using these geometrical aspects, what we have to do is finally to find $A > 0$ and $S_0 > 0$ such that for each $s_0 \geq S_0$, $g \in H$ with $\|g\|_\infty \leq \frac{1}{s_0^2}$, we can find $(d_0, d_1) \in \mathbb{R}^2$ so that $\forall s \geq s_0$,

$$q_{d_0,d_1}(s) \in V_A(s). \tag{35}$$

Let us explain briefly the general ideas of the proof.

-In a first part, we will reduce the problem of controlling all the components of q in V_A to a problem of controlling $(q_0, q_1)(s)$. That is, we reduce an infinite dimensional problem to a finite dimensional one.

-In a second part, we solve the finite dimensional problem, that is to find $(d_0, d_1) \in \mathbb{R}^2$ such that $(q_0, q_1)(s)$ satisfies certain conditions. We will proceed by contradiction and use dynamics in dimension 2 of $(q_0, q_1)(s)$ to reach a topological obstruction (using Index Theory).

The constant C now denotes a universal one independent of variables, only depending upon constants of the problem such as p.

Part I: Reduction to a finite dimensional problem

In this section, we show that finding $(d_0, d_1) \in \mathbb{R}^2$ such that $\forall s \geq s_0$ $q(s) \in V_A(s)$ is equivalent to finding $(d_0, d_1) \in \mathbb{R}^2$ such that $|q_m(s)| \leq \frac{A}{s^2}$ $\forall s \geq s_0$, $\forall m \in \{0, 1\}$. For this purpose, we give the following definition:

Definition 3.3 *For each $A > 0$, for each $s > 0$ we define $\hat{V}_A(s)$ as being the set $[-\frac{A}{s^2}, \frac{A}{s^2}]^2 \subset \mathbb{R}^2$.*
For each $A > 0$, we define $\hat{V}_A$ as being the set of all (q_0, q_1) in $C([s_0, +\infty), \mathbb{R}^2)$ satisfying $(q_0, q_1)(s) \in \hat{V}_A(s)$ $\forall s \geq s_0$.

Step 1: Reduction for initial data

Let us show that for a given A (to be chosen later), for $s_0 \geq s_1(A)$, the control of $q(s_0)$ in $V_A(s_0)$ is equivalent to the control of $(q_0, q_1)(s_0)$ in $\hat{V}_A(s_0)$.

Lemma 3.1 *i) For each $A > 0$, there exists $s_1(A) > 0$ such that for each $s_0 \geq s_1(A)$, $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \hat{V}_A(s_0)$, then*

$$\begin{aligned} |q_2(s_0)| &\leq (\log s_0){s_0}^{-2}, \\ |q_-(y, s_0)| &\leq C(1 + |y|^3){s_0}^{-2}, \\ \|q_e(., s_0)\|_{L^\infty} &\leq {s_0}^{-\frac{1}{2}} \end{aligned}$$

ii) There exists $A_1 > 0$ such that for each $A \geq A_1$, there exists $s_1(A) > 0$ such that for each $s_0 \geq s_1(A)$, $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, we have the following equivalence:

$$q(s_0) \in V_A(s_0) \text{ if and only if } (q_0, q_1)(s_0) \in \hat{V}_A(s_0).$$

Proof:
We first note that part *ii*) of the lemma follows immediately from part *i*) and definition 3.1. We prove then only part *i*).
Let $A > 0$, $s_0 > 0$ and $g \in H$ such that $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$. Let $(d_0, d_1) \in \mathbb{R}^2$.
We write initial data (Cf (33)) as

$$q(y, s_0) = q^0(y, s_0) + q^1(y, s_0) + q^2(y, s_0) + q^3(y, s_0)$$

where $q^0(y,s_0) = d_0F(\frac{y}{\sqrt{s_0}})$, $q^1(y,s_0) = d_1\frac{y}{\sqrt{s_0}}F(\frac{y}{\sqrt{s_0}})$, $q^2(y,s_0) = -\frac{\kappa}{2ps_0}$, $q^3(y,s_0) = g(\frac{y}{\sqrt{s_0}})$ and $F(\frac{y}{\sqrt{s_0}}) = (p-1+\frac{(p-1)^2}{4ps_0}y^2)^{-\frac{p}{p-1}}$.

We decompose all the q^i as suggested by (29).

-From $\|g\|_{L^\infty} \le \frac{1}{s_0^2}$ we derive that $|q_0^3(s_0)|+|q_1^3(s_0)|+|q_2^3(s_0)|+\|q_e^3(s_0)\|_{L^\infty} \le \frac{C}{s_0^2}$, and then, $|q_-^3(y,s_0)| \le \frac{C}{s_0^2}(1+|y|^3)$.

-Using simple calculations we obtain $|q_0^2(s_0)| \le \frac{C}{s_0}$,
$q_1^2(s_0) = 0$, $|q_2^2(s_0)| \le Ce^{-s_0}$, $|q_-^2(y,s_0)| \le Cs_0^{-2}(1+|y|^3)$ and $\|q_e^2(s_0)\|_{L^\infty} \le Cs_0^{-1}$.

-For q^0, we have $q_0^0(s_0) = d_0\int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}}) \sim d_0C(p)$ $(s_0 \to \infty)$,
$q_1^0(s_0) = 0$, $q_2^0(s_0) = d_0\int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}})\frac{z^2-2}{8} \sim d_0\frac{C'(p)}{s_0}$ $(s_0 \to \infty)$,
$|q_-^0(y,s_0)| \le d_0\frac{C}{s_0}(1+|y|^3)$ and $\|q_e^0(s_0)\|_{L^\infty} \le Cd_0$.

All theses last bounds are simple to obtain, perhaps except that for q_-^0. Indeed, we write $q_-^0(y,s_0) =$
$d_0\chi_{s_0}F(\frac{y}{\sqrt{s_0}}) - d_0\int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}}) - d_0\int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}})\frac{z^2-2}{8}(y^2-2)$. The last term can be bounded by $\frac{Cd_0}{s_0}(1+|y|^3)$. We write the first term as $d_0\Big\{\chi_{s_0}(y)F(\frac{y}{\sqrt{s_0}}) - \chi_{s_0}(0)F(0) - \int d\mu(z)(\chi_{s_0}F(\frac{z}{\sqrt{s_0}}) - \chi_{s_0}(0)F(0))\Big\}$. Using a Lipschitz property, we have $|\chi_{s_0}(y)F(\frac{y}{\sqrt{s_0}}) - \chi_{s_0}(0)F(0)| \le \frac{Cy^2}{s_0}$, and the conclusion follows.

-Similarly, we obtain for q^1, $q_0^1(s_0) = 0$, $q_1^1(s_0) = \frac{d_1}{\sqrt{s_0}}\int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}})\frac{z}{2}z \sim d_1\frac{C''(p)}{\sqrt{s_0}}$ $(s_0 \to \infty)$, $q_2^1(s_0) = 0$, $|q_-^1(y,s_0)| \le d_1\frac{C}{s_0^{3/2}}(1+|y|^3)$ and $\|q_e^1(s_0)\|_{L^\infty} \le C\frac{d_1}{\sqrt{s_0}}$.

Hence, by linearity, we write

$$\begin{aligned} q_0(s_0) &= d_0a_0(s_0) + b_0(g,s_0) \\ q_1(s_0) &= d_1a_1(s_0) + b_1(g,s_0) \end{aligned} \tag{36}$$

with $a_0(s_0) \sim C(p)$, $a_1(s_0) \sim \frac{C''(p)}{\sqrt{s_0}}$, $|b_0(g,s_0)| \le \frac{C}{s_0}$ and $|b_1(g,s_0)| \le \frac{C}{s_0^2}$. Therefore, we see that if (d_0,d_1) is chosen such that $(q_0,q_1)(s_0) \in \hat{V}_A(s_0)$ and if $s_0 \ge s_1(A)$, we obtain $|d_m| \le \frac{C}{s_0}$ for $m \in \{0,1\}$. Using linearity and the above estimates, we obtain $|q_2(s_0)| \le \frac{C}{s_0^2}$, $|q_-(y,s_0)| \le \frac{C}{s_0^2}(1+|y|^3)$ and $\|q_e(s_0)\| \le \frac{C}{s_0}$. Taking $s_1(A)$ larger we conclude the proof of lemma 3.1.

Step 2: A priori estimates

This step is the crucial one in the proof of Theorem 1. Here, we will show through a priori estimates that for $s \geq s_0$, the control of q in $V_A(s)$ reduces to the control of (q_0, q_1) in $\hat{V}_A(s)$. Indeed, this result will imply that if for $s_* \geq s_0$, $q(s_*) \in \partial V_A(s_*)$, then $(q_0(s_*), q_1(s_*)) \in \partial \hat{V}_A(s_*)$. (Compare with definition 3.1).

Remark 3.1
We shall note here that for each initial data $q(s_0)$, equation (15) has a unique solution on $[s_0, S]$ with either $S = +\infty$ or $S < +\infty$ and $\|q(s)\|_{L^\infty} \to +\infty$, when $s \to S$. Therefore, in the case where $S < +\infty$, there exists $s_* > s_0$ such that $q(s_*) \notin V_A(s_*)$ and the solution is in particular defined up to s_*.

Proposition 3.2 (Control of q by (q_0, q_1) in V_A) *There exists $A_2 > 0$ such that for each $A \geq A_2$, there exists $s_2(A) > 0$ such that for each $s_0 \geq s_2(A)$, for each $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, we have the following property:*
-if (d_0, d_1) is chosen so that $(q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0)$, and,
-if for $s_1 \geq s_0$, we have $\forall s \in [s_0, s_1]$, $q(s) \in V_A(s)$,
then $\forall s \in [s_0, s_1]$,

$$
\begin{aligned}
|q_2(s)| &\leq A^2 s^{-2} \log s - s^{-3} \\
|q_-(y,s)| &\leq \frac{A}{2}(1 + |y|^3) s^{-2} \\
\|q_e(s)\|_{L^\infty} &\leq \frac{A^2}{2\sqrt{s}}.
\end{aligned}
$$

Proof: see Proof of Proposition 3.2 below.

Step 3: Transversality
Using now the fact that (q_0, q_1) controls the evolution of q in V_A, we show a transversality condition of (q_0, q_1) on $\partial \hat{V}_A(s_*)$.

Lemma 3.2 *There exists $A_3 > 0$ such that for each $A \geq A_3$, there exists $s_3(A)$ such that for each $s_0 \geq s_3(A)$, we have the following properties:*
i) Assume there exists $s_* \geq s_0$ such that $q(s_*) \in V_A(s_*)$ and $(q_0, q_1)(s_*) \in \partial \hat{V}_A(s_*)$, then there exists $\delta_0 > 0$ such that $\forall \delta \in (0, \delta_0)$, $(q_0, q_1)(s_* + \delta) \notin \hat{V}_A(s_* + \delta)$.
ii) If $q(s_0) \in V_A(s_0)$, $q(s) \in V_A(s)$ $\forall s \in [s_0, s_*]$ and $q(s_*) \in \partial V_A(s_*)$ then there exists $\delta_0 > 0$ such that $\forall \delta \in (0, \delta_0)$, $q(s_* + \delta) \notin V_A(s_* + \delta)$.

Proof:
Part *ii)* follows from Step 2 and part *i)*.

To prove part $i)$, we will show that for each $m \in \{0,1\}$, for each $\epsilon \in \{-1,1\}$, if $q_m(s_*) = \epsilon \frac{A}{s_*^2}$, then $\frac{dq_m}{ds}(s_*)$ has the opposite sign of $\frac{d}{ds}(\frac{\epsilon A}{s^2})(s_*)$ so that (q_0,q_1) actually leaves $\hat{V}_A$ at s_* for $s_* \geq s_0$ where s_0 will be large. Now, let us compute $\frac{dq_0}{ds}(s_*)$ and $\frac{dq_1}{ds}(s_*)$ for $q(s_*) \in V_A(s_*)$ and $(q_0(s_*), q_1(s_*)) \in \partial\hat{V}_A(s_*)$. First, we note that in this case, $\|q(s_*)\|_{L^\infty} \leq \frac{CA^2}{\sqrt{s_*}}$ and $|q_b(y,s_*)| \leq CA^2 \frac{\log s_*}{s_*^2}(1+|y|^3)$ (Provided $A \geq 1$). Below, the classical notation $O(l)$ stands for a quantity whose absolute value is bounded precisely by l and not Cl.

For $m \in \{0,1\}$, we derive from equation (15) and (22): $\int d\mu \chi(s_*) \frac{\partial q}{\partial s} k_m =$

$$\int d\mu \chi(s_*) \mathcal{L} q k_m + \int d\mu \chi(s_*) V q k_m + \int d\mu \chi(s_*) B(q) k_m + \int d\mu \chi(s_*) R(s_*) k_m.$$

We now estimate each term of this identity:

a) $|\int d\mu \chi(s_*) \frac{\partial q}{\partial s} k_m - \frac{dq_m}{ds}| = |\int d\mu \frac{d\chi}{ds} q k_m| \leq |\int d\mu \frac{d\chi}{ds} q k_m| \leq \int d\mu |\frac{d\chi}{ds}| \frac{CA^2}{\sqrt{s_*}} |k_m|$ $\leq Ce^{-s_*}$ if $s_0 \geq s_3(A)$.

b) Since $\mathcal{L}$ is self-adjoint on $L^2(\mathbb{R}, d\mu)$, we write

$$\int d\mu \chi(s_*) \mathcal{L} q k_m = \int d\mu \mathcal{L}(\chi(s_*) k_m) q.$$

Using $\mathcal{L}(\chi(s_*)k_m) = (1-\frac{m}{2})\chi(s_*)k_m + \frac{\partial^2 \chi}{\partial s^2} k_m + \frac{\partial \chi}{\partial y}(2\frac{\partial k_m}{\partial y} - \frac{y}{2}k_m)$, we obtain $\int d\mu \chi(s_*) \mathcal{L} q k_m = (1-\frac{m}{2}) q_m(s_*) + O(CAe^{-s_*})$.

c) We then have from (16): $\forall y, |V(y,s)| \leq \frac{C}{s}(1+|y|^2)$. Therefore,

$$|\int d\mu \chi(s_*) V q k_m| \leq \int d\mu \frac{C}{s_*}(1+|y|^5) \frac{CA^2 \log s_*}{s_*^2} |k_m| \leq \frac{CA^2 \log s_*}{s_*^3}$$

d) A standard Taylor expansion combined with the definition of V_A shows that $|\chi(y,s_*) B(q(y,s_*))| \leq C|q|^2 \leq C(|q_b|^2 + |q_e|^2) \leq \frac{CA^4 (\log s_*)^2}{s_*^4}(1+|y|^3)^2 + 1_{\{|y| \geq K\sqrt{s_*}\}}(y) \frac{A^2}{\sqrt{s_*}}$. Thus, $|\int d\mu \chi(s_*) B(q) k_m| \leq \frac{CA^4(\log s_*)^2}{s_*^4} + Ce^{-s_*}$.

e) A direct calculus yields $|\int d\mu \chi(s_*) R(s_*) k_m| \leq \frac{C(p)}{s_*^2}$ (Actually it is equal to 0 if $m=1$). Indeed, in the case $m=0$, we start from (18) and (14) and expand each term up to the second order when $s \to \infty$. Since $\varphi(y,s) = f(\frac{y}{\sqrt{s}}) + \frac{\kappa}{2ps}$, we derive:

1) $\int d\mu \chi(s)(-\frac{\varphi}{p-1}) = -\frac{1}{p-1}(\kappa - \frac{\kappa}{2ps} + \frac{\kappa}{2ps} + O(Cs^{-2})) = -\frac{\kappa}{p-1} + O(Cs^{-2})$,

2) $\int d\mu \chi(s) \varphi^p = \int d\mu f^p + \frac{\kappa}{2ps} \int d\mu p f^{p-1} + O(Cs^{-2}) = \frac{\kappa}{p-1} - \frac{\kappa}{2(p-1)s} + \frac{\kappa}{2ps}\frac{p}{p-1} + O(Cs^{-2}) = \frac{\kappa}{p-1} + O(Cs^{-2})$,

3) $\varphi_s(y,s) = \frac{p-1}{4ps^2}y^2f^p - \frac{\kappa}{2ps^2}$ and then $\int d\mu\chi(s)(-\varphi_s) = O(Cs^{-2})$,
4) $\varphi_y(y,s) = -\frac{p-1}{2ps}yf^p$ and then $\int d\mu\chi(s)(-\frac{1}{2}y\varphi_y) = \frac{\kappa}{2ps} + O(Cs^{-2})$,
5) $\varphi_{yy}(y,s) = -\frac{p-1}{2ps}f^p + \frac{(p-1)^2}{4ps^2}y^2f^{2p-1}$, then $\int d\mu\chi(s)\varphi_{yy} = -\frac{\kappa}{2ps} + O(Cs^{-2})$.
Adding all these expansions, we obtain $\int d\mu\chi_{s_*}R(s_*) = O(C(p)s_*^{-2})$. Concluding steps a) to e), we obtain

$$\frac{dq_m}{ds}(s_*) = (1-\frac{m}{2})\frac{\epsilon A}{s_*^2} + O(\frac{C(p)}{s_*^2}) + O(CA^4\frac{\log s_*}{s_*^3})$$

whenever $q_m(s_*) = \frac{\epsilon A}{s_*^2}$. Let us now fix $A \geq 2C(p)$, and then we take $s_3(A)$ larger so that for $s_0 \geq s_3(A)$, $\forall s \geq s_0$, $\frac{C(p)}{s^2} + O(CA^4\frac{\log s}{s^3}) \leq \frac{3C(p)}{2s^2}$. Hence, if $\epsilon = -1$, $\frac{dq_m}{ds}(s_*) < 0$, if $\epsilon = 1$, $\frac{dq_m}{ds}(s_*) > 0$. This concludes the proof of lemma 3.2.

