Refined uniform estimates at blow-up and applications for nonlinear heat equations

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1 introduction

We are interested in the following nonlinear heat equation:

$$\left\{ \begin{array}{rcl} u_t & = & \Delta u + u^p \\ u(0) & = & u_0 \geq 0, \end{array} \right. \tag{1}$$

where u is defined for $(x,t) \in \mathbb{R}^N \times [0,T)$, $1 < p$, $(N-2)p < N+2$ and $u_0 \in H^1(\mathbb{R}^N)$.

In this paper, we deal with blow-up solutions of equation (1) $u(t)$ which blow-up in finite time $T > 0$: this means that u exists for all $t \in [0,T)$, $\lim_{t \to T} \|u(t)\|_{H^1} = +\infty$ and $\lim_{t \to T} \|u(t)\|_{L^\infty} = +\infty$. Let us consider such a solution. We aim at studying the blow-up behavior of $u(t)$ as $t \to T$. In particular, we are interested in obtaining uniform estimates on $u(t)$ and deducing from these estimates the asymptotic shape of the singularities.

One can show that in this case, $u(t)$ has at least one blow-up point, that is $x_0 \in \mathbb{R}^N$ such that there exists $(x_n, t_n)_{n \in \mathbb{N}}$ satisfying $(x_n, t_n) \to (x_0, T)$ and $|u(x_n, t_n)| \to +\infty$ as $n \to +\infty$.

For each $a \in \mathbb{R}^N$, we introduce the following self-similar transformation:

$$\begin{array}{rcl} y & = & \frac{x-a}{\sqrt{T-t}} \\ s & = & -\log(T-t) \\ w_a(y,s) & = & (T-t)^{\frac{1}{p-1}} u(x,t). \end{array} \tag{2}$$

Then, we see that $w_a = w$ satisfies $\forall s \geq -\log T$, $\forall y \in \mathbb{R}^N$:

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + w^p. \tag{3}$$

The study of $u(t)$ near (x_0, T) where x_0 is a blow-up point is equivalent to the study of the long time behavior of w_{x_0} as $s \to +\infty$.
Giga and Kohn prove in [10] that there exists $\epsilon_0 > 0$ such that

$$\forall s \geq -\log T,\ \epsilon_0 \leq \|w_{x_0}(s)\|_{L^\infty} \leq \frac{1}{\epsilon_0}$$

or equivalently:

$$\forall t \in [0, T),\ \epsilon_0 (T-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty} \leq \frac{1}{\epsilon_0}(T-t)^{-\frac{1}{p-1}}.$$

At this level, no other uniform estimates were known.

In [16], we proved the following Liouville Theorem for equation (3):

Let w be a nonnegative solution of (3) defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that $w \in L^\infty(\mathbb{R}^N \times \mathbb{R})$. Then, necessarily one of the following cases occurs:

$$w \equiv 0 \text{ or } w \equiv \kappa \text{ or } \exists s_0 \in \mathbb{R} \text{ such that } w(y,s) = \varphi(s - s_0) \qquad (4)$$

$$\text{where } \varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}} \text{ and } \kappa = (p-1)^{-\frac{1}{p-1}}.$$

From this theorem we derived in [16] the following uniform estimates of order zero:

Consider a solution w of (3) defined for $s \geq -\log T$ (such that $u(t)$ blows- up at time T). Then,

$$\|w(s)\|_{L^\infty} \to \kappa \text{ and } \|\nabla w(s)\|_{L^\infty} + \|\Delta w(s)\|_{L^\infty} \to 0 \text{ as } s \to +\infty. \qquad (5)$$

We also derived from this result the following localization theorem:
$\forall \epsilon > 0$, $\exists C_\epsilon > 0$ *such that* $\forall t \in [\frac{T}{2}, T)$, $\forall x \in \mathbb{R}^N$,

$$\left| \frac{\partial u}{\partial t} - u^p \right| \leq \epsilon u^p + C_\epsilon. \qquad (6)$$

These estimates are still insufficient to yield precise estimates on blow-up profile. But, we have a compactness property on $w_a(s)$ uniformly with respect to $a \in \mathbb{R}^N$, which allows us to claim the following result from linearization around the limit set as $s \to +\infty$:

Theorem 1 (Refined L^∞ estimates for $w(s)$ and $u(t)$ at blow-up) *There exist positive constants C_1, C_2 and C_3 such that if u is a solution of*

(1) which blows-up at time $T>0$ and satisfies $u(0)\in H^1(\mathbb{R}^N)$, then $\forall\epsilon>0$, there exists $s_0(\epsilon)\geq -\log T$ such that

i) $\forall s\geq s_0$, $\forall a\in\mathbb{R}^N$,

$$\begin{array}{rclrcl} \|w_a(s)\|_{L^\infty} & \leq & \kappa+(\frac{N\kappa}{2p}+\epsilon)\frac{1}{s}, & \|\nabla w_a(s)\|_{L^\infty} & \leq & \frac{C_1}{\sqrt{s}}, \\ \|\nabla^2 w_a(s)\|_{L^\infty} & \leq & \frac{C_2}{s}, & \|\nabla^3 w_a(s)\|_{L^\infty} & \leq & \frac{C_3}{s^{3/2}}, \end{array}$$

where $\kappa=(p-1)^{-\frac{1}{p-1}}$,

ii) $\forall t\geq T-e^{-s_0}$,

$$\begin{array}{rcl} \|u(t)\|_{L^\infty} & \leq & \left(\kappa+(\frac{N\kappa}{2p}+\epsilon)\frac{1}{|\log(T-t)|}\right)(T-t)^{-\frac{1}{p-1}}, \\ \|\nabla^i u(t)\|_{L^\infty} & \leq & C_i\frac{(T-t)^{-(\frac{1}{p-1}+\frac{i}{2})}}{|\log(T-t)|^{i/2}} \end{array}$$

for $i=1,2,3$.

Remark: If $v:\mathbb{R}^N\to\mathbb{R}$ is regular, $\nabla^i v$ stands for the differential of order i of v. For all $y\in\mathbb{R}^N$, we define $|\nabla v(y)|^2=\sum_{j=1}^N(\partial_j v(y))^2$, $|\nabla^2 v(y)|=\sup_{z\in\mathbb{R}^N}\frac{\left|z^T\nabla^2 v(y)z\right|}{|z|^2}$ and $|\nabla^3 v(y)|=\sup_{\alpha,\beta,\gamma\in\mathbb{R}^N}\left|\sum_{i,j,k}\frac{\alpha_i}{|\alpha|}\frac{\beta_j}{|\beta|}\frac{\gamma_k}{|\gamma|}\partial^3_{i,j,k}v(y)\right|$.
In addition, $\|v\|_{L^\infty}=\sup_{y\in\mathbb{R}^N}|v(y)|$ and $\|\nabla^i v\|_{L^\infty}=\sup_{y\in\mathbb{R}^N}|\nabla^i v(y)|$.

In fact, we can see from the proof of Theorem 1 that $s_0(\epsilon)$ depends only on the size of initial data. We have the following result:

Theorem 1' (Compactness) *Consider $(u_n)_{n\in\mathbb{N}}$ a sequence of nonnegative solutions of equation (1) such that for some $T>0$ and for all $n\in\mathbb{N}$, u_n is defined on $[0,T)$ and blows-up at time T. Assume also that $\|u_n(0)\|_{H^2(\mathbb{R}^N)}$ is bounded uniformly in n. Then, $\forall\epsilon>0$, there exists $t_0(\epsilon)<T$ such that* $\forall t\in[t_0(\epsilon),T)$, $\forall n\in\mathbb{N}$,

$$\begin{array}{rcl} \|u_n(t)\|_{L^\infty} & \leq & \left(\kappa+(\frac{N\kappa}{2p}+\epsilon)\frac{1}{|\log(T-t)|}\right)(T-t)^{-\frac{1}{p-1}}, \\ \|\nabla^i u_n(t)\|_{L^\infty} & \leq & C_i\frac{(T-t)^{-(\frac{1}{p-1}+\frac{i}{2})}}{|\log(T-t)|^{\frac{i}{2}}} \end{array}$$

where C_i are defined in Theorem 1.

Remark: In the case $N=1$, Herrero and Velázquez [12] (Filippas and Kohn [6] also) prove some estimates related to Theorem 1, using a Sturm

property first used by Chen and Matano [4] (the space oscillations number is a decreasing function of time).
Remark: The constant $\frac{N\kappa}{2p}$ appearing in the term of order one in the estimates on $\|w(s)\|_{L^\infty}$ and $\|u(t)\|_{L^\infty}$ is optimal. Indeed, there exist solutions of equation (3) such that $\|w(s)\|_{L^\infty} = \kappa + \frac{N\kappa}{2ps} + o\left(\frac{1}{s}\right)$ as $s \to +\infty$ (see Bricmont and Kupiainen [3], Filippas and Kohn [6], Merle and Zaag [17]).
Remark: From the local (in time) regularity of the solution to the Cauchy problem, we can obtain with the same proof an analogous compactness result when the blow-up times T_n are not the same. The assumptions are $\|u_n(0)\|_{H^2(\mathbb{R}^N)} + T_n$ is bounded uniformly in n. The conclusion is there is $t_0'(\epsilon)$ such that $\forall n \in \mathbb{N}$, $\forall t \in [T_n - t_0'(\epsilon), T_n)$, the inequalities hold.
Remark: Other compactness results can be shown considering for example equations of the type:

$$\frac{\partial u}{\partial t} = \Delta u + b(x)u^p$$

where $b \in C^3(\mathbb{R}^N)$ (see [16]).

These estimates are in fact crucial for the understanding of the solution at blow-up, especially, the shape of the singularity. Let us recall some results on this question.

Let us consider $x_0 \in \mathbb{R}^N$ a blow-up point of $u(t)$, a solution of (1), that is a point $x_0 \in \mathbb{R}^N$ such that there exists $(x_n, t_n) \to (x_0, T)$ such that $u(x_n, t_n) \to +\infty$ as $n \to +\infty$. The question is to see whether $u(t)$ (or $w_{x_0}(s)$ defined in (2)) has a universal behavior as $t \to T$ (or $s \to +\infty$).

First, Giga and Kohn prove in [10] and [11] (see also [9]) that for a given blow-up point $x_0 \in \mathbb{R}^N$,

$$\lim_{s\to+\infty} w_{x_0}(y,s) = \lim_{t\to T}(T-t)^{\frac{1}{p-1}} u(x_0 + y\sqrt{T-t}, t) = \kappa \qquad (7)$$

uniformly on compact subsets of $\mathbb{R}^N$. The result is pointwise in x_0. Besides, for a.e. y, $\lim_{s\to+\infty} \nabla w_{x_0}(y,s) = 0$.
Filippas and Liu [7] (see also Filippas and Kohn [6]) and Velázquez [18], [19] (see also Herrero and Velázquez [12], [14]) classify the behavior of $w(y,s)$ $(= w_{x_0}(y,s))$ for $|y|$ bounded. They prove that one of the following cases occurs:

- Case 1: non degenerate rate of blow-up:

there exists $k \in \{0, 1, ..., N-1\}$ and a $N \times N$ orthonormal matrix Q such that

$$\forall R>0,\ \sup_{|y|\le R}\left|w_{x_0}(y,s)-\left[\kappa+\frac{\kappa}{2ps}\left((N-k)-\frac{1}{2}y^TA_ky\right)\right]\right|=O\left(\frac{1}{s^{1+\delta}}\right) \tag{8}$$

as $s\to+\infty$ where $\delta>0$,

$$A_k=Q\begin{pmatrix} I_{N-k} & 0\\ 0 & 0\end{pmatrix}Q^{-1} \tag{9}$$

and I_{N-k} is the $(N-k)\times(N-k)$ identity matrix,

- Case 2: degenerate rate of blow-up: $\forall R>0,\ \sup_{|y|\le R}|w(y,s)-\kappa|\le C(R)e^{-\epsilon_0 s}$ for some $\epsilon_0>0$.

This yields a blow-up behavior classification in a small range scale. In some sense and from a physical point of view, these results do not show the transition between the singular zone ($w\ge\alpha$ where $\alpha>0$) and the regular one ($w\simeq 0$) well.

Using the renormalization theory, Bricmont and Kupiainen showed in [3] the existence of a solution of (3) such that

$$\forall s\ge s_0,\ \forall y\in\mathbb{R}^N,\ \left|w(y,s)-f_0\left(\frac{y}{\sqrt{s}}\right)\right|\le\frac{C}{\sqrt{s}} \tag{10}$$

where $f_0(z)=\left(p-1+\frac{(p-1)^2}{4p}|z|^2\right)^{-\frac{1}{p-1}}$ (see also [1]). We show in [17] the same result through a reduction to a finite dimensional problem. We also obtained there a stability result of this behavior with respect to initial data. This gives a result in an intermediate scale $z=\frac{y}{\sqrt{s}}$, which is more satisfactory since it separates the blow-up region ($w>\alpha>0$) and non-blow-up ones ($w\simeq 0$).

In [20], the second author showed that the behavior in the initial variable x is known in the case where (10) occurs. More precisely, $u(x,t)\to u^*(x)$ as $t\to T$ uniformly on compact sets of $\mathbb{R}^N\backslash\{0\}$ and

$$u^*(x)\sim\left[\frac{8p|\log|x||}{(p-1)^2|x|^2}\right]^{\frac{1}{p-1}}\quad\text{as } x\to 0. \tag{11}$$

Therefore, except in the small range variable (which does not precise from a physical or analytical point of view the singular behavior), no result of classification was known.

In a first step, we use the estimates of Theorem 1 on ∇w and $\nabla^2 w$ in a crucial way, and the results of Filippas and Liu, and Velázquez concerning the classification of blow-up behaviors for $|y|$ bounded to establish a blow-up profile classification theorem in the variable $z = \frac{y}{\sqrt{s}}$ (which is the intermediate scale that separates the regular and singular parts in the non degenerate case):

Theorem 2 (Existence of a blow-up profile in the intermediate scale for solutions of (1))
Let $u(t)$ be a solution of (1) which blows-up at time $T > 0$ and satisfies $u(0) \in H^1(\mathbb{R}^N)$. Let x_0 be a blow-up point of $u(t)$. Then, there exist $k \in \{0, 1, .., N\}$ and an orthonormal $N \times N$ matrix Q such that

$$\forall K_0 > 0, \quad \sup_{|z| \le K_0} |w_{x_0}(z\sqrt{s}, s) - f_k(z)| \to 0 \text{ as } s \to +\infty, \tag{12}$$

where

$$f_k(z) = \left(p - 1 + \frac{(p-1)^2}{4p} z^T A_k z\right)^{-\frac{1}{p-1}} \tag{13}$$

and A_k is defined in (9).

Remark: Velázquez in [19] obtained a related profile existence result. He extended the $|y|$ bounded convergence of [18] to the larger set $|y| \le K_0\sqrt{s}$, by estimating the effect of the convective term $-\frac{1}{2}y.\nabla w$ in the equation (3), in L^p spaces with a Gaussian measure. However, the convergence that he obtains depends strongly on the considered blow-up point x_0. Let us point out that the convergence we have in Theorem 2 can be shown to be independent of x_0. Indeed, by using the uniform estimates of Theorem 1, we can give a uniform version of the result of [7] and [18], and obtain thanks to our techniques a convergence independent of x_0 in Theorem 2. However, we use the result of [7] and [18] in this paper, since this shortens the proof. We also notice that the proof yields that if the case (12) occurs, then (8) occurs with the same A_k (if $k = N$, then take $A_N = 0$) and conversely. See also Theorem 3.
Remark: In the case $k = N$, this theorem yields κ as asymptotic "profile" of $w(s)$ in the variable $z = \frac{y}{\sqrt{s}}$: this is a degenerate blow-up behavior. Indeed, in this case, the scale $\frac{y}{\sqrt{s}}$ is not good for describing the blow-up behavior. One must refine this scale and exhibit other blow-up profiles in different scales $y \simeq \exp\left[\left(\frac{k-1}{2k}\right) s\right]$ for $k = 2, 3, ...$ (see for instance [3], [18]). However, we suspect these profiles to be unstable with respect to initial data.

One interesting problem that follows from Theorem 2 is to find a relationship between the different notions of profile in the scales: $|y| \le C$,

$z = \frac{|y|}{\sqrt{s}} \leq C$ and $|x - x_0|$ small. We show in the following theorem that all these descriptions are equivalent in the case of a solution $u(t)$ of (1) that blows-up at some point $x_0 \in \mathbb{R}^N$ in a non degenerate way (which is supposed to be the generic case):

$$k = 0 \text{ and } A_k = I_N.$$

This answers many questions which were underlined on this problem in preceding works.

Theorem 3 (Equivalence of different notions of blow-up profiles)
Let $x_0 \in \mathbb{R}^N$ be an isolated blow-up point of $u(t)$ solution of (1) such that $u_0 \in H^1(\mathbb{R}^N)$. The following blow-up behaviors of $u(t)$ near x_0 or $w(s) = w_{x_0}(s)$ (defined in (2)) are equivalent:

(A) $\forall R > 0,\ \sup_{|y| \leq R} \left| w(y,s) - \left[\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)\right]\right| = o\left(\frac{1}{s}\right)$ *as* $s \to +\infty$ *where* $\kappa = (p-1)^{-\frac{1}{p-1}}$,

(B) $\exists \epsilon_0 > 0$ *such that* $\left\| w(y,s) - f_0(\frac{y}{\sqrt{s}}) \right\|_{L^\infty(|y| \leq \epsilon_0 e^{s/2})} \to 0$ *as* $s \to +\infty$ *with* $f_0(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$,

(C) $\exists \epsilon_0 > 0$ *such that if* $|x - x_0| < \epsilon_0$, *then* $u(x,t) \to u^*(x)$ *as* $t \to T$ *and* $u^*(x) \sim \left[\frac{8p|\log|x-x_0||}{(p-1)^2|x-x_0|^2}\right]^{\frac{1}{p-1}}$ *as* $x \to x_0$.

Remark: In [19], Velázquez shows that (A) $\Longrightarrow$ (B) $\Longrightarrow$ (C) by estimating the local effect to the term $-\frac{1}{2}y.\nabla w$ in equation (3) in L^p with Gaussian measure. The classification of [19] also yields that (C) $\Longrightarrow$ (A). Let us point that the estimates in our proof are quite elementary and rely on localization effect and uniform estimates. In addition, one can show from our proof and our uniform techniques that the convergence speeds in (A), (B) and (C) depend only on each other and on a bound on the C^2 norm of initial data (and not on the initial data itself).

