On growth rate near the blow-up surface for semilinear wave equations

Frank Merle
Université de Cergy Pontoise
Hatem Zaag
CNRS UMR 8553 Ecole Normale Supérieure

January 23, 2005

Abstract: In this paper, we find the optimal growth estimate near the blow-up surface for the semilinear wave equation with a power nonlinearity. The techniques are based on local energy estimates of our earlier work [16] and [17], which extend to the present situation. The exponent p is superlinear and less or equal to $1+\dfrac{4}{N-1}$ if $N\geq 2$.

AMS Classification: 35L05, 35L67

Keywords: Wave equation, finite time blow-up, blow-up rate.

1 Introduction

This paper is devoted to the study of blow-up solutions for the following semilinear wave equation

$$\begin{cases} \partial^2_{tt} u = \Delta u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}$.
The space $\mathrm{L}^2_{\mathrm{loc,u}}$ is the set of all v in $\mathrm{L}^2_{\mathrm{loc}}$ such that

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}} \equiv \sup_{a\in\mathbb{R}^N} \left(\int_{|x-a|<1} |v(x)|^2 dx \right)^{1/2} < +\infty,$$

and the space $\mathrm{H}^1_{\mathrm{loc,u}} = \{\mathrm{v} \mid \mathrm{v}, \nabla \mathrm{v} \in \mathrm{L}^2_{\mathrm{loc,u}}\}$.
We assume in addition that

$$1 < p \leq p_c \equiv 1 + \frac{4}{N-1}. \tag{2}$$

The Cauchy problem for equation (1) in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$ follows from the finite speed of propagation and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$, whenever $1 < p < 1 + 4/(N-2)$. See for instance Lindblad and Sogge [11], Shatah and Struwe [18] and their references (for the local in time wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$). The existence of blow-up solutions for equation (1) is a consequence of the finite speed of propagation and ODE techniques (see for example John [8]). In [2], Antonini and Merle find a new blow-up criterion (see Proposition 2.1 below). More blow-up results can be found in Caffarelli and Friedman [4], Alinhac [1], Kichenassamy and Litman [9], [10].

If u is a blow-up solution of (1), we define (see for example Alinhac [1]) a continuous surface Γ as the graph of a function $x \to T(x)$ such that u cannot be extended beyond the set

$$D_u = \{(x,t) \mid t < T(x)\}. \tag{3}$$

The set D_u is called the maximal influence domain of u. Moreover, from the finite speed of propagation, T is a 1-Lipschitz function. Let $\bar{T}$ be the minimum of $T(x)$ for all $x \in \mathbb{R}^N$. The time $\bar{T}$ and the surface Γ are called (respectively) the blow-up time and the blow-up surface of u. We have in addition

$$\|u(t)\|_{\mathrm{H}^1_{\mathrm{loc,u}}} + \|\partial_t u(t)\|_{\mathrm{L}^2_{\mathrm{loc,u}}} \to +\infty \text{ as } t \to \bar{T}.$$

We have proved in [16] and [17] that the blow-up rate of u is given by the associated ODE

$$v'' = v^p, \quad v(\bar{T}) = +\infty.$$

that is

$$v(t) \sim \kappa(\bar{T} - t)^{-\frac{2}{p-1}} \text{ where } \kappa = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}.$$

More precisely,

For all $t \in [\bar{T}(1 - 1/e), \bar{T})$,

$$0 < \epsilon_0 \leq (\bar{T} - t)^{\frac{2}{p-1}} \|u(t)\|_{\mathrm{L}^2_{\mathrm{loc,u}}} + (\bar{T} - t)^{\frac{2}{p-1}+1} \left(\|u_t\|_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\nabla u\|_{\mathrm{L}^2_{\mathrm{loc,u}}}\right) \leq K$$

for some $\epsilon_0 \equiv \epsilon_0(N,p) > 0$ and a constant K which depends only on N, p and on bounds on $\bar{T}$ and the initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$.