Now, let us fix $A \geq \sup(A_2, A_3)$.

Part II: Topological argument
Now, we reduce the problem to studying a two-dimensional one. Let us study now this problem. We give its initialization in the following lemma:

Lemma 3.3 (Initialization of the finite dimensional problem) *There exists $s_4(A) > 0$ such that for each $s_0 \geq s_4(A)$, for each $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, there exists a set $\mathcal{D}_{g,s_0} \subset I\!R^2$ topologically equivalent to a square with the following property:*

$$q(d_0,d_1,s_0) \in V_A(s_0) \textit{ if and only if } (d_0,d_1) \in \mathcal{D}_{g,s_0}.$$

Proof:
As stated by lemma 3.1 (ii), if we take $s_0 > s_1(A)$ and $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, then it is enough to prove that there exists a set $\mathcal{D}_{g,s_0}$ topologically equivalent to a square satisfying

$$(q_0,q_1)(s_0) \in \hat{V}_A(s_0) \text{ if and only if } (d_0,d_1) \in \mathcal{D}_{g,s_0}.$$

If we refer to the calculus of $q_m(s_0)$ (Cf (36) and what follows), and take $s_4(A) \geq s_0(A)$ and $s_4(A)$ large enough, then this concludes the proof of lemma 3.3.

Now, we fix $S_0 > \sup(s_1(A), s_2(A), s_3(A), s_4(A))$ and take $s_0 \geq S_0$. Then we start the proof of Theorem 1 for A and $s_0(A)$ and a given $g \in H$ with

$\|g\|_{L^\infty} \le \frac{1}{s_0^2}$.
We argue by contradiction: According to lemma 3.3, for each $(d_0, d_1) \in \mathcal{D}_{g,s_0}$ $q(d_0, d_1, s_0) \in V_A(s_0)$. We suppose then that for each $(d_0, d_1) \in \mathcal{D}_{g,s_0}$, there exists $s > s_0$ such that $q(d_0, d_1, s) \notin V_A(s)$. Let $s_*(d_0, d_1)$ be the infimum of all these s. (Note here that $s_*(d_0, d_1)$ exists because of remark 3.1).

Applying proposition 3.2, we see that $q(d_0, d_1, s_*(d_0, d_1))$ can leave $V_A(s_*(d_0, d_1))$ only by its first two components, hence,

$$(q_0, q_1)(d_0, d_1, s_*(d_0, d_1)) \in \partial \hat{V}_A(s_*(d_0, d_1)).$$

Therefore, we can define the following function:

$$\begin{array}{rcl} \Phi_g : \mathcal{D}_{g,s_0} & \longrightarrow & \partial \mathcal{C} \\ (d_0, d_1) & \longrightarrow & \dfrac{s_*(d_0, d_1)^2}{A}(q_0, q_1)(d_0, d_1, s_*(d_0, d_1)) \end{array}$$

where $\mathcal{C}$ is the unit square of $\mathbb{R}^2$.
Now, we claim

Proposition 3.3 *i) Φ_g is a continuous mapping from $\mathcal{D}_{g,s_0}$ to $\partial\mathcal{C}$.*
ii) The restriction of Φ_g to $\partial\mathcal{D}_{g,s_0}$ is homeomorphic to identity.

From that, a contradiction follows (Index Theory). This means that there exists $(d_0(g), d_1(g))$ such that $\forall s \ge s_0$, $q(d_0, d_1, s) \in V_A(s)$, that is $q \in V_A$. In particular,

$$\|q(s)\|_{L^\infty} \le \frac{C(A)}{\sqrt{s}}.$$

Using Proposition 3.1, this concludes the proof of Theorem 1.

Proof of Proposition 3.3:
Step 1: i)
We have $(q_0, q_1)(s)$ is a continuous function of $(w(s_0), s) \in H \times [s_0, +\infty)$ where $w(s_0)$ is initial data for equation (4). Since $w(s_0)$ ($= q(y, s_0) + \varphi(y, s_0)$, Cf (33) and (14)) is continuous in (d_0, d_1) (it is linear), we have $(q_0, q_1)(s)$ is continuous with respect to (d_0, d_1, s). Now, using the transversality property of (q_0, q_1) on $\partial\hat{V}_A$ (lemma 3.2), we claim that $s_*(d_0, d_1)$ is continuous. Therefore, Φ_g is continuous.
Step 2: ii)
If $(d_0, d_1) \in \partial\mathcal{D}_{g,s_0}$, then, according to the proof of lemma 3.3, $(q_0, q_1)(s_0) \in \partial\hat{V}_A(s_0)$. Therefore, using $q(s_0) \in V_A(s_0)$ (lemma 3.1), we have $q(s_0) \in$

$\partial V_A(s_0)$. Applying *ii)* of lemma 3.2 with s_0 and $s_* = s_0$ yields $\delta_0 > 0$ such that $\forall \delta \in (0, \delta_0)$, $q(s_0+\delta) \notin V_A(s_0+\delta)$. Hence,

$$s_*(d_0, d_1) = s_0,$$

and $\Phi_g(d_0, d_1) = \frac{s_0^2}{A}(q_0, q_1)(s_0)$. Formulas (36) show then that $\Phi_{g|\partial\mathcal{D}_{\},\{}}$ is homeomorphic to identity. This concludes the proof of Proposition 3.3. Let us now prove Proposition 3.2.

3.3 Proof of Proposition 3.2

For further purpose, we are going to prove a more general proposition which implies Proposition 3.2.

Proposition 3.4 *For each $\tilde{A} > 0$ There exists $\tilde{A}_2(\tilde{A}) > 0$ such that for each $A \geq \tilde{A}_2(\tilde{A})$, there exists $\tilde{s}_2(\tilde{A}, A) > 0$ such that for each $s_0 \geq \tilde{s}_2(\tilde{A}, A)$, for each solution q of equation (15), we have the following property:*
-if

$$\begin{aligned} |q_m(s_0)| &\leq As_0^{-2}, m = 0, 1 \\ |q_2(s_0)| &\leq \tilde{A}s_0^{-2}\log s_0, \\ |q_-(y, s_0)| &\leq \tilde{A}s_0^{-2}(1 + |y|^3), \\ \|q_e(s)\|_{L^\infty} &\leq \tilde{A}s_0^{-1/2}, \end{aligned} \tag{37}$$

-if for $s_1 \geq s_0$, we have $\forall s \in [s_0, s_1]$, $q(s) \in V_A(s)$, then $\forall s \in [s_0, s_1]$,

$$\begin{aligned} |q_2(s)| &\leq A^2 s^{-2}\log s - s^{-3} \\ |q_-(y, s)| &\leq \frac{A}{2}(1 + |y|^3)s^{-2} \\ \|q_e(s)\|_{L^\infty} &\leq \frac{A^2}{2\sqrt{s}}. \end{aligned}$$

Proposition 3.4 implies Proposition 3.2. Indeed, referring to Lemma 3.1, we apply proposition 3.4 with $\tilde{A} = \max(1, C)$. This gives $\tilde{A}_2 > 0$, and for each $A \geq \tilde{A}_2$, $\tilde{s}_2(\tilde{A}, A)$. If we take $s_2(A) = \max(\tilde{s}_2(\max(1, C), A), s_1(A))$ (Cf Lemma 3.1), then, applying proposition 3.4 and Lemma 3.1, one easily checks that Proposition 3.2 is valid for these values.

Proof of Proposition 3.4
The proof is divided in two parts:

In a first part, we give a priori estimates on $q(s)$ in $V_A(s)$: assume that for given $A > 0$ large, $\tilde{A} > 0$, $\rho > 0$ and initial time $s_0 \geq s_5(A, \tilde{A}, \rho)$, we have $q(s) \in V_A(s)$ for each $s \in [\sigma, \sigma + \rho]$, where $\sigma \geq s_0$. Using the equation satisfied by q, we then derive new bounds on q_2, q_- and q_e in $[\sigma, \sigma + \rho]$ (involving A, $\tilde{A}$ and ρ).

In a second part, we will use these new bounds to conclude the proof of Proposition 3.4.

Step 1: A priori estimates of q.
Let us recall the integral equation satisfied by q (Cf (19)):

$$q(s) = K(s,\sigma)q(\sigma) + \int_\sigma^s d\tau K(s,\tau)B(q(\tau)) + \int_\sigma^s d\tau K(s,\tau)R(\tau), \qquad (38)$$

where

$$\begin{aligned} B(q) &= |\varphi + q|^{p-1}(\varphi + q) - \varphi^p - p\varphi^{p-1}q, \\ R(y,s) &= \Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{1}{p-1}\varphi + \varphi^p - \frac{\partial\varphi}{\partial s}, \end{aligned}$$

and K is the fundamental solution of $\mathcal{L}_\mathcal{V}$ (Cf (16)).
We now assume that for each $s \in [\sigma, \sigma + \rho]$, $q(s) \in V_A(s)$. Using (38), we derive new bounds on the three terms in the right hand side of (38), and then on q.
In the case $\sigma = s_0$, from initial data properties, it turns out that we obtain better estimates for $s \in [s_0, s_0 + \rho]$.
More precisely, we have the following lemma:

Lemma 3.4 *There exists $A_5 > 0$ such that for each $A \geq A_5$, $\tilde{A} > 0$, $\rho^* > 0$, there exists $s_5(A, \tilde{A}, \rho^*) > 0$ with the following property:*
$\forall s_0 \geq s_5(A, \tilde{A}, \rho^*)$, $\forall \rho \leq \rho^*$, assume $\forall s \in [\sigma, \sigma + \rho]$, $q(s) \in V_A(s)$ with $\sigma \geq s_0$.

I)Case $\sigma \geq s_0$:
we have $\forall s \in [\sigma, \sigma + \rho]$,
i) **(linear term)**

$$\begin{aligned} |\alpha_2(s)| &\leq A^2 \frac{\log \sigma}{s^2} + (s - \sigma)CAs^{-3}, \\ |\alpha_-(y,s)| &\leq C(e^{-\frac{1}{2}(s-\sigma)}A + e^{-(s-\sigma)^2}A^2)(1 + |y|^3)s^{-2}, \\ \|\alpha_e(s)\|_{L^\infty} &\leq C(A^2 e^{-\frac{(s-\sigma)}{p}} + Ae^{(s-\sigma)})s^{-\frac{1}{2}}, \end{aligned}$$

where

$$K(s,\sigma)q(\sigma)=\alpha(y,s)=\sum_{m=0}^{2}\alpha_m(s)h_m(y)+\alpha_-(y,s)+\alpha_e(y,s).$$

ii) **(nonlinear term)**

$$\begin{aligned}|\beta_2(s)| &\leq \frac{(s-\sigma)}{s^{3+1/2}},\\ |\beta_-(y,s)| &\leq (s-\sigma)(1+|y|^3)s^{-2-\epsilon},\\ \|\beta_e(s)\|_{L^\infty} &\leq (s-\sigma)s^{-\frac{1}{2}-\epsilon},\end{aligned}$$

where

$$\epsilon=\epsilon(p)>0,$$

and

$$\int_\sigma^s d\tau K(s,\tau)B(q(\tau))=\beta(y,s)=\sum_{m=0}^{2}\beta_m(s)h_m(y)+\beta_-(y,s)+\beta_e(y,s).$$

iii) **(corrective term)**

$$\begin{aligned}|\gamma_2(s)| &\leq (s-\sigma)Cs^{-3},\\ |\gamma_-(y,s)| &\leq (s-\sigma)C(1+|y|^3)s^{-2},\\ \|\gamma_e(s)\|_{L^\infty} &\leq (s-\sigma)s^{-3/4},\end{aligned}$$

where

$$\int_\sigma^s d\tau K(s,\tau)R(.,\tau)=\gamma(y,s)=\sum_{m=0}^{2}\gamma_m(s)h_m(y)+\gamma_-(y,s)+\gamma_e(y,s).$$

II)Case $\sigma=s_0$:
Assume in addition that $q(s_0)$ satisfies (37). Then, $\forall s\in[s_0,s_0+\rho]$,
i) **(linear term)**

$$\begin{aligned}|\alpha_2(s)| &\leq \tilde{A}\frac{\log s_0}{s^2}+C\max(A,\tilde{A})(s-s_0)s^{-3},\\ |\alpha_-(y,s)| &\leq C\tilde{A}(1+|y|^3)s^{-2},\\ \|\alpha_e(s)\|_{L^\infty} &\leq C\tilde{A}(1+e^{(s-s_0)})s^{-\frac{1}{2}}.\end{aligned}$$

We will give the proof of this lemma later.

Step 2: Lemma 3.4 implies Proposition 3.4
Let $\tilde{A}$ be an arbitrary positive number. Let $A > \tilde{A}_2(\tilde{A})$ where $\tilde{A}_2(\tilde{A})$ will be defined later. Let $s_0 > 0$ to be chosen larger than $\tilde{s}_2(A)$ (where $\tilde{s}_2(A)$ will be defined later). Let q be a solution of equation (15) satisfying (37), and $s_1 \geq s_0$. Assume in addition that $\forall s \in [s_0, s_1]$, $q(s) \in V_A(s)$.
We want to prove that $\forall s \in [s_0, s_1]$

$$|q_2(s)| \leq A^2 \frac{\log s}{s^2} - \frac{1}{s^3}, |q_-(y,s)| \leq \frac{A}{2s^2}(1+|y|^3), \|q_e(s)\|_{L^\infty} \leq \frac{A^2}{2\sqrt{s}}. \quad (39)$$

Let $\rho_1 \geq \rho_2$ two positive numbers (to be fixed in terms of A later). It is then enough to prove (39), on one hand for $s - s_0 \leq \rho_1$, and on the other hand for $s - s_0 \geq \rho_2$. In both cases, we use lemma 3.4. Hence, we suppose $A \geq A_5$, $s_0 \geq \max(s_5(A, \tilde{A}, \rho_1), s_5(A, \tilde{A}, \rho_2))$.

Case 1: $s - s_0 \leq \rho_1$.
Since we have $\forall \tau \in [s_0, s]$, $q(\tau) \in V_A(\tau)$, we apply lemma 3.4 (IIi), Iii), iii)) with A, $\rho^* = \rho_1$ and $\rho = s - s_0$. From (38), we obtain:

$$\begin{aligned}
|q_2(s)| &\leq \tilde{A}\frac{\log s_0}{s^2} + C_1(\max(A,\tilde{A})+1)(s-s_0)s^{-3} + (s-s_0)s^{-3-1/2} \\
|q_-(y,s)| &\leq (C_1\tilde{A} + C_1(s-s_0))(1+|y|^3)s^{-2} + (s-s_0)(1+|y|^3)s^{-2-\epsilon} \\
\|q_e(s)\|_{L^\infty} &\leq (C_1\tilde{A} + C_1\tilde{A}e^{s-s_0})s^{-\frac{1}{2}} + (s-s_0)s^{-3/4} + (s-s_0)s^{-\frac{1}{2}-\epsilon}. \quad (40)
\end{aligned}$$

To have (39), it is enough to satisfy

$$\begin{aligned}
\tilde{A}\frac{\log s_0}{s^2} &\leq \frac{A^2}{2}\frac{\log s}{s^2} \quad (41) \\
C_1\tilde{A}s^{-2} + C_1(s-s_0)s^{-2} &\leq \frac{A}{4}s^{-2} \\
C_1\tilde{A}s^{-1/2} + C_1\tilde{A}e^{s-s_0}s^{-1/2} &\leq \frac{A^2}{4}s^{-\frac{1}{2}},
\end{aligned}$$

on one hand, and

$$\begin{aligned}
C_1(\max(A,\tilde{A})+1)(s-s_0)s^{-3} + (s-s_0)s^{-3-1/2} &\leq \frac{A^2}{2}\frac{\log s}{s^2} - s^{-3} \quad (42) \\
(s-s_0)s^{-2-\epsilon} &\leq \frac{A}{4}s^{-2} \\
(s-s_0)s^{-3/4} + (s-s_0)s^{-\frac{1}{2}-\epsilon} &\leq \frac{A^2}{4}s^{-\frac{1}{2}}
\end{aligned}$$

on the other hand.
If we restrict ρ_1 to satisfy $C_1\rho_1 \le \frac{A}{8}$, $C_1\tilde{A}e^{\rho_1} \le \frac{A^2}{8}$, (which is possible if we fix $\rho_1 = \frac{3}{2}\log A$ for A large), and A to satisfy $\tilde{A} \le A$, $\tilde{A} \le \frac{A^2}{2}$, $C_1\tilde{A} \le \frac{A}{8}$ and $C_1\tilde{A} \le \frac{A^2}{8}$ (that is $A \ge A_6(\tilde{A})$), then, since $s - s_0 \le \rho_1$, (41) is satisfied.
With this value of ρ_1, (42) will be satisfied if the following is true:

$$\begin{aligned} C_1(A+1)\frac{3}{2}\log A s^{-3} + \frac{3}{2}\log A s^{-3-1/2} &\le \frac{A^2}{2}\frac{\log s}{s^2} - s^{-3} \\ \frac{3}{2}\log A s^{-2-\epsilon} &\le \frac{A}{4}s^{-2} \\ \frac{3}{2}\log A s^{-3/4} + \frac{3}{2}\log A s^{-\frac{1}{2}-\epsilon} &\le \frac{A^2}{4}s^{-\frac{1}{2}}, \end{aligned}$$

which is possible, if $s_0 \ge s_6(A)$.
This concludes Case 1.