Remark: In fact, (A) (or (B) or (C)) imply that x_0 is an isolated blow-up point. It is conjectured that the equivalence holds (in the case of the (supposed to be) generic blow-up rate).

Remark: The techniques we introduce in the proof of Theorem 3 allow us to obtain the same results as Velázquez in the case where (8) occurs with $k < N$.

Section 2 is devoted to the proof of the uniform estimates on w (Theorems 1 and 1'). Section 3 deals with results on profiles (Theorems 2 and 3).

2 L^∞ estimates of order one for solutions of (3)

2.1 Formulation and reduction of the problem

We prove Theorems 1 and 1' in this section. Let us first show Theorem 1. Theorem 1' follows from similar arguments.

Proof of Theorem 1: We consider $u(t)$ a blow-up solution of (1) which blows-up at time $T > 0$.

We can assume from regularizing effect of the heat flow that $T < 1$, $u_0 \in C^3(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. We are interested in finding L^∞ estimates of order one for w_0 $(= w)$ defined in (2). In [16], we have already proved L^∞ estimates of order zero for w stated in (5). Note that with obvious simple adaptations of the proof of (5), we have the following result:

$$\|w(s)\|_{L^\infty} \to \kappa \text{ and } \|\nabla w(s)\|_{L^\infty} + \|\nabla^2 w(s)\|_{L^\infty} + \|\nabla^3 w(s)\|_{L^\infty} \to 0 \quad (14)$$

as $s \to +\infty$.

We now want to refine the estimates (14). More precisely, we want to show that there exist positive constants C_1, C_2 and C_3 depending only on p such that $\forall \epsilon > 0$, $\exists s_0(\epsilon)$ such that $\forall s \geq s_0(\epsilon)$,

$$\begin{array}{rclrcl} \|w(s)\|_{L^\infty} & \leq & \kappa + (\frac{N\kappa}{2p} + (N+1)\epsilon)\frac{1}{s}, & \|\nabla w(s)\|_{L^\infty} & \leq & \frac{C_1}{\sqrt{s}} \\ \|\nabla^2 w(s)\|_{L^\infty} & \leq & \frac{C_2}{s}, & \|\nabla^3 w(s)\|_{L^\infty} & \leq & \frac{C_3}{s^{3/2}}. \end{array} \quad (15)$$

For this purpose, we take an arbitrary $\epsilon \in (0, \epsilon_0)$ (where $\epsilon_0 \leq 1$ is small enough) that we consider as fixed now, and introduce the following definitions:

Definition 2.1 *For all $A > 0$ and $s \geq -\log T$, we define $V_A(s)$ as being the set of all $w \in W^{3,\infty}(\mathbb{R}^N)$ satisfying:*

$$\begin{array}{rclrcl} \|w\|_{L^\infty} & \leq & \kappa + \frac{c_0}{s}, & \|\nabla w\|_{L^\infty} & \leq & \frac{c_1}{\sqrt{s}} \\ \|\nabla^2 w\|_{L^\infty} & \leq & \frac{A}{s}, & \|\nabla^3 w\|_{L^\infty} & \leq & \frac{A^{5/4}}{s^{3/2}}. \end{array}$$

and

$$\forall a \in \mathbb{R}^N, \ -\frac{c_2}{s} I_N \leq \int_{\mathbb{R}^N} \nabla^2 w(y+a)\rho(y)dy$$

in the sense of symmetric $N \times N$ matrices, where the norms are introduced in the remark after Theorem 1,

$$c_0(\epsilon) = \frac{N\kappa}{2p} + (N+1)\epsilon, \ c_1(\epsilon) = \frac{\kappa}{\sqrt{p}} + 2\epsilon\sqrt{p}, \ c_2(\epsilon) = \frac{\kappa}{2p} + \epsilon, \quad (16)$$

$$I_N \text{ is the } N\times N \text{ identity matrix and } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}. \tag{17}$$

Definition 2.2 *For all $s \geq -\log T$, we define*
$\hat{V}_A(s) = \{w \in C([-\log T, s), W^{3,\infty}(\mathbb{R}^N)) \mid \forall \tau \in [-\log T, s),\ w(\tau) \in V_A(\tau)\}$.

Let us remark that condition (16) is in some sense a lower bound on $\nabla^2 w(a)$. Indeed, if $w \in V_A(s)$, then we have $\forall a \in \mathbb{R}^N$,

$$|\int_{\mathbb{R}^N} \nabla^2 w(y+a)\rho(y)dy - \nabla^2 w(a)| \leq C^*(N)\|\nabla^3 w\|_{L^\infty} \tag{18}$$

$$\text{and} \qquad \frac{A}{s} I_N \geq \nabla^2 w(a) \geq -\left[\frac{c_2}{s} + C^*(N)\frac{A^{5/4}}{s^{3/2}}\right] I_N \tag{19}$$

where $C^*(N) = \int |y|\rho(y)dy$.
Proof of (18) and (19): Using a Taylor expansion, we have: $\forall y \in \mathbb{R}^N$, $\nabla^2 w(y+a) - \nabla^2 w(a) = \int_0^1 \nabla^3 w(a+ty)(y)dt$. Hence,
$|\nabla^2 w(y+a) - \nabla^2 w(a)| \leq |y|\|\nabla^3 w\|_{L^\infty} \leq |y|\frac{A^{5/4}}{s^{3/2}}$. This yields (18) and (19) by integration (use $\int \rho(y)dy = 1$). ■

Notice that the lower bound on $\nabla^2 w(a)$ is (consider the order $\frac{1}{s}$) independent of A, which will be crucial in the proof.

Theorem 1 is in fact a consequence of the following proposition:

Proposition 2.1 (Reduction) *There exist $A(p) > 0$ and $\epsilon_0(p) \in (0,1)$ such that for all $\epsilon \in (0, \epsilon_0)$, there exists $S(A,\epsilon)$ so that the following property is true:*
Assume that w is a solution of (3) defined for all time $s \geq -\log T$ and satisfying $w(-\log T) \in H^1(\mathbb{R}^N)$. Assume in addition that $w \in \hat{V}_A(\hat{s})$ for some $\hat{s} \geq S(A,\epsilon)$, then:
i) $w(\hat{s}) \notin \partial V_A(\hat{s})$,
ii) $\forall s \geq -\log T$, $w(s) \in \hat{V}_A(s)$.

Proposition 2.1 implies Theorem 1:

Let $\epsilon \in (0, \epsilon_0)$, $A = A(p)$ and $S(A,\epsilon)$ defined in Proposition 2.1. Our strategy is to find $n_0(\epsilon) = n_0 \in \mathbb{N}$ such that $\forall s \geq -\log T$, $w(s+n_0) \in V_A(s)$. Indeed, one can easily check the following result:

Lemma 2.1 *Assume for all $\epsilon \in (0,1)$, there exists $n_0(\epsilon) \in \mathbb{N}$ such that $\forall s \geq -\log T$, $w(s+n_0) \in V_A(s)$. Then, (15) is satisfied with*

$$C_1 = \frac{\kappa}{2p} + 4\sqrt{p},\ C_2 = 2A(p) \text{ and } C_3 = 2A(p)^{5/4}.$$

Let us consider $W = w(.+n)$. Then, W satisfies (3) for all $s \geq -\log T$ and $W(-\log T) = w(n - \log T) \in H^1(\mathbb{R}^N)$ from the solving of the initial value problem for w.

We claim the following: for n large, we have $w(.+n) \in \hat{V}_A(S(A,\epsilon))$. Indeed, let

$$\delta = \frac{1}{4(1+C^*(N))} \min\left(\frac{c_0}{S(A,\epsilon)}, \frac{c_1}{\sqrt{S(A,\epsilon)}}, \frac{c_2}{S(A,\epsilon)}, \frac{A}{S(A,\epsilon)}, \frac{A^{5/4}}{S(A,\epsilon)^{3/2}}\right) \tag{20}$$

where $C^*(N)$ is defined in (19). (14) implies that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\forall s \in [-\log T, S(A,\epsilon)]$, $\|w(s+n)\|_{L^\infty} \leq \kappa + \delta \leq \kappa + \frac{c_0}{4s}$, $\|\nabla w(s+n)\|_{L^\infty} \leq \delta \leq \frac{c_1}{4\sqrt{s}}$, $\|\nabla^2 w(s+n)\|_{L^\infty} \leq \delta \leq \frac{A}{4s}$ and $\|\nabla^3 w(s+n)\|_{L^\infty} \leq \delta \leq \frac{A^{5/4}}{4s^{3/2}}$.

Let $s \in [-\log T, S(A,\epsilon)]$ and $a \in \mathbb{R}^N$. According to (18), we have $\int_{\mathbb{R}^N} \nabla^2 w(y+a, s+n)\rho(y)dy$
$\geq -\left(|\nabla^2 w(a,s+n)| + C^*(N)\|\nabla^3 w(s+n)\|_{L^\infty}\right) I_N \geq -\left(\delta + C^*(N)\delta\right) I_N \geq -\frac{c_2}{4s_2} I_N$. Thus, $w(.+n_0) \in \hat{V}_A(S(A,\epsilon))$. Applying Proposition 2.1, we see from $ii)$ that

$$\forall s \in [-\log T, +\infty),\ w(s+n_0) \in V_A(s).$$

This concludes the proof of Theorem 1. ∎

Proof of Theorem 1':

For all $n \in \mathbb{N}$, we introduce $w_n = w_{n,0}$ defined from u_n by (2). Then, by simple obvious adaptations of the proof of Theorem 1' in [16], we claim that $\sup_{n\in\mathbb{N}} \|w_n(s)\|_{L^\infty} \to \kappa$ and $\sup_{n\in\mathbb{N}} \|\nabla^i w_n(s)\|_{L^\infty} \to 0$ as $s \to +\infty$ for $i = 1, 2$ and 3.

Hence, there exists $n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $\forall s \in [-\log T, S(A,\epsilon)]$, $\|w_n(s+n_0)\|_{L^\infty} \leq \kappa + \delta$ and $\|\nabla^i w_n(s+n_0)\|_{L^\infty} \leq \delta$ for $i = 1, 2, 3$ where δ is defined in (20). Hence, as for the proof of Theorem 1, we get $\forall n \in \mathbb{N}$, $w_n(.+n_0) \in \hat{V}_A(S(A,\epsilon))$. Thus,

$$\forall n \in \mathbb{N},\ \forall s \in [-\log T, +\infty),\ w_n(s+n_0) \in V_A(s)$$

by $ii)$ of Proposition (2.1). This concludes the proof of Theorem 1'. ∎

Therefore, the question reduces to prove Proposition 2.1.

Proof of Proposition 2.1:

$i) \Longrightarrow ii)$: By contradiction, we assume that there exists $s \geq -\log T$ such that $w(s) \notin V_A(s)$. Let s' be the lowest s satisfying this. Then, $s' \geq \hat{s} \geq S(A,\epsilon)$, $w \in \hat{V}_A(s')$ and $w(s') \in \partial V_A(s')$. This contradicts $i)$.

Proof of i): Let us argue by contradiction. We suppose that for all $A > 0$, there is a sequence $s_n \to +\infty$ and a solution of (3) w_n defined for all $s \geq -\log T$ such that $w_n(-\log T) \in H^1(\mathbb{R}^N)$, $\forall s \in [-\log T, s_n]$, $w_n(s) \in V_A(s)$ and $w_n(s_n) \in \partial V_A(s_n)$.
Let us denote w_n by w to simplify the notations. We claim the following

Proposition 2.2 (Characterization of $\partial V_A(s_n)$) *There exists $y_n \in \mathbb{R}^N$ such that one of the following cases must occur:*
Case 1: $w(y_n, s_n) = \kappa + \frac{c_0}{s_n}$,
Case 2: $|\nabla w(y_n, s_n)| = \frac{c_1}{\sqrt{s_n}}$,
Case 3: there exists a unitary $\varphi_n \in \mathbb{R}^N$ such that
$\varphi_n^T \int_{\mathbb{R}^N} \nabla^2 w(y + y_n, s_n)\rho(y)dy\varphi_n = -\frac{c_2}{s_n}$,
Case 4: $|\nabla^2 w(y_n, s_n)| = \frac{A}{s_n}$,
Case 5: $|\nabla^3 w(y_n, s_n)| = \frac{A^{5/4}}{s_n^{3/2}}$.

Proof:
Let us remark that since $w(-\log T) \in H^1(\mathbb{R}^N)$, we can assume from the regularizing effect of the heat flow that $w(-\log T, y) \to 0$ and $\nabla^i w(-\log T, y) \to 0$ as $|y| \to +\infty$ for $i = 1, 2$ and 3. Hence, we have by classical estimates $w(y, s) \to 0$ and $\nabla^i w(y, s) \to 0$ as $|y| \to +\infty$ uniformly in $s \in [s_n, s_n + 1]$. Hence, by Lebesgue's Theorem, $\int \nabla^2 w(y + a, s)\rho(y)dy \to 0$ as $|a| \to +\infty$.
This insures that one of the five cases of Proposition 2.2 occurs. ■

We now use the classification of Proposition 2.2 and consider in the following subsection all the five cases in order to reach a contradiction.
Let us notice that we reduce to the case

$$y_n = 0.$$

Indeed, from (2) and the translation invariance of (1), we define for all $y \in \mathbb{R}^N$ and $s \geq -\log T$:

$$W(y, s) = w(y + y_n e^{\frac{s - s_n}{2}}, s). \tag{21}$$

We still have:
- W is solution of (3) defined for $s \in [-\log T, +\infty)$,
- $W(s) \in V_A(s)$ for all $s \in [-\log T, s_n]$,
- $W(s_n) \in \partial V_A(s_n)$.

We will denote W by w and φ_n by φ.
We now claim that there exist $\epsilon_0(p) > 0$ and $A(p) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, there is $S(A, \epsilon)$ such that all the cases 1, 2, 3, 4 and 5 do not occur if $s_n \geq S(A, \epsilon)$, which will conclude the proof of Proposition 2.1.

2.2 Proof of the boundary estimates

There exist $\epsilon_0(p)$ and $A_0(p)$ such that $\forall \epsilon \in (0, \epsilon_0)$, $\forall A \geq A_0(p)$, $\exists S = S(A, \epsilon)$ such that Cases 1,2,3,4 and 5 do not occur if $s_n \geq S(A, \epsilon)$.

Let us show the following lemma

Lemma 2.2 (Taylor expansions) *Assume that $w(s) \in V_A(s)$. Then, $\forall y \in \mathbb{R}^N$:*

$$-\frac{1}{2}|y|^2 \left(\frac{c_2}{s} + C^*(N) \frac{A^{5/4}}{s^{3/2}} \right) \leq w(y,s) - w(0,s) - y.\nabla w(0,s) \leq \frac{1}{2}|y|^2 \frac{A}{s}, \tag{22}$$

$$\left| w(y,s) - w(0,s) - y.\nabla w(0,s) - \frac{1}{2} y^T \nabla^2 w(0,s) y \right| \leq \frac{1}{6}|y|^3 \frac{A^{5/4}}{s^{3/2}}, \tag{23}$$

$$\left| \int \nabla w(y,s) \rho(y) dy - \nabla w(0,s) \right| \leq C^*(N) \frac{A}{s}, \tag{24}$$

$$\text{and } |w(y,s) - w(0,s)| \leq \frac{c_1}{\sqrt{s}} |y| \tag{25}$$

where $C^*(N) = \int |y| \rho(y) dy$.

Proof: By a Taylor expansion of $w(y,s)$ to the second order near $y = 0$, we write: $w(y,s) - w(0,s) - y.\nabla w(0,s) = \int_0^1 (1-t) y^T \nabla^2 w(ty,s) y dt$. Using (19) we get the first inequality.

The second and the forth inequalities are obtained in the same way by expanding $w(y,s)$ respectively until the third and the first order, and using $\|\nabla^3 w(s)\|_{L^\infty} \leq \frac{A^{5/4}}{s^{3/2}}$ and $\|\nabla w(s)\|_{L^\infty} \leq \frac{c_1}{\sqrt{s}}$.

For the third inequality, we write for all $y \in \mathbb{R}^N$, $\nabla w(y,s) - \nabla w(0,s) = y. \int_0^1 \nabla^2 w(ty,s) dt$. Using $\|\nabla^2 w(s)\|_{L^\infty} \leq \frac{A}{s}$, we obtain $|\nabla w(y,s) - \nabla w(0,s)| \leq |y| \frac{A}{s}$. Integrating this inequality with respect to ρdy, we get the conclusion. ■

Case 1: $w(s_n)$ can not reach $\kappa + \frac{c_0}{s_n}$

For all $\epsilon > 0$ and $A > 0$, there exists $S_1(A, \epsilon)$ such that if $s_n \geq S_1(A, \epsilon)$, Case 1 in Proposition 2.2 does not occur.

Proof: This estimate is in fact crucial and it follows from a blow-up argument.

Assume that

$$w(0, s_n) = \kappa + \frac{c_0}{s_n}. \tag{26}$$

Since $w(s_n) \in V_A(s_n)$, we have $\|w(s_n)\|_{L^\infty} \le \kappa + \frac{c_0}{s_n}$ and 0 is a global maximum for $w(s_n)$. Therefore, $\nabla w(0, s_n) = 0$. Hence, (22) yields
$w(y, s_n) \ge \kappa + \frac{c_0}{s_n} - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)|y|^2$ and
$\int w(y, s_n)\rho(y)dy \ge \kappa + \frac{c_0}{s_n} - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)\int |y|^2\rho(y)dy$
$= \kappa + \frac{c_0 - Nc_2}{s_n} - NC^*(N)\frac{A^{5/4}}{s_n^{3/2}} = \kappa + \frac{\epsilon}{s_n} - NC^*(N)\frac{A^{5/4}}{s_n^{3/2}} > \kappa$ for s_n large ($s_n \ge S_1(A, \epsilon) = \frac{2N^2C^*(N)^2A^{5/2}}{\epsilon^2}$).
This contradicts the global (in time) existence of w. Indeed, we have the following blow-up criterion for nonnegative solutions of (3):

Lemma 2.3 (A blow-up criterion for nonnegative solutions of (3)) *Consider $W \ge 0$ a solution of (3) and suppose that for some $s_0 \in \mathbb{R}$, $\int W(y, s_0)\rho(y)dy > \kappa$, then W blows-up in finite time $S > s_0$.*

Proof: See Proposition 3.5 in [16]. ■

Therefore, w blows-up in finite time S, which is a contradiction for $s_n \ge S_1(A, \epsilon)$.
Thus, Case 1 of Proposition 2.2 can not occur.