Let us mention that Bizón, Chmaj and Tabor [3] obtained a numerical confirmation of this result in the wider range $1 < p < 1 + 4/(N-2)$ which can suggest that condition (2) is only a technical limitation of our method. In [4], the authors prove this result for $N \leq 3$ and under restrictive conditions on initial data that ensure that

$$u \geq 0 \text{ and } \partial_t u > |\nabla u|.$$

Unlike previous work where the considered question was to construct blow-up solutions with explicit blow-up behavior (for example, the authors construct in [9] and [10] a solution that blows up on a prescribed analytic space-like hypersurface), the question we address in this paper is about classification of blow-up behavior (see for example Giga and Kohn [5], [6], [7], and Merle and Zaag [15] for the semilinear heat equation, see Merle and Raphaël [13], [14], [12] for the critical nonlinear Schrödinger equation (NLS)). By classification, we mean that we consider an arbitrary blow-up solution and we want to know about its properties, in particular, the blow-up rate in H^1.

Since the notion of singular surface is an artifact of the finite speed of propagation, it is natural to ask if the result obtained in [16] and [17] in the region $\{(x,t) \mid t < \bar{T}\}$ extends to the region $\{(x,t) \mid t < T(x)\}$. In this paper, we find the growth estimate near the space-time blow-up surface in any dimension for general initial data:

Theorem 1 (Growth estimate near the blow-up surface for solutions of equation (1))] *If u is a solution of (1) with blow-up surface $\Gamma : \{x \to T(x)\}$, then, for all $x_0 \in \mathbb{R}^N$ and $t \in [\frac{3}{4}T(x_0), T(x_0))$:*

$$(T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{N/2}}$$
$$+(T(x_0)-t)^{\frac{2}{p-1}+1}\left(\frac{\|u_t(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{N/2}}\right) \le K$$

where the constant K depends only on N, p, and on an upper bound on $T(x_0)$, $1/T(x_0)$ and the initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$.

Remark: Since we know from the finite speed of propagation that the backward light cone with vertex $(x_0, T(x_0))$ is included in the maximal influence domain D_u whose boundary is the blow-up surface Γ, it holds that

$$\frac{\sqrt{2}}{2}(T(x_0)-t) \le d_0 = \mathrm{dist}\left((x_0,t),\Gamma\right) \le T(x_0)-t \tag{4}$$

where

$$d_0 = \mathrm{dist}\left((x_0,t),\Gamma\right).$$

This yields the following weaker version of Theorem 1:

$$d_0^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,\frac{d_0}{2}))}}{d_0^{N/2}} + d_0^{\frac{2}{p-1}+1}\left(\frac{\|u_t(t)\|_{L^2(B(x_0,\frac{d_0}{2}))}}{d_0^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,\frac{d_0}{2}))}}{d_0^{N/2}}\right) \le K.$$

Remark: This theorem holds if one replaces the integration domain by $B(x_0, \eta d_0)$ or $B\left(x_0, \eta(T(x_0)-t)\right)$ for any $\eta \in (0,1)$. In that case, the constant K we obtain in the proof depends on η too. It is not clear whether we can have a constant independent of η or not (except in the situation of Theorem 2 below).

Remark: In [4], when $N \le 3$ under strong restrictions on initial data, the authors handle strong solutions and obtain similar but pointwise growth estimates. In our work, considering general initial data in $\mathrm{H}^1_{\mathrm{loc}} \times \mathrm{L}^2_{\mathrm{loc}}$, we work with weak solutions, hence, we naturally get estimates on $\mathrm{L}^2_{\mathrm{loc}}$ means of the solution and its first derivatives. In the case $N = 1$, however, this gives a pointwise estimate from the Sobolev injection.

Remark: The result holds in the vector valued case with the same proof. The critical value for p in our theorem ($p = 1 + \frac{4}{N-1}$) is also critical for the existence of a conformal transformation for equation (1). Note that our proof strongly relies on the fact that $p \le p_c$. In particular, we don't give any answer in the range $p_c < p < 1 + 4/(N-2)$ (subcritical with respect to the Sobolev injection of H^1).