Case 2: $s - s_0 \ge \rho_2$.
Since we have $\forall \tau \in [\sigma, s]$, $q(\tau) \in V_A(\tau)$, we apply Part I) of lemma 3.4 with A, $\rho = \rho^* = \rho_2$, $\sigma = s - \rho_2$. From (38), we derive:

$$\begin{aligned} |q_2(s)| &\le A^2\frac{\log(s-\rho_2)}{s^2} + C_2A\rho_2 s^{-3} + C_2\rho_2 s^{-3} + \rho_2 s^{-3-1/2} \qquad (43) \\ |q_-(y,s)| &\le C_2(e^{-\frac{1}{2}\rho_2}A + e^{-\rho_2{}^2}A^2 + \rho_2)(1+|y|^3)s^{-2} + \rho_2(1+|y|^3)s^{-2-\epsilon} \\ \|q_e(s)\|_{L^\infty} &\le C_2(A^2e^{-\frac{\rho_2}{p}} + Ae^{\rho_2})s^{-\frac{1}{2}} + \rho_2 s^{-3/4} + \rho_2 s^{-\frac{1}{2}-\epsilon}, \end{aligned}$$

To obtain (39), it is enough to have:

$$\begin{aligned} f_{A,\rho_2}(s) &\ge 0 \qquad (44) \\ C_2(e^{-\frac{1}{2}\rho_2}A + e^{-\rho_2{}^2}A^2 + \rho_2) &\le \frac{A}{4} \\ C_2(A^2e^{-\frac{\rho_2}{p}} + Ae^{\rho_2}) &\le \frac{A^2}{4}, \end{aligned}$$

with

$$f_{A,\rho_2}(s) = A^2\frac{\log s}{s^2} - s^{-3} - [A^2\frac{\log(s-\rho_2)}{s^2} + C_2(A+1)\rho_2 s^{-3} + \rho_2 s^{-3-1/2}]$$

on one hand, and

$$\begin{aligned} \rho_2 s^{-2-\epsilon} &\le \frac{A}{4}s^{-2} \qquad (45) \\ \rho_2 s^{-3/4} + \rho_2 s^{-\frac{1}{2}-\epsilon} &\le \frac{A^2}{4}s^{-\frac{1}{2}}, \end{aligned}$$

on the other hand.
Now, it is convenient to fix the value of ρ_2 such that $C_2 A e^{\rho_2} = \frac{A^2}{8}$, that is $\rho_2 = \log \frac{A}{8C_2}$. The conclusion follows from this choice, for A large. Indeed, for arbitrary A, we write

$$|f_{A,\log \frac{A}{8C_2}}(s) - s^{-3}(A^2 \log \frac{A}{8C_2} - 1 - C_2(A+1) \log \frac{A}{8C_2})| \leq \frac{CA^2}{s^{3+1/2}}(\log \frac{A}{8C_2})^2.$$

Then, we take $A \geq A_7$ such that

$$\begin{array}{rcl} (A^2 \log \frac{A}{8C_2} - 1 - C_2(A+1) \log \frac{A}{8C_2}) & \geq & 1 \\ C_2((\frac{A}{8C_2})^{-1/2} A + e^{-(\log \frac{A}{8C_2})^2} A^2 + \log \frac{A}{8C_2}) & \leq & \frac{A}{4} \\ C_2(A^2 (\frac{A}{8C_2})^{-1/p} + A \frac{A}{8C_2}) & \leq & \frac{A^2}{4}. \end{array}$$

After, we introduce $s_7(A) > 0$ such that for $s \geq s_0 \geq s_7(A)$, we have $s^{-3-1/2} C A^2 (\log \frac{A}{8C_2})^2 \leq \frac{1}{2} s^{-3}$ and (45) satisfied.
This way, (44) and (45) are satisfied, for $A \geq A_7$ and $s_0 \geq s_7(A)$, which concludes Case 2.

We remark that for $A \geq A_8$, we have $\rho_1 = \frac{3}{2} \log A \geq \rho_2 = \log \frac{A}{8C_2}$.
If now we take $A_2 = \sup(A_5, A_6(\tilde{A}), A_7, A_8)$, and then
$s_2 = \max(s_5(A, \tilde{A}, \rho_1(A)), s_5(A, \tilde{A}, \rho_2(A)), s_6(A), s_7(A))$, then this concludes the proof of Proposition 3.2.

Proof of Lemma 3.4
Let $A \geq A_5$ with $A_5 > 0$ to be fixed later. Let $\tilde{A} > 0$, $\rho^* > 0$. We take $\rho \leq \rho^*$ and $s_0 \geq s_5(A, \tilde{A}, \rho^*)$. We consider $\sigma \geq s_0$ such that $\forall s \in [\sigma, \sigma + \rho]$, $q(s) \in V_A(s)$. For each part $Ii), ii), iii)$ and $IIi)$, we want to find $s_5(A, \tilde{A}, \rho_0)$ such that the concerned part holds for $s_0 \geq s_5(A, \tilde{A}, \rho^*)$.
The proof is given in two steps:
-In a first step, we give various estimates on different terms appearing in the equation (19).
-In a second step, we use these estimates to conclude the proof.

Step 1: Estimates for equation (38)
i) Estimates on K:

Lemma 3.5 (Bricmont-Kupiainen) .
a) $\forall s \geq \tau \geq 1$ *with* $s \leq 2\tau$, $\forall y, x \in I\!R$,
$|K(s,\tau,y,x)| \leq Ce^{(s-\tau)\mathcal{L}}(y,x)$, *with*
$e^{\theta\mathcal{L}}(y,x) = \frac{e^{\theta}}{\sqrt{4\pi(1-e^{-\theta})}} \exp[-\frac{(ye^{-\theta/2}-x)^2}{4(1-e^{-\theta})}]$.

b) For each $A' > 0$, $A'' > 0$, $A''' > 0$, $\rho^* > 0$, *there exists* $s_9(A', A'', A''', \rho^*)$ *with the following property:*
$\forall s_0 \geq s_9$, *assume that for* $\sigma \geq s_0$,

$$\begin{aligned} |q_m(\sigma)| &\leq A'\sigma^{-2}, m = 0, 1, \\ |q_2(\sigma)| &\leq A''(\log\sigma)\sigma^{-2}, \\ |q_-(y,\sigma)| &\leq A'''(1+|y|^3)\sigma^{-2}, \\ \|q_e(\sigma)\|_{L^\infty} &\leq A''\sigma^{-\frac{1}{2}}, \end{aligned} \tag{46}$$

then, $\forall s \in [\sigma, \sigma+\rho^*]$

$$\begin{aligned} |\alpha_2(s)| &\leq A''\frac{\log\sigma}{s^2} + (s-\sigma)C\max(A', A''')s^{-3}, \\ |\alpha_-(y,s)| &\leq C(e^{-\frac{1}{2}(s-\sigma)}A''' + e^{-(s-\sigma)^2}A'')(1+|y|^3)s^{-2}, \\ \|\alpha_e(s)\|_{L^\infty} &\leq C(A''e^{-\frac{(s-\sigma)}{p}} + A'''e^{(s-\sigma)})s^{-\frac{1}{2}}, \end{aligned}$$

where

$$K(s,\sigma)q(\sigma) = \alpha(y,s) = \sum_{m=0}^{2} \alpha_m(s)h_m(y) + \alpha_-(y,s) + \alpha_e(y,s). \tag{47}$$

c) $\forall \rho^* > 0$, $\exists s_{10}(\rho^*)$ *such that* $\forall \sigma \geq s_{10}(\rho^*)$, $\forall s \in [\sigma, \sigma+\rho^*]$,

$$\begin{aligned} |\gamma_2(s)| &\leq (s-\sigma)Cs^{-3}, \\ |\gamma_-(y,s)| &\leq (s-\sigma)C(1+|y|^3)s^{-2}, \end{aligned}$$

where

$$\int_\sigma^s d\tau K(s,\tau)R(\tau) = \gamma(y,s) = \sum_{m=0}^{2} \gamma_m(s)h_m(y) + \gamma_-(y,s) + \gamma_e(y,s).$$

Proof:
see Appendix A.
Using the above lemma and simple calculation, we derive the following:

Corollary 3.1 *$\forall s \geq \tau \geq 1$ with $s \leq 2\tau$,*

$$|\int K(s,\tau,y,x)(1+|x|^m)dx| \leq C\int e^{(s-\tau)\mathcal{L}}(y,x)(1+|x|^m)dx \leq e^{s-\tau}(1+|y|^m). \quad (48)$$

ii) Estimates on B:

Lemma 3.6 *$\forall A>0$, $\exists s_{11}(A)$ such that $\forall \tau \geq s_{11}(A)$, $q(\tau) \in V_A(\tau)$ implies*

$$|\chi(y,\tau)B(q(y,\tau))| \leq C|q|^2 \quad (49)$$

and

$$|B(q)| \leq C|q|^{\bar{p}} \quad (50)$$

with $\bar{p} = \min(p,2)$.

Proof: Let $A>0$. If $q(\tau) \in V_A(\tau)$, then $\|q(\tau)\|_{L^\infty} \leq C(A)\tau^{-1/2} \leq \frac{1}{2}f(2K_0)$, if $\tau \geq s_{11}(A)$ (Cf Definition 3.2, (7) for f and (26) for K_0).
(49) and (50) are equivalent to 1), 2) and 3), with

1) $p \geq 2$ and $|B(q)| \leq C|q|^2$,
2) $p<2$ and $|\chi(y,\tau)B(q(y,\tau))| \leq C|q|^2$,
3) $p<2$ and $|B(q)| \leq C|q|^p$.

We prove 1), 2) and 3).
For 1), we Taylor expand $B(q)$, and use the boundedness of $|\varphi|$ and $|q|$.
2) holds if $\chi(y,\tau)=0$. Otherwise, we have $|y| \leq 2K_0\sqrt{\tau}$. Again, we Taylor expand $B(q)$: $\chi(y,\tau)|B(q)| \leq C\chi(y,\tau)|q|^2\int_0^1(1-\theta)|\varphi+\theta q|^{p-2}d\theta$, and conclude writing $\chi(y,s)|\varphi+\theta q|^{p-2} \leq \chi(y,s)(|\varphi|-|q|)^{p-2} \leq (f(2K_0)-\frac{1}{2}f(2K_0))^{p-2} = C$.
For 3), we write $\frac{B(q)}{|q|^p} = \frac{|1+\xi|^{p-1}(1+\xi)-1-p\xi}{|\xi|^p}$ by setting $\xi = \frac{q}{\varphi}$. We easily check that this expression is bounded for $\xi \to 0$ and $\xi \to \infty$.

iii) Estimate on R:

Lemma 3.7 *$\exists s_{12}>0$ $\forall \tau \geq s_{12}$,*

$$|R(y,\tau)| \leq \frac{C}{\tau}. \quad (51)$$

Proof:

From (18) and (14), we compute: $\varphi_{yy} = -\frac{p-1}{2p\tau}f^p + \frac{(p-1)^2}{4p\tau^2}y^2f^{2p-1}$, $\varphi_s = -\frac{p-1}{4p\tau^2}y^2f^p + \frac{\kappa}{2p\tau^2}$, and $\varphi^p - \frac{\varphi}{p-1} - \frac{1}{2}y\varphi_y = [f+\frac{\kappa}{2p\tau}]^p - \frac{\kappa}{2p(p-1)\tau} - \frac{f}{p-1} +$

$\frac{p-1}{4p\tau}y^2f^p = -\frac{\kappa}{2p(p-1)\tau} + [f + \frac{\kappa}{2p\tau}]^p - f^p$, using a Lipschitz property and simple calculations, the conclusion follows.

iv) Estimates on q in V_A:
From Definition 3.2, we simply derive the following:

Lemma 3.8 *$\exists s_{13} > 0$ $\forall A > 0$, $\forall \tau \geq s_{13}$, if $q(\tau) \in V_A(\tau)$, then*

$$|q(y,\tau)| \leq CA^2\tau^{-2}\log\tau(1+|y|^3) \tag{52}$$

and

$$|q(y,\tau)| \leq CA^2\tau^{-1/2}. \tag{53}$$

Step 2: Conclusion of the Proof of Lemma 3.4

We choose $s_0 \geq \rho^*$ in all cases so that if $s_0 \leq \sigma \leq \tau \leq \sigma+\rho$ and $\rho \leq \rho^*$, we have $\sigma^{-1} \leq 2s^{-1}$ and $\tau^{-1} \leq 2s^{-1}$.

Ii) linear term in I) :
We apply b) of lemma 3.5 with $A' = A$, $A'' = A^2$ and $A''' = A$. Take $s_5(A,\rho^*) = s_9(A, A^2, A, \rho^*)$.

IIi) linear term in II) :
We apply b) of lemma 3.5 with $A' = A$, $A'' = \tilde{A}$ and $A''' = \tilde{A}$.

Iii) nonlinear term:
-$\beta_2(s)$:
By definition, $\beta_2(s) = \int d\mu(y)k_2(y)\chi(y,s)\beta(y,s)$.
$= \int d\mu(y)k_2(y)\chi(y,s)\int_\sigma^s d\tau \int K(s,\tau,y,x)B(q(x,\tau))dx = I + II$, where
$I = \int d\mu(y)k_2(y)\chi(y,s)\int_\sigma^s d\tau \int K(s,\tau,y,x)\chi(x,\tau)B(q(x,\tau))dx$, and
$II = \int d\mu(y)k_2(y)\chi(y,s)\int_\sigma^s d\tau \int K(s,\tau,y,x)(1-\chi(x,\tau))B(q(x,\tau))dx$.
For I we write:
$|I| \leq \int d\mu(y)|k_2(y)|\int_\sigma^s d\tau \int |K(s,\tau,y,x)|\chi(x,\tau)|B(q(x,\tau))|dx$
$\leq C\int d\mu(y)|k_2(y)|\int_\sigma^s d\tau \int |K(s,\tau,y,x)||q(x,\tau)|^2dx$ (Cf (49))
$\leq C\int d\mu(y)|k_2(y)|\int_\sigma^s d\tau \int |K(s,\tau,y,x)|A^4\tau^{-4}(\log\tau)^2(1+|x|^6)dx$ (Cf (52))
$\leq CA^4\int d\mu(y)|k_2(y)|\int_\sigma^s d\tau\tau^{-4}(\log\tau)^2e^{s-\tau}(1+|y|^6)$ (Cf corollary 3.1)
$\leq CA^4\int d\mu(y)|k_2(y)|(1+|y|^6)(s-\sigma)\sigma^{-4}(\log s)^2e^{s-\sigma}$
$\leq CA^4(s-\sigma)e^{s-\sigma}(\frac{s}{2})^{-4}(\log s)^2$ (we take $s_0 \geq \rho^*$ so that $s \leq \sigma+\rho^* \leq \sigma+s_0 \leq \sigma+\sigma = 2\sigma$)
For II, we use (50) and (53) to have:
$|II| \leq C\int e^{-\frac{y^2}{4}}dy\chi(y,s)|k_2(y)|\int_\sigma^s d\tau\int dx(1-\chi(x,\tau))$

$\frac{e^{s-\sigma}}{\sqrt{4\pi(1-e^{-(s-\tau)})}}\exp[-\frac{(ye^{-(s-\tau)/2}-x)^2}{4(1-e^{-(s-\tau)})}]A^{2\bar{p}}\tau^{-\bar{p}/2}$.

Now, we have $e^{\frac{1}{2}[-\frac{y^2}{4}-\frac{(ye^{-t/2}-x)^2}{4(1-e^{-t})}]}\leq e^{-c(K_0)s}\leq e^{-Cs}$, for $|y|\leq 2K_0\sqrt{s}$ and $|x|\geq K_0\sqrt{\tau}$ (if $s_0\geq\rho^*$). Hence, we derive

$|II|\leq C\int e^{-\frac{y^2}{8}}dy|k_2(y)|\int_\sigma^s d\tau\int dx(1-\chi(x,\tau))$

$\frac{e^{s-\sigma}}{\sqrt{4\pi(1-e^{-(s-\tau)})}}\exp[-\frac{1}{2}\frac{(ye^{-(s-\tau)/2}-x)^2}{4(1-e^{-(s-\tau)})}]e^{-Cs}A^{2\bar{p}}\tau^{-\bar{p}/2}$.

Using a variable change in x, and carrying all calculation, we bound $|II|$ by $(s-\sigma)e^{-Cs}$, for $s\geq s_{14}(A,\rho^*)$. Adding the bounds for I and II, and taking $\sigma\geq s_{15}(A,\rho^*)$, we obtain the estimate for $\beta_2(s)$.