Case 2: $|\nabla w(s_n)|$ can not reach $\frac{c_1}{\sqrt{s_n}}$

There exist $\epsilon_2(p) > 0$ such that $\forall \epsilon \in (0, \epsilon_2(p))$, $\forall A > 0$, $\exists S_2(A, \epsilon)$ such that if $s_n \ge S_2(A, \epsilon)$, then Case 2 in Proposition 2.2 can not occur.

Proof: It follows from the bounds of $w(s_n)$ and $\nabla^2 w(s_n)$.
In this case, $|\nabla w(0, s_n)| = \frac{c_1}{\sqrt{s_n}}$. Using (22) with
$\hat{y}_n = (2\sqrt{p} + \epsilon)\sqrt{s_n}\frac{\nabla w(y_n, s_n)}{|\nabla w(y_n, s_n)|}$, we get:
$w(\hat{y}_n, s_n) \ge 0 + (2\sqrt{p} + \epsilon)\sqrt{s_n}\frac{c_1}{\sqrt{s_n}} - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)(2\sqrt{p} + \epsilon)^2 s_n$
$= \kappa + 2p\epsilon + O\left(\epsilon^2\right) + O\left(\frac{1}{\sqrt{s_n}}\right)$ as $n \to +\infty$. Therefore, if $\epsilon \le \epsilon_2(p)$ for some $\epsilon_2(p) > 0$, then $w(\hat{y}_n, s_n) \ge \kappa + p\epsilon + O\left(\frac{1}{\sqrt{s_n}}\right)$. Hence,

$$\kappa + \frac{c_0}{s_n} \ge \|w(s_n)\|_{L^\infty} \ge \kappa + p\epsilon + O\left(\frac{1}{\sqrt{s_n}}\right),$$

which is a contradiction if $s_n \ge S_2(A, \epsilon)$ for some $S_2(A, \epsilon)$.
Thus, Case 2 of Proposition 2.2 can not occur.

Case 3: $\varphi^T \int_{\mathbb{R}^N} \nabla^2 w(y, s_n)\rho(y)dy\varphi > -\frac{c_2}{s_n}$

$\forall \epsilon > 0$, $\forall A \geq 0$, $\exists S_3(A,\epsilon)$ such that if $s_n \geq S_3(A,\epsilon)$, then Case 3 in Proposition 2.2 does not occur.

Proof: We assume that $\varphi^T \int_{\mathbb{R}^N} \nabla^2 w(y,s_n)\rho(y)dy\varphi = -\frac{c_2}{s_n}$ for some unitary $\varphi \in \mathbb{R}^N$. We proceed in two steps: in Step 1, we derive a differential equation on $\int \nabla^2 w(y,s)\rho(y)dy$. In Step 2, we conclude the proof by a contradiction between this equation and the fact that w is globally defined in time.

Step 1: Equation on $\int \nabla^2 w(y,s)\rho(y)dy$

We recall that w is a solution of

$$\frac{\partial w}{\partial s} = \mathcal{L}w - \frac{p}{p-1}w + w^p \tag{27}$$

where $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$ is a self-adjoint operator on $\mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N)$ with ρ defined in (17). The spectrum of $\mathcal{L}$ consists of eigenvalues

$$\text{spec } \mathcal{L} = \{1 - \frac{m}{2} \mid m \in \mathbb{N}\}.$$

Let us recall that in dimension 1, the eigenvalues are simple and the eigenfunction corresponding to $1-\frac{m}{2}$ is

$$h_m(y) = \sum_{j=0}^{[\frac{m}{2}]} \frac{1}{j!(m-2j)!}(-1)^j y^{m-2j} \tag{28}$$

where h_m satisfies $\int h_m h_j \rho dy = \frac{2^m}{m!}\delta_{mj}$.
In dimension N, we write the spectrum of $\mathcal{L}$ as

$$\text{spec } \mathcal{L} = \{1 - \frac{m_1 + ... + m_N}{2} \mid m_1, ..., m_N \in \mathbb{N}\}. \tag{29}$$

For $(m_1, ..., m_N) \in \mathbb{N}$, the eigenfunction corresponding to $1 - \frac{m_1+...+m_N}{2}$ is

$$y \to h_{m_1}(y_1)...h_{m_N}(y_N). \tag{30}$$

Since the eigenfunctions of $\mathcal{L}$ constitute a total orthonormal family of $L^2_\rho(\mathbb{R}^N)$, we can write

$$w(y,s) = w_0(s) + w_1(s).y + \left(\frac{1}{2}y^T w_2(s) y - tr w_2(s)\right) + w_-(y,s) \tag{31}$$

where:
- $w_0(s) = \int w(y,s)\rho(y)dy \in \mathbb{R}$ (eigenvalue 1),

- $w_1(s) = \int w(y,s)\frac{y}{2}\rho(y)dy \in \mathbb{R}^N$ (eigenvalue $\frac{1}{2}$),
- $w_2(s) = \int w(y,s)M(y)\rho(y)dy$ is a $N \times N$ symmetric matrix (eigenvalue 0)

$$\text{with } M_{i.j}(y) = \frac{1}{4}y_iy_j - \frac{1}{2}\delta_{i,j}, \tag{32}$$

- $w_- = P_-(w)$ and P_- is the L^2_ρ projector on the negative subspace of $\mathcal{L}$.

Our purpose is to write an equation satisfied by $w_2(s)$. We claim the following:

Lemma 2.4 (Equation satisfied by $w_2(s)$) *For n large enough, we have:*
i) $w_1(s) = \int \nabla w(y,s)\rho(y)dy$ and $w_2(s) = \int \nabla^2 w(y,s)\rho(y)dy$,
ii) $|w_1(s_n)| \le \frac{c_1}{\sqrt{s_n}}$, $|w_2(s_n)| \le \frac{A}{s_n}$, $\forall y \in \mathbb{R}^N$, $|w_-(y,s_n)| \le C(N)\frac{A^{5/4}}{s_n^{3/2}}(1+|y|^3)$ and $\delta_0 \le w_0(s_n) \le \kappa$ where $\delta_0 = \frac{c_2^3}{128C(N)^2A^{5/4}}$ for some $C(N) > 0$.
iii)

$$\begin{aligned} w_2'(s_n) &= \left(pw_0(s_n)^{p-1} - \frac{p}{p-1}\right)w_2(s_n) \\ &+ p(p-1)w_0(s_n)^{p-2}\left[2w_2(s_n)^2 + w_1(s_n)\otimes w_1(s_n)\right] \\ &+ O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right) + O\left(\frac{1}{s_n^{5/2}}\right). \end{aligned} \tag{33}$$

Proof: see Appendix A.
Remark: - If u and v are in $\mathbb{R}^N$, then we recall that $u\otimes v$ is the $N\times N$ matrix such that $(u\otimes v)_{i,j} = u_iv_j$ and $O(f)$ stands for a function which is bounded by $C(A,p,\epsilon)f$ as $n \to +\infty$.

Step 2: Conclusion for Case 3
Let $m(s) = \varphi^T w_2(s)\varphi$. Then, m is C^1, and since $w(s) \in V_A(s)$ for all $s \in [-\log T, s_n]$, we have: $m(s_n) = -\frac{c_2}{s_n}$ and $\forall s \in [-\log T, s_n]$, $m(s) \ge -\frac{c_2}{s}$. Thus,

$$m(s_n) = -\frac{c_2}{s_n} \text{ and } m'(s_n) \le \frac{c_2}{s_n^2}. \tag{34}$$

Multiplying (33) by φ^T on the left and φ on the right, we find:
$m'(s_n) = \left(pw_0(s_n)^{p-1} - \frac{p}{p-1}\right)m(s_n)$
$+p(p-1)w_0(s_n)^{p-2}\left[2m(s_n)^2 + (w_1(s_n).\varphi)^2\right] + O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right) + O\left(\frac{1}{s_n^{5/2}}\right)$.
Therefore, since $(w_1(s_n).\varphi)^2 \ge 0$, we have
$pw_0(s_n)^{p-1} - \frac{p}{p-1} \ge \frac{-1}{m(s_n)}\Big(- m'(s_n) + 2p(p-1)w_0(s_n)^{p-2}m(s_n)^2$

$+O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right)+O\left(\frac{1}{s_n^{5/2}}\right)\Bigg)$.
With (34), we obtain

$$\begin{aligned} w_0(s_n) \quad &\geq \quad \Bigg(\frac{1}{p-1}+\frac{s_n}{c_2}\Bigg(-\frac{c_2}{ps_n^2}+2(p-1)w_0(s_n)^{p-2}\frac{c_2^2}{s_n^2} \\ &+ \quad O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right)+O\left(\frac{1}{s_n^{5/2}}\right)\Bigg)\Bigg)^{\frac{1}{p-1}}. \end{aligned} \tag{35}$$

Now, we claim that the following lemma yields the conclusion:

Lemma 2.5 *There exists positive constants $C(A,p,\epsilon)$ and $C'(A,p,\epsilon)$ such that*

$$|w(0,s_n)-\kappa| \leq \frac{C(A,p,\epsilon)}{s_n} \tag{36}$$

$$\textit{and} \qquad |w_1(s_n)| \leq \frac{C'(A,p,\epsilon)}{s_n}. \tag{37}$$

Indeed, if we inject (36) and (37) in (35), then we get
$w_0(s_n) \geq \left(\frac{1}{p-1}+\frac{s_n}{c_2}\left(-\frac{c_2}{ps_n^2}+2(p-1)\kappa^{p-2}\frac{c_2^2}{s_n^2}+o\left(\frac{1}{s_n^2}\right)\right)\right)^{\frac{1}{p-1}}$, which yields
$w_0(s_n) \geq \kappa + 2\left(c_2-\frac{\kappa}{2p}\right)\frac{1}{s_n}+o\left(\frac{1}{s_n}\right)$. Since $c_2 > \frac{\kappa}{2p}$, we obtain

$$w_0(s_n) > \kappa$$

for s_n large enough, which contradicts by lemma 2.3 the fact that w is globally defined on $[-\log T, +\infty)$.

Proof of lemma 2.5:
We derive from *ii)* of lemma 2.4 and (35): $w_0(s_n) \geq \left(\frac{1}{p-1}+O\left(\frac{1}{s_n}\right)\right)^{\frac{1}{p-1}} = \kappa + O\left(\frac{1}{s_n}\right)$. Since w is globally defined for $s \in [-\log T, +\infty)$, lemma 2.3 gives $w_0(s_n) \leq \kappa$. Hence,

$$w_0(s_n) = \kappa + O\left(\frac{1}{s_n}\right). \tag{38}$$

Integrating (22) with respect to ρdy, we obtain: $|w_0(s_n) - w(0,s_n)| \leq O\left(\frac{1}{s_n}\right)$. Together with (38), this gives (36).

Now, we claim that $|\nabla w(0,s_n)| \leq \frac{B}{s_n}$ with
$B = \sqrt{2c_2(3c_0 + C(A,p,\epsilon))}$. Indeed, if not, then we use the left inequality

of (22) and write for $\hat{y}_n = \frac{B}{c_2}\frac{\nabla w(0,s_n)}{|\nabla w(0,s_n)|}$:
$w(\hat{y}_n, s_n) \geq w(0, s_n) + \hat{y}_n.\nabla w(0, s_n) - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)|\hat{y}_n|^2$
$\geq \kappa - \frac{C(A,p,\epsilon)}{s_n} + \frac{B^2}{c_2 s_n} - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)\frac{B^2}{c_2^2}$
$= \kappa + \frac{3c_0}{s_n} + O\left(\frac{1}{s_n^{3/2}}\right)$. Therefore,

$$\kappa + \frac{c_0}{s_n} \geq \|w(s_n)\|_{L^\infty} \geq \kappa + \frac{2c_0}{s_n}$$

if s_n is large enough, which is a contradiction. Hence, $|\nabla w(0, s_n)| \leq \frac{B}{s_n}$. Using (24), we find $|w_1(s_n)| \leq \frac{C'(A,p,\epsilon)}{s_n}$ with $C'(A, p, \epsilon) = B + C^*(N)A$. This concludes the proof of lemma 2.5. ■

Thus, Case 3 can not occur.

Case 4: $\|\nabla^2 w(s_n)\|_{L^\infty}$ can not reach $\frac{A}{s_n}$

There exists $A_4(p)$ such that for all $A \geq A_4$, and $\epsilon > 0$, $\exists S_4(A, \epsilon)$ such that if $s_n \geq S_4(A, \epsilon)$, then Case 4 of Proposition 2.2 can not occur.

Proof: It follows from the bounds on w and $\nabla^3 w$. We have $|\nabla^2 w(0, s_n)| = \frac{A}{s_n}$. Hence, there exists $\eta_0 \in \{-1, 1\}$ and a unitary vector $\psi_n \in \mathbb{R}^N$ such that $\psi_n^T \nabla^2 w(0, s_n)\psi_n = \eta_0 \frac{A}{s_n}$. Let us notice that if $A > \frac{\kappa}{2p}$, then we have from (19) $\eta_0 = 1$ for n large enough.

Using (23) with $\hat{y}_n = \eta_1 \frac{\sqrt{s_n}}{A^{1/4}}\psi_n$ where $\eta_1 \in \{-1, 1\}$ is chosen so that $\hat{y}_n.\nabla w(0, s_n) \geq 0$, we write:
$w(\hat{y}_n, s_n) \geq w(0, s_n) + \hat{y}_n.\nabla w(0, s_n) + \frac{1}{2}\hat{y}_n^T \nabla^2 w(0, s_n)\hat{y}_n - \frac{1}{6}|\hat{y}_n|^3 \frac{A^{5/4}}{s_n^{3/2}}$
$\geq 0 + 0 + \frac{s_n}{2\sqrt{A}}\frac{A}{s_n} - \frac{s_n^{3/2}}{6A^{3/4}}\frac{A^{5/4}}{s_n^{3/2}} = \frac{\sqrt{A}}{3}$. If $A \geq 36\kappa^2$, then we have

$$\kappa + \frac{c_0}{s_n} \geq \|w(s_n)\|_{L^\infty} \geq 2\kappa$$

which is a contradiction for s_n large enough. ■

Thus, Case 4 can not occur.

Case 5: $\|\nabla^3 w(s_n)\|_{L^\infty}$ can not reach $\frac{A^{5/4}}{s_n^{3/2}}$

We first give a crucial uniform ODE comparison result for w in $V_A(s)$. Such a result has been shown in [16] for a fixed solution (see (6)). We claim that these estimates are in fact uniform for $w \in \hat{V}_A(s)$.

We have the following proposition:

Proposition 2.3 (ODE like behavior in $V_A(s)$) *For a given $A > 0$, $\forall \eta > 0$, $\exists C_\eta > 0$ such that for all $s^* \geq -\log T$, for all solution w of (3) defined for all $s \geq -\log T$ and satisfying $w \in \hat{V}_A(s^*)$, we have $\forall x \in \mathbb{R}^N$, $\forall t \in [0, t^*]$,*

$$\left| \frac{\partial u}{\partial t}(x,t) - u(x,t)^p \right| \leq \eta u(x,t)^p + C_\eta$$

where $t^* = T - e^{-s^*}$ and $u(x,t) = (T-t)^{\frac{1}{p-1}} w\left(\frac{x}{\sqrt{T-t}}, -\log(T-t)\right)$.

Proof: It is mainly the same as in [16] (Theorem 3), and it uses a compactness procedure. See Appendix B. ■

Now, we begin the treatment of Case 5.

We have

$$|\nabla^3 w(0, s_n)| = \frac{A^{5/4}}{s_n^{3/2}}, \text{ and } \forall s \in [-\log T, s_n],\ w(s) \in V_A(s). \tag{39}$$

Since $w(s_n) \in V_A(s_n)$, we have $0 \leq w(0, s_n) \leq \kappa + \frac{c_0}{s_n}$. Therefore, we can assume that

$$w(0, s_n) \to a \in [0, \kappa] \text{ as } n \to +\infty.$$

We will consider the case where a is small in Part I, and let the case where it is not small for Part II. We first claim the following lemma:

Lemma 2.6 *$\forall S > 0$, $\sup_{s \in [s_n - S, s_n]} |w(0,s) - \varphi_a(s - s_n)| \to 0$ as $n \to +\infty$ where φ_a is the solution of*

$$\begin{cases} \varphi_a'(s) &= -\frac{\varphi_a(s)}{p-1} + \varphi_a(s)^p \\ \varphi_a(0) &= a, \end{cases}$$

$$\textit{that is } \varphi_a(s) = \kappa \left(1 + \left(\frac{a^{1-p}}{p-1} - 1\right) e^s\right)^{-\frac{1}{p-1}} \textit{ if } a > 0, \textit{ and } \varphi_0(s) \equiv 0. \tag{40}$$

Proof: Let $z_n(s) = w(0,s)$, then we have from (3) $\forall s \in [s_n - S, s_n]$

$$\begin{cases} z_n'(s) + \frac{z_n(s)}{p-1} - z_n(s)^p = \Delta w(0,s) \\ z_n(s_n) \to a. \end{cases}$$

Since $\forall s \in [s_n - S, s_n]$, $w(s) \in V_A(s)$, we get $|\Delta w(0,s)| \leq N \|\nabla^2 w(s)\|_{L^\infty} \leq \frac{NA}{s}$. Hence, $\forall \eta > 0$, we have for n large enough and $s \in [s_n - S, s_n]$:

$$\begin{cases} \left| z_n'(s) + \frac{z_n(s)}{p-1} - z_n(s)^p \right| \leq \eta \\ |z_n(s_n) - a| \leq \eta. \end{cases}$$

Therefore, by classical continuity arguments on ordinary differential equations, $w(0,s) = z_n(s) \to \varphi_a(s - s_n)$ as $n \to +\infty$, uniformly on $[s_n - S, s_n]$. This concludes the proof of lemma 2.6. ■

Part I: Case where $a \le \delta(p)$

There exists $\delta(p) \in (0, \kappa)$ *and* $S_5(p)$ *such that if* $A \ge 1$, $s_n \ge S_5(p)$ *and* $a \le \delta(p)$, *then Case 5 of Proposition 2.2 can not occur.*

This result follows from local estimates in new variables (ξ, τ) defined below and scaling arguments. We assume $a \le \delta(p)$ where $\delta(p)$ will be fixed later small enough, lower than $\frac{\kappa}{5}$.