This theorem does not give the blow-up rate, since we are unable to give a lower estimate near all points of the blow-up surface. However, under a non degeneracy condition, we can get such an estimate, which determines locally the blow-up rate. Let us first introduce for all $x \in \mathbb{R}^N$, $t \le T(x)$ and $\delta > 0$, the cone

$$\mathcal{C}_{x,t,\delta} = \{(\xi,\tau) \neq (x,t) \mid 0 \le \tau \le t - \delta|\xi - x|\}. \tag{5}$$

Our non degeneracy condition is the following: x_0 is a non characteristic point if

$$\exists \delta_0 = \delta_0(x_0) \in (0,1) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0,T(x_0),\delta_0}. \tag{6}$$

Condition (6) is equivalent to the fact that the Lipschitz constant of the blow-up surface at x_0 is bounded away from 1. Note that [4] assumes the same thing, uniformly on compact sets. It is an open problem to tell whether condition (6) holds for all space-time blow-up points. In fact, one sees that Theorem 1 is a first step in the understanding of this degeneracy condition.

For a non characteristic point x_0, we have the following lower and upper bound:

Theorem 2 (Growth rate for solutions of equation (1) at a non characteristic point) *If u is a solution of (1) with blow-up surface $\Gamma : \{x \to T(x)\}$ and if $x_0 \in \mathbb{R}^N$ is non characteristic (in the sense (6)), then, for all $t \in [\frac{3}{4}T(x_0), T(x_0))$,*

$$0 < \epsilon_0(N,p) \le \quad (T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}}$$

$$+(T(x_0)-t)^{\frac{2}{p-1}+1} \quad \left(\frac{\|u_t(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} \right) \le K$$

where the constant K depends only on N, p, and on an upper bound on $T(x_0)$, $1/T(x_0)$, $\delta_0(x_0)$ and the initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$.

Now, if we know that all points are uniformly non characteristic in some ball, then all estimates are uniform in that ball. More precisely,

Corollary 3 (Growth rate near the blow-up set for solutions of equation (1)) *If u is a solution of (1) with blow-up surface $\Gamma : \{x \to T(x)\}$ and if all points in $B(x_0, R_0)$ for some x_0 and $R_0 \in (0,1)$ are non characteristic (in the sense (6)) with*

$$\hat{\delta}_0(x_0) \equiv \sup_{|x-x_0|<R_0} \delta_0(x) < 1,$$

then, for all $x \in B(x_0,R_0)$ and $t \in [\frac{3}{4}T(x), T(x))$,

$$0 < \epsilon_0(N,p) \quad \le (T(x)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x,T(x)-t))}}{(T(x)-t)^{N/2}}$$

$$+(T(x)-t)^{\frac{2}{p-1}+1} \left(\frac{\|u_t(t)\|_{L^2(B(x,T(x)-t))}}{(T(x)-t)^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x,T(x)-t))}}{(T(x)-t)^{N/2}} \right) \le K$$

where the constant K depends only on N, p, $\hat{\delta}_0(x_0)$ and on an upper bound on $T(x_0)$, $1/T(x_0)$ and the initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$.

Our method relies on the one hand on the estimates in similarity variables introduced in Antonini and Merle [2] and used in [16] and [17], and on the other hand on covering techniques adapted to the geometric shape of the blow-up surface, which are the new ingredients of this paper.

Indeed, given some (x_0, T_0) such that $0 < T_0 \le T(x_0)$, we introduce the following self-similar change of variables:

$$w_{x_0,T_0}(y,s) = (T_0 - t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t). \tag{7}$$

This change of variables transforms the backward light cone with vertex (x_0, T_0) into the infinite cylinder $(y,s) \in B \times [-\log T_0, +\infty)$ where $B = B(0,1)$. The key idea is to obtain estimates on w_{x_0,T_0} as $s \to \infty$, uniformly with respect to the scaling point. Note that in [16] and [17], we took always $T_0 = \bar{T}$. Note also that it will be essential to allow T_0 to vary in the interval $(0, T(x_0)]$ and not just equal to $T(x_0)$.