-$\beta_-(y,s)$:

Using (50), (52), and (48), and computing as before yields $|\beta(y,s)|\leq CA^{2\bar{p}}(s-\sigma)e^{(s-\sigma)}(1+|y|^3)^{\bar{p}}(\frac{\log s}{s^2})^{\bar{p}}$. If we multiply this term by $\chi(s)$ and bound in it $|y|^{3\bar{p}-3}$ by $(\sqrt{s})^{3\bar{p}-3}$, we obtain $|\beta_b(y,s)|\leq CA^{2\bar{p}}(s-\sigma)e^{(s-\sigma)}(1+|y|^3)(\sqrt{s})^{3\bar{p}-3}(\frac{\log s}{s^2})^{\bar{p}}$, hence $|\beta_b(y,s)|\leq CA^{2\bar{p}}(s-\sigma)e^{(s-\sigma)}(1+|y|^3)\frac{(\log s)^{\bar{p}}}{s^{(\bar{p}+3)/2}}$, which implies simply the estimate for β_- (for $\sigma\geq s_{16}(\rho^*)$ and some $\epsilon_1(p)$).

-$\beta_e(y,s)$*:*

Using (50), (53), and (48), and computing as before yields $|\beta(y,s)|\leq CA^{2\bar{p}}(s-\sigma)e^{(s-\sigma)}s^{-\frac{1}{2}\bar{p}}$. From this, we derive directly the estimate for β_e (for $\sigma\geq s_{17}(\rho^*)$ and some $\epsilon_2(p)$).

Finally, we take $\sigma\geq\max(s_{15},s_{16},s_{17}))=s_5(A,\rho^*)$ and $\epsilon=\min(\epsilon_1,\epsilon_2)$ to have the conclusion.

iii) corrective term:

For γ_2 and γ_-, we use $c)$ of lemma 3.5. For γ_e, we start from (51) and write $\gamma_e(y,s)=(1-\chi(y,s))\gamma(y,s)=(1-\chi)\int_\sigma^s d\tau\int dxK(s,\tau,y,x)R(x,\tau)$, and then as in *ii*), $|\gamma_e(y,s)|\leq C\int_\sigma^s d\tau\int dxe^{(s-\tau)\mathcal{L}}(y,x)\frac{C}{\tau}=C\int_\sigma^s\frac{d\tau}{t}e^{s-\tau}\leq\frac{C}{s}(s-\sigma)e^{s-\sigma}\leq(s-\sigma)s^{-\frac{3}{4}}$, if $\sigma\geq s_{10}(\rho^*)$.

4 Stability

In this section, we give the proof of Theorem 2. As in section 3, we consider $N=1$ for simplicity, but the same proof holds in higher dimension. We will mention at the end of the section how to adapt the proof to the case $N\geq 2$.

4.1 Case $N = 1$:

Let us consider $\hat{u}_0$ an initial data in H, constructed in Theorem 1. Let $\hat{u}(t)$ be the solution of equation (1):

$$u_t = \Delta u + |u|^{p-1}u, u(0) = \hat{u}_0.$$

Let $\hat{T}$ be its blow-up time and $\hat{a}$ be its blow-up point.

We know from (35) that there exists $\hat{A} > 0$, $\hat{s}_0 > \log \hat{T}$ such that $\forall s \geq \hat{s}_0$, $\hat{q}_{\hat{T},\hat{a}}(s) \in V_{\hat{A}}(s)$, where $\hat{q}_{\hat{T},\hat{a}}$ is defined in (13) by:

$$\hat{q}_{\hat{T},\hat{a}}(y,s) = e^{-\frac{s}{p-1}}\hat{u}(\hat{a} + ye^{-\frac{s}{2}}, \hat{T} - e^{-s}) - [\frac{\kappa}{2ps} + (p - 1 + \frac{(p-1)^2}{4ps}y^2)^{-\frac{1}{p-1}}].$$

Remark: Following Remark 1.2, we can consider a more general $\hat{u}_0$, that is $\hat{u}_0$ with the following property:
$\exists(\hat{T}, \hat{a})$, $\exists \hat{A}, \hat{s}_0$ such that $\forall s \geq \hat{s}_0$, $\hat{q}_{\hat{T},\hat{a}}(s) \in V_{\hat{A}}(s)$. From Definition 3.2, the definition of $\hat{q}_{\hat{T},\hat{a}}(s)$, and Proposition 3.1, $\hat{u}(t)$ blows up at time $\hat{T}$ at one single point $\hat{a}$, and behaves as the conclusion of Theorem 1.

We want to prove that there exists a neighborhood $\mathcal{V}\prime$ of $\hat{u}_0$ in H with the following property:

$\forall u_0 \in \mathcal{V}\prime$, $u(t)$ blows-up in finite time T at only one blow-up point a, where $u(t)$ is the solution of equation (1) with initial data $u(0) = u_0$. Moreover, $u(t)$ satisfies:

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z\ , t) = f(z) \tag{54}$$

uniformly in $z \in \mathbb{R}$, with

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p}z^2)^{-\frac{1}{p-1}}.$$

The proof relays strongly on the same ideas as the proof of Theorem 1: use of finite dimensional parameters, reduction to a finite dimensional problem and continuity. For Theorem 2, we introduce a one-parameter group, defined by:

$$(T, a) \longrightarrow q_{T,a},$$

where $q_{T,a}$ is defined by (13), for a given solution $u(t)$ of equation (1) with initial data u_0. This one-parameter group has an important property: $\forall (T, a)$, $q_{T,a}$ is a solution of equation (15). Therefore, our purpose is to fine-tune the parameter (T, a) in order to get $(T(u_0), a(u_0))$ such that $q_{T(u_0),a(u_0)}(s) \in V_{A_0}(s)$, for $s \geq s_0$, A_0 and s_0 are to be fixed later. Hence, through the reduction to a finite dimensional problem, we give a geometrical interpretation of our problem, since we deal with finite dimensional functions depending on finite dimensional parameters through a one-parameter group.

As indicated in the formulation of the problem in section 2 and used in section 3 (Definitions 3.1 and 3.2), it is enough to prove the following:

Proposition 4.1 (Reduction) *There exist $A_0 > 0$, $s_0 > 0$, D_0 neighborhood of $(\hat{T}, \hat{a})$ in $I\!R^2$, and $\mathcal{V}_\prime$ neighborhood of $\hat{u}_0$ in H with the following property:*
$\forall u_0 \in \mathcal{V}_\prime, \exists (\mathcal{T}, \dashv) \in \mathcal{D}_\prime$ such that $\forall s \geq s_0$, $q_{T,a}(s) \in V_{A_0}(s)$, where $q_{T,a}$ is defined by (13), and $u(t)$ is the solution of equation (1) with initial data $u(0) = u_0$. (We keep here the $(T,\ a)$ dependence for clearness).

Indeed, once this proposition is proved, (54) follows directly from (3), (13) and definitions 3.1, 3.2. Proposition 3.1 applied to $u(x - a, t)$ then shows directly that $u(t)$ blows-up at time T at one single point: $x = a$.

The proof relays strongly on the same ideas as those developed in section 3, and geometrical interpretation of T and a. Let us explain briefly its main ideas:

-In a first part, as before, we reduce the control of all the components of q to a problem of control $(q_0, q_1)(s)$, uniformly for $u_0 \in \mathcal{V}_\infty$ and $(T, a) \in D_1$ (where $\mathcal{V}_\infty$ and D_1 are respectively neighborhoods of $\hat{u}_0$ and $(\hat{T}, \hat{a})$).

-In a second part, we focus on the finite dimensional variable $(q_0, q_1)(s)$, and try to control it. We study the behavior of $\hat{q}_{T,a}$ under perturbations in (T, a) near $(\hat{T}, \hat{a})$ (and some topological structure related to these). We then extend the properties of $\hat{q}$ to q, for u_0 near $\hat{u}_0$. We conclude the proof proceeding by contradiction to reach a topological obstruction (using Index Theory).

The constant C again denotes a universal one independent of variables, only depending upon constants of the problem such as p.

For each initial data u_0, $u(t)$ denotes the solution of (1) satisfying $u(0) = u_0$, and for each $(T, a) \in \mathrm{I\!R}^2$, $w_{T,a}$ and $q_{T,a}$ denote the auxiliary functions derived from u by transformations (3) and (13).

Part I: Initialization and reduction to a finite dimensional problem

In this section, we first use continuity arguments to show that for A, s_0 large enough (to be fixed later), for (u_0, T, a) close to $(\hat{u}_0, \hat{T}, \hat{a})$, $q_{T,a}$ is defined at $s = s_0$, and satisfies $q_{T,a}(s_0) \in V_A(s_0)$ (Step 1). After, we aim at finding (T, a) such that $q_{T,a}(s)$ in $V_A(s)$ for $s \geq s_0$. For this purpose, we reduce through a priori estimates the control of $q_{T,a}(s)$ in $V_A(s)$ to the control of $(q_{0,T,a}, q_{1,T,a})(s)$ in $\hat{V}_A(s)$ for $s \geq s_0$ (Step 2).

Step 1: Initialization

We use here the fact that $\hat{q}_{\hat{T},\hat{a}}(s) \in V_{\hat{A}}(s)$ for any $s \geq \hat{s}_0$, and the continuity of $q_{T,a}$ with respect to initial data u_0 and (T, a), to insure that for fixed $s_0 \geq \hat{s}_0$, $q_{T,a}(s_0) \in V_{2\tilde{A}}(s_0)$, for (u_0, T, a) close to $(\hat{u}_0, \hat{T}, \hat{a})$. Hence, if A is large enough, we have $q_{T,a}(s_0) \in V_A(s_0)$ and $q_{T,a}(s_0)$ is "small" in a way.

Lemma 4.1 (Initialization) *For each $s_0 > \hat{s}_0$ there exist $\mathcal{V}_\infty$ neighborhood of $\hat{u}_0$ in H and $D_1(s_0)$ neighborhood of $(\hat{T}, \hat{a})$ in $\mathbb{R}^2$, such that for each $u_0 \in \mathcal{V}_\infty$, $(T, a) \in D_1(s_0)$, $q(T, a, s)$ is defined (at least) for $s \in (-\log T, s_0]$, and $q_{T,a}(s_0) \in V_{2\hat{A}}(s_0)$.*

Proof of Lemma 4.1:

$\forall T > 0$, $\forall a \in \mathbb{R}$, $q_{T,a}(s)$ is defined on:

$(-\log T, +\infty)$, if $T \leq \hat{T}$, or $(-\log T, -\log(T - \hat{T}))$, if $T > \hat{T}$.

Therefore, $q_{T,a}(s)$ is defined on $(-\log T, s_0]$ for T near $\hat{T}$.

i) Reduction to the continuity of $q_{T,a}(s_0) \in L^\infty(\mathbb{R})$

Let $s_0 > \hat{s}_0$. It is enough to prove that $\forall \epsilon > 0$, there exist $\mathcal{V}$ and D such that $\forall u_0 \in \mathcal{V}$, $(T, a) \in D$,

$$\|q_{T,a}(s_0) - \hat{q}_{\hat{T},\hat{a}}(s_0)\|_{L^\infty(\mathbb{R})} \leq \epsilon. \tag{55}$$

Indeed, if it is the case, then,

$$\forall m \in \{0, 1, 2\}, |q_{m,T,a}(s_0) - \hat{q}_{m,\hat{T},\hat{a}}(s_0)| \leq C\epsilon, \tag{56}$$

$$|q_{-,T,a}(y, s_0) - \hat{q}_{-,\hat{T},\hat{a}}(y, s_0)| \leq C\epsilon(1 + |y|^2), \tag{57}$$

$$\|q_{e,T,a}(s_0) - \hat{q}_{e,\hat{T},\hat{a}}(s_0)\|_{L^\infty(\mathbb{R})} \leq C\epsilon. \tag{58}$$

(56) and (58) follow directly from (55). For (57), write $q_-(y, s) = \chi(y, s)q(y, s) - \sum_{m=0}^{2} q_m(s)h_m(y)$, and use (55) and (56). Using $\hat{q}_{\hat{T},\hat{a}}(s_0) \in V_{\hat{A}}(s_0)$ and taking $\epsilon > 0$ small enough yields the conclusion of lemma 4.1.

ii) Continuity of $q_{T,a}(s_0) \in L^\infty(\mathbb{R})$

We have:

$$q_{T,a}(y, s_0) - \hat{q}_{\hat{T},\hat{a}}(y, s_0) = w_{T,a}(y, s_0) - \hat{w}_{\hat{T},\hat{a}}(y, s_0)$$

$$
\begin{aligned}
&= e^{-\frac{s_0}{p-1}}\{u(e^{-s_0/2}y+a, T-e^{-s_0}) \quad - \quad \hat{u}(e^{-s_0/2}y+\hat{a}, \hat{T}-e^{-s_0})\} \\
&= e^{-\frac{s_0}{p-1}}\{u(e^{-s_0/2}y+a, T-e^{-s_0}) \quad - \quad \hat{u}(e^{-s_0/2}y+a, T-e^{-s_0})\} \\
&\ +e^{-\frac{s_0}{p-1}}\{\hat{u}(e^{-s_0/2}y+a, T-e^{-s_0}) \quad - \quad \hat{u}(e^{-s_0/2}y+\hat{a}, T-e^{-s_0})\} \\
&\ +e^{-\frac{s_0}{p-1}}\{\hat{u}(e^{-s_0/2}y+\hat{a}, T-e^{-s_0}) \quad - \quad \hat{u}(e^{-s_0/2}y+\hat{a}, \hat{T}-e^{-s_0})\}.
\end{aligned}
$$

Since $u_0 \to u(t) \in \mathcal{C}^\infty([\frac{T-e^{-s_0}}{2}, T-\frac{e^{-s_0}}{2}], \mathcal{C}^\infty(\mathbb{R}))$ is defined and continuous (for u_0 near $\hat{u}_0$), we have the conclusion.

Step 2: Uniform finite dimensional reduction

This step is similar to Step 2 of Part 1 in the proof of Theorem 1. Here we show that for A and s_0 to be fixed later, if $q_{T,a}(s_0)$ is "small" in $V_A(s_0)$, then, the control of $q_{T,a}(s)$ in $V_A(s)$ for $s \geq s_0$ reduces to the control of $(q_{0,T,a}, q_{1,T,a})(s)$ in $\hat{V}_A(s)$.

Lemma 4.2 (Control of q by (q_0, q_1) in V_A) *There exists $A_2 > 2\hat{A}$ such that for each $A \geq A_2$, there exists $s_2(A) > 0$ such that for each $s_0 \geq s_2(A)$, we have the following properties:*

i) For any q, solution of equation (15), satisfying

- $q(s_0) \in V_{2\hat{A}}(s_0)$ and,

- for $s_1 \geq s_0$, $\forall s \in [s_0, s_1]$, $q(s) \in V_A(s)$,

we have: $\forall s \in [s_0, s_1]$,

$$
\begin{aligned}
|q_2(s)| &\leq A^2 s^{-2}\log s - s^{-3} \\
|q_-(y,s)| &\leq \frac{A}{2}(1+|y|^3)s^{-2} \\
\|q_e(s)\|_{L^\infty} &\leq \frac{A^2}{2\sqrt{s}}.
\end{aligned}
$$

Moreover,

ii) For any q, solution of equation (15), satisfying

- $q(s_0) \in V_{2\hat{A}}(s_0)(\subset V_A(s_0))$,

- For $s_* > s_0$, $q(s) \in V_A(s)$ $\forall s \in [s_0, s_*]$, and

-$q(s_*) \in \partial V_A(s_*)$,

we have $(q_0, q_1)(s_*) \in \partial \hat{V}_A(s_*)$, and there exists $\delta_0 > 0$ such that $\forall \delta \in (0, \delta_0)$, $(q_0, q_1)(s_* + \delta) \notin \hat{V}_A(s_* + \delta)$, (hence, $q(s_* + \delta) \notin V_A(s_* + \delta)$).

Proof:

i) We apply Proposition 3.4 with $\tilde{A} = max(2\hat{A}, (2\hat{A})^2)$, and take $A_2 = max(\tilde{A}_2, 2\hat{A})$, and $s_2(A) = \max(\hat{s}_0 + 1, \tilde{s}_2(\tilde{A}, A))$ to have the conclusion.

ii) We apply $i)$ with $s_1 = s_*$, and use Definition 3.1. Then, we apply lemma 3.2.

Part II: Topological argument

Below, we use the notations $q_{T,a}(s) = q(T,a,s)$, $q_{T,a}(y,s) = q(T,a,y,s)$, $q_{m,T,a}(s) = q_m(T,a,s)$.

In Part 1, we have reduced the problem to a finite dimensional one: for each u_0 close to $\hat{u}_0$, we have to find a parameter $(T,a) = (T(u_0), a(u_0))$ near $(\hat{T},\hat{a})$ such that $(q_0,q_1)(T,a,s) \in V_A(s)$ for $s \geq s_0$. We first study the behavior of $\hat{q}(T,a)$ for (T,a) close to $(\hat{T},\hat{a})$. Then, we show a stability result on this behavior for u_0 near $\hat{u}_0$. Therefore, for a given u_0, we proceed by contradiction to prove Proposition 4.1, which implies Theorem 2.

Step 1: Study of $\hat{q}(T,a)$

We study the behavior of $\hat{q}(T,a)$ for (T,a) close to $(\hat{T},\hat{a})$ in $\mathbb{R}^2$.