Step 1: Setting of the problem:

For each $n \in \mathbb{N}$, we introduce $s'_n = \max\{\frac{s_n}{2}\} \cup \{s \in [\frac{s_n}{2}, s_n] \mid w(0,s) \ge \frac{\kappa}{2}\}$. Let us remark that $w(0, s'_n) \le \kappa$ and if $s'_n > \frac{s_n}{2}$, then $w(0, s'_n) = \frac{\kappa}{2}$.

We have the following lemma:

Lemma 2.7 *There exists* $S(\delta) \to +\infty$ *as* $\delta \to 0$ *such that for* n *large enough,* $S(\delta) \le s_n - s'_n \le \frac{s_n}{2}$.

Proof: Since $s'_n \ge \frac{s_n}{2}$, we have $s_n - s'_n \le \frac{s_n}{2}$.
We get from (40) $S > 0$ such that $\forall s \in [-S, 0]$, $a \le \varphi_a(s) \le \frac{\kappa}{5}$ and $S \to +\infty$ as $a \to 0$. Hence, $S \to +\infty$ as $\delta \to 0$, since $a \le \delta$.

Since $w(0,s) \to \varphi_a(s - s_n)$ as $n \to +\infty$ uniformly on $[s_n - S, s_n]$ by lemma 2.6, we obtain for n large enough $\forall s \in [s_n - S, s_n]$, $w(0,s) \le \frac{\kappa}{4}$. Thus, $s'_n \le s_n - S$. This concludes the proof of lemma 2.7. ■

Let us define for each $n \in \mathbb{N}$, $\xi \in \mathbb{R}^N$ and $\tau \in [-1, 1)$,

$$\begin{aligned} v_n(\xi, \tau) &= e^{-\frac{s'_n}{p-1}} u\left(\xi e^{-\frac{s'_n}{2}}, T + (\tau - 1)e^{-s'_n}\right) \\ &= (1-\tau)^{-\frac{1}{p-1}} w\left(\frac{\xi}{\sqrt{1-\tau}}, s'_n - \log(1-\tau)\right) \end{aligned} \tag{41}$$

where u is defined from w by (2) (take $a = 0$), and introduce $\tau_n \in [0, 1]$ defined by $s'_n - \log(1 - \tau_n) = s_n$. Then, v_n satisfies: $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [-1, 1)$

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + v_n^p. \tag{42}$$

From (39) and the definition of s'_n, we get: $v_n(\xi, 0) = w(\xi, s'_n)$,

$$\left\{ \begin{array}{rclrcl} v_n(0,0) & \le & \frac{\kappa}{2}, & \|\nabla v_n(0)\|_{L^\infty} & \le & \frac{c_1}{\sqrt{s'_n}}, \\ \|\nabla^2 v_n(0)\|_{L^\infty} & \le & \frac{A}{s'_n}, & \|\nabla^3 v_n(0)\|_{L^\infty} & \le & \frac{A^{5/4}}{s'^{3/2}_n}. \end{array} \right. \tag{43}$$

Note that if $s'_n > \frac{s_n}{2}$, then $v_n(0,0) = \frac{\kappa}{2}$.

Step 2: Estimates in v variable

We claim the following lemmas:

Lemma 2.8 (First estimate) *For n large enough, we have:*
i) $\forall \tau \in [-1, \tau_n]$, $\forall |\xi| \le 2 s_n'^{1/4}$: $v_n(\xi, \tau) \le C(p)$.
ii) For all $i = 1, 2, 3$, $\forall \tau \in [-\frac{1}{4}, \tau_n]$, $\forall |\xi| \le \frac{3}{2} s_n'^{1/4}$, $|\nabla^i v_n(\xi, \tau)| \le C'(p)$.

Lemma 2.9 (Refined estimate) *Assume that* $s'_n > \frac{s_n}{2}$. *Then,*
i) $\forall \tau \in [0, \tau_n]$, $\forall |\xi| \le s_n'^{1/4}$, $\frac{\kappa}{4} \le v_n(\xi, \tau) \le C(p)$.
ii) There exist positive constants $C_6(p)$, $C_7(p)$ *and* $C_8(p)$ *such that if* $A \ge 1$ *then* $\forall \tau \in [0, \tau_n]$:

$$\forall |\xi| \le \frac{s_n'^{1/4}}{4}, |\nabla v_n(\xi, \tau)| \le \frac{C_6(p)}{\sqrt{s'_n}}, \tag{44}$$

$$\forall |\xi| \le \frac{s_n'^{1/4}}{4^2}, |\nabla^2 v_n(\xi, \tau)| \le \frac{C_7(p) A}{s'_n}, \tag{45}$$

$$\forall |\xi| \le \frac{s_n'^{1/4}}{4^3}, |\nabla^3 v_n(\xi, \tau)| \le \frac{C_8(p) A^{5/4}}{s_n'^{3/2}}. \tag{46}$$

Proof of lemma 2.8:
i) By Proposition 2.3, we have: $\forall \eta > 0$, $\forall x \in \mathbb{R}^N$, $\forall t \in [0, T - e^{-s_n})$

$$\left| \frac{\partial u}{\partial t}(x,t) - u(x,t)^p \right| \le \eta u(x,t)^p + C_\eta.$$

Therefore, we get from (41): $\forall \eta > 0$, we have for n large enough: $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [-1, \tau_n]$

$$\left| \frac{\partial v_n}{\partial \tau} - v_n(\xi, \tau)^p \right| \le \eta v_n(\xi, \tau)^p + C_\eta e^{-\frac{p s'_n}{p-1}} \le \eta \left(v_n(\xi, \tau)^p + 1 \right). \tag{47}$$

Using a Taylor expansion and (43), we get for n large enough: $\forall |\xi| \le 2 s_n'^{1/4}$

$$|v_n(\xi, 0) - v_n(0,0)| \le \frac{2c_1}{s_n'^{1/4}} \text{ and } v_n(\xi, 0) \le \frac{3\kappa}{4}. \tag{48}$$

We take $\eta = \eta(p) > 0$ small enough such that $v_\eta(\tau)$ and $V_\eta(\tau)$ defined by

$$v_\eta(0) = V_\eta(0) = \frac{3\kappa}{4},\ v'_\eta = (1+\eta) v_\eta^p + \eta,\ \text{and } V'_\eta = (1-\eta) V_\eta^p - \eta.$$

are well defined for all $\tau \in [-1,1]$ and satisfy $\max\left(V_\eta(\tau), v_\eta(\tau)\right) \le 2v_0(1) = C(p)$.

Hence, for n large enough: $\forall |\xi| \le 2s_n'^{1/4}$,

$$\forall \tau \in [0,\tau_n],\ v_n(\xi,\tau) \le v_\eta(\tau),\ \text{and } \forall \tau \in [-1,0],\ v_n(\xi,\tau) \le V_\eta(\tau). \quad (49)$$

Therefore, $v_n(\xi,\eta) \le C(p)$ for all $\tau \in [-1,\tau_n]$. This concludes the proof of i).

ii) We use a classical result (see Theorem 3 p. 406 in Friedman [8], see also Douglis and Nirenberg [5]):

Lemma 2.10 *Assume that h solves*

$$\frac{\partial h}{\partial \tau} = \Delta h + a(\xi,\tau)h$$

for $(\xi,\tau) \in D$ where $D = B(0,3) \times (-\tau_0, \tau_*)$ and $\tau_0, \tau_* \in [\frac{1}{2}, 1]$. Assume in addition that $\|a\|_{L^\infty} + |a|_{\alpha,D}$ is finite, where

$$|a|_{\alpha,D} = \sup_{(\xi,\tau),(\xi',\tau') \in D} \frac{|a(\xi,\tau) - a(\xi',\tau')|}{\left(|\xi - \xi'| + |\tau - \tau'|^{1/2}\right)^\alpha}$$

and $\alpha \in (0,1)$. Then,

$$\|h\|_{C^2(D')} + |\nabla^2 h|_{\alpha,D'} \le K\|h\|_{L^\infty(D)}$$

where $K = K\left(\|a\|_{L^\infty(D)} + |a|_{\alpha,D}\right)$ and $D' = B(0,1) \times [-\tau_0 + \frac{1}{4}, \tau_*)$.

Since v_n is bounded on $B(0, 2s_n'^{1/4}) \times [-1,\tau_n]$ (see i)), and since v_n and ∇v_n satisfy

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + a_1(\xi,\tau)v_n$$

and

$$\frac{\partial \nabla v_n}{\partial \tau} = \Delta\left(\nabla v_n\right) + a_2(\xi,\tau)\nabla v_n$$

for all $(\xi,\tau) \in B(0, 2s_n'^{1/4}) \times [-1,\tau_n]$, with $a_2 = pa_1 = pv_n^{p-1}$, it is enough to prove that $|v_n|_{1, B(0,2s_n'^{1/4}) \times (-\frac{3}{4}, \tau_n)}$ is finite and to apply lemma 2.10 successively to v_n and ∇v_n in order to conclude the proof of ii).

For this purpose and from translation invariance, we restrict ourselves to $|\xi| < 3$ and write for all $(\xi,\tau) \in D = B(0,3) \times (-1,\tau_n)$, $v_n = h_1 + h_2$ where:

- h_1 is a solution of

$$\begin{cases} \frac{\partial h_1}{\partial \tau} &= \Delta h_1 \text{ for } (\xi,\tau) \in D \\ h_1(\xi,\tau) &= v_n(\xi,\tau) \text{ for } |\xi| = 3 \text{ and } \tau \in (-1,\tau_n) \\ h_1(\xi,-1) &= v_n(\xi,-1) \text{ for } |\xi| < 3, \end{cases}$$

- h_2 is a solution of

$$\begin{cases} \frac{\partial h_2}{\partial \tau} &= \Delta h_2 + f(x,t) \text{ for } (\xi,\tau) \in \mathbb{R}^N \times (-1,\tau_n) \\ h_2(\xi,-1) &= 0 \text{ for all } \xi \in \mathbb{R}^N \end{cases} \tag{50}$$

with

$$f(\xi,\tau) = v_n(\xi,\tau)^p 1_{\{(\xi,\tau)\in D\}} \leq C(p). \tag{51}$$

From maximum principle, h_2 is bounded by $C(p)$ on $\mathbb{R}^N$, hence on D. Therefore, h_1 is bounded by $C(p)$ also. Applying lemma 2.10 with $h = h_1$ and $a = 0$, we see that in particular $|h_1|_{1,D'} \leq C(p)$ where $D' = B(0,1)\times[-\frac{3}{4},\tau_n)$.

We have from (50): $\forall(\xi,\tau) \in \mathbb{R}^N \times [-1,\tau_n)$,

$$h_2(\xi,\tau) = \int_{-1}^{\tau} e^{(\tau-\sigma)\Delta} f(\sigma) d\sigma. \tag{52}$$

We claim that

$$|h_2|_{1,\mathbb{R}^N\times[-1,\tau_n)} \leq C(p), \tag{53}$$

which concludes the proof.

Proof of (53):

Let us recall that for all $\varphi \in L^\infty(\mathbb{R}^N)$: $\|e^{\tau\Delta}\varphi\|_{L^\infty} \leq \|\varphi\|_{L^\infty}$,

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^\infty} \leq \frac{C}{\sqrt{\tau}}\|\varphi\|_{L^\infty}, \text{ and } \|\frac{\partial}{\partial\tau} e^{\tau\Delta}\varphi\|_{L^\infty} \leq \frac{C}{\tau}\|\varphi\|_{L^\infty}. \tag{54}$$

In order to prove (53), it is enough to estimate $|\nabla h_2(\xi,\tau)|$ and $\frac{|h_2(\xi,\tau_1)-h_2(\xi,\tau_2)|}{|\tau_1-\tau_2|^{1/2}}$ for all $\xi \in \mathbb{R}^N$ and $\tau,\tau_1,\tau_2 \in [-1,\tau_n)$.

By (52), (54) and (51), we have:

$|\nabla h_2(\xi,\tau)| = \left|\int_{-1}^{\tau} \nabla e^{(\tau-\sigma)\Delta} f(\sigma)d\sigma\right| \leq \int_{-1}^{\tau} \frac{C}{\sqrt{\tau-\sigma}}\|f(\sigma)\|_{L^\infty} d\sigma$
$\leq 2C(p)\sqrt{\tau+1} \leq C(p)$.

Now, we take $\tau_2 < \tau_1$ and introduce $\tau_3 = \max\left(-1, \tau_2 - \sqrt{\tau_1-\tau_2}\right)$. Then,
$\frac{|h(\xi,\tau_1)-h(\xi,\tau_2)|}{\sqrt{\tau_1-\tau_2}} = (\tau_1-\tau_2)^{-\frac{1}{2}} \left|\int_{-1}^{\tau_1} e^{(\tau_1-\sigma)\Delta} f(\sigma)d\sigma - \int_{-1}^{\tau_2} e^{(\tau_2-\sigma)\Delta} f(\sigma)d\sigma\right|$
$\leq I + II + III$ with $I = (\tau_1-\tau_2)^{-\frac{1}{2}} \int_{\tau_3}^{\tau_1} \|e^{(\tau_1-\sigma)\Delta} f(\sigma)\|_{L^\infty} d\sigma$,

$II=(\tau_1-\tau_2)^{-\frac{1}{2}}\int_{\tau_3}^{\tau_2}\|e^{(\tau_2-\sigma)\Delta}f(\sigma)\|_{L^\infty}d\sigma$ and
$III=\int_{-1}^{\tau_3}\left|e^{(\tau_1-\sigma)\Delta}f(\sigma)-e^{(\tau_2-\sigma)\Delta}f(\sigma)\right|d\sigma.$

From (54) and (51), we have:
$I\le(\tau_1-\tau_2)^{-\frac{1}{2}}\int_{\tau_3}^{\tau_1}C(p)d\sigma=C(p)(\tau_1-\tau_2)^{-\frac{1}{2}}(\tau_1-\tau_3)$
$\le C(p)(\tau_1-\tau_2)^{-\frac{1}{2}}(\tau_1-\tau_2+\sqrt{\tau_1-\tau_2})\le C(p).$
Similarly, $II\le C(p)$. ■

For III, we write
$III=(\tau_1-\tau_2)^{-\frac{1}{2}}\int_{-1}^{\tau_3}d\sigma\left|\int_{\tau_2-\sigma}^{\tau_1-\sigma}\frac{\partial}{\partial\sigma_1}e^{\sigma_1\Delta}f(\sigma)d\sigma_1\right|$
$\le(\tau_1-\tau_2)^{-\frac{1}{2}}\int_{-1}^{\tau_3}d\sigma\int_{\tau_2-\sigma}^{\tau_1-\sigma}\frac{C}{\sigma_1}d\sigma_1$ by (54),
$\le(\tau_1-\tau_2)^{-\frac{1}{2}}\int_{-1}^{\tau_3}d\sigma\frac{C(\tau_1-\tau_2)}{(\tau_2-\sigma)}$
$\le C(\tau_1-\tau_2)^{\frac{1}{2}}(\tau_3+1)(\tau_2-\tau_3)^{-1}\le C(\tau_1-\tau_2)^{\frac{1}{2}}\times 2\times(\sqrt{\tau_1-\tau_2})^{-1}=C.$

Thus, $\forall\xi\in\mathbb{R}^N$, $\forall\tau_1,\tau_2\in[-1,\tau_n)$,
$|h_2(\xi,\tau_1)-h_2(\xi,\tau_2)|\le C|\tau_1-\tau_2|^{\frac{1}{2}}.$

This concludes the proof of (53) and the proof of lemma 2.8 also. ■

Proof of lemma 2.9:
In this case, $v_n(0,0)=\frac{\kappa}{2}$.
i) As in lemma 2.6, (47) and (48) yield $\sup_{|\xi|\le s_n'^{1/4},\tau\in[0,\tau_n]}|v_n(\xi,\tau)-v(\tau)|\to 0$ as $n\to+\infty$, where v is the solution of

$$v'(\tau)=v(\tau)^p,\ v(0)=\frac{\kappa}{2},\ \text{that is } v(\tau)=\kappa\left(2^{p-1}-\tau\right)^{-\frac{1}{p-1}}.$$

Since $\forall\tau\in[0,1]$, $v(\tau)\ge\frac{\kappa}{2}$, we have for n large enough:

$$\forall|\xi|\le s_n'^{1/4},\ \forall\tau\in[0,1),\ \frac{\kappa}{4}\le v_n(\xi,\tau). \tag{55}$$

i) of lemma 2.8 yields the upper bound.

ii) Let us recall the following lemma:

Lemma 2.11 *Assume that* $z(\xi,\tau)$ *satisfies* $\forall|\xi|\le 4B_1$, $\forall\tau\in[0,\tau_*]$:

$$\left\{\begin{array}{l}\frac{\partial z}{\partial\tau}\le\Delta z+\lambda z+\mu,\\ z(\xi,0)\le z_0,\ z(\xi,\tau)\le B_2\end{array}\right. \tag{56}$$

where $\tau_*\le 1$. *Then,* $\forall|\xi|\le B_1$, $\forall\tau\in[0,\tau_*]$,

$$z(\xi,\tau)\le e^{\lambda\tau}\left(z_0+\mu+CB_2e^{-\frac{B_1^2}{4}}\right).$$

Proof: See Appendix C. ∎

Estimate on $\nabla v_n(\xi,\tau)$:
We estimate $h(\xi,\tau) = |\nabla v_n(\xi,\tau).\alpha|$ where α is a unitary vector of $\mathbb{R}^N$.
From (42), Kato's inequality, (43) and lemma 2.8, we see that $\forall|\xi| \le s_n'^{1/4}$, $\forall\tau \in [0,\tau_n]$,

$$\left\{\begin{array}{l} \frac{\partial h}{\partial \tau} \le \Delta h + p v_n^{p-1} h \le \Delta h + pC(p)^{p-1} h, \\ h(\xi,0) \le \frac{c_1}{\sqrt{s_n'}},\ h(\xi,\tau) \le C'(p). \end{array}\right. \tag{57}$$

Using lemma 2.11, we get: $\forall|\xi| \le \frac{s_n'^{1/4}}{4}$, $\forall\tau \in [0,\tau_n]$,

$$h(\xi,\tau) \le e^{pC(p)^{p-1}} \left(\frac{c_1}{\sqrt{s_n'}} + CC'(p) e^{-\frac{s_n'^{1/2}}{4}} \right)$$

which yields (44) since $c_1 \le \frac{\kappa}{\sqrt{p}} + 2\sqrt{p}$.