The function w_{x_0,T_0} (we write w for simplicity) satisfies the following equation for all $y \in B$ and $s \ge -\log T_0$:

$$\partial^2_{ss} w - \frac{1}{\rho} \operatorname{div}(\rho \nabla w - \rho (y.\nabla w) y) + \frac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w = -\frac{p+3}{p-1} \partial_s w - 2y.\nabla \partial_s w \tag{8}$$

$$\begin{aligned} &\text{where} \quad \rho(y) = (1 - |y|^2)^\alpha \text{ with } \alpha = \tfrac{2}{p-1} - \tfrac{N-1}{2} > 0 \quad \text{if } p < p_c \\ &\text{and} \quad \rho \equiv 1 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } p = p_c \end{aligned} \tag{9}$$

(note that one could use the first line of (9) as a definition for $\rho(y)$ for all $p \le p_c$; indeed, $\alpha = 0$ when $p = p_c$). In similarity variables (7), Theorem 2 can be restated as follows:

Theorem 2' (Uniform bounds on solutions of (8)) *If u is a solution of (1) with blow-up surface $\Gamma : \{x \to T(x)\}$ and if $x_0 \in \mathbb{R}^N$ is non characteristic (in the sense (6)), then for all $s \ge -\log \frac{T(x_0)}{4}$,*

$$0 < \epsilon_0(N,p) \le \|w_{x_0,T(x_0)}(s)\|_{\mathrm{H}^1(B)} + \|\partial_s w_{x_0,T(x_0)}(s)\|_{\mathrm{L}^2(B)} \le K \tag{10}$$

where $w_{x_0,T(x_0)}$ is defined in (7), B is the unit ball of $\mathbb{R}^N$ and K depends only on N, p and on an upper bound on $T(x_0)$, $1/T(x_0)$, $\delta_0(x_0)$ and the initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$.

Now, the proof of the main result relies on a geometrical adaptation of the ideas already used in [16] (the existence of a Lyapunov functional for equation (7) and some energy estimates related to this structure, the improvement of regularity estimates by interpolation and some Gagliardo-Nirenberg type argument). Note that while in our previous work we used the invariance of equation (1) with respect to translations in space, we need use here the invariance with respect to translations in space and in time in order to deal with the changing blow-up time (extra parameter T_0). In addition, new covering arguments adapted to the geometry of the blow-up surface have to be introduced.

Since Theorem 2 and Corollary 3 immediately follow from Theorem 2' by the self-similar transformation (7) and inequality (4), we only prove Theorem 1 and Theorem 2'. In section 2, we recall dispersive and energy-type estimates from [16].

In section 3, we introduce the new covering techniques adapted to the geometry of the blow-up set and conclude the proof in the subcritical case ($p < p_c$).
In the last section, we show how to derive the result in the critical case ($p = p_c$), relying on energy estimates of [17] together with the covering technique of the subcritical case.

Acknowledgment. We would like to thank the referee for his valuable suggestions which made our paper much clearer. The first author would like to thank the Institute for Advanced Study where part of this work has been done.

2 Dispersive and energy-type estimates in similarity variables for subcritical p

In this section, we deal with the subcritical case. No geometrical information is needed, in particular, condition (6) is not needed. We first recall energy estimates from [16] and then use new covering techniques, adapted to the geometry of the blow-up surface. In the critical case, dispersion with w is degenerate in a certain sense, and the proof is rather different. See the last section.

2.1 Dispersion in similarity variables

Throughout this section, w stands for any solution of equation (8), defined for all $(y, s) \in B \times [s_0, \infty)$ for some $s_0 \in \mathbb{R}$, where B is the unit ball of $\mathbb{R}^N$.

Let the functional $E(w)$ be

$$E(w) = \int_B \left(\frac{1}{2}\partial_s w^2 + \frac{1}{2}|\nabla w|^2 - \frac{1}{2}(y.\nabla w)^2 + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy. \qquad (11)$$

Recall from (9) that

$$\text{where} \quad \rho(y) = (1-|y|^2)^\alpha \text{ with } \alpha = \tfrac{2}{p-1} - \tfrac{N-1}{2} > 0 \quad \text{if } p < p_c$$

$$\text{and} \quad \rho \equiv 1 \quad \text{if } p = p_c.$$

We recall also from [2] and [16] the following identities (note that in previous work, w was defined in the whole space $\mathbb{R}^N$, however, only its values in the unit ball were needed).