Proposition 4.2 (Behavior of $\hat{q}(T,a)$ near $(\hat{T},\hat{a})$) *There exists $A_4 > 0$ such that for each $A \geq A_4$, there exists $s_4(A) > 0$ with the following property:*
For each $s_0 \geq s_4(A)$, there exists $D_4(s_0)$ neighborhood of $(\hat{T},\hat{a})$ such that for each $(T,a) \in D_4(s_0) \backslash \{(\hat{T},\hat{a})\}$,
i) $\hat{q}(T,a,s)$ is defined for $s \in (-\log T, s_0]$ and $\hat{q}(T,a,s_0) \in V_A(s_0)$,
ii) $\exists s_*(T,a) > s_0$ such that $\forall s \in [s_0, s_*(T,a)]$, $\hat{q}(T,a,s) \in V_A(s)$ and $\hat{q}(T,a,s_*(T,a)) \in \partial V_A(s_*(T,A))$, and if we define

$$
\begin{array}{rcl}
\Psi_{\hat{u}_0} : D_4(s_0)\backslash\{(\hat{T},\hat{a})\} & \longrightarrow & \mathbb{R}^2 \\
(T,a) & \longrightarrow & \dfrac{\hat{s}_*(T,a)^2}{A}(\hat{q}_0,\hat{q}_1)(T,a,\hat{s}_*(T,a))
\end{array}
\tag{59}
$$

then $Im(\Psi_{\hat{u}_0}) \subset \partial\mathcal{C}$, where $\mathcal{C}$ is the unit square of $\mathbb{R}^2$.
Moreover,
iii)$\Psi_{\hat{u}_0}$ is continuous,
iv) $\forall \epsilon > 0$, there exists a curve $\Gamma_\epsilon \in D_4(s_0)$ such that $d(\Gamma_\epsilon, \Psi_{\hat{u}_0}, 0) = -1$, and $\forall (T,a) \in \Gamma_\epsilon$, $|(T,a) - (\hat{T},\hat{a})| \leq \epsilon$.

Proof:
In order to prove $i)$, $ii)$, and $iii)$, we take $A \geq A_5$ with $A_5 = \max(2\hat{A}, A_2, A_3)$, $s_0 \geq s_5(A) = \max(\hat{s}_0 + 1, s_2(A), s_3(A))$, $D_5(s_0) = D_1(s_0)$ (with the notations of lemma 4.1). For such A and s_0, we can apply lemma 4.1, and lemma 4.2.

Proof of i):

By lemma 4.1, $\forall (T,a) \in D_5(s_0)$, $\hat{q}(T,a,s)$ is defined (at least) for $s \in (-\log T, s_0]$ and $\hat{q}(T,a,s_0) \in V_{2\hat{A}}(s_0) \subset V_A(s_0)$, which proves $i)$.

Proof of ii):

We claim that $\forall (T,a) \in D_5(s_0)\backslash\{(\hat{T},\hat{a})\}$, $\exists s(T,a) > s_0$such that $\hat{q}(T,a,s) \notin V_A(s)$. Indeed:

Case 1: $T > \hat{T}$:

Since $\hat{q}(T,a,y,s) = e^{-\frac{s}{p-1}}\hat{u}(a+ye^{-\frac{s}{2}}, T-e^{-s}) - \varphi(y,s)$, $\hat{q}(T,a,s)$ is defined on $[s_0, -\log(T-\hat{T}))$ and not after. Suppose that $\hat{q}(T,a,s)$ does not leave $V_A(s)$ for $s \in [s_0, -\log(T-\hat{T}))$, then, $\forall y \in \mathbb{R}$, $\forall s \in [s_0, -\log(T-\hat{T}))$, $|\hat{q}(T,a,y,s)| \le \frac{C(A)}{\sqrt{s}}$ (Cf Definition 3.2).

Since $\hat{u}(x,t) = (T-t)^{-\frac{1}{p-1}}(\hat{q}(T,a,\frac{x-a}{\sqrt{T-t}}, -\log(T-t)) + \varphi(\frac{x-a}{\sqrt{T-t}}, -\log(T-t)))$, $\limsup_{t\to\hat{T}} \|\hat{u}(t)\|_{L^\infty(\mathbb{R})} \le C_{T,\hat{T},A} < +\infty$. This contradicts the fact that $\hat{u}(t)$ blows up at time $\hat{T}$.

Case 2: $T \le \hat{T}$ and $(T,a) \ne (\hat{T},\hat{a})$:

$\hat{q}(T,a,s)$ is defined on $[s_0,+\infty)$. Suppose that $\hat{q}(T,a,s)$ does not leave $V_A(s)$ for $s \in [s_0,+\infty)$. Then, $\forall y \in \mathbb{R}$, $\forall s \in [s_0,+\infty)$, $|\hat{q}(T,a,y,s)| \le \frac{C(A)}{\sqrt{s}}$ (Cf Definition 3.2). Hence, by (13),

$\lim_{t\to T} \|(T-t)^{\frac{1}{p-1}}u(a+\sqrt{(T-t)|\log(T-t)|}z,t) - f(z)\|_{L^\infty} = 0$, and from Proposition 3.1, $u(t)$ blows up at time T at one single point; $x=a$. Since $(T,a) \ne (\hat{T},\hat{a})$, we have a contradiction. Therefore, $\hat{q}(T,a,s)$ leaves $V_A(s)$ for $s \ge s_0$.

In conclusion, we derive: $\forall (T,a) \in D\backslash\{(\hat{T},\hat{a})\}$, $\exists s_*(T,a) > s_0$ such that $\forall s \in [s_0, s_*(T,a)]$, $\hat{q}(T,a,s) \in V_A(s)$ and $\hat{q}(T,a,s_*(T,a)) \in \partial V_A(s_*(T,A))$. ($\hat{s}_*(T,a) > s_0$ since $\hat{q}(T,a,s)$ is in $V_{2\hat{A}}(s_0)$ which is strictly included in $V_A(s_0)$). If now we define $\Psi_{\hat{u}_0}$ by (59), then we see from lemma 4.2 that $Im(\Psi_{\hat{u}_0}) \subset \partial\mathcal{C}$.

Proof of iii):

Let $(T,a) \in D_5(s_0)\backslash(\hat{T},\hat{a})$. We have explicitly for $m=0,1$:

$\hat{q}_m(T,a,s) = \int d\mu k_m(y)\chi(y,s)\hat{q}(T,a,y,s)$

$= \int d\mu k_m(y)\chi(y,s)e^{-\frac{s}{p-1}}\hat{u}(a+ye^{-s/2}, T-e^{-s}) - \int d\mu k_m(y)\chi(y,s)\varphi(y,s)$.

From the continuity of $u(x,t)$ with respect to (x,t), and $ii)$ of lemma 4.2, $\hat{s}_*(T,a)$ and $\frac{\hat{s}_*(T,a)^2}{A}(q_0,q_1)(T,a,\hat{s}_*(T,a))$ are continuous with respect to (T,a).

Proof of iv):

Let $\epsilon > 0$. We now construct Γ_ϵ satisfying $d(\Gamma_\epsilon, \Psi_{\hat{u}_0}, 0) = -1$ and $\forall (T,a) \in \Gamma_{A,s_1}$, $|(T,a) - (\hat{T},\hat{a})| \le \epsilon$. This will be implied by the following:

Lemma 4.3 *There exists* $A_6 > 0$ *such that* $\forall A \geq A_6$, $\exists s_6(A) > 0$ *satisfying the following property:*
$\forall s_0 \geq s_6(A)$, $\exists D_6(s_0)$ *neighborhood of* $(\hat{T}, \hat{a})$ *such that* $\forall \epsilon > 0$,
$\exists s_1(A, \epsilon, s_0) > s_0$, $\exists \Gamma_\epsilon$, *a 1-manifold in* $D_6(s_0)$ *satisfying:*
$\forall (T, a) \in \Gamma_\epsilon$, $|(T, a) - (\hat{T}, \hat{a})| \leq \epsilon$
$\forall s \in [s_0, s_1]$, $\hat{q}(T, a, s) \in V_A(s)$,

$$(\hat{q}_0, \hat{q}_1)(T, a, s_1) \in \partial \hat{V}_A(s_1),$$

$$d(\Gamma_\epsilon, (\hat{q}_0, \hat{q}_1)(., ., s_1), 0) = -1. \quad (60)$$

a) Proof of lemma 4.3: The proof is not difficult, but it is a bit technical. See Appendix B for more details.

b) Lemma 4.3 implies iv):
Let $A_4 = \max(A_5, A_6)$, and $A \geq A_4$. Let $s_4(A) = \max(s_5(A), s_6(A))$, and $s_0 \geq s_4(A)$. Let $D_4(s_0) = D_5(s_0) \cap D_6(s_0)$, and $\epsilon > 0$.
Then, according to the beginning of Proof of Proposition 4.2, *i) ii)* and *iii)* hold. We take now $s_1 = s_1(A, \epsilon, s_0)$ and Γ_ϵ. By lemma 4.3, we see that $\forall (T, a) \in \Gamma_\epsilon$, $s_*(T, a) = s_1$, and $\Psi_{\hat{u}_0}(T, a) = \frac{s_1^2}{A}(\hat{q}_0, \hat{q}_1)(T, a, s_1)$. From (60), we derive, $d(\Gamma_\epsilon, \Psi_{\hat{u}_0}, 0) = -1$, which concludes the proof of Proposition 4.2.

Step 2: Behavior of $q(T, a)$ for u_0 near $\hat{u}_0$.
Now, we fix $A_0 = 1 + \sup(2\hat{A}, A_2, A_3, A_4)$, and then
$s_0 = s_0(A_0) = \sup(\hat{s}_0, s_2(A_0), s_3(A_0), s_4(A_0))$. Applying lemma 4.1 gives us $\mathcal{V}_\infty$, and $D_1(s_0)$. We then fix $D_0 = D_1(s_0) \cap D_4(s_0)$. Applying proposition 4.2 with s_0 and $\epsilon_0 > 0$ small enough gives us the curve $\Gamma_0 = \Gamma_{\epsilon_0}$, included in D_0. We consider now Γ_0 as fixed.

Our purpose is to show that for u_0 near $\hat{u}_0$, the behavior of $q(T, a)$ on the curve $\Gamma_0 = \Gamma(\hat{u}_0)$ is the same as $\hat{q}(T, a)$. More precisely, we have:

Proposition 4.3 (Stability result on the behavior on Γ_0, for u_0 near $\hat{u}_0$) $\forall \epsilon > 0$, $\exists \mathcal{V}_\epsilon \subset \mathcal{V}_\infty$, *neighborhood of* $\hat{u}_0$ *such that* $\forall u_0 \in \mathcal{V}_\epsilon$, $\forall (T, a) \in \Gamma_0$,
i) $q(T, a, s)$ *is defined for* $s \in (-\log T, s_0]$ *and* $q(T, a, s_0) \in V_{A_0}(s_0)$,
ii) $\exists s_*(T, a) > s_0$ *such that* $\forall s \in [s_0, s_*(T, a)]$, $q(T, a, s) \in V_{A_0}(s)$, *and* $(q_0, q_1)(T, a, s_*(T, a)) \in \partial \hat{V}_{A_0}(s_*(T, a))$. *Then we can define*

$$\begin{aligned} \Psi_{u_0} : \gamma &\longrightarrow \partial \mathcal{C} \\ (T, a) &\longrightarrow \frac{s_*(T, a)^2}{A_0}(q_0, q_1)(T, a, s_*(T, a)) \end{aligned} \quad (61)$$

where $\mathcal{C}$ is the unit square of $I\!R^2$.
Moreover,
iii) Ψ_{u_0} is a continuous mapping from Γ_0 to $\partial\mathcal{C}$,
iv) $\|\Psi_{u_0|\Gamma_0} - \Psi_{\hat{u}_0|\Gamma_0}\|_{L^\infty(\Gamma_0)} \leq \epsilon$

Proof:
We first show a local result, then by compactness arguments we conclude the proof. We claim the following:

Lemma 4.4 (Punctual stability on Γ_0) *$\forall\epsilon > 0$, $\forall(T,a) \in \Gamma_0$, $\exists D_{\epsilon,T,a}$ neighborhood of (T,a) in D_0, $\exists\mathcal{V}_{\epsilon,\mathcal{T},\dashv}$ neighborhood of $\hat{u}_0$ in $\mathcal{V}$ such that:*
$\forall(T',a') \in D_{\epsilon,T,a}$, $\forall u_0 \in \mathcal{V}_{\epsilon,\mathcal{T},\dashv}$,
i) $q(T',a',s)$ is defined (at least) for $s \in (-\log T, s_0]$ and $q(T',a',s_0) \in V_{A_0}(s_0)$,
ii)$\exists s_*(T',a') > s_0$ such that $\forall s \in [s_0, s_*(T',a')]$, $q(T',a',s) \in V_{A_0}(s)$, and $(q_0,q_1)(T',a',s_*(T',a')) \in \partial\hat{V}_{A_0}(s_*(T',a'))$.
Moreover,

$$|\frac{s_*(T',a')^2}{A_0}(q_0,q_1)(T',a',s_*(T',a')) - \frac{s_*(T',a')^2}{A_0}(\hat{q}_0,\hat{q}_1)(T',a',\hat{s}_*(T',a'))| \leq \epsilon. \tag{62}$$

We remark that Proposition 4.3 follows from lemma 4.4. Indeed, for $\epsilon > 0$, from lemma we write:

$$\Gamma_0 \subset \cup_{(T,a)\in\Gamma_0} D_{\epsilon,T,a},$$

and using the compactness of Γ_0, we have the conclusion.

Proof of Lemma 4.4
We have explicitly for $u_0 \in H$, $s \in (-\log T, -\log(T-\hat{T}))$ if $T > \hat{T}$, otherwise $s \in (-\log T, +\infty)$, and $m = 0,1$
$q_{m,T,a}(s) = \int d\mu k_m(y)\chi(y,s)q(T,a,y,s)$
$= \int d\mu k_m(y)\chi(y,s)e^{-\frac{s}{p-1}}u(a+ye^{-s/2}, T-e^{-s}) - \int d\mu k_m(y)\chi(y,s)\varphi(y,s)$.
Therefore, using the continuity of $u(x,t)$ with respect to (u_0,x,t), $(q_0,q_1)(T,a,s)$ is a continuous function of (u_0,T,a,s). Using this fact and the transversality of $(\hat{q}_0,\hat{q}_1)(T,a,\hat{s}_*(T,a))$ on $\hat{V}_{A_0}(s_*(T,a))$ (lemma 4.2 *ii*)), *i*) and *ii*) follow then easily.
This concludes the proof of Proposition 4.3.

Step 3: The conclusion of the proof

From continuity properties of the topological degree, there exists $\epsilon_1 > 0$ such that $\forall \Psi \in \mathcal{C}(-\prime, \mathrm{IR}^{\in})$ satisfying $\|\Psi - \Psi_{\hat{u}_0}\|_{L^\infty(\Gamma_0)} \leq \epsilon_1$, we have $d(\Gamma_0, \Psi, 0) = -1$.
Applying Proposition 4.3, with $\epsilon = \epsilon_1$, we have $\forall u_0 \in \mathcal{V}_{\epsilon_\infty}$, $d(\Gamma_0, \Psi_{u_0}, 0) = -1$.
We claim that the conclusion of Proposition 4.1 follows with A_0, s_0, D_0 and $\mathcal{V}_\prime = \mathcal{V}_{\epsilon_\infty}$. Indeed, by contradiction as in section 3: suppose that for $u_0 \in \mathcal{V}_\prime$, we have $\forall (T, a) \in D_0$, there exists $s \geq s_0$, $q(T, a, s) \notin V_{A_0}(s)$. Let $s_*(T, a)$ be the infimum of all these s. We now remark that Ψ_{u_0} is defined on D_0 (lemma 4.1 and lemma 4.2). Ψ_{u_0} is continuous from D_0 to $\partial\mathcal{C}$ (see proof of Proposition 4.2 *iii*), and $d(\Gamma_0, \Psi_{u_0}, 0) = 0$, which is a contradiction. Hence Proposition 4.1 is proved, which concludes the proof of Theorem 2.

4.2 Case $N \geq 2$:

Let us consider $\hat{u}_0$ an initial data in H, constructed in Theorem 1. Let $\hat{u}(t)$ be the solution of equation (1):

$$u_t = \Delta u + |u|^{p-1}u, u(0) = \hat{u}_0.$$

Let $\hat{T}$ be its blow-up time and $\hat{a}$ be its blow-up point.

Although the proof of Theorem 1 was given in 1 dimension, we know that there exists $\hat{A} > 0$, $\hat{s}_0 > \log \hat{T}$ such that $\forall s \geq \hat{s}_0$, $\hat{q}_{\hat{T},\hat{a}}(s) \in V_{\hat{A}}(s)$, where:
- $\hat{q}_{\hat{T},\hat{a}}$ is defined in (13) by:

$$q_{\hat{T},\hat{a}}(y, s) = e^{-\frac{s}{p-1}}\hat{u}(\hat{a} + ye^{-\frac{s}{2}}, \hat{T} - e^{-s}) - [\frac{N\kappa}{2ps} + (p - 1 + \frac{(p-1)^2}{4ps}|y|^2)^{-\frac{1}{p-1}}],$$

and
-Definitions 3.1 and 3.2 are still good to define $V_{\hat{A}}(s)$, if we understand $q_m(s)$ to be a vector valued function, as defined in section 2 (see (27) and (28)), and $|q_m(s)|$ to be the supremum of of all coordinates of $q_m(s)$. (By the same way, the definition of $\hat{V}_A(s)$ given in 3.3 is good here).

With these adaptations, our purpose is summarized in the following Proposition, analogous to Proposition 4.1:

Proposition 4.4 (Reduction) *There exist $A_0 > 0$, $s_0 > 0$, D_0 neighborhood of $(\hat{T}, \hat{a})$ in $\mathrm{I\!R}^{1+N}$, and $\mathcal{V}_\prime$ neighborhood of $\hat{u}_0$ in H with the following property:*

$\forall u_0 \in \mathcal{V}_\prime, \exists (\mathcal{T}, \dashv) \in \mathcal{D}_\prime$ such that $\forall s \geq s_0$, $q_{T,a}(s) \in V_{A_0}(s)$, where $q_{T,a}$ is defined by (13), and $u(t)$ is the solution of equation (1) with initial data $u(0) = u_0$.