Estimate on $\nabla^2 v_n(\xi,\tau)$:
We estimate $\theta(\xi,\tau) = |\alpha^T \nabla^2 v_n(\xi,\tau)\alpha|$ where α is a unitary vector in $\mathbb{R}^N$.
From (42) and Kato's inequality, we have: $\forall\xi \in \mathbb{R}^N$, $\forall\tau \in [0,1)$,

$$\frac{\partial \theta}{\partial \tau} \le \Delta\theta + p v_n^{p-1}\theta + p(p-1) v_n^{p-2} |\nabla v_n|^2.$$

Using (44), lemma 2.8, $i)$ of lemma 2.9 and (43), we claim that $\forall|\xi| \le \frac{s_n'^{1/4}}{4}$, $\forall\tau \in [0,\tau_n]$,

$$\left\{\begin{array}{l} \frac{\partial \theta}{\partial \tau} \le \Delta\theta + C(p)\theta + C(p)\frac{C_6(p)^2}{s_n'}, \\ \theta(\xi,0) \le \frac{A}{s_n'},\ \theta(\xi,\tau) \le C'(p) \end{array}\right.$$

By lemma 2.11, we obtain, $\forall|\xi| \le \frac{s_n'^{1/4}}{4^2}$, $\forall\tau \in [0,\tau_n]$,

$$\theta(\xi,\tau) \le e^{C(p)} \left(\frac{A}{s_n'} + C(p)\frac{C_6(p)^2}{s_n'} + CC'(p) e^{-\frac{s_n'^{1/2}}{4^3}} \right).$$

Since $A \ge 1$, this yields (45).

Estimate on $\nabla^3 v_n(\xi,\tau)$:
We estimate $\nu(\xi,\tau) = |\nabla^3 v_n(\xi,\tau)(\alpha,\beta,\gamma)|$ where α, β and γ are unitary vectors in $\mathbb{R}^N$.
From (42) and Kato's inequality, we have: $\forall\xi \in \mathbb{R}^N$, $\forall\tau \in [0,1)$,

$$\frac{\partial \nu}{\partial \tau} \le \Delta\nu + p v_n^{p-1}\nu + 3p(p-1) v_n^{p-2} |\nabla v_n||\nabla^2 v_n| + p(p-1)|p-2| v_n^{p-3} |\nabla v_n|^3.$$

Using (44), (45), lemma 2.8, *i*) of lemma 2.9 and (43), we get: $\forall |\xi| \leq \frac{s_n'^{1/4}}{4^2}$, $\forall \tau \in [0, \tau_n]$.

$$\begin{cases} \frac{\partial \nu}{\partial \tau} \leq \Delta \nu + C(p)\nu + C(p)\frac{\left(C_6(p)^3 + C_6(p)C_7(p)\right)}{s_n'^{3/2}}, \\ \nu(\xi, o) \leq \frac{A^{5/4}}{s_n'^{3/2}}, \nu(\xi,\tau) \leq C'(p). \end{cases}$$

Applying again lemma 2.11, we obtain: $\forall |\xi| \leq \frac{s_n'^{1/4}}{4^3}$, $\forall \tau \in [0, 1)$,

$$\nu(\xi,\tau) \leq e^{C(p)} \left(\frac{A^{5/4}}{s_n'^{3/2}} + C(p)\frac{(C_6(p)^3 + C_6(p)C_7(p))}{s_n'^{3/2}} + CC'(p)e^{-\frac{s_n'^{1/2}}{4^5}} \right).$$

Since $A \geq 1$, this yields (46).

This concludes the proof of lemma 2.9. ■

Step 3: Conclusion of the proof

From (41), we have

$$\nabla^3 w(0, s_n) = (1 - \tau_n)^{\left(\frac{1}{p-1} + \frac{3}{2}\right)} \nabla^3 v_n(0, \tau_n), \tag{58}$$

where τ_n is defined by $s_n' - \log(1 - \tau_n) = s_n$.

- If $s_n' = \frac{s_n}{2}$, then $1 - \tau_n = e^{s_n' - s_n} = e^{-\frac{s_n}{2}}$. Hence, (58) and lemma 2.8 yield:

$$|\nabla^3 w(0, s_n)| \leq e^{-\frac{s_n}{2}\left(\frac{1}{p-1} + \frac{3}{2}\right)} C'(p).$$

This contradicts (39) for s_n large enough.

- If $s_n' > \frac{s_n}{2}$, then we have by lemma 2.7 $s_n' - s_n \leq -S(\delta)$ for n large enough. Therefore, (58) and lemma 2.9 yield

$$\begin{aligned} |s_n^{3/2} \nabla^3 w(0, s_n)| &\leq A^{5/4} C_8(p) e^{(s_n' - s_n)\left(\frac{1}{p-1} + \frac{3}{2}\right)} \left(\frac{s_n}{s_n'}\right)^{\frac{3}{2}} \\ &\leq A^{5/4} C_8(p) e^{-S(\delta)\left(\frac{1}{p-1} + \frac{3}{2}\right)} \left(\frac{s_n}{s_n/2}\right)^{3/2}. \end{aligned} \tag{59}$$

Since $S(\delta) \to +\infty$ as $\delta \to 0$, we fix $\delta(p) > 0$ such that

$$C_8(p) e^{-S(\delta)\left(\frac{1}{p-1} + \frac{3}{2}\right)} 2^{3/2} \leq \frac{1}{2}.$$

Therefore, (59) yields $|s_n^{3/2} \nabla^3 w(0, s_n)| \leq \frac{A^{5/4}}{2}$. This contradicts (39).

Thus, Case 5 can not occur if $a \le \delta(p)$.

Part II: Case where $a \ge \delta(p)$

There exists $A_6(p) > 0$ and $S_6(p)$ such that for all $A \ge A_6(p)$, if $s_n \ge S_6(p)$, then Case 5 of Proposition 2.2 can not occur if $a \ge \delta(p)$.

This follows from linear estimates on w, for the spectrum of the linear part of the equation on $\nabla^3 w$ is fully negative.

Let us remark that in this case, we have:

$$\forall s \in [s_n - 1, s_n],\ \forall |y| \le \frac{\delta\sqrt{s}}{4c_1}, \frac{\delta}{4} \le w(y,s) \le \kappa + 1. \tag{60}$$

Indeed, the upper bound follows from the fact that $w(s) \in V_A(s)$. For the lower bound, we notice that since $a \ge \delta$, we have from lemma 2.6 and (40): $\forall s \in [s_n - 1, s_n]$, $w(0,s) \ge \frac{\delta}{2}$ for s_n large enough. Therefore, we have by (25): $w(y,s) \ge w(0,s) - \frac{c_1}{\sqrt{s}}|y| \ge \frac{\delta}{2} - \frac{\delta\sqrt{s}}{4c_1}\frac{c_1}{\sqrt{s}} = \frac{\delta}{4}$.

From (39), we have the existence of α, β, $\gamma \in \mathbb{R}^N$ such that $|\alpha| = |\beta| = |\gamma| = 1$ and

$$|\nabla^3 w(0,s_n)(\alpha,\beta,\gamma)| = \frac{A^{5/4}}{s_n^{3/2}}. \tag{61}$$

Our strategy is to derive from (3) an equation on $g(y,s) = \nabla^3 w(y,s)(\alpha,\beta,\gamma)$ and to do a priori estimates on it in order to contradict (61). We in fact define

$$G(y,s) = F(y,s)\chi(y,s), \qquad F(y,s) = |g(y,s)| = |\nabla^3 w(y,s)(\alpha,\beta,\gamma)|, \tag{62}$$

$$\chi(y,s) = \chi_0\left(\frac{8c_1|y|}{\delta\sqrt{s}}\right) \tag{63}$$

and $\chi_0 \in C^\infty([0,+\infty),\mathbb{R}^+)$ satisfies $\chi_0(z) = 1$ for $|z| \le 1$, $\chi_0(z) = 0$ for $|z| \ge 2$.

From (3), we see that

$$\begin{aligned}
\frac{\partial g}{\partial s} &= \left(\mathcal{L} - \frac{3}{2} + pw(y,s)^{p-1} - \frac{p}{p-1}\right) g \\
&+ p(p-1)w(y,s)^{p-2}\Big((\alpha.\nabla w)(\beta^T\nabla^2 w\gamma) + (\beta.\nabla w)(\gamma^T\nabla^2 w\alpha) \\
&+ (\gamma.\nabla w)(\alpha^T\nabla^2 w\beta)\Big) \\
&+ p(p-1)(p-2)w(y,s)^{p-3}(\alpha.\nabla w)(\beta.\nabla w)(\gamma.\nabla w).
\end{aligned}$$

We see from (39) and definition 2.1 that for s_n large enough, we have:
- $\forall y \in \mathbb{R}^N$, $\forall s \in [s_n - 1, s_n]$, $pw(y,s)^{p-1} - \frac{p}{p-1} \le \frac{1}{4}$,
- $0 < \frac{\kappa}{\sqrt{p}} \le c_1(\epsilon) \le \frac{\kappa}{\sqrt{p}} + 2\sqrt{p}$ for all $\epsilon \in (0,1)$.
Therefore, F satisfies the following inequality: $\forall y \in \mathbb{R}^N$, $\forall s \in [s_n - 1, s_n]$,

$$\frac{\partial F}{\partial s} \le (\mathcal{L} - \frac{5}{4})F + \frac{C(p)A}{s^{3/2}} w(y,s)^{p-2} + \frac{C(p)}{s^{3/2}} w(y,s)^{p-3}.$$

Hence, by (63) and (60), G satisfies the following inequality: $\forall y \in \mathbb{R}^N$, $\forall s \in [s_n - 1, s_n]$
$\frac{\partial G}{\partial s} \le (\mathcal{L} - \frac{5}{4})G + \frac{C(p)A}{s^{3/2}} \chi w(y,s)^{p-2} + \frac{C(p)}{s^{3/2}} \chi w(y,s)^{p-3} + F(\frac{\partial \chi}{\partial s} + \Delta\chi + \frac{1}{2} y.\nabla\chi) - 2\nabla.(F\nabla\chi)$.
$\le (\mathcal{L} - \frac{5}{4})G + C(p)\frac{(A+1)}{s^{3/2}} + C(p)\frac{A^{5/4}}{s^{3/2}} 1_{\{|y| \ge \frac{\delta\sqrt{s}}{8c_1}\}} - 2\nabla.(F\nabla\chi)$.

Using an integral formulation of this inequality between $s_n - \eta$ and s_n where $\eta(p)$ is fixed such that

$$\eta \in (0,1) \text{ and } \frac{\delta^2}{512 c_1^2 (1 - e^{-\eta})} \ge \frac{\delta^2}{512 c_1(1)^2 (1 - e^{-\eta})} > \frac{1}{4}, \tag{64}$$

we obtain

$$G(0, s_n) \le I + II + III + IV \tag{65}$$

where
$I = \left[e^{\eta(\mathcal{L} - \frac{5}{4})} G(s_n - \eta) \right](0)$,
$II = \left[\int_{s_n - \eta}^{s_n} dt e^{(s_n - t)(\mathcal{L} - \frac{5}{4})} C(p) \frac{(A+1)}{t^{3/2}} \right](0)$,
$III = \left[\int_{s_n - \eta}^{s_n} dt e^{(s_n - t)(\mathcal{L} - \frac{5}{4})} C(p) \frac{A^{5/4}}{t^{3/2}} 1_{\{|x| \ge \frac{\delta\sqrt{t}}{8c_1}\}} \right](0)$ and
$IV = \left[-2 \int_{s_n - \eta}^{s_n} dt e^{(s_n - t)(\mathcal{L} - \frac{5}{4})} \nabla.(F\nabla\chi) \right](0)$.

Let us recall that the kernel of $\mathcal{L}$ is: $\forall s > 0$,

$$e^{s(\mathcal{L} - \frac{5}{4})}(y,x) = \frac{e^{-\frac{s}{4}}}{(4\pi(1 - e^{-s}))^{N/2}} \exp\left(-\frac{|ye^{-\frac{s}{2}} - x|^2}{4(1 - e^{-s})} \right) \tag{66}$$

and that for all $\varphi \in L^\infty(\mathbb{R}^N)$,

$$\|e^{s(\mathcal{L} - \frac{5}{4})}\varphi\|_{L^\infty} \le e^{-\frac{s}{4}} \|\varphi\|_{L^\infty}, \ \|e^{s(\mathcal{L} - \frac{5}{4})}\nabla\varphi\|_{L^\infty} \le \frac{C(N)}{\sqrt{1 - e^{-s}}} \|\varphi\|_{L^\infty}. \tag{67}$$

From (67), (62) and (39), we have
$I \le e^{-\frac{\eta}{4}} \|G(s_n - \eta)\|_{L^\infty} \le e^{-\frac{\eta}{4}} \frac{A^{5/4}}{(s_n - \eta)^{3/2}}$.

Again, by (67), we have
$II \leq \int_{s_n-\eta}^{s_n} dt e^{-\frac{(s_n-t)}{4}} C(p)\frac{(A+1)}{t^{3/2}} \leq C(p)\frac{(A+1)}{(s_n-\eta)^{3/2}}\eta \leq C(p)\frac{(A+1)}{(s_n-\eta)^{3/2}}$ by (64).

By (66), we have:
$III = \int_{s_n-\eta}^{s_n} dt \frac{e^{-\frac{(s_n-t)}{4}}}{\left(4\pi(1-e^{-(s_n-t)})\right)^{N/2}} \int_{\{|x|\geq\frac{\delta\sqrt{t}}{8c_1}\}} dx \exp\left(-\frac{|x|^2}{4(1-e^{-(s_n-t)})}\right) C(p)\frac{A^{5/4}}{t^{3/2}}.$
For $|x| \geq \frac{\delta\sqrt{t}}{8c_1}$ and $t \in [s_n-\eta, s_n]$, we have
$\exp\left(-\frac{|x|^2}{4(1-e^{-(s_n-t)})}\right) = \exp\left(-\frac{|x|^2}{8(1-e^{-(s_n-t)})}\right)\exp\left(-\frac{|x|^2}{8(1-e^{-(s_n-t)})}\right)$
$\leq \exp\left(-\frac{\delta^2 t}{512c_1^2(1-e^{-\eta})}\right)\exp\left(-\frac{|x|^2}{8(1-e^{-(s_n-t)})}\right) \leq e^{-\frac{t}{4}}\exp\left(-\frac{|x|^2}{8(1-e^{-(s_n-t)})}\right)$ from (64).
Therefore,
$III \leq C(p)\frac{A^{5/4}e^{-\frac{s_n}{4}}}{(s_n-\eta)^{3/2}}\int_{s_n-\eta}^{s_n} dt \int \frac{dx}{\left(4\pi(1-e^{-(s_n-t)})\right)^{N/2}}\exp\left(-\frac{|x|^2}{8(1-e^{-(s_n-t)})}\right)$
$= C(p)\frac{A^{5/4}e^{-\frac{s_n}{4}}}{(s_n-\eta)^{3/2}}\int_{s_n-\eta}^{s_n} dt \int dX e^{-|X|^2}$
$= C(p)\frac{A^{5/4}e^{-\frac{s_n}{4}}}{(s_n-\eta)^{3/2}}\eta \leq C(p)\frac{A^{5/4}e^{-\frac{s_n}{4}}}{(s_n-\eta)^{3/2}}$ by (64).

From (66) and integration by parts, we have:
$IV \leq C(N)\int_{s_n-\eta}^{s_n}\frac{dt}{\sqrt{1-e^{-(s_n-t)}}}\|F(t)\nabla\chi(t)\|_{L^\infty}$. From (62), (39) and (63), we have $F(x,t) \leq \frac{A^{5/4}}{t^{3/2}}$ and $|\nabla\chi| \leq \frac{C}{\sqrt{t}}$. Therefore,
$IV \leq \frac{CA^{5/4}}{(s_n-\eta)^2}\int_{s_n-\eta}^{s_n}\frac{dt}{\sqrt{1-e^{-(s_n-t)}}} \leq C\frac{A^{5/4}}{(s_n-\eta)^2}C\sqrt{\eta} \leq C\frac{A^{5/4}}{(s_n-\eta)^2}$ by (64).

From (65) and (62), we then get:
$|g(0,s_n)| = G(0,s_n) \leq \frac{A^{5/4}}{(s_n-\eta)^{3/2}}\left(e^{-\frac{\eta}{4}} + C(p)e^{-\frac{s_n}{4}} + \frac{C}{\sqrt{s_n-\eta}}\right) + C(p)\frac{(A+1)}{(s_n-\eta)^{3/2}}.$

Now, we take $A \geq A_5(p)$ such that $C(p)(A+1) \leq \left(e^{-\frac{\eta}{5}} - e^{-\frac{\eta}{4}}\right)A^{5/4}$, and $s_n \geq S_5(p)$ such that
$\frac{1}{(s_n-\eta)^{3/2}}\left(e^{-\frac{\eta}{5}} + C(p)e^{-\frac{s_n}{4}} + \frac{C}{\sqrt{s_n-\eta}}\right) \leq \frac{e^{-\frac{\eta}{6}}}{s_n^{3/2}}$. If $s_n \geq S_5(p)$, then we have :
$|\nabla^3 w(0,s_n)(\alpha,\beta,\gamma)| = |g(0,s_n)| \leq e^{-\frac{\eta}{6}}\frac{A^{5/4}}{s_n^{3/2}} < \frac{A^{5/4}}{s_n^{3/2}}$. This contradicts (61).

Thus, Case 5 can not occur if $a \geq \delta(p)$.

3 Blow-up profile notions for equation (1)

In this section, we prove Theorems 2 and 3.
Let us first show the existence of a profile in the intermediate variable $z = \frac{y}{\sqrt{s}}$.

Proof of Theorem 2:
The theorem is a consequence of:

- the behavior of the solution $w(y,s)$ for y bounded,
- the pointwise estimates on $\Delta w(y,s)$ in Theorem 1, which will enable us to treat this term in equation (3) as a perturbation.

Let $u(t)$ be a solution of (1) which blows-up at time $T>0$ and satisfies $u(0)\in H^1(\mathbb{R}^N)$. Let x_0 be a blow-up point of $u(t)$ and consider w_{x_0} defined by (2). We just write w for w_{x_0}.