Proposition 2.1
i) **(E is a Lyapunov functional)***: For all $s_2 \ge s_1 \ge s_0$,*

$$E(w(s_2)) - E(w(s_1)) = -2\alpha \int_{s_1}^{s_2} \int_B \partial_s w(y,s)^2 (1-|y|^2)^{\alpha-1} dy ds.$$

ii) **(A blow-up criterion for equation (8))** *If $E(w(s_1)) < 0$ for some $s_1 \in \mathbb{R}$, then w blows up in finite time $S^* > s_1$.*
iii) **(Bounds on E and its dissipation)** *For all $s \ge s_0$, $s_2 \ge s_1 \ge s_0$,*

$$0 \le E(w(s)) \le E(w(s_0)), \qquad (12)$$

$$\int_{s_1}^{s_2} \int_B \partial_s w(y,s)^2 (1-|y|^2)^{\alpha-1} dy ds \quad \le \quad \frac{E(w(s_0))}{2\alpha}. \qquad (13)$$

Proof: See Lemma 2.1 page 1145 in [2], Corollary 2.3 page 1151 in [16]. ■

2.2 Energy estimates in similarity variables

We recall now the main result of section 2 of [16] in the following:

Proposition 2.2 (Bounds on Sobolev norms of solutions of equation (8)) *Consider w a solution of (8) defined for all $(y,s) \in B \times [s_0,\infty)$ for some $s_0 > 0$, where B is the unit ball of $\mathbb{R}^N$. Then, for all $s \geq s_0 + 1$,*

$$\begin{array}{lrcl} (i) & \displaystyle\int_s^{s+1} \int_B & & \left(|w|^{p+1}\rho + \partial_s w^2 (1-|y|^2)^{\alpha-1} + |\nabla w|^2 (1-|y|^2)^{\alpha+1}\right) dyds' \\ & & \leq & C\left(E(w(s_0))+1\right), \qquad (14) \\ & \displaystyle\int_B w(y,s)^2 \rho dy & \leq & C\left(E(w(s_0))+1\right), \\ (ii) & \displaystyle\int_s^{s+1} \int_{B_{1/2}} & & \left(\partial_s w^2 + |\nabla w|^2 + |w|^{p+1}\right) dyds \leq C\left(E(w(s_0))+1\right), \\ & \displaystyle\int_{B_{1/2}} w^2 dy & \leq & C\left(E(w(s_0))+1\right), \\ (iii) & \displaystyle\int_{B_{1/2}} |w|^r dy & \leq & C\left(E(w(s_0))+1\right)^\gamma, \end{array}$$

where $B_{1/2} \equiv B(0,1/2)$, $r = \frac{p+3}{2}$ and $\gamma = 1$ if $N \geq 2$, and $r = p+1$ and $\gamma = p$ if $N = 1$.

Proof: See Section 2 and Corollary 2.6 page 1156 in [16] for (i) and (ii), see the proof of Proposition 3.1 page 1156 in [16] for (iii). ■

Consider u a solution of (1) with blow-up surface $\Gamma : \{t = T(x)\}$, and consider its self-similar transformation w_{x_0,T_0} defined at some scaling point (x_0,T_0) by (7) where $T_0 \leq T(x_0)$. Because of the finite speed of propagation, w_{x_0,T_0} is defined on the cylinder $B \times [-\log T_0, \infty)$. Using the definition (7) of w_{x_0,T_0}, we write

$$E(w_{x_0,T_0}(-\log T_0)) \leq \left(T_0^{\frac{4}{p-1}-N} + T_0^{\frac{2(p+1)}{p-1}}\right) C\left(\|(u_0,u_1)\|_{\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}}\right) \leq \Phi(T_0) \qquad (15)$$

where

$$\Phi(T_0) \text{ is a generic constant of the type } C\left(T_0^{-\gamma} + T_0^{\gamma}\right) \text{ for some } \gamma > 0. \qquad (16)$$

Applying Proposition 2.2 to w_{x_0,T_0}, we derive the following Corollary:

Corollary 2.3 (Bounds on Sobolev norms of self-similar transformations of equation (1)) *Consider u a solution of (1) with blow-up surface $\Gamma : \{t = T(x)\}$. For all $x_0 \in \mathbb{R}^N$, $T_0 \leq T(x_0)$ and $s \geq -\log T_0 + 1$,*

$$\begin{array}{rcl} \displaystyle\int_s^{s+1} \int_{B_{1/2}} \left(\partial_s w_{x_0,T_0}^2 + |\nabla w_{x_0,T_0}|^2 + |w_{x_0,T_0}|^{p+1}\right) dyds' & \leq & C\Phi(T_0), \\ \displaystyle\int_{B_{1/2}} |w_{x_0,T_0}(y,s)|^r dy & \leq & C\Phi(T_0), \end{array}$$

where w_{x_0,T_0} is defined in (7), $r = \frac{p+3}{2}$ if $N \geq 2$ and $r = p+1$ if $N = 1$, and C depends only on N, p, and an upper bound on the norm of initial data for u in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$.