Indeed, once this proposition is proved, from (3), (13) and definitions 3.1, 3.2, we have:

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z\ , t) = f(z)$$

uniformly in $z \in \mathbb{R}^N$, with

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}.$$

Proposition 3.1 (which is true in N dimensions) applied to $u(x-a,t)$ then shows directly that $u(t)$ blows-up at time T at one single point: $x = a$.

Formally, the proof in the case $N \geq 2$ and in the case $N = 1$ have exactly the same steps with the same statements of Propositions and lemmas, under the following obvious changes:

-$(\hat{T}, \hat{a})$, (T,a) and (T', a') are in $\mathbb{R}^{1+N}$, and every neighborhood of such a point is a neighborhood in $\mathbb{R}^{1+N}$.

-In Part 2, $\mathcal{C}$ denotes the unit (1+N)-cube of $\mathbb{R}^{1+N}$, Γ (and Γ_ϵ, Γ_0,...) is a Lipschitz N-submanifold of $\mathbb{R}^{1+N}$, forming the boundary of a bounded connected Lipschitz open set of $\mathbb{R}^{1+N}$, and all introduced topological degrees different from zero are equal to $(-1)^N$.

Moreover, the proofs can be adapted without difficulty to the case $N \geq 2$, even:

-the proof of Proposition 4.2, which relays on results of section 3 (subsection 3.3 and lemma 3.2) that are true in N dimensions (In particular, the lemma 3.5 of Bricmont and Kupiainen, with the adaptation $\mathbb{R} \to \mathbb{R}^N$).

-the construction of Γ_ϵ given in Appendix B can be simply adapted to the case $N \geq 2$.

A Proof of lemma 3.5

In this appendix, we prove lemma 3.5. Equation (15) has been studied in [3], hence, our analysis will be very close to [3] (the proof is essentially the same as in [3]). Lemma 3.5 relays mainly on the understanding of the behavior of the kernel $K(s,\sigma,y,x)$ (see (20)). This behavior follows from a perturbation method around $e^{(s-\sigma)\mathcal{L}}(y,x)$.

Step 1: Perturbation formula for $K(s,\sigma,y,x)$

Since $\mathcal{L}$ is conjugated to the harmonic oscillator $e^{-x^2/8}\mathcal{L}e^{x^2/8} = \partial^2 - \frac{x^2}{16} + \frac{1}{4} + 1$, we use the definition (20) of K and give a Feynman-Kac representation for K:

$$K(s,\sigma,y,x) = e^{(s-\sigma)\mathcal{L}}(y,x)\int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{s-\sigma}V(\omega(\tau),\sigma+\tau)d\tau} \tag{63}$$

where $d\mu_{yx}^{s-\sigma}$ is the oscillator measure on the continuous paths $\omega : [0, s-\sigma] \to \mathbb{R}$ with $\omega(0) = x$, $\omega(s-\sigma) = y$, i.e. the Gaussian probability measure with covariance kernel $\Gamma(\tau,\tau')$

$$= \omega_0(\tau)\omega_0(\tau') + 2(e^{-\frac{1}{2}|\tau-\tau'|} - e^{-\frac{1}{2}|\tau+\tau'|} + e^{-\frac{1}{2}|2(s-\sigma)-\tau'+\tau|} - e^{-\frac{1}{2}|2(s-\sigma)-\tau'-\tau|}, \tag{64}$$

which yields $\int d\mu_{yx}^{s-\sigma}\omega(\tau) = \omega_0(\tau)$ with
$\omega_0(\tau) = (\sinh\frac{s-\sigma}{2})^{-1}(y\sinh\frac{\tau}{2} + x\sinh\frac{s-\sigma-\tau}{2})$.

We have in addition

$$e^{\theta\mathcal{L}}(y,x) = \frac{e^\theta}{\sqrt{4\pi(1-e^{-\theta})}}\exp[-\frac{(ye^{-\theta/2}-x)^2}{4(1-e^{-\theta})}].$$

Now, we derive from (63) a simplified expression for $K(s,\sigma,y,x)$ considered as a perturbation of $e^{(s-\sigma)\mathcal{L}}(y,x)$. In order to simplify the notation, we write from now on (ψ,φ) for $\int d\mu(y)\psi(y)\varphi(y)$.

Lemma A.1 (Bricmont-Kupiainen) *$\forall s \geq \sigma \geq 1$ with $s \leq 2\sigma$, the kernel $K(s,\sigma,y,x)$ satisfies*

$$K(s,\sigma,y,x) = e^{(s-\sigma)\mathcal{L}}(y,x)(1 + \frac{1}{s}P_1(s,\sigma,y,x) + P_2(s,\sigma,y,x))$$

where P_1 is a polynomial

$$P_1(s,\sigma,y,x) = \sum_{m,n\geq 0, m+n\leq 2} p_{m,n}(s,\sigma)y^m x^n$$

with $|p_{m,n}(s,\sigma)| \leq C(s-\sigma)$ and

$$|P_2(s,\sigma,y,x)| \leq C(s-\sigma)(1+s-\sigma)s^{-2}(1+|y|+|x|)^4.$$

Moreover, $|(k_2,(K(s,\sigma)-(\sigma s^{-1})^2)h_2)| \leq C(s-\sigma)(1+s-\sigma)s^{-2}$.

Proof: See lemma 5 in [3].

Step 2: Conclusion of the proof of lemma 3.5

Proof of a): From (16), it follows easily that $V(y,s) \leq Cs^{-1}$. Using this estimate and (63), we write:
$|K(s,\tau,y,x)| \leq e^{(s-\tau)\mathcal{L}}(y,x) \int d\mu_{yx}^{s-\tau}(\omega) e^{\int_0^{s-\tau} C(\tau+t)^{-1}dt}$
$\leq e^{(s-\tau)\mathcal{L}}(y,x) \int d\mu_{yx}^{s-\tau}(\omega)(s\tau^{-1})^C \leq Ce^{(s-\tau)\mathcal{L}}(y,x)$ since $s \leq 2\tau$ and $d\mu_{yx}^{s-\tau}$ is a probability.

Proof of c): See lemma 2 in [3].

Proof of b): We consider $A' > 0$, $A'' > 0$, $A''' > 0$ and $\rho^* > 0$. Let $s_0 \geq \rho^*$, $\sigma \geq s_0$ and $q(\sigma)$ satisfying (46). We want to estimate some components of $\alpha(y,s) = K(s,\sigma)q(\sigma)$ (see (47)) for each $s \in [\sigma, \sigma+\rho^*]$.

Since $\sigma \geq s_0 \geq \rho^*$, we have: $\forall \tau \in [\sigma, s]$, $\tau \leq s \leq 2\tau$. Therefore, up to a multiplying constant, any power of any $\tau \in [\sigma, s]$ will be bounded systematically by the same power of s during the proof.

i) Estimate of $\alpha_2(s)$:

$\alpha_2(s) = (k_2, \chi(.,s)K(s,\sigma)q(\sigma))$
$= \sigma^2 s^{-2} q_2(\sigma) + (k_2, (\chi(.,s) - \chi(.,\sigma))\sigma^2 s^{-2} q(\sigma))$
$+(k_2, \chi(.,s)(K(s,\sigma) - \sigma^2 s^{-2})q(\sigma))$.

From (46), (21) and (26), we have $|\sigma^2 s^{-2} q_2(\sigma)| \leq A'' s^{-2} \log\sigma$ and
$|(k_2, (\chi(.,s) - \chi(.,\sigma))\sigma^2 s^{-2} q(\sigma))| \leq Ce^{-C\sigma}\sigma^{-3/2}(s-\sigma)\sigma^2 s^{-2} \frac{\max(A'',A''')}{\sqrt{\sigma}}$
$\leq CA'(s-\sigma)s^{-3}$ for $\sigma \geq s_0 \geq s_1(A', A'', A''', \rho^*)$.

We write $(k_2, \chi(.,s)(K(s,\sigma) - \sigma^2 s^{-2})q(\sigma))$ as $\sum_{r=0}^{2} b_r + b_- + b_e$ where
$b_r = (k_2, \chi(.,s)(K(s,\sigma) - \sigma^2 s^{-2})h_r)q_r(\sigma)$,
$b_- = (k_2, \chi(.,s)(K(s,\sigma) - \sigma^2 s^{-2})q_-(\sigma))$ and $b_e = (k_2, \chi(.,s)(K(s,\sigma) - \sigma^2 s^{-2})q_e(\sigma))$.

For $r = 0$ or 1, we use lemma A.1, corollary 3.1, (21), (46), the fact that $e^{(s-\sigma)\mathcal{L}} h_r = e^{(1-r/2)(s-\sigma)} h_r$ and $(k_2, h_r) = 0$, and derive $|b_r| =$
$|(k_2, \chi(.,s)(K(s,\sigma) - e^{(s-\sigma)\mathcal{L}})h_r)q_r(\sigma) + (k_2, \chi(.,s)(e^{(s-\sigma)\mathcal{L}} - \sigma^2 s^{-2})h_r)q_r(\sigma)|$
$\leq CA'(s-\sigma)s^{-3} + Ce^{-Cs}(s-\sigma) \leq CA'(s-\sigma)s^{-3} \leq CA'(s-\sigma)s^{-3}$.

We have by lemma A.1 and the same arguments $|b_2| = |(k_2, (K(s,\sigma) - \sigma^2 s^{-2})h_2)q_2(\sigma) + (k_2, (-1+\chi(.,s))(K(s,\sigma) - \sigma^2 s^{-2})h_2)q_2(\sigma)| \leq C(s-\sigma)(1+s-\sigma)s^{-2} A'' s^{-2} \log s + Ce^{-Cs}(s-\sigma) \leq CA'(s-\sigma)s^{-3}$
if $\sigma \geq s_0 \geq s_2(A', A'', \rho^*)$.

b_- can be treated exactly as b_0, it is bounded by $C(s-\sigma)A''' s^{-3}$.

Since $K(s,\sigma)-\sigma^2 s^{-2} = K(s,\sigma)-e^{(s-\sigma)\mathcal{L}}+(e^{(s-\sigma)\mathcal{L}}-1)+(1-\sigma^2 s^{-2})$, we write $b_e = b_{e,1}+b_{e,2}+b_{e,3}$ with $b_{e,1} = (k_2, \chi(.,s)(K(s,\sigma)-e^{(s-\sigma)\mathcal{L}})q_e(\sigma))$, $b_{e,2} = (k_2, \chi(.,s)\int_0^{s-\sigma} d\tau \mathcal{L} e^{\tau\mathcal{L}} q_e(\sigma))$, $b_{e,3} = (k_2, \chi(.,s)(1-\sigma^2 s^{-2})q_e(\sigma))$.

From (46), we bound $b_{e,3}$ by $C(s-\sigma)s^{-1}A''\sigma^{-1/2}e^{-C\sigma} \le C(s-\sigma)A's^{-3}$ if $\sigma \ge s_0 \ge s_3(A', A'', \rho^*)$. Since $\mathcal{L}$ is self-adjoint, $|b_{e,2}| \le$ $\int \frac{e^{-y^2/4}}{\sqrt{4\pi}} dy \mathcal{L}(k_2\chi(.,s))(y) \int_0^{s-\sigma} d\tau \int dx \frac{e^{(s-\sigma)}}{\sqrt{4\pi(1-e^{-1})}} \exp[-\frac{(ye^{-\tau/2}-x)^2}{4(1-e^{-\tau})}]A''\sigma^{-1/2}$.

Now, we have $e^{\frac{1}{2}[-\frac{y^2}{4}-\frac{(ye^{-\tau/2}-x)^2}{4(1-e^{-\tau})}]} \le e^{-C(K_0)s} \le e^{-2s}$, for $|y| \le 2K_0\sqrt{s}$ and $|x| \ge K_0\sqrt{\sigma}$ (if K_0 is big enough and $s_0 \ge \rho^*$). Hence, $|b_{e,2}| \le$ $CA''s^{-1/2}\int e^{-y^2/8}dy \int_0^{s-\sigma} d\tau \int dx \frac{e^{-s}}{\sqrt{4\pi(1-e^{-1})}} \exp[-\frac{1}{2}\frac{(ye^{-\tau/2}-x)^2}{4(1-e^{-\tau})}]$
$\le CA''s^{-1/2}(s-\sigma)e^{-s} \le CA'(s-\sigma)s^{-3}$ if $\sigma \ge s_0 \ge s_4(A', A'', \rho^*)$.

Using these techniques and lemma A.1 we bound $b_{e,1}$ in the same way.

Adding all these bounds yields the bound for $|\alpha_2(s)|$.

ii) Estimate of $\alpha_-(y,s)$:

By definition, $\alpha_-(y,s)$

$$\begin{aligned} &= P_-(\chi(.,s)K(s,\sigma)q(\sigma)) = P_-(\chi(.,s)K(s,\sigma)q_-(\sigma)) \\ &+ \sum_{r=0}^{2} q_r(\sigma)P_-(\chi(.,s)K(s,\sigma)h_r) + P_-(\chi(.,s)K(s,\sigma)q_e(\sigma)) \end{aligned} \quad (65)$$

where P_- is the $L^2(\mathbb{R}, d\mu)$ projector on the negative subspace of $\mathcal{L}$ (see subsection 2.2). In order to bound the first term, we proceed as in [3]

$$K(s,\sigma)q_-(\sigma) = \int dx e^{x^2/4} K(s,\sigma,.,x)f(x) \quad (66)$$

where $f(x) = e^{-x^2/4}q_-(x,\sigma)$. From Step 1, we have $e^{x^2/4}K(s,\sigma,y,x) = N(y,x)E(y,x)$ with

$$N(y,x) = [4\pi(1-e^{-(s-\sigma)}]^{-1/2} e^{s-\sigma} e^{x^2/4} e^{-\frac{(y-e^{-(s-\sigma)/2}x)^2}{4(1-e^{-(s-\sigma)})}} \quad (67)$$

and $E(y,x) = \int d\mu_{yx}^{s-\sigma}(\omega) e^{\int_0^{s-\sigma} V(\omega(\tau),\sigma+\tau)d\tau}$. Let $f^0 = f$ and for $m \ge 1$, $f^{(-m-1)}(y) = \int_{-\infty}^{y} dx f^{(-m)}(x)$. From (46) and the following lemma, we can bound $f^{(-m)}$:

Lemma A.2 $|f^{(-m)}(y)| \le CA'''s^{-2}(1+|y|^3)^{3-m}e^{-y^2/4}$.

Proof: See lemma 6 in [3].

By integrating by parts, we rewrite (66) as:

$$\begin{aligned}(K(s,\sigma)q_-(\sigma))(y) &= \sum_{r=0}^{2}(-1)^{r+1}\int \partial_x^r N(y,x)\partial_x E(y,x) f^{(-r-1)}(x)dx \\ &- \int \partial_x^3 N(y,x)E(y,x)f^{(-3)}(x)dx. \qquad (68)\end{aligned}$$

From (67), we get for $s-\sigma\geq 1$ and $r\in\{0,1,2,3\}$
$|\partial_x^r N(y,x)| \leq Ce^{-\frac{r(s-\sigma)}{2}}(1+|y|+|x|)^r e^{x^2/4}e^{(s-\sigma)\mathcal{L}}(y,x)$.

Using the integration by parts formula for Gaussian measures (see [11]), we have:

$$\begin{aligned}\partial_x E(y,x) &= \frac{1}{2}\int_0^{s-\sigma}\int_0^{s-\sigma} d\tau d\tau' \partial_x\Gamma(\tau,\tau')\int d\mu_{yx}^{s-\sigma}(\omega)V'(\omega(\tau),\sigma+\tau) \\ & V'(\omega(\tau'),\sigma+\tau')e^{\int_0^{s-\sigma} d\tau'' V(\omega(\tau''),\sigma+\tau'')} \qquad (69) \\ + \ & \frac{1}{2}\int_0^{s-\sigma} d\tau\partial_x\Gamma(\tau,\tau)\int d\mu_{yx}^{s-\sigma}(\omega)V''(\omega(\tau),\sigma+\tau)e^{\int_0^{s-\sigma} d\tau'' V(\omega(\tau''),\sigma+\tau'')}.\end{aligned}$$

By (16), we have $V(y,s)\leq Cs^{-1}$ and $|\frac{d^nV}{dy^n}|\leq Cs^{-n/2}$ for $n=0,1,2$. Combining this with (64) and using $s\leq 2\sigma$ we have
$\int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{s-\sigma} d\tau'' V(\omega(\tau''),\sigma+\tau'')}\leq C$ and $|\partial_x E(y,x)|\leq Cs^{-1}(s-\sigma)(1+s-\sigma)(|y|+|x|)$.

Using (46), (68) and all these bounds, we get
$|(K(s,\sigma)q_-(\sigma))(y)| \leq CA'''s^{-2}e^{-(s-\sigma)/2}(1+|y|^3)$ if $\sigma\geq s_0\geq s_5(\rho^*)$ and $s-\sigma\geq 1$. This yields $|(P_-\chi(.,s)K(s,\sigma)q_-(\sigma))(y)|\leq CA'''s^{-2}e^{-(s-\sigma)/2}(1+|y|^3)$ if $s-\sigma\geq 1$. For $s-\sigma\leq 1$, we use directly lemma A.1, corollary 3.1, (46) and $C\leq e^{-(s-\sigma)/2}$ to get the same estimate.