The proof is in two steps:

Step 1: Reduction of the problem

According to Filippas and Liu [7] and Velázquez [18],
- either $\forall R>0,\ \sup\limits_{|y|\le R}|w(y,s)-\kappa|\le C(R)e^{-\delta s}$ for some $\delta>0$,
- or there exists $k\in\{0,...,N-1\}$ and a $N\times N$ orthonormal matrix Q such that $\forall R>0,\ \sup\limits_{|y|\le R}\left|w(y,s)-\left[\kappa+\frac{\kappa}{2ps}\left((N-k)-\frac{1}{2}y^TA_ky\right)\right]\right|=o\left(\frac{1}{s}\right)$ as $s\to+\infty$ where

$$A_k=Q\begin{pmatrix} I_{N-k} & 0\\ 0 & 0\end{pmatrix}Q^{-1} \tag{68}$$

and I_{N-k} is the $(N-k)\times(N-k)$ identity matrix.
By direct calculations, we summarize both cases by:

$$\forall R>0,\ \ \sup_{|y|\le R}|w(y,s)-f_k\left(\frac{y}{\sqrt{s}}\right)-\frac{a}{s}|=o\left(\frac{1}{s}\right) \tag{69}$$

where

$$f_k(z)=\left(p-1+\frac{(p-1)^2}{4p}z^TA_kz\right)^{-\frac{1}{p-1}},\ \ a=\frac{\kappa(N-k)}{2p}, \tag{70}$$

$k\in\{0,1,...,N\}$ and A_k is defined in (68) (take $A_N=0$).

We claim now that (69) implies that the convergence is uniform on larger sets:

Proposition 3.1 (Convergence extension to space-time parabolas) *Assume that w is a solution of (3) which satisfies (69). Then, $\forall K_0>0$,*

$$\sup_{|z|\le K_0}|w(z\sqrt{s},s)-f_k(z)|\to 0\ \textit{as}\ s\to+\infty.$$

It is immediate that Theorem 2 is a direct consequence of Proposition 3.1. Thus, we now focus on the proof of Proposition 3.1.

The main feature in the proof is an a priori estimate on

$$q(y,s) = w(y,s) - f_k(\frac{y}{\sqrt{s}}). \tag{71}$$

We consider the equation satisfied by q as a perturbation of a hyperbolic equation (the size of the perturbation is crucially controlled by Theorem 1). We claim the following result:

Proposition 3.2 (Hyperbolic estimate on $q(y,s)$ for $A \le |y| \le K_0\sqrt{s}$) *Assume (69). Then, for any $K_0 > 0$, there exist $A_0(K_0) > 0$ and $B(K_0) > 0$ such that for all $A \ge A_0$, there exists $S_0(K_0, A)$ with the following property: If $\omega \in S^{N-1}$, $s_0 \ge S_0$, then*

$$\forall s \in [s_0, s_1],\ |q(Ae^{\frac{s-s_0}{2}}\omega, s)| \le B\frac{e^{s-s_0}}{s_0}$$

where $s_1 \ge s_0$ is defined by

$$Ae^{\frac{s_1-s_0}{2}} = K_0\sqrt{s_1}. \tag{72}$$

Let us first show how this proposition concludes the proof of Proposition 3.1.
Remark: We notice that it directly follows from Proposition 3.2 that for S_0 larger, we have

$$\forall s \in [s_0, s_1],\ |q(Ae^{\frac{s-s_0}{2}}\omega, s)| \le \frac{2BK_0^2}{A^2}. \tag{73}$$

Indeed, we have $\forall s \in [s_0, s_1]$, $Ae^{\frac{s-s_0}{2}} \le K_0\sqrt{s} \le K_0\sqrt{s_1}$. Therefore, $|q(Ae^{\frac{s-s_0}{2}}\omega, s)| \le B\frac{e^{s-s_0}}{s_0} \le \frac{BK_0^2}{A^2}\frac{s_1}{s_0}$. If K_0 and A are fixed, then it is easy to see that $s_1 \sim s_0$ as $s_0 \to +\infty$. One might take $S_0(K_0, A)$ larger to have $\frac{s_1}{s_0} \le 2$, which yields $|q(Ae^{\frac{s-s_0}{2}}\omega, s)| \le \frac{2BK_0^2}{A^2}$.
We now prove Proposition 3.1.
Let $K_0 > 0$ and $\epsilon > 0$. Fix $A \ge A_0(K_0)$ so that $\frac{2BK_0^2}{A^2} \le \epsilon$.
By (69), there exists $s_{02}(\epsilon)$ such that

$$\forall s \ge s_{02},\ \forall |y| \le A, |q(y,s)| \le \epsilon. \tag{74}$$

Let $s_{03}(K_0, A) \ge S_0$ be defined by $Ae^{\frac{s_{03}-S_0}{2}} = K_0\sqrt{s_{03}}$. We claim that $\forall s \ge \max(s_{02}(\epsilon), s_{03}(K_0, A))$, $\forall |y| \le K_0\sqrt{s}$, $|q(y,s)| \le \epsilon$.

Indeed, if $|y| \le A$, then the conclusion follows from (74). If $A \le |y| \le K_0\sqrt{s}$, we define $s_0(|y|,s)$ by $|y| = Ae^{\frac{s-s_0}{2}}$. By construction of $s_{03}(K_0,A)$, we have $s_0(|y|,s) \ge S_0(K_0,A)$. We also have $s_0 \le s \le s_1$, since $Ae^{\frac{s-s_0}{2}} = |y| \le K_0\sqrt{s}$ and $Ae^{\frac{s_1-s_0}{2}} = K_0\sqrt{s_1}$. Applying the remark (73) coming after Proposition 3.2 gives $|q(y,s)| = |q(Ae^{\frac{s-s_0}{2}}\frac{y}{|y|},s)| \le \frac{2BK_0^2}{A^2} \le \epsilon$. This is the conclusion of Proposition 3.1 and that of Theorem 2 also. Let us now prove proposition 3.2.

Step 2: Hyperbolic estimates: *Proof of Proposition 3.2*:
Define

$$B(K_0) = 3(|a|+1+C_4)\left[1+\frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}} \tag{75}$$

with $C_4 = C_5 + \frac{1}{2}\|z.\nabla f(z)\|_{L^\infty}$, C_5 is the constant given by Theorem 1 such that $\|\Delta w(s)\|_{L^\infty} \le \frac{C_5}{s}$ and a is defined in (70).

We consider $A \ge A_0(K_0)$ and $s_0 \ge S_0(K_0,A)$ ($A_0(K_0)$ and $S_0(K_0,A)$ will be defined later).

Let $\omega \in S^{N-1}$ and introduce

$$y(A,\omega,s_0,s) = Ae^{\frac{s-s_0}{2}}\omega \text{ and } h(A,\omega,s_0,s) = q\left(y(A,\omega,s_0,s),s\right). \tag{76}$$

For simplicity, we will just write $y(s)$ and $h(s)$. Let us define $s_{04}(K_0,A)$ (independent of w) such that $\forall s_0 \ge s_{04}(K_0,A)$, s_1 (introduced in (72)) is well defined and satisfies $s_1 \le 2s_0$, and

$$|h(s_0)| = |q(A\omega,s_0)| \le \frac{|a|+1}{s_0} < \frac{B}{s_0} \tag{77}$$

by definition of $B(K_0)$ (This follows directly from (69)).

The proof of Proposition 3.2 reduces now to prove that $\forall s_0 \ge S_0(K_0,A)$, $\forall s \in [s_0,s_1]$, $|h(s)| \le B\frac{e^{s-s_0}}{s_0}$. We proceed by a priori estimates.

We suppose by contradiction the existence of some $s_* \in [s_0,s_1]$ such that

$$\forall s \in [s_0,s_*),\ |h(s)| < \frac{Be^{s-s_0}}{s_0} \text{ and } |h(s_*)| = \frac{Be^{s_*-s_0}}{s_0}. \tag{78}$$

Since f_k is a solution of $0 = -\frac{1}{2}y.\nabla f_k(z) - \frac{f_k(z)}{p-1} + f_k(z)^p$, we derive from (71) and (3) an equation satisfied by q: $\forall y \in \mathbb{R}^N$, $\forall s \ge -\log T$:

$$\frac{\partial q}{\partial s} = -\frac{1}{2}y.\nabla q + \left(pf_k\left(\frac{y}{\sqrt{s}}\right) - \frac{1}{p-1}\right)q + N(q) + r(y,s)$$

where $N(q) = (f_k+q)^p - f_k^p - pf_k^{p-1}q$ and $r(y,s) = \frac{1}{2}\frac{y}{s^{3/2}}\nabla f_k\left(\frac{y}{\sqrt{s}}\right) + \Delta w(y,s)$.

Therefore, we derive from (76) an equation satisfied by h:

$$\frac{dh}{ds} = \left(pf_k\left(\frac{y(s)}{\sqrt{s}}\right)^{p-1} - \frac{1}{p-1}\right)h(s) + N(h) + r(y(s),s).$$

From (75) and homogeneity, we write $\forall s\in[s_0,s_*]$, $|N(h)|\le C^*(K_0)|h|^2 \le C^*(K_0)\frac{Be^{s-s_0}}{s_0}|h|$ and $|r(y(s),s)|\le\frac{C_4}{s}$.

Therefore, if $g(s)=|h(s)|$, then $g(s)$ satisfies:

$$\left\{\begin{array}{rcl} \forall s\in[s_0,s_*],\ g'(s) & \le & \alpha(s)g(s)+\frac{C_4}{s},\\ g(s_0) & \le & \frac{(|a|+1)}{s_0}\end{array}\right. \tag{79}$$

with

$$\alpha(s) = pf_k\left(\frac{y(s)}{\sqrt{s_1}}\right)^{p-1} - \frac{1}{p-1} + C^*(K_0)\frac{Be^{s-s_0}}{s_0}. \tag{80}$$

Using Gronwall's inequality, we write

$$\forall s\in[s_0,s_*],\ g(s)\le I+II$$

where

$$I = \exp\left(\int_{s_0}^s\alpha\right)g(s_0) \text{ and } II = C_4\int_{s_0}^s\frac{d\sigma}{\sigma}\exp\left(\int_\sigma^s\alpha\right). \tag{81}$$

We estimate in the following lemma $\exp\left(\int_\sigma^s\alpha\right)$ for $s_0\le\sigma\le s\le s_1$.

Lemma 3.1 *There exists $A_1(K_0)>0$ such that $\forall A\ge A_1(K_0)$, $\exists s_{05}(K_0,A)$ such that $\forall s_0\ge s_{05}(K_0,A)$, if $s_0\le\sigma\le s\le s_1$, then*

$$\exp\left(\int_\sigma^s\alpha\right)\le\frac{3}{2}e^{s-\sigma}\left[1+\frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}}.$$

We let the proof of this lemma to the end, and finish the proof of Proposition 3.2.

Now, we define $A_0(K_0)=A_1(K_0)$ and for each $A\ge A_0(K_0)$, $S_0(K_0,A)=\max(s_{04}(K_0,A),s_{05}(K_0,A))$. For $A\ge A_0(K_0)$ and $s_0\ge S_0(K_0,A)$, we use (79) and lemma 3.1 to bound I and II (see (81)) for $s\in[s_0,s_*]$:

$I\le(|a|+1)\frac{3}{2}\left[1+\frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}}\frac{e^{s-s_0}}{s_0}$ and

$II\le C_4\frac{3}{2}\left[1+\frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}}\int_{s_0}^s\frac{d\sigma}{\sigma}e^{s-\sigma}\le\frac{3}{2}C_4\left[1+\frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}}\frac{e^{s-s_0}}{s_0}.$

Hence, for $s = s_*$,

$|h(s_*)| = g(s_*) \le I + II \le \frac{3}{2}(|a| + 1 + C_4)\left[1 + \frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}} \frac{e^{s_*-s_0}}{s_0} = \frac{B(K_0)}{2}\frac{e^{s_*-s_0}}{s_0}$ (see (75)).

This contradicts (78) and concludes the proof of Proposition 3.2, Proposition 3.1 and Theorem 2 also.

Proof of lemma 3.1:

From (80), (70) and (76), we have

$\alpha(s) = \frac{p}{p-1+b(\omega)A^2\frac{e^{s-s_0}}{s_1}} - \frac{1}{p-1} + C^*(K_0)B\frac{e^{s-s_0}}{s_0}$ with

$$b(\omega) = b\omega^T A_k \omega \text{ and } b = \frac{(p-1)^2}{4p}. \qquad (82)$$

Therefore,

$\int_\sigma^s \alpha(\tau)d\tau = \left[\tau + \ln\left(p-1+b(\omega)A^2\frac{e^{\tau-s_0}}{s_1}\right)^{-\frac{p}{p-1}} + C^*(K_0)\frac{Be^{\tau-s_0}}{s_0}\right]_\sigma^s$

$= s-\sigma+\ln\left(\frac{p-1+b(\omega)A^2\frac{e^{\sigma-s_0}}{s_1}}{p-1+b(\omega)A^2\frac{e^{s-s_0}}{s_1}}\right)^{\frac{p}{p-1}} + C^*(K_0)\frac{B}{s_0}\left(e^{s-s_0}-e^{\sigma-s_0}\right)$. This implies that

$\exp\left(\int_\sigma^s\right) = e^{s-\sigma}\left(\frac{p-1+b(\omega)A^2\frac{e^{\sigma-s_0}}{s_1}}{p-1+b(\omega)A^2\frac{e^{s-s_0}}{s_1}}\right)^{\frac{p}{p-1}} \exp\left(C^*(K_0)\frac{B}{s_0}\left(e^{s-s_0}-e^{\sigma-s_0}\right)\right)$.

Since $\sigma \le s \le s_1$ and $Ae^{\frac{s_1-s_0}{2}} = K_0\sqrt{s_1}$, we have $Ae^{\frac{\sigma-s_0}{2}} \le K_0\sqrt{s_1}$ and $Ae^{\frac{s-s_0}{2}} \le K_0\sqrt{s_1}$. Therefore,

$\exp\left(\int_\sigma^s\right) \le e^{s-\sigma}\left[1+\frac{bK_0^2}{p-1}\right]^{\frac{p}{p-1}} \exp\left(\frac{C^*(K_0)K_0^2Bs_1}{A^2 s_0}\right)$ (note that $b(\omega) \le b$, see (82)).

We now introduce $A_1(K_0) > 0$ such that for all $A \ge A_1(K_0)$,

$\exp\left(\frac{2C^*(K_0)K_0^2B}{A^2}\right) \le \frac{3}{2}$ and consider $A \ge A_1(K_0)$. Then, we introduce $s_{05}(K_0, A)$ such that for all $s_0 \ge s_{05}(K_0, A)$, $s_1 \le 2s_0$. Then, for $s_0 \ge s_{05}(K_0, A)$, we have

$\exp\left(\int_\sigma^s\right) \le \frac{3}{2}e^{s-\sigma}\left[1+\frac{bK_0^2}{p-1}\right]^{\frac{p}{p-1}}$, which concludes the proof of lemma 3.1. ■

Proof of Theorem 3:

The proof will follow from Proposition 3.1 and localization estimates. We consider $u(t)$ a solution of (1) which blows-up at time $T > 0$ at some point $x_0 \in \mathbb{R}^N$. By translation invariance, we take $x_0 = 0$. We assume 0 to be an isolated blow-up point of $u(t)$. Therefore, there exists $\epsilon_0 > 0$ such that 0 is the unique blow-up point $u(t)$ in $B(0, 2\epsilon_0)$.

We aim at proving the equivalence of the following behaviors for $u(t)$ near 0 and for w_0 (=w) defined in (2):

(A) $\forall R > 0$, $\sup_{|y|\le R}\left|w(y,s) - \left[\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)\right]\right| = o\left(\frac{1}{s}\right)$ as $s \to +\infty$,

(B) $\exists \epsilon_0 > 0$ such that $\|q_0(y,s)\|_{L^\infty(|y|\le \epsilon_0 e^{s/2})} \to 0$ as $s \to +\infty$ where

$$q_0(y,s) = w(y,s) - f_0(\frac{y}{\sqrt{s}}) \qquad (83)$$

and

$$f_0(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}, \qquad (84)$$

(C) $\exists \epsilon_0 > 0$ such that if $|x| \le \epsilon_0$, then $u(x,t) \to u^*(x)$ as $t \to T$ and $u^*(x) \sim U(x)$ as $x \to 0$ where

$$U(x) = \left[\frac{8p|\log|x||}{(p-1)^2|x|^2}\right]^{\frac{1}{p-1}}. \qquad (85)$$

For further purpose, we introduce a weaker version of (B) (which will be in fact equivalent):

(B') $\forall K_0 > 0$, $\|q_0(y,s)\|_{L^\infty(|y|\le K_0\sqrt{s})} \to 0$ as $s \to +\infty$.

The proof will be over if we prove the following implications:

$$\text{(A)} \Longrightarrow \text{(B')} \Longrightarrow \text{(C)} \Longrightarrow \text{(B)} \Longrightarrow \text{(A)}.$$

We first prove some useful technical estimates. We then use them to prove the different implications.

Part I: Preliminary results for subcritical values of w ($w < \kappa$)

We crucially use the localization result proved in [16].