3 Proof in the subcritical case

We prove Theorem 1 and Theorem 2' here in the case $p < p_c$. Please note that Theorem 2 and Corollary 3 immediately follow by the self-similar transformation (7). From now on, the proofs are different from [16], except for the lower bound.

3.1 Lower bound

As mentioned in [16], the lower bound in Theorem 2 follows from the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$, finite speed of propagation, the fact that x_0 is not characteristic and scaling arguments. In this subsection, we prove the following:

Lemma 3.1 (Lower bound on the growth rate at a non characteristic point) *If u is a solution of (1) with blow-up surface $\Gamma : \{x \to T(x)\}$ and if $x_0 \in \mathbb{R}^N$ is non characteristic (in the sense (6)), then, for all $t \in [\frac{3}{4}T(x_0), T(x_0))$,*

$$\begin{aligned} 0 < \epsilon_0(N,p) \quad &\leq (T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} \\ &+ (T(x_0)-t)^{\frac{2}{p-1}+1} \left(\frac{\|u_t(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} \right). \end{aligned}$$

Proof: We proceed by contradiction and assume that there is some non characteristic point $x_0 \in \mathbb{R}^N$ (in the sense (6)) and $t \in [\frac{3}{4}T(x_0), T(x_0))$ such that

$$\begin{aligned} &(T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} \\ + \quad &(T(x_0)-t)^{\frac{2}{p-1}+1} \left(\frac{\|u_t(t)\|_{L^2(B(x,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{N/2}} \right) \leq \epsilon_0 \end{aligned}$$

where ϵ_0 will be fixed small enough in terms of N and p. Since u is defined on the cone $\mathcal{C}_{x_0,T(x_0),\delta_0}$ where $\delta_0 = \delta_0(x_0) < 1$ by (6), and since u, u_t and ∇u are in $L^2_{loc}\{x \mid (x,t) \in D_u\}$, there exists $\rho^* = \rho^*(x_0,t,\epsilon_0) > 1$ such that $\bar{B}(x_0, \rho^*(T(x_0)-t)) \times \{t\} \subset\subset C_{x_0,T(x_0),\delta_0}$ and

$$\begin{aligned} &[\rho^*(T(x_0)-t)]^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,\rho^*(T(x_0)-t)))}}{[\rho^*(T(x_0)-t)]^{N/2}} + \\ &[\rho^*(T(x_0)-t)]^{\frac{2}{p-1}+1} \left(\frac{\|u_t(t)\|_{L^2(B(x_0,\rho^*(T(x_0)-t)))}}{[\rho^*(T(x_0)-t)]^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,\rho^*(T(x_0)-t)))}}{[\rho^*(T(x_0)-t)]^{N/2}} \right) \leq 2\epsilon_0. \end{aligned} \tag{17}$$

Let $T^* = t + \rho^*(T(x_0)-t) > T(x_0)$, and define for all $\xi \in B(0,1)$ and $\tau \in [0\frac{1}{\rho^*})$

$$v^*(\xi,\tau) = (T^*-t)^{\frac{2}{p-1}} u(x_0 + \xi(T^*-t), t + \tau(T^*-t)). \tag{18}$$

Then, v^* is a solution of equation (1) and we see from (17) that at $\tau = 0$,

$$\|v^*(0)\|_{H^1(B(0,1))} + \|\partial_\tau v^*(0)\|_{L^2(B(0,1))} \leq 2\epsilon_0.$$