Now, we consider the second term in (65) ($r=0,1,2$). From corollary 3.1, lemma A.1, and the fact that $|y|\leq 2K_0 s^{1/2}$, we obtain:

$$\begin{aligned}&|q_r(\sigma)(\chi(.,s)K(s,\sigma)h_r)(y) - q_r(\sigma)e^{(s-\sigma)(1-r/2)}(\chi(.,s)h_r)(y)| \\ &\leq C\max(A',A'')s^{-3+1/2}\log s.(s-\sigma)(1+s-\sigma)e^{s-\sigma}(1+|y|^3) \quad (70)\end{aligned}$$

Hence $P_-\{q_r(\sigma)(\chi(.,s)K(s,\sigma)h_r)(y) - q_r(\sigma)e^{(s-\sigma)(1-r/2)}(\chi(.,s)h_r)(y)\}$ satisfies the same bound. Since $P_-h_r=0$ and $|(1-\chi(.,s))h_r|\leq Cs^{-1/2}(1+|y|^3)$, we can bound $q_r(\sigma)e^{(s-\sigma)(1-r/2)}P_-(\chi(.,s)h_r)$ by (70). Hence, the second term of (65) is bounded by $CA'''s^{-2}e^{-(s-\sigma)/2}(1+|y|^3)$ if $\sigma\geq s_0\geq s_6(A',A'',A''',\rho^*)$.

For the last term in (65), we use (46) and a) of lemma 3.5 to get

$$\|(1+|y|^3)^{-1}\chi(.,s)K(s,\sigma)q_e(\sigma)\|_{L^\infty} \le CA''e^{s-\sigma}s^{-1/2}\sup_{y,x}(1+|y|^3)^{-1}$$

$$.\exp[-\frac{1}{2}\frac{(x-ye^{-(s-\sigma)/2})^2}{4(1-e^{-(s-\sigma)})}]\chi(y,\sigma+(s-\sigma))(1-\chi(x,\sigma))$$

$$\le \begin{cases} CA''s^{-2} & s-\sigma \le t_0 \\ e^{-s} & s-\sigma \ge t_0 \end{cases}$$

for a suitable constant t_0. This yields a bound on the last term in (65) which can be written as $CA''e^{-(s-\sigma)^2}s^{-2}(1+|y|^3)$ for $\sigma \ge s_0$ large enough.

Hence, combining all bounds for terms in (65), we have

$$|\alpha_-(y,s)| \le Cs^{-2}(A'''e^{-(s-\sigma)/2} + A''e^{-(s-\sigma)^2})(1+|y|^3).$$

Estimate of $\alpha_e(y,s)$:

We write $\alpha_e(y,s) = (1-\chi(y,s))K(s,\sigma)q(\sigma) = (1-\chi(y,s))K(s,\sigma)(q_b(\sigma) + q_e(\sigma))$. From (46) and corollary 3.1, we have $|q_b(y,s)| \le CA'''\sigma^{-1/2}$ and $\|(1-\chi(y,s))K(s,\sigma)q_b(\sigma)\|_{L^\infty} \le A'''e^{s-\sigma}s^{-1/2}$ if $\sigma \ge s_0 \ge s_7(A',A'',A''')$. Using (46) and the following lemma from [3]:

Lemma A.3 $\|K(s,\sigma)(1-\chi(\sigma))\|_{L^\infty} \le Ce^{-(s-\sigma)/p}$

we have $\|(1-\chi(y,s))K(s,\sigma)q_e(\sigma)\|_{L^\infty} \le A''e^{-(s-\sigma)/p}s^{-1/2}$, which yields the conclusion.

This concludes the proof of lemma 3.5.

B Proof of lemma 4.3

Let us recall lemma 4.3:

Lemma B.1 *There exists* $A_6 > 0$ *such that* $\forall A \ge A_6$, $\exists s_6(A) > 0$ *satisfying the following property:*
$\forall s_0 \ge s_6(A)$, $\exists D_6(s_0)$ *neighborhood of* $(\hat{T},\hat{a})$ *such that* $\forall \epsilon > 0$,
$\exists s_1(A,\epsilon,s_0) > s_0$, $\exists \Gamma_\epsilon$, *a 1-manifold in* $D_6(s_0)$ *satisfying:*
$\forall (T,a) \in \Gamma_\epsilon$, $|(T,a)-(\hat{T},\hat{a})| \le \epsilon$
$\forall s \in [s_0,s_1]$, $\hat{q}(T,a,s) \in V_A(s)$,

$$(\hat{q}_0,\hat{q}_1)(T,a,s_1) \in \partial\hat{V}_A(s_1), \tag{71}$$

$$d(\Gamma_\epsilon,(\hat{q}_0,\hat{q}_1)(.,.,s_1),0) = -1. \tag{72}$$

In this lemma, we want to control the evolution of $\hat{q}(T,a,s)$ in $V_A(s)$, for (T,a) close to $(\hat{T},\hat{a})$. Hence, in a first step, we use $\hat{q}_{\hat{T},\hat{a}}(s)\in V_{\hat{A}}(s)$ $\forall s\geq \hat{s}_0$, to give estimates on different components of $\hat{q}_{T,a}(s)$, for (T,a) near $(\hat{T},\hat{a})$. From these estimates, we introduce a function $(\tilde{q}_0,\tilde{q}_1)(T,a,s)$ close to $(\hat{q}_0,\hat{q}_1)(T,a,s)$, but much more simple, and show that $(\tilde{q}_0,\tilde{q}_1)$ satisfies properties analogous to (71) and (72). Therefore, we extend this result to $(\hat{q}_0,\hat{q}_1)$, by continuity, and then finish the proof of lemma 4.3.

Step 1: Asymptotic development of $\hat{q}(T,a)$ for (T,a) near $(\hat{T},\hat{a})$ Applying (13) and (3), one time to $(\hat{T},\hat{a})$ and one time to (T,a), we write:

$$\begin{aligned}\hat{q}(T,a,y,s) &= \{(1-\tau)^{-\frac{1}{p-1}}\hat{q}(\hat{T},\hat{a},\frac{y+\alpha}{\sqrt{1-\tau}},s-\log(1-\tau))\} \\ + & \quad \{(1-\tau)^{-\frac{1}{p-1}}(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}} \\ - & \quad (p-1+\frac{(p-1)^2y^2}{4ps})^{-\frac{1}{p-1}}\} \\ + & \quad \{(1-\tau)^{-\frac{1}{p-1}}\frac{\kappa}{2p(s-\log(1-\tau))}-\frac{\kappa}{2ps}\},\end{aligned} \tag{73}$$

with $\tau=(T-\hat{T})e^s$, and $\alpha=(a-\hat{a})e^{s/2}$. Now, we use $\hat{q}(\hat{T},\hat{a},s)\in V_{\hat{A}}(s)$ for $s\geq\hat{s}_0$, to give a development of $\hat{q}_{T,a}(y,s)$, when $|\tau|\leq\frac{1}{2}$, and $|\alpha|\leq\frac{1}{2}$.

Lemma B.2 (development of $\hat{q}(T,a)$ near $(\hat{T},\hat{a})$) *There exists $s_7>0$ such that $\forall s\geq s_7$, $\forall (T,a)\in I\!R^2$ satisfying $|(T-\hat{T})e^s|\leq\frac{1}{2}$ and $|(a-\hat{a})e^{\frac{s}{2}}|\leq\frac{1}{2}$, we have:*

$$\hat{q}_0(T,a,s) = \tilde{q}_0(T,a,s)+O(\frac{\log s}{s^{5/2}}+\frac{\tau}{\sqrt{s}}+\tau^2+\alpha^2\frac{1}{s}) \tag{74}$$

$$\hat{q}_1(T,a,s) = \tilde{q}_1(T,a,s)+O(\frac{\alpha\log s}{s^2}+\frac{\alpha^2}{s}+\frac{\tau}{s}+\frac{\log s}{s^3})$$

$$\frac{\partial\hat{q}_0}{\partial T}(T,a,s) = \frac{\partial\tilde{q}_0}{\partial T}(T,a,s)+e^s(O(\tau+s^{-1/2})), \tag{75}$$

$$\frac{\partial\hat{q}_0}{\partial a}(T,a,s) = \frac{\partial\tilde{q}_0}{\partial a}(T,a,s)+e^{s/2}O(\frac{\log s}{s^2}+\frac{|\alpha|}{s}), \tag{76}$$

$$\frac{\partial\hat{q}_1}{\partial T}(T,a,s) = \frac{\partial\tilde{q}_1}{\partial T}(T,a,s)+e^sO(\frac{1}{\sqrt{s}}), \tag{77}$$

$$\frac{\partial\hat{q}_1}{\partial a}(T,a,s) = \frac{\partial\tilde{q}_1}{\partial a}(T,a,s)+e^{s/2}O(\frac{|\tau|}{s}+\frac{1}{s^2}+\frac{|\alpha|}{s}) \tag{78}$$

with

$$\begin{aligned}\tilde{q}_0(T,a,s) &= -\frac{5\kappa}{8ps^2}+\tau\frac{\kappa}{p-1} \\ \tilde{q}_1(T,a,s) &= -\frac{\alpha}{s}\frac{\kappa}{2p},\end{aligned} \tag{79}$$

and $\tau=(T-\hat{T})e^s$ *and* $\alpha=(a-\hat{a})e^{\frac{s}{2}}$.
Moreover,

$$\begin{aligned}|\hat{q}_2(T,a,s)| &\leq C\frac{\log s}{s^2}+C\frac{|\tau|}{s}+C\tau^2 \\ |\hat{q}_-(T,a,y,s)| &\leq C(1+|y|^3)(\frac{1}{s^2}+\frac{|\tau|+|\alpha|}{s^{3/2}}) \\ |\hat{q}_e(T,a,y,s)| &\leq \frac{C}{\sqrt{s}}.\end{aligned}$$

Proof of lemma B.2:
The idea is simple: for $s\geq\hat{s}_0$, , we try to express each component of $\hat{q}(T,a)$ in terms of the corresponding component of $\hat{q}(\hat{T},\hat{a})$, and bound the residual terms using $\hat{q}(\hat{T},\hat{a},s)\in V_{\hat{A}}(s)$ and other estimates that follow from.
Hence, we first give various estimates following from $\hat{q}(\hat{T},\hat{a},s)\in V_{\hat{A}}(s)$, and then , we prove only some of the estimates in lemma B.2, since the other estimates can be obtained in the same way.

i) We write the estimates following from $\hat{q}(\hat{T},\hat{a},s)\in V_{\hat{A}}(s)$.

Lemma B.3 (Consequences of $\hat{q}(\hat{T},\hat{a},s)\in V_{\hat{A}}(s)$) *$\exists s_{16}>0$, $\forall s\geq s_{16}$,*

$$|\hat{q}(\hat{T},\hat{a},y,s)|\leq\frac{C}{\sqrt{s}}, \tag{80}$$

$$|\hat{q}_b(\hat{T},\hat{a},y,s)|\leq\frac{C\log s}{s^2}(1+|y|^3), \tag{81}$$

$$\hat{q}_0(\hat{T},\hat{a},s)=-\frac{5\kappa}{8ps^2}+o(\frac{1}{s^2}),|\frac{\partial\hat{q}_0}{\partial s}(\hat{T},\hat{a},s)|\leq\frac{C}{s^2}, \tag{82}$$

$$|\hat{q}_1(\hat{T},\hat{a},s)|\leq C\frac{\log s}{s^3}, \tag{83}$$

$$|\frac{\partial\hat{q}}{\partial s}(\hat{T},\hat{a},y,s)|\leq C\frac{1+|y|}{\sqrt{s}}, \tag{84}$$

Proof of lemma B.3:

(80) and (81) follow directly from Definition 3.2.

After some simple calculations, we show that $\int d\mu\chi(y,s)R(y,s) = \frac{5\kappa}{8ps^2} + O(s^{-3})$. As in the proof of lemma 3.2, we write the equation satisfied by $q_0(s)$:

$$\frac{d\hat{q}_0}{ds}(\hat{T},\hat{a},s) = \hat{q}_0(\hat{T},\hat{a},s) + \frac{5\kappa}{8ps^2} + O(\frac{\log s}{s^3}),$$

which implies (82).

By the same way, we write:

$$\frac{d\hat{q}_1}{ds}(\hat{T},\hat{a},s) = \frac{1}{2}\hat{q}_1(\hat{T},\hat{a},s) + O(\frac{\log s}{s^3}),$$

which yields (83).

From (80), we derive that $r = \frac{\partial q}{\partial s}$ satisfies

$$\frac{\partial r}{\partial s} = \frac{\partial^2 r}{\partial y^2} - \frac{1}{2}y\frac{\partial r}{\partial y} + A(y,s)r + D(y,s),$$

with $|A(y,s)| \leq C$ and, if $p \geq \frac{3}{2}$ $|D(y,s)| \leq \frac{C}{s}$, otherwise, $|D(y,s)| \leq \frac{C}{s^{p-\frac{1}{2}}}$. By parabolic regularity, (84) follows.

ii) Proof of some estimates in lemma B.2: (74) and (75)
(The other estimates follow from similar techniques).

From (73), we have: $\hat{q}_0(T,a,s) = I_1 + I_2 + I_3$, with

$I_1 = (1-\tau)^{-\frac{1}{p-1}} \int d\mu(y)\chi(y,s)\hat{q}(\hat{T},\hat{a},\frac{y+\alpha}{\sqrt{1-\tau}}, s-\log(1-\tau))$,

$I_2 = (1-\tau)^{-\frac{1}{p-1}} \int d\mu(y)\chi(y,s)(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}}$
$-\int d\mu(y)\chi(y,s)(p-1+\frac{(p-1)^2y^2}{4ps})^{-\frac{1}{p-1}}$

$I_3 = \int d\mu(y)\chi(y,s)\{(1-\tau)^{-\frac{1}{p-1}}\frac{\kappa}{2p(s-\log(1-\tau))} - \frac{\kappa}{2ps}\}$.

-I_3*:* We have easily: $|I_3| \leq C|\tau|s^{-1}$.

-I_2*:* Since all quantities appearing in I_2 are bounded, we can write:

$I_2 = O(e^{-s}) + \int d\mu(y)\{(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}} - (p-1+\frac{(p-1)^2y^2}{4ps})^{-\frac{1}{p-1}}\}$
$+\frac{\tau}{p-1}\int d\mu(y)(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}} + O(\tau^2)$,
$= O(e^{-s}) + O(\tau^2)$
$+\int d\mu(y)\{\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))} - \frac{(p-1)^2y^2}{4ps}\}\frac{-1}{p-1}(p-1+\frac{(p-1)^2y^2}{4ps})^{-1-\frac{1}{p-1}}$
$+O(\int d\mu(y)\{\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))} - \frac{(p-1)^2y^2}{4ps}\}^2)$

$+\frac{\tau\kappa}{p-1}+\frac{\tau}{p-1}\{\int d\mu(y)(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}}-\int d\mu(y)\kappa\}$, hence,
$|I_2-\frac{\tau\kappa}{p-1}|\le Ce^{-s}+C\tau^2+C|\tau|s^{-1}+C\alpha^2s^{-1}+C\tau^2s^{-2}+C\alpha^2s^{-2}+C\alpha^4s^{-2}+C|\tau|s^{-1}$. Therefore,
$|I_2-\frac{\tau\kappa}{p-1}|\le Ce^{-s}+C\tau^2+C|\tau|s^{-1}+C\alpha^2s^{-1}$.

-*I_1:* Using (80), we write:
$I_1=O(\tau s^{-1/2})+\int d\mu(y)\chi(y,s)\hat{q}(\hat{T},\hat{a},\frac{y+\alpha}{\sqrt{1-\tau}},s-\log(1-\tau))$. If we introduce a new integration variable: $z=\frac{y+\alpha}{\sqrt{1-\tau}}$, we obtain: $I_1=O(\tau s^{-1/2})+L_1+L_2$ with
$L_1=\int\chi(z,s-\log(1-\tau))\hat{q}(\hat{T},\hat{a},z,s-\log(1-\tau))\frac{\exp(-\frac{(z\sqrt{1-\tau}-\alpha)^2}{4})}{4\pi}dz$, and
$L_2=\int\{\chi(z\sqrt{1-\tau}-\alpha,s)-\chi(z,s-\log(1-\tau))\}\hat{q}(\hat{T},\hat{a},z,s-\log(1-\tau))$
$\frac{\exp(-\frac{(z\sqrt{1-\tau}-\alpha)^2}{4})}{4\pi}dz$.