Lemma 3.2 *Assume that 0 is the only blow-up point of $u(t)$ in $B(0, 2\epsilon_0)$ for some $\epsilon_0 > 0$. Consider (y_n, s_n) a sequence in $\mathbb{R}^N \times [-\log T, +\infty)$ satisfying $|y_n| \le \epsilon_0 e^{s_n/2}$ and suppose that $w(y_n, s_n) \to l \in (0,\kappa)$ and $s_n \to +\infty$ as $n \to +\infty$.*
If $x_n = y_n e^{-s_n/2}$ and $z_n = \frac{|y_n|}{\sqrt{s_n}}$, then:

i) $x_n \to 0$ as $n \to +\infty$,

ii) $\forall n \in \mathbb{N}$, $u(x_n, t) \to u^*(x_n)$ as $t \to T$ and

$$u^*(x_n) \sim \left[\frac{|x_n|^2}{2z_n^2\left|\log\frac{|x_n|}{z_n}\right|}\right]^{-\frac{1}{p-1}} (l^{1-p} - p + 1)^{-\frac{1}{p-1}} \text{ as } n \to +\infty.$$

Proof: We proceed by contradiction in order to prove that $x_n \to 0$ as $n \to +\infty$. If not, then we have $x_{n'} > \delta > 0$ for some subsequence $x_{n'}$. Since $u(t)$ does not blow-up for $\delta \le |x| \le \epsilon_0$, there exists $C(\delta) > 0$ such that if $t \in [\frac{T}{2}, T)$ and $\delta \le |x| \le \epsilon_0$, then $|u(x,t)| \le C(\delta)$. Therefore, (2) implies that $0 \le w(y_{n'}, s_{n'}) \le e^{-\frac{s_{n'}}{p-1}} C(\delta) \to 0$ as $n \to +\infty$, which contradicts the fact that $l > 0$. Thus, $x_n \to 0$ as $n \to +\infty$.
Let us find an equivalent of $u^*(x_n)$.
We define for each $(\xi, \tau) \in \mathbb{R}^N \times [0,1)$

$$\begin{aligned} v_n(\xi,\tau) &= e^{-\frac{s_n}{p-1}} u(x_n + \xi e^{-\frac{s_n}{2}}, T + (\tau - 1)e^{-s_n}) \\ &= (1-\tau)^{\frac{1}{p-1}} w(\frac{y_n + \xi}{\sqrt{1-\tau}}, s_n - \log(1-\tau)). \end{aligned} \tag{86}$$

Then v_n satisfies: $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [0,1)$

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + v_n^p.$$

According to (6), $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ such that

$$\begin{cases} \left| \frac{\partial v_n}{\partial \tau}(0,\tau) - v_n(0,\tau)^p \right| \le \epsilon v_n(0,\tau)^p + C_\epsilon e^{-\frac{p s_n}{p-1}}, \forall \tau \in [0,1), \\ v_n(0,0) \to l. \end{cases}$$

Let us define first $v(\tau)$ as the solution of

$$v'(\tau) - v(\tau)^p = 0, \; v(0) = l,$$

that is $v(\tau) = \left(l^{1-p} - \tau(p-1)\right)^{-\frac{1}{p-1}}$.
Thus, if we denote $v_n(0,\tau)$ by $y_n(\tau)$, we have: $\forall \epsilon > 0$, there exists $n_0(\epsilon)$ such that $\forall n \ge n_0(\epsilon)$

$$\begin{cases} |y_n'(\tau) - y_n(\tau)^p| &\le \epsilon(y_n^p + 1), \; \forall \tau \in [0,1) \\ |y_n(0) - l| &\le \epsilon. \end{cases}$$

Since $v(0) \le v(\tau) \le v(1) < +\infty$, it follows from continuity results on ordinary differential equations that $\sup\limits_{\tau \in [0,1)} |y_n(\tau) - v(\tau)| \le \delta(\epsilon)$ with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. In particular,
$\lim\limits_{\tau \to 1} v_n(0,\tau) = \lim\limits_{t \to T} y_n(t) \to v(1) = \left(l^{1-p} - p + 1\right)^{-\frac{1}{p-1}}$ as $n \to +\infty$.

From (86), we have $u^*(x_n) = \lim\limits_{t \to T} u(x_n, t) = \lim\limits_{\tau \to 1} e^{\frac{s_n}{p-1}} v_n(0,\tau)$. Therefore,

$$e^{-\frac{s_n}{p-1}} u^*(x_n) \sim \left(l^{1-p} - p + 1\right)^{-\frac{1}{p-1}} \text{ as } n \to +\infty. \tag{87}$$

Since

$$\frac{|x_n|}{z_n} = \sqrt{s_n e^{-s_n}}, \tag{88}$$

we get

$$s_n \sim 2\left|\log \frac{|x_n|}{z_n}\right| \tag{89}$$

and then

$$e^{\frac{s_n}{p-1}} \sim \left[\frac{|x_n|^2}{2z_n^2\left|\log\frac{|x_n|}{z_n}\right|}\right]^{-\frac{1}{p-1}} \quad \text{as } n \to +\infty.$$

Combining this with (87) concludes the proof of lemma 3.2. ■

Corollary 3.1 *Under the assumptions of lemma 3.2, if* $\frac{u^*(x_n)}{U(x_n)} \to 1$ *as* $n \to +\infty$*, then* $w(y_n, s_n) - f_0(\frac{y_n}{\sqrt{s_n}}) \to 0$*, where* f_0 *is defined in (84).*

Proof: Let us show that $\frac{u^*(x_n)}{U(x_n)} \to 1$ implies that $f_0(\frac{y_n}{\sqrt{s_n}}) \to l$. From (85) and lemma 3.2, we get

$$\frac{l^{1-p} - p + 1}{z_n^2\left|\log\frac{|x_n|}{z_n}\right|} \sim \frac{(p-1)^2}{4p\left|\log|x_n|\right|} \quad \text{as } n \to +\infty. \tag{90}$$

We claim that

$$\left|\log|x_n|\right| \sim \frac{s_n}{2}. \tag{91}$$

Indeed, (90) and (89) imply that $z_n \sim \frac{C(p,l)}{\sqrt{s_n}}\left|\log|x_n|\right|$. Using (88), we get from this $|x_n| e^{\frac{s_n}{2}} \sim C(p,l)\sqrt{\left|\log|x_n|\right|}$ which gives $\left|\log|x_n|\right| \sim \frac{s_n}{2}$. Combining (90), (89) and (91) gives

$$z_n^2 \to \frac{4p(l^{1-p} - p + 1)}{(p-1)^2},$$

that is $f_0(z_n) \to l$ as $n \to +\infty$ (by (84)). ■

Part II: Proof of Theorem 3

Now, we are able to prove the equivalence.

(A) $\Longrightarrow$ (B'):

One can easily see from (84) that $\forall R > 0$,

$$\sup_{|y|\le R}\left|f_0(\frac{y}{\sqrt{s}}) - \left(\kappa - \frac{\kappa}{4ps}|y|^2\right)\right| = O\left(\frac{1}{s^2}\right).$$

By (A), it follows that $\forall R>0,\ \sup_{|y|\le R}\left|w(y,s)-f_0(\frac{y}{\sqrt{s}})-\frac{N\kappa}{2ps}\right|=o\left(\frac{1}{s}\right)$.

Proposition 3.1 applied with $k=0$ (and $A_k=I_N$) yields by (83): $\forall K_0>0$, $\|q_0(y,s)\|_{L^\infty(|y|\le K_0\sqrt{s})}\to 0$ as $s\to+\infty$, which is (B').

(B') $\Longrightarrow$ (C):

Since 0 is the only blow-up point of u in $B(0,2\epsilon_0)$, we can define $u^*(x)=\lim_{t\to T}u(x,t)$ for all $0<|x|\le\epsilon_0$. Let (x_n) be any sequence tending to zero in $\mathbb{R}^N$. Let us prove that $u^*(x_n)\sim U(x_n)$ as $n\to+\infty$ where U is defined in (85).

Fix $r_0>0$. If n is large enough, we can uniquely define $s_n\to+\infty$ and y_n by $r_0e^{-s_n/2}\sqrt{s_n}=|x_n|$ and $y_n=x_ne^{s_n/2}$. Since $z_n=\frac{|y_n|}{\sqrt{s_n}}=r_0>0$, it follows from (B') and (83) that $w(y_n,s_n)\to f_0(r_0)\in(0,\kappa)$. Applying lemma 3.2 yields

$$u^*(x_n)\sim\left[\frac{|x_n|^2}{2r_0^2\left|\log\frac{|x_n|}{r_0}\right|}\right]^{-\frac{1}{p-1}}\left(f_0(r_0)^{1-p}-p+1\right)^{-\frac{1}{p-1}}.$$

From (84), we have $f_0(r_0)^{1-p}-(p-1)=\frac{(p-1)^2}{4p}r_0^2$. Therefore,

$$u^*(x_n)\sim\left[\frac{(p-1)^2|x_n|^2}{8p\left|\log\frac{|x_n|}{r_0}\right|}\right]^{-\frac{1}{p-1}}$$

which is equivalent to $U(x_n)$ by (85).

(C) $\Longrightarrow$ (B):

We want to prove that $\|q_0(y,s)\|_{L^\infty(|y|\le\epsilon_0e^{s/2})}\to 0$ as $s\to+\infty$. We proceed by contradiction and assume the existence of $\epsilon>0$, $s_n\to+\infty$ and $|y_n|\le\epsilon_0e^{s_n/2}$ such that

$$|q_0(y_n,s_n)|\ge\epsilon \text{ as } n\to+\infty. \tag{92}$$

We can assume that $w(y_n,s_n)\to l_1$ and $f_0\left(\frac{y_n}{\sqrt{s_n}}\right)\to l_2$. According to Theorem 1 and (84), $l_1,l_2\in[0,\kappa]$. Note that (92) yields

$$|l_1-l_2|\ge\epsilon. \tag{93}$$

Let us consider three cases:

Case 1: $l_1 \in (0,\kappa)$. From (93), $w(y_n,s_n) - f_0\left(\frac{y_n}{\sqrt{s_n}}\right)$ does not go to 0. Hence, from lemma 3.2 and corollary 3.1, $x_n = y_n e^{-s_n/2} \to 0$ and $\frac{u^*(x_n)}{U(x_n)}$ does not go to 1 as $n \to +\infty$. This contradicts (C).

Case 2: $l_1 = \kappa$. Note that (93) implies that $l_2 \le \kappa - \epsilon$. We claim the existence of y_n' such that

$$|y_n| \le |y_n'| \text{ and } w(y_n', s_n) = \frac{1}{2}\left(f_0\left(\frac{y_n'}{\sqrt{s_n}}\right) + \kappa\right) \tag{94}$$

for large n. Indeed, w and f_0 are continuous, and we have

$$w(y_n,s_n) - \frac{1}{2}\left(f_0\left(\frac{y_n}{\sqrt{s_n}}\right) + \kappa\right) > 0$$

and

$$w(\frac{y_n}{|y_n|}\epsilon_0 e^{s_n/2}, s_n) - \frac{1}{2}\left(f_0\left(\frac{y_n}{|y_n|}\epsilon_0 \frac{e^{s_n/2}}{\sqrt{s_n}}\right) + \kappa\right) < 0$$

for large n (use (84) and write $w(\frac{y_n}{|y_n|}\epsilon_0 e^{s_n/2}, s_n) = e^{-\frac{s_n}{p-1}} u(\frac{y_n}{|y_n|}\epsilon_0, T - e^{-s_n})$ $\le C(\epsilon_0) e^{-\frac{s_n}{p-1}}$ since $u(t)$ does not blow-up for $|x| = \epsilon_0$).

We can assume that $w(y_n', s_n) \to l_1' \in [0,\kappa]$ (Theorem 1) and $f_0\left(\frac{y_n'}{\sqrt{s_n}}\right) \to l_2' \in [0,\kappa]$. Since f_0 is decreasing and $|y_n| \le |y_n'|$, we get $l_2' \le l_2 < \kappa$. Using (94), we get $l_1' = \frac{1}{2}(l_2' + \kappa) \in [\frac{\kappa}{2}, \kappa)$ and $|l_2' - l_1'| = \frac{1}{2}|\kappa - l_2'| > 0$. Therefore, $w(y_n', s_n) - f_0\left(\frac{y_n'}{\sqrt{s_n}}\right)$ does not go to 0. Hence, from lemma 3.2 and corollary 3.1, $x_n' = y_n' e^{-s_n/2} \to 0$ and $\frac{u^*(x_n')}{U(x_n')}$ does not go to 1 as $n \to +\infty$. This contradicts (C).

Case 3: $l_1 = 0$. Note that (93) implies that $l_2 \ge \epsilon$. We claim the existence of y_n' such that

$$|y_n| \ge |y_n'| \text{ and } w(y_n', s_n) = \frac{1}{2} f_0\left(\frac{y_n'}{\sqrt{s_n}}\right) \tag{95}$$

for large n. Indeed, w and f_0 are continuous,

$$\lim_{n\to+\infty}\left[w(y_n,s_n) - \frac{1}{2} f_0\left(\frac{y_n}{\sqrt{s_n}}\right)\right] = -\frac{l_2}{2} \le -\frac{\epsilon}{2},$$

and

$$w(0,s_n) - \frac{1}{2} f_0(0) \to \frac{\kappa}{2}$$

$(w(0,s_n) \to \kappa$ according to (7), since 0 is a blow-up point for $u(t))$. We can assume that $w(y'_n,s_n) = \frac{1}{2}f_0\left(\frac{y'_n}{\sqrt{s_n}}\right) \to l'_1 \le \frac{\kappa}{2}$ as $n \to +\infty$. Since $|y_n| \ge |y'_n|$, we have $f_0\left(\frac{y'_n}{\sqrt{s_n}}\right) \ge f_0\left(\frac{y_n}{\sqrt{s_n}}\right)$ and $2l'_1 \ge l_2 \ge \epsilon > 0$. Therefore, $l'_1 \in (0,\frac{\kappa}{2})$ and $w(y'_n,s_n) - f_0\left(\frac{y'_n}{\sqrt{s_n}}\right) \to -l'_1 < 0$. According to lemma 3.2 and corollary 3.1, $x'_n = y'_n e^{-s_n/2} \to 0$ and $\frac{u^*(x'_n)}{U(x'_n)}$ does not go to 1 as $n \to +\infty$. This contradicts (C).

(B) $\Longrightarrow$ (A):

According to (69), there exists $k \in \{0,1,...,N\}$ and a $N\times N$ orthonormal matrix Q such that

$$\forall R > 0, \ \sup_{|y|\le R} |w(y,s) - f_k\left(\frac{y}{\sqrt{s}}\right) - \frac{a}{s}| = o\left(\frac{1}{s}\right) \tag{96}$$

where f_k and a are defined in (70).

Applying Proposition 3.1, we see that $\forall K_0 > 0$, $\sup_{|z|\le K_0} \left|w(z\sqrt{s},s) - f_k(z)\right| \to 0$ as $s \to +\infty$.

Together with (B), this gives $f_k \equiv f_0$. Therefore, $k=0$ and $a = \frac{N\kappa}{2p}$. Thus, (96) yields (A).

This concludes the proof of Theorem 3. ■

A Proof of lemma 2.4

$i)$: - According to (32), $\forall i,j \in \{1,...,N\}$, $w_{2,i,j}(s) = \int w(y,s)\left(\frac{1}{4}y_iy_j - \frac{1}{2}\delta_{i,j}\right)\rho(y)dy$.
Just remark that $\left(\frac{1}{4}y_iy_j - \frac{1}{2}\delta_{i,j}\right)\rho(y) = \frac{\partial^2\rho}{\partial y_i \partial y_j}$ and do two integrations by parts to get $w_2(s) = \int \nabla^2 w(y,s)\rho(y)dy$. The estimate for w_1 is similar.

$ii)$: The estimates on w_1 and w_2 follow directly from $i)$ since $\|\nabla w(s_n)\|_{L^\infty} \le \frac{c_1}{\sqrt{s_n}}$ and $\|\nabla^2 w(s_n)\|_{L^\infty} \le \frac{A}{s_n}$.

- By (23), we write: $\forall y \in \mathbb{R}^N$, $w(y,s_n) = w(0,s_n) + y.\nabla w(0,s_n) + \frac{1}{2}y^T\nabla^2 w(0,s_n)y + \phi(y,s_n)$ where

$$|\phi(y,s_n)| \le \frac{1}{6}\frac{A^{5/4}}{s_n^{3/2}}|y|^3. \tag{97}$$

According to (31), (30) and (28),

$$w_- = P_-(w) = P_-(\phi) = \phi_- \tag{98}$$

with notations similar to (31). From (31) and (97), we have $|\phi_m(s_n)| \le C'(N)\frac{A^{5/4}}{s_n^{3/2}}$ for $m = 0, 1, 2$. Therefore, (31) yields $|\phi_-(y, s_n)| \le C(N)\frac{A^{5/4}}{s_n^{3/2}}(1+|y|^3)$. Using (98), we get $|w_-(y, s_n)| \le C(N)\frac{A^{5/4}}{s_n^{3/2}}(1 + |y|^3)$.

- Since $w(s)$ is well defined for all $s \ge -\log T$ and satisfies (3), lemma 2.3 implies that $w_0(s_n) \le \kappa$. Let us show that $w_0(s_n) \ge \delta_0 = \frac{c_2^3}{128C(N)^2A^{5/2}}$. We proceed by contradiction and assume that $w_0(s_n) < \delta_0$.
Consider $\hat{y}_n = \eta\frac{c_2\sqrt{s_n}}{4C(N)A^{5/4}}\varphi$ where φ is unitary and satisfies $\varphi^T w_2(s_n)\varphi = -\frac{c_2}{s_n}$ (use i) and Proposition 2.2), and $\eta \in \{-1, 1\}$ is chosen so that $w_1(s_n).\hat{y}_n \le 0$. Therefore, from (31) and the bounds on w_0, w_1, w_2 and w_-, we get:
$w(\hat{y}_n, s_n) = w_0(s_n) + w_1(s_n).\hat{y}_n + \left(\frac{1}{2}\hat{y}_n^T w_2(s_n)\hat{y}_n - trw_2(s_n)\right) + w_-(y, s_n)$
$\le \delta_0 + 0 - \frac{1}{2}\frac{c_2^2 s_n}{16C(N)^2A^{5/2}}\frac{c_2}{s_n} + \frac{C''(N)A}{s_n} + \frac{C(N)A^{5/4}}{s_n^{3/2}}\left(1 + \frac{c_2^3 s_n^{3/2}}{64C(N)^3A^{15/4}}\right)$
$= \delta_0 - \frac{c_2^3}{64C(N)^2A^{5/2}} + O\left(\frac{1}{s_n}\right) = -\delta_0 + O\left(\frac{1}{s_n}\right) < 0$ for s_n large enough. This contradicts the fact that w is nonnegative. Thus, $w_0(s_n) \ge \delta_0$.

iii): Since $M(y)$ defined in (32) is the matrix of eigenfunctions corresponding to the null eigenvalue of $\mathcal{L}$, we find the following equation if we multiply (27) by $M(y)\rho(y)$, integrate the expression over $\mathbb{R}^N$ and use (31):

$$w_2'(s_n) = -\frac{p}{p-1}w_2(s_n) + \int w(y, s_n)^p M(y)\rho(y)dy.$$

Thus, we focus on the computation of $\int w(y, s_n)^p M(y)\rho(y)dy$. Since $0 < \delta_0 \le w_0(s_n) \le \kappa$ and $0 \le w(y, s_n) \le \kappa + 1$, we can Taylor expand $w(y, s_n)$ around $w_0(s_n)$ until the third order and use (31) to write:
$\int w(y, s_n)^p M(y)\rho(y)dy = I + II + III + IV + V + VI$ where
$I = \int w_0(s_n)^p M(y)\rho(y)dy = 0$,
$II = \int p w_0(s_n)^{p-1} V(y, s_n)M(y)\rho(y)dy$,
$III = \int \frac{p(p-1)}{2} w_0(s_n)^{p-2} V(y, s_n)^2 M(y)\rho(y)dy$,
$IV = \int \frac{p(p-1)(p-2)}{6} w_0(s_n)^{p-3} V(y, s_n)^3 M(y)\rho(y)dy$
$V = O\left(\int |V(y, s_n)|^4 |M(y)|\rho(y)dy\right)$ and

$$V(y, s_n) = w_1(s_n).y + \left(\frac{1}{2}y^T w_2(s_n)y - trw_2(s_n)\right) + w_-(y, s_n). \tag{99}$$

Using (99), the orthogonality (in $L^2_\rho(\mathbb{R}^N)$) of y and $M(y)$ on one hand, and $M(y)$ and $w_-(y, s_n)$ on the other, we write:

$II = pw_0(s_n)^{p-1} \int \left(\frac{1}{2}y^T w_2(s_n)y - trw_2(s_n)\right) M(y)\rho(y)dy$
$= pw_0(s_n)^{p-1}w_2(s_n)$ by integration by parts.