Using the finite speed of propagation and the local in time wellposedness in $H^1 \times L^2$ for equation (1), we see that if we fix $\epsilon_0 = \epsilon_0(N,p) > 0$ small enough, then v^* is defined for all (ξ,τ) in the cone $\mathcal{C}_{0,1,1}$ and

$$\sup_{\tau\in[0,1)} \frac{\|v^*(\tau)\|_{H^1(B(0,1-\tau))} + \|\partial_\tau v^*(\tau)\|_{L^2(B(0,1-\tau))}}{(1-\tau)^{\frac{N}{2}}} \le 1,$$

which implies by (18) that u is defined in the cone $\mathcal{C}_{x_0,T^*,1}$ which strictly contains $\mathcal{C}_{x_0,T(x_0),1}$. This contradicts the fact that u can not be extended beyond the surface $\Gamma = \{t = T(x)\}$. This concludes the proof of Lemma 3.1. ■

3.2 Covering technique

In this section, we introduce a new covering technique to extend the estimate of any known L^q norm of w, $\partial_s w$ or ∇w from $B_{1/2}$ to the whole unit ball. Here, we strongly need the following local space-time generalization of the notion of characteristic point: a point $(x_0,T_0) \in \bar{D}_u$ is δ_0-non characteristic with respect to t where $\delta_0 \in (0,1)$ if

$$u \text{ is defined on } \mathcal{D}_{x_0,T_0,t,\delta_0} \tag{19}$$

where

$$\mathcal{D}_{x_0,T_0,t,\delta_0} = \{(\xi,\tau) \neq (x_0,T_0) \mid t \le \tau \le T_0 - \delta_0|\xi - x_0|\}. \tag{20}$$

We claim the following:

Lemma 3.2 *Consider $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$ and $t \in [T_0(1-1/e), T_0)$ such that $\mathcal{D}_{x_0,T_0,t,\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$.*
(i) Then, for all x such that $|x - x_0| \le (T_0 - t)/\delta_0$, $\mathcal{D}_{x,T^*(x),t,\delta_0} \subset D_u$ where

$$T^*(x) = T_0 - \delta_0|x - x_0|. \tag{21}$$

(ii) Moreover,

$$t \in [T^*(x)(1-1/e), T^*(x)), \;\; -\log(T^*(x) - t) \ge -\log T^*(x) + 1, \tag{22}$$

and

$$\Phi(T^*(x)) \le C(\delta_0)\Phi(T_0). \tag{23}$$

Remark: The point $(x, T^*(x))$ is on the lateral boundary of $\mathcal{D}_{x_0,T_0,t,\delta_0}$.
Proof: Consider $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$ and $t \in [T_0(1-1/e), T_0)$ such that $\mathcal{D}_{x_0,T_0,t,\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$. Consider x such that $|x - x_0| \le (T_0 - t)/\delta_0$.

(i) For any $(\xi,\tau) \in \mathcal{D}_{x,T^*(x),t,\delta_0}$, we have $t \le \tau \le T^*(x) - \delta_0|\xi - x| = T_0 - \delta_0|x - x_0| - \delta_0|\xi - x| \le T_0 - \delta_0|\xi - x_0|$. Thus, $\mathcal{D}_{x,T^*(x),t,\delta_0} \subset \mathcal{D}_{x_0,T_0,t,\delta_0} \subset D_u$.

(ii) Since $T_0 \ge T^*(x) = T_0 - \delta_0|x - x_0| \ge t \ge T_0(1-1/e) \ge T^*(x)(1-1/e)$, this yields (22), and by (16), (23) follows. This concludes the proof of Lemma 3.2. ■

Unlike in [16], we find it easier to work in the $u(x,t)$ setting, in order to respect the geometry of the blow-up set. We claim the following:

Proposition 3.3 (Covering technique) *Consider $\eta \geq 0$, $q \geq 1$ and f a function defined in D_u, the domain of u such that for all $t > 0$, $f(.,t) \in L^q_{loc}(x \mid (x,t) \in D_u)$. Then, for all $x_0 \in \mathbb{R}^N$, $T_0 \leq T(x_0)$ and $t \leq T_0$ such that $\mathcal{D}_{x_0,T_0,t,\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$,*

(i) $f(.,t)$ is defined on $B(x, T^*(x) - t)$, for any x such that $|x - x_0| \leq \frac{T_0-t}{\delta_0}$.