$L_1=\int\chi(z,s-\log(1-\tau))\hat{q}(\hat{T},\hat{a},z,s-\log(1-\tau))\frac{\exp(-\frac{z^2}{4})}{4\pi}\exp(\frac{\tau z^2}{4})$
$\exp(\frac{2\alpha z\sqrt{1-\tau}-\alpha^2}{4})dz$
$=O(\tau s^{-1/2})+\int\chi(z,s-\log(1-\tau))\hat{q}(\hat{T},\hat{a},z,s-\log(1-\tau))$
$\frac{\exp(-\frac{z^2}{4})}{4\pi}\{1+\frac{2\alpha z\sqrt{1-\tau}-\alpha^2}{4}+\frac{1}{2}(\frac{2\alpha z\sqrt{1-\tau}-\alpha^2}{4})^2\int_0^1\exp(\xi(\frac{2\alpha z\sqrt{1-\tau}-\alpha^2}{4}))d\xi\}dz$.
Using (81), we obtain: $L_1=O(\tau s^{-1/2})+\hat{q}_0(\hat{T},\hat{a},s-\log(1-\tau))+\alpha\hat{q}_1(\hat{T},\hat{a},s-\log(1-\tau))+O(\alpha^2s^{-2}\log s)$. By (82) and (83), we have:
$L_1=-\frac{5\kappa}{8ps^2}+O(\tau s^{-1/2})+O(s^{-3})+O(\alpha^2s^{-2}\log s)$.
$|L_2|\le C\int|\frac{z\sqrt{1-\tau}-\alpha}{\sqrt{s}}-\frac{z}{\sqrt{s-\log(1-\tau)}}|(|\hat{q}_b(\hat{T},\hat{a},z,s-\log(1-\tau))|+|\hat{q}_e(\hat{T},\hat{a},z,s-\log(1-\tau))|)\exp(-Cz^2)dz$.
Using (81) for q_b, (80) for q_e, and the fact that $q_e\equiv 0$ for $|z|\le K_0\sqrt{s}$ yields:
$L_2\le C\{|\tau|s^{-1/2}+|\alpha|s^{-1/2}\}(s^{-2}\log s+e^{-s})$. In conclusion,
$I_1=-\frac{5\kappa}{8ps^2}+O(\tau s^{-1/2})+O(s^{-5/2})+O(\alpha^2s^{-2}\log s)$. Adding I_1, I_2 and I_3 yields (74).

We compute $\frac{\partial\hat{q}_0}{\partial\tau}$ instead of $\frac{\partial\hat{q}_0}{\partial T}$, and then we use $\frac{\partial\hat{q}_0}{\partial T}=e^s\frac{\partial\hat{q}_0}{\partial\tau}$ to conclude. With the previous notations, we write:
$\frac{\partial\hat{q}_0}{\partial\tau}(T,a,s)=\frac{\partial I_1}{\partial\tau}+\frac{\partial I_2}{\partial\tau}+\frac{\partial I_3}{\partial\tau}$.

$\frac{\partial I_3}{\partial\tau}$:
$\frac{\partial I_3}{\partial\tau}=\frac{1}{p-1}(1-\tau)^{-1-\frac{1}{p-1}}\frac{\kappa}{2p(s-\log(1-\tau))}(1-\frac{1}{s-\log(1-\tau)})$, and $|\frac{\partial I_3}{\partial\tau}|\le Cs^{-1}$.

$\frac{\partial I_2}{\partial\tau}$:
$\frac{\partial I_2}{\partial\tau}$

$= \frac{1}{p-1}(1-\tau)^{-1-\frac{1}{p-1}} \int d\mu(y)\chi(y,s)(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}} + (1-\tau)^{-\frac{1}{p-1}}$
$\int d\mu(y)\chi(y,s)\frac{1}{p-1}\frac{(p-1)^2(y+\alpha)^2(1-(s-\log(1-\tau)))}{4p(1-\tau)^2(s-\log(1-\tau))^2}(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-1-\frac{1}{p-1}}$.
Computing as for I_2, we obtain: $\frac{\partial I_2}{\partial \tau} = O(\tau) + \frac{\kappa}{p-1} + O(s^{-1})$.

$\frac{\partial I_1}{\partial \tau}$:
$\frac{\partial I_1}{\partial \tau} = M_1 + M_2 + M_3$ with
$M_1 = \frac{1}{p-1}(1-\tau)^{-1-\frac{1}{p-1}} \int d\mu(y)\chi(y,s)\hat{q}(\hat{T},\hat{a},\frac{y+\alpha}{\sqrt{1-\tau}}, s-\log(1-\tau))$,
$M_2 = \frac{1}{p-1}(1-\tau)^{-\frac{1}{p-1}} \int d\mu(y)\chi(y,s)\frac{y+\alpha}{2(1-\tau)^{3/2}}\frac{\partial \hat{q}}{\partial y}(\hat{T},\hat{a},\frac{y+\alpha}{\sqrt{1-\tau}}, s-\log(1-\tau))$,
$M_3 = \frac{1}{p-1}(1-\tau)^{-\frac{1}{p-1}} \int d\mu(y)\chi(y,s)\frac{1}{1-\tau}\frac{\partial \hat{q}}{\partial s}(\hat{T},\hat{a},\frac{y+\alpha}{\sqrt{1-\tau}}, s-\log(1-\tau))$,
From (80), (84), and integration by parts we derive: $|\frac{\partial I_1}{\partial \tau}| \le |M_1| + |M_2| + |M_3| \le Cs^{-1/2}$.
this concludes the proof of lemma B.2.

Step 2: Behavior of $(\hat{q}_0, \hat{q}_1)$ near blow-up
We use the explicit asymptotic development given in lemma B.2 to construct a 1-manifold $\tilde{\Gamma}$ that is mapped by $(\hat{q}_0, \hat{q}_1)$ into $\partial \hat{V}_A(s)$.

Lemma B.4 (Behavior of $(\hat{q}_0, \hat{q}_1)$) $\exists C_0 = C_0(p)$, $\exists A_9 > 0$ $\forall A \ge A_9$, $\exists s_9(A) > 0$ $\forall s \ge s_9(A)$, $\exists \Gamma_{A,s}$ *rectangle in*
$D_{A,s} = (\hat{T},\hat{a}) + (-C_0Ae^{-s}s^{-2}, C_0Ae^{-s}s^{-2}) \times (-C_0Ae^{-\frac{s}{2}}s^{-1}, C_0Ae^{-\frac{s}{2}}s^{-1})$
such that $\forall (T,a) \in \Gamma_{A,s}$, $(\hat{q}_0,\hat{q}_1)(T,a,s) \in \partial \hat{V}_A(s)$,
and $d(\Gamma_{A,s}, (\hat{q}_0,\hat{q}_1)(.,.,s), 0) = -1$.

Proof:
Since $(\tilde{q}_0, \tilde{q}_1)$ given in (79) is almost the linear part of $(\hat{q}_0, \hat{q}_1)$ (see lemma B.2), we can first show for $(\tilde{q}_0, \tilde{q}_1)$ an analogous version of lemma B.4, then use lemma B.2 to conclude. We use scaling arguments to get uniform estimates in s. Indeed, let us introduce:

$$\begin{aligned} \tilde{Q} = (\tilde{Q}_0, \tilde{Q}_1) : (-C_0A, C_0A)^2 &\longrightarrow \mathbb{R}^2 \\ (\tilde{\tau}, \tilde{\alpha}) &\longrightarrow \frac{1}{A}(-\frac{5\kappa}{8p} + \tilde{\tau}\frac{\kappa}{p-1}, -\tilde{\alpha}\frac{\kappa}{4p}), \end{aligned} \tag{85}$$

and

$$\begin{aligned} \hat{Q}_s = (\hat{Q}_0, \hat{Q}_1)_s : (-C_0A, C_0A)^2 &\longrightarrow \mathbb{R}^2 \\ (\tilde{\tau}, \tilde{\alpha}) &\longrightarrow \frac{s^2}{A}(\hat{q}_0, \hat{q}_1)(\hat{T} + \frac{\tilde{\tau}}{e^s s^2}, \hat{a} + \frac{\tilde{\alpha}}{e^{\frac{s}{2}} s^1}, s), \end{aligned} \tag{86}$$

where $C_0 = C_0(p)$. Note that $\tilde{Q}$ is independent of s, and that

$$\begin{aligned}(\tilde{q}_0, \tilde{q}_1)(T, a, s) &= \frac{A}{s^2}(\tilde{Q}_0, \tilde{Q}_1)((T - \hat{T})e^s s^2, (a - \hat{a})e^{\frac{s}{2}} s). \\ (\hat{q}_0, \hat{q}_1)(T, a, s) &= \frac{A}{s^2}(\hat{Q}_0, \hat{Q}_1)_s((T - \hat{T})e^s s^2, (a - \hat{a})e^{\frac{s}{2}} s).\end{aligned}$$

The conclusion of lemma B.4 follows if we show that there exists a 1-manifold $\tilde{\Gamma}$ in $(-C_0A, C_0A)^2$ such that $\forall (\tilde{\tau}, \tilde{\alpha}) \in \tilde{\Gamma}$, $\hat{Q}_s(\tilde{\tau}, \tilde{\alpha}) \in \partial\mathcal{C}$, and $d(\tilde{\Gamma}, \hat{Q}_s, 0) = -1$. From lemma B.2, we compute for $s \geq s_{17}(A)$: $\|\tilde{Q} - \hat{Q}_s\|_{\mathcal{C}^1((-CA,CA)^2)} \leq \frac{C \log s}{A\sqrt{s}} \to 0$ when $s \to +\infty$.
It is easy to see that $\forall \eta \in [0, 1)$, $\exists \tilde{\Gamma}_\eta$ rectangle such that $\forall (\tilde{\tau}, \tilde{\alpha}) \in \tilde{\Gamma}_\eta$, $\tilde{Q}(\tilde{\tau}, \tilde{\alpha}) \in (1 + \eta)\partial\mathcal{C}$, and $d(\tilde{\Gamma}_\eta, \tilde{Q}, 0) = -1$.
From the continuity of topological degree, we know that there exist $\eta_0 > 0$, $\epsilon_0 > 0$ such that for each curve $\tilde{\Gamma}$ (indexed by $\partial\mathcal{C}$) satisfying $\|\tilde{\Gamma} - \tilde{\Gamma}_0\|_{L^\infty(\partial C)} \leq \eta_0\sqrt{2}$ ($\tilde{\Gamma}_0$ itself is indexed by $\partial\mathcal{C}$), for each continuous function $Q : (-C_0A, C_0A)^2 \longrightarrow \mathbb{R}^2$ satisfying $\|\tilde{Q} - Q\|_{L^\infty((-C_0A,C_0A)^2)} \leq \epsilon_0$, we have: $d(\tilde{\Gamma}, Q, 0) = -1$.
Since we have $\|\tilde{Q} - \hat{Q}_s\|_{L^\infty((-C_0A,C_0A)^2)} \leq \frac{C \log s}{A\sqrt{s}}$, and from (85) Jac $\tilde{Q} = -\frac{\kappa^2}{4p(p-1)A^2} < 0$, we can take s large enough, ($s \geq s_{11}(A, \epsilon_0, \eta_0)$) so that:

$$-\forall (\tilde{\tau}, \tilde{\alpha}) \in \tilde{\Gamma}_{\eta_0}, \hat{Q}_s(\tilde{\tau}, \tilde{\alpha}) \in ext(1 + \frac{\eta_0}{2})\mathcal{C}, \tag{87}$$

$$-\forall (\tilde{\tau}, \tilde{\alpha}) \in (-C_0A, C_0A)^2, Jac\hat{Q}_s(\tilde{\tau}, \tilde{\alpha}) < 0, \tag{88}$$

-$\forall \omega \in Im\hat{Q}_s \cap Im\tilde{Q}$, if $\omega = \hat{Q}_s(\xi)$ then

$$|\xi - \tilde{Q}^{-1}(\omega)| \leq \eta_0, \tag{89}$$

$$-\|\tilde{Q} - \hat{Q}_s\|_{L^\infty((-C_0A,C_0A)^2)} \leq \epsilon_0. \tag{90}$$

By (90) and (87), we have $d(\tilde{\Gamma}_{\eta_0}, \hat{Q}_s, 0) = -1$. Therefore, by (87), $\forall \omega \in (1 + \frac{\eta_0}{4})\mathcal{C}$, $d(\tilde{\Gamma}_{\eta_0}, \hat{Q}_s, \omega) = -1$ (the degree is the same in the same component of $\mathbb{R}^2 \backslash \hat{Q}_s(\tilde{\Gamma}_{\eta_0})$). Combining this with (88) and the definition of topological degree for $\mathcal{C}^1$ functions yields $\forall \omega \in (1 + \frac{\eta_0}{4})\mathcal{C}$, there exists a unique$(\tilde{\tau}, \tilde{\alpha}) \in \mathbb{R}^2$ such that $\hat{Q}_s(\tilde{\tau}, \tilde{\alpha}) = \omega$. Hence, $\hat{Q}_s$ is a diffeomorphism from $(\hat{Q}_s)^{-1}((1 + \frac{\eta_0}{4})\mathcal{C})$ onto $(1 + \frac{\eta_0}{4})\mathcal{C}$. Thus there exists a piecewise $\mathcal{C}^1$ 1-manifold $\tilde{\Gamma}$ interior to $\tilde{\Gamma}_{\eta_0}$, such that $\hat{Q}_s$ maps $\tilde{\Gamma}$ onto ∂C ($\tilde{\Gamma}$ is diffeomorphic to $\partial\mathcal{C}$). By (89), $|\tilde{\Gamma} - \tilde{\Gamma}_{0,A}| \leq \eta_0$. Therefore, we derive: $d(\tilde{\Gamma}, \hat{Q}_s, 0) = -1$. This concludes the proof of lemma B.4.

Step 3: Conclusion of the proof of lemma 4.3

We take $A \geq A_9$, $s_0 \geq \max(\hat{s}_0+1, s_7, s_9(A))$ and $\epsilon > 0$. $\forall s_1 > s_0$, we consider D_{A,s_1} and Γ_{A,s_1} given by lemma B.4. If $s_1 \geq s_{12}(A,\epsilon,s_0)$, then $\forall (T,a) \in \Gamma_{A,s_1}$, $|(T,a)-(\hat{T},\hat{a})| \leq \epsilon$, and $(T,a) \in D_1(s_0)$ (with the notations of lemma 4.1). Therefore, for such s_1, we have $|(T-\hat{T})e^{s_1}| \leq \frac{CA}{s_1^2}$ and $|(a-\hat{a})e^{\frac{s_1}{2}}| \leq \frac{CA}{s_1}$.
This implies $\forall s \in [s_0,s_1]$, $|(T-\hat{T})e^{s}| \leq \frac{CA}{s^2}$ and $|(a-\hat{a})e^{\frac{s}{2}}| \leq \frac{CA}{s}$.
What we want to do now is to show that $\forall s \in [s_0,s_1]$, $\hat{q}(T,a,s) \in V_A(s)$.
By lemma B.2, we have:
For $s_0 \geq s_{13}(A)$, $\forall (T,a) \in \Gamma_{A,s_1}$, $\forall s \in [s_0,s_1]$:

$$|\hat{q}_0(T,a,s)| \leq \frac{CA}{s^2} \quad (91)$$

$$|\hat{q}_1(T,a,s)| \leq \frac{CA}{s^2} \quad (92)$$

$$|\hat{q}_2(T,a,s)| \leq C\frac{\log s}{s^2} \quad (93)$$

$$|\hat{q}_-(T,a,y,s)| \leq C(1+|y|^3)\frac{1}{s^2} \quad (94)$$

$$|\hat{q}_e(T,a,y,s)| \leq \frac{C}{\sqrt{s}}. \quad (95)$$

Therefore, if $A \geq A_{14}$,

$$|\hat{q}_2(T,a,s)| \leq A^2\frac{\log s}{s^2}, |\hat{q}_-(T,a,y,s)| \leq A(1+|y|^3)\frac{1}{s^2}, |\hat{q}_e(T,a,y,s)| \leq \frac{A^2}{\sqrt{s}}. \quad (96)$$

It remains for us to show that $|\hat{q}_m(T,a,s)| \leq \frac{A}{s^2}$, for $m=0,1$.
Following the proof of lemma 3.2, we easily prove:

Lemma B.5 (Transversality property) $\exists A_{15} > 0$, $\forall A \geq A_{15}$, $\exists s_{15}(A)$ *such that* $\forall s_0 \geq s_{15}(A)$, $\forall s_1 > s_0$, *for any solution* q *of (15), satisfying:*
-Properties (91) to (95), for $s \in [s_0,s_1]$,
-$\exists s \in (s_0,s_1]$ *such that* $(q_0,q_1)(s) \in \partial\hat{V}_A(s)$,
we have the following property:
$\exists \delta > 0$ *such that* $\forall s_- \in (s-\delta,s)$, $(q_0,q_1)(s_-) \in int(\hat{V}_A(s_-))$.

If $A \geq A_{15}$ and $s_0 \geq s_{15}(A)$, then by lemma B.5, $\forall (T,a) \in \Gamma_{A,s_1}$

$$\forall s \in [s_0,s_1), (\hat{q}_0,\hat{q}_1)(T,a,s) \in int(\hat{V}_A(s)). \quad (97)$$

Indeed, this follows if we apply lemma B.5 to s_1 ($(\hat{q}_0, \hat{q}_1)(s_1) \in \partial \hat{V}_A(s_1)$ by lemma B.4) and to $s \in (s_0, s_1]$, and use
$I = \{s \in [s_0, s_1) | \forall s' \in [s, s_1), (\hat{q}_0, \hat{q}_1)(T, a, s') \in int(\hat{V}_A(s'))\}$.
The conclusion of lemma 4.3 follows for $A \geq A_6 = \max(A_9, A_{14}, A_{15})$,
$s_0 \geq \max(\hat{s}_0 + 1, s_7, s_9(A), s_{13}(A), s_{15}(A)), D_6(s_0) = D_1(s_0)$, and for $\epsilon > 0$,
$s_1 = s_{12}(A, \epsilon, s_0)$ and $\Gamma_\epsilon = \Gamma_{A,s_1}$.

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