From (99), we have:
$III = \frac{p(p-1)}{2} w_0(s_n)^{p-2} \int \Big[(w_1(s_n).y)^2 + \left(\frac{1}{2}y^T w_2(s_n)y - trw_2(s_n)\right)^2$
$+w_-(y,s_n)^2 + 2w_1(s_n).y\left(\frac{1}{2}y^T w_2(s_n)y - trw_2(s_n)\right) + 2w_1(s_n).yw_-(y,s_n)$
$+2\left(\frac{1}{2}y^T w_2(s_n)y - trw_2(s_n)\right) w_-(y,s_n)\Big] M(y)\rho(y)dy.$
Using *ii*), parity and simple but long calculations (based on integration by parts, (32) and (17)) that we omit, we find:
$IV = \frac{p(p-1)}{2} w_0(s_n)^{p-2} \Big[2w_1(s_n) \otimes w_1(s_n) + 4w_2(s_n)^2 + O\left(\frac{1}{s_n^3}\right) + 0$
$+O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right) + O\left(\frac{1}{s_n^{5/2}}\right)\Big]$. Hence,
$III = p(p-1)w_0(s_n)^{p-2}\left[w_1(s_n) \otimes w_1(s_n) + 2w_2(s_n)^2\right] + O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right)$
$+O\left(\frac{1}{s_n^{5/2}}\right).$

As for III, one can expand $V(y,s_n)^3$ and $V(y,s_n)^4$, and use *i*) to get:
$IV = O\left(\frac{1}{s_n^{5/2}}\right) + O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right)$ and $V = O\left(\frac{1}{s_n^{5/2}}\right) + O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right)$.

Gathering all the previous bounds on I, II, III, IV and V yields *iii*). This concludes the proof of lemma 2.4. ■

B Proof of Proposition 2.3

In [16], the same result has been proved in the case of one fixed solution (Theorem 3). Hence, we should adopt here the same strategy as for the proof of Theorem 3 in [16]. In fact, we will focus only on points which are different from [16] (energy estimates and a compactness procedure), and summarize the other arguments. We give the proof in two steps. We first use a compactness procedure and then proceed by contradiction in a second step in order to conclude the proof.

Step 1: Compactness Procedure
We proceed by contradiction and assume that for some $\eta_0 > 0$ and for all $k \in \mathbb{N}$, there are $s_k^* \geq -\log T$, w_k solution of (3) defined for all $s \geq -\log T$ and satisfying $w_k \in \hat{V}_A(s_k^*)$, $x_k \in \mathbb{R}^N$ and $t_k \in [0, t_k^*]$ such that $|\Delta u_k(x_k,t_k)| \geq \eta_0 u_k(x_k,t_k)^p + k$ where $t_k^* = T - e^{-s_k^*}$ and $u_k(x,t) = (T-t)^{\frac{1}{p-1}} w_k\left(\frac{x}{\sqrt{T-t}}, -\log(T-t)\right)$.

Let us introduce $U_k(x,t) = u_k(x+x_k,t)$ and $W_k(y,s) = e^{-\frac{s}{p-1}}U_k(ye^{-\frac{s}{2}}, T-e^{-s})$. Therefore, U_k is a solution of (1), W_k is a solution of (3),

$$\forall s \in [-\log T, s_k^*],\ W_k(s) \in V_A(s), \tag{100}$$

$$\text{and } |\Delta U_k(0,t_k)| \geq \eta_0 U_k(0,t_k)^p + k \tag{101}$$

where $t_k \in [0, t_k^*]$.

We first notice that

$$t_k \to T \text{ as } k \to +\infty.$$

Indeed, if not, then $t_{k'} \leq T - \delta_0$ where $\delta_0 > 0$ for some subsequence $t_{k'}$. Therefore, (100) implies that $|\Delta U_k(0,t_{k'})| \leq C(T-\delta_0)$ for k' large enough, which contradicts (101).

From (101) and (100), we have
$U_k(0,t_k) \leq \left(\frac{\Delta U_k(0,t_k)}{\eta_0}\right)^{\frac{1}{p}} \leq \left(\frac{A}{\eta_0}\right)^{\frac{1}{p}} \frac{(T-t_k)^{-\frac{1}{p-1}}}{|\log(T-t_k)|^{\frac{1}{p}}}$. Therefore,

$$W_k(0,s_k) = (T-t_k)^{\frac{1}{p-1}} U_k(0,t_k) \to 0 \text{ as } k \to +\infty \tag{102}$$

where $s_k = -\log(T-t_k)$. From Definition 2.1, (100) and compactness procedure, we derive the existence of U solution of (1) in $C^2(\mathbb{R}^N \times [0,T))$ such that $U_k \to U$ as $k \to +\infty$ in $C^2(K)$ for all compact subset of $\mathbb{R}^N \times [0,T)$.

Step 2: Energy estimates on U
We claim that U blows-up at time T at the point $x = 0$.
Let us first introduce the following localized energy for u:

$$\begin{aligned} \mathcal{E}_{a,t}(u) &= t^{\frac{2}{p-1}-\frac{N}{2}+1} \int \left[\frac{1}{2}|\nabla u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right] \rho\left(\frac{x-a}{\sqrt{t}}\right) dx \\ &+ \frac{1}{2(p-1)} t^{\frac{2}{p-1}-\frac{N}{2}} \int |u(x)|^2 \rho\left(\frac{x-a}{\sqrt{t}}\right) dx \end{aligned} \tag{103}$$

where ρ is introduced in (17).
It was proved in [11] that if the energy is small at some point $a \in \mathbb{R}^N$, then u does not blow-up at a. More precisely,

Proposition B.1 (Giga-Kohn) *Let u be a solution of equation (1).*
i) If for all $x \in B(x_0,\delta)$, $\mathcal{E}_{x,T-t_0}(u(t_0)) \leq \sigma$, then $\forall x \in B(x_0,\frac{\delta}{2})$, $\forall t \in (\frac{t_0+T}{2}, T)$, $|u(t,x)| \leq \eta(\sigma)(T-t)^{-\frac{1}{p-1}}$ where $\eta(\sigma) \leq c\sigma^\theta$, $\theta > 0$, and c and θ depend only on p.

ii) **(Merle)** *Assume in addition that* $\forall x \in B(x_0, \delta)$, $|u(\frac{t_0+T}{2}, x)| \le M$. *There exists* $\sigma_0 = \sigma_0(p) > 0$ *such that if* $\sigma \le \sigma_0$, *then* $\forall x \in B(x_0, \frac{\delta}{4})$, $\forall t \in (\frac{t_0+T}{2}, T)$, $|u(t,x)| \le M^*$ *where* M^* *depends only on* M, δ, T *and* t_0.

Proof: see Proposition 3.5 and Theorem 2.1 in [11] (see also [15]). ■

Suppose that $U(x,t)$ does not blow-up at $(x,t) = (0,T)$, then [11] shows that $\mathcal{E}_{0,T-t}(U(t)) \to 0$ as $t \to T$. Therefore, we choose $t_0 > T$ such that $\mathcal{E}_{0,T-t_0}(U(t_0)) \le \frac{\sigma_0}{4}$ where σ_0 is introduced in Proposition B.1. From a continuity argument in x, there is $R_0 > 0$ such that if $|x| \le R_0$, then $\mathcal{E}_{x,T-t_0}(U(t_0)) \le \frac{\sigma_0}{2}$.
Since $U_k(t_0) \to U(t_0)$ as $k \to 0$ in $C^2(K)$ for all K compact subset and $\|U_k(t_0)\|_{W^{1,\infty}} \le C(t_0)$ by (100), we have for all $|x| \le R_0$, $\mathcal{E}_{x,T-t_0}(U_k(t_0)) \le \sigma_0$ for k large enough. From (100), we have $\|U_k\left(\frac{t_0+T}{2}\right)\|_{L^\infty} \le C(t_0)$ for k large enough.
Applying Proposition B.1, we get for k large enough: $\forall |x| \le R_0$, $\forall t \in (\frac{t_0+T}{2}, T)$, $|U_k(x,t)| \le M(t_0, R_0)$. By parabolic regularity (see lemma 2.10 and its proof for a sketch of the technique), we get

$$\forall t \in (\frac{3t_0+T}{4}, T),\ |\Delta U_k(0,t)| \le M'(t_0)$$

for k large enough, which contradicts (101). Therefore, U blows-up at time T at $x = 0$.

Step 3: Conclusion of the proof

We now follow the same ideas as for the Theorem 3 in [16]. We claim the existence of $t'_k < t_k$ such that

$$t'_k \to T \text{ and } W_k(0, s'_k) = (T - t'_k)^{\frac{1}{p-1}} U_k(0, t'_k) = \kappa_0 \tag{104}$$

where $s'_k = -\log(T - t'_k)$, $\kappa_0 \in (0, \kappa)$ satisfies $\forall t > 0$, $\forall a \in \mathbb{R}^N$, $\mathcal{E}_{a,t}(\kappa_0 t^{-\frac{1}{p-1}}) = \frac{\kappa_0^2}{2(p-1)} - \frac{\kappa_0^{p+1}}{p+1} \le \frac{\sigma_0}{2}$ and σ_0 is defined in Proposition B.1. Since U blows-up at $x = 0$, $U(0,t)(T-t)^{\frac{1}{p-1}} \to \kappa$ as $t \to T$ by [11]. Hence, if $\delta > 0$ is small enough, then $\delta^{\frac{1}{p-1}} U(0, T-\delta) \ge \frac{3\kappa+\kappa_0}{4}$. Since $U_k(0, T-\delta) \to U(0, T-\delta)$ as $k \to +\infty$, we get $\delta^{\frac{1}{p-1}} U_k(0, T-\delta) \ge \frac{\kappa+\kappa_0}{2}$ for k large enough.

By (102) and continuity arguments, we have the existence of $t'_{\delta,k} \in [T - \delta, t_k]$ such that $(T - t'_{\delta,k})^{\frac{1}{p-1}} U_k(0, t'_{\delta,k}) = \kappa_0$. The existence of t'_k follows then from a diagonal process.

Let us define for all $\xi \in \mathbb{R}^N$ and $\tau \in [0,1)$,

$$v_k(\xi,\tau) = (T-t'_k)^{\frac{1}{p-1}} U_k\left(\xi\sqrt{T-t'_k}, t'_k + \tau(T-t'_k)\right). \tag{105}$$

Then, v_k is a solution of (1), and $v_k(\xi,0) = (T-t'_k)^{\frac{1}{p-1}} U_k(\xi\sqrt{T-t'_k}, t'_k) = W_k(\xi, s'_k)$, where $s'_k = -\log(T-t'_k) \le s^*_k$. Since $t'_k + \frac{3}{4}(T-t'_k) \le t_k \le t^*_k$ (the second estimate is true by construction, and the first follows from (102), (104) and techniques similar to those in lemma 2.7), it follows from (100) and (104) that $v_k(0,0) = \kappa_0$ and

$$\forall \tau \in [0,\frac{3}{4}],\ \|\nabla v_k(\tau)\|_{L^\infty} \le \frac{C(p)c_1}{\sqrt{|\log(T-t'_k)|}},\ \|\nabla^2 v_k(\tau)\|_{L^\infty} \le \frac{C(p)A}{|\log(T-t'_k)|}, \tag{106}$$

and for k large enough and for all $|\xi| \le 4|\log(T-t'_k)|^{1/4}$, $\mathcal{E}_{\xi,1}(v_k(0)) \le 2\mathcal{E}_{\xi,1}(\kappa_0) \le \sigma_0$. Therefore, from Proposition B.1 (applied with $\delta = 1$ and using translation invariance), we have $\forall \tau \in [\frac{1}{2},1]$, $\forall |\xi| \le 2|\log(T-t'_k)|^{1/4}$, $|v_k(\xi,\tau)| \le M(p)$.

Now using arguments similar to those of lemma 2.8, we get

$$\forall \tau \in [\frac{3}{4},1),\ \forall |\xi| \le |\log(T-t'_k)|^{1/4},\ |v_k| + |\nabla v_k| + |\nabla^2 v_k| \le M(p). \tag{107}$$

By arguments similar to those of lemma 2.9, we get from (106) and (107) for k large enough,

$$\sup_{\tau\in[0,1]} |\Delta_\xi v_k(0,\tau)| \to 0 \text{ as } k \to +\infty.$$

Therefore, since v_k is a solution of (1), we have

$$\forall \tau \in [0,1),\ v_k(0,\tau) \ge \frac{\kappa_0}{2}$$

for k large enough. Hence

$$\forall \tau \in [0,1),\ |\Delta_\xi v_k(0,\tau)| \le \frac{\eta_0}{2} v_k(0,\tau)^p \tag{108}$$

for k large enough, and this yields a contradiction.
Indeed, taking $\tau_k = \frac{t_k - t'_k}{T-t'_k}$, we get from (108) and (105): $\forall k \ge k_0$,
$|\Delta U_k(0,t_k)| = (T-t'_k)^{-\frac{p}{p-1}} |\Delta_\xi v_k(0,\tau_k)|$
$\le \frac{\eta_0}{2}(T-t'_k)^{-\frac{p}{p-1}} v_k(0,\tau_k)^p = \frac{\eta_0}{2} U_k(0,t_k)^p$, which contradicts (101).

This concludes the proof of Proposition 2.3.

C Proof of lemma 2.11

Define $\chi_1(\xi) = \chi_0(\frac{\xi}{2B_1})$ where χ_0 is defined in (63). Then, $\forall \xi \in \mathbb{R}^N$,

$$|\nabla \chi_1(\xi)| \leq \frac{C}{B_1} 1_{\{|\xi| \geq 2B_1\}} \text{ and } |\Delta \chi_1(\xi)| \leq \frac{C}{{B_1}^2} 1_{\{|\xi| \geq 2B_1\}}. \tag{109}$$

Let $Z(\xi, \tau) = \chi_1(\xi) e^{-\lambda t} z(\xi, \tau)$. Then, we have from (56): $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [0, \tau_*]$,

$$\begin{cases} \frac{\partial Z}{\partial \tau} \leq \Delta Z + \mu + z e^{-\lambda \tau} \Delta \chi_1 - 2 e^{-\lambda \tau} \nabla . (z \nabla \chi_1), \\ Z(\xi, 0) \leq z_0, \ z(\xi, \tau) \leq B_2. \end{cases} \tag{110}$$

We now take $|\xi| \leq B_1$ and use an integral formulation of (110) to write $Z(\xi, \tau) \leq I + II + III + IV$ where
$I = \left(e^{\tau \Delta} Z(0)\right)(\xi)$, $II = \int_0^\tau ds e^{(\tau - s)\Delta} \mu$, $III = \int_0^\tau ds e^{(\tau - s)\Delta} e^{-\lambda s} z(s) \Delta \chi_1$ and $IV = -2 \int_0^\tau ds e^{(\tau - s)\Delta} e^{-\lambda s} \nabla . (z(s) \nabla \chi_1)$.

From the maximum principle and (110), we have $I \leq z_0$ and $II \leq \mu \int_0^\tau ds \leq \mu$.

The treatment of III and IV is similar. However, handling IV is a bit more delicate.
By an integration by parts, we have:
$IV = 2 \int_0^\tau ds e^{-\lambda s} \nabla e^{(\tau - s)\Delta} z(s) \nabla \chi_1$
$= 2 \int_0^\tau ds e^{-\lambda s} \int dx \left(-\frac{(\xi - x)}{2(\tau - s)}\right) \frac{e^{-\frac{|\xi - x|^2}{4(\tau - s)}}}{(4\pi(\tau - s))^{N/2}} z(x, s) \nabla \chi_1(x)$.
From (110) and (109), we obtain:
$IV \leq \int_0^\tau ds \int_{\{|x| \geq 2B_1\}} dx \frac{|\xi - x|}{\tau - s} \frac{e^{-\frac{|\xi - x|^2}{4(\tau - s)}}}{(4\pi(\tau - s))^{N/2}} \frac{CB_2}{{B_1}^2}$.
Since $|\xi| \leq B_1$, $|x| \geq 2B_1$ and $0 \leq \tau - s \leq 1$, we have $e^{-\frac{|\xi - x|^2}{4(\tau - s)}} = e^{-\frac{|\xi - x|^2}{8(\tau - s)}} e^{-\frac{|\xi - x|^2}{8(\tau - s)}} \leq e^{-\frac{|\xi - x|^2}{8(\tau - s)}} e^{-\frac{{B_1}^2}{8}}$. Therefore,
$IV \leq \frac{CB_2}{B_1} e^{-\frac{{B_1}^2}{8}} \int_0^\tau \frac{ds}{\sqrt{\tau - s}} \int_{\{|x| \geq 2B_1\}} dx \frac{|\xi - x|}{\sqrt{\tau - s}} \frac{e^{\frac{-|\xi - x|^2}{8(\tau - s)}}}{(4\pi(\tau - s))^{N/2}}$
$\leq \frac{CB_2}{B_1} e^{-\frac{{B_1}^2}{8}} \int_0^\tau \frac{ds}{\sqrt{\tau - s}} \int |X| e^{-|X|^2} dX \leq CB_2 e^{-\frac{{B_1}^2}{4}}$.

Similarly, we obtain: $III \leq CB_2 e^{-\frac{{B_1}^2}{4}}$.

Combining the bounds on I, II, III and IV, we get the conclusion of lemma 2.11. ■

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