(ii)
$$\sup_{\{x \mid |x-x_0| \leq \frac{T_0-t}{\delta_0}\}} (T^*(x) - t)^\eta \int_{B(x,T^*(x)-t)} |f(\xi,t)|^q d\xi < +\infty,$$

(iii)
$$\begin{aligned} &\sup_{\{x \mid |x-x_0| \leq \frac{T_0-t}{\delta_0}\}} (T^*(x) - t)^\eta \int_{B(x,T^*(x)-t)} |f(\xi,t)|^q d\xi \\ \leq\ & C(\delta_0,\eta) \sup_{\{x \mid |x-x_0| \leq \frac{T_0-t}{\delta_0}\}} (T^*(x) - t)^\eta \int_{B(x,\frac{T^*(x)-t}{2})} |f(\xi,t)|^q d\xi \end{aligned}$$

where $T^*(x)$ is defined in (21) and $\delta_0 = \delta_0(x_0, T_0)$ is defined in (19).

Remark: Note that the supremum is taken over the basis of $\mathcal{D}_{x_0,T_0,t,\delta_0}$. Note also that $\{(x, T^*(x)) \neq (x_0, T_0) \mid T^*(x) = T_0 - \delta_0|x - x_0| \geq t\}$ is the lateral boundary of $\mathcal{D}_{x_0,T_0,t,\delta_0}$, which is in the interior of the domain D_u.
Proof:
(i) and (ii): For all x such that $|x - x_0| \leq \frac{T_0-t}{\delta_0}$, we have

$$T^*(x) - t \leq T_0 - t \text{ and } B(x, T^*(x) - t) \subset B\left(x_0, \frac{T_0 - t}{\delta_0}\right), \tag{24}$$

the basis of $\mathcal{D}_{x_0,T_0,t,\delta_0}$. Indeed, if $|y - x| < T^*(x) - t$, then we have from (21) $|y - x_0| \leq |y-x|+|x-x_0| < T^*(x)-t+|x-x_0| = T_0-t+(1-\delta_0)|x-x_0| \leq (T_0-t)\left(1 + \frac{1-\delta_0}{\delta_0}\right) = \frac{T_0-t}{\delta_0}$. Thus, (24) and then (i) hold.
Using (24), we write

$$(T^*(x) - t)^\eta \int_{B(x,T^*(x)-t)} |f(\xi,t)|^q d\xi \leq (T_0 - t)^\eta \int_{B(x_0,\frac{T_0-t}{\delta_0})} |f(\xi,t)|^q d\xi$$

which is finite since $f(.,t) \in L^q_{loc}(x \mid (x,t) \in D_u)$ and there exists $r_0 > 0$ such that $B(x_0, \frac{T_0-t}{\delta_0} + r_0) \times \{t\} \subset D_u$ (this is true because D_u is open and $\bar{B}(x_0, \frac{T_0-t}{\delta_0}) \times \{t\} \subset D_u$). Thus, the supremum exists and (ii) is proved.

(iii) Consider x^* such that $|x^* - x_0| \leq \frac{T_0-t}{\delta_0}$ and

$$(T^*(x^*)-t)^\eta \int_{B(x^*,T^*(x^*)-t)} |f(\xi,t)|^q d\xi \geq \frac{1}{2} \sup_{|x-x_0| \leq \frac{T_0-t}{\delta_0}} (T^*(x)-t)^\eta \int_{B(x,T^*(x)-t)} |f(\xi,t)|^q d\xi.$$

It is enough to prove that

$$\begin{aligned} (T^*(x^*) - t)^\eta &\int_{B(x^*,T^*(x^*)-t)} |f(\xi,t)|^q d\xi \\ &\leq C(\delta_0,\eta) \sup_{|x-x_0| \leq \frac{T_0-t}{\delta_0}} (T^*(x) - t)^\eta \int_{B(x,\frac{T^*(x)-t}{2})} |f(\xi,t)|^q d\xi \end{aligned} \tag{25}$$

in order to conclude. In the following, we will prove (25).

Note that we can cover $B(x^*, T^*(x^*)-t)$ by $k(\delta_0)$ balls $B(x_i, \frac{1-\delta_0}{2}(T^*(x^*)-t))$ where $|x_i - x^*| \le T^*(x^*)-t$. Indeed, this number does not change by scaling and is thus the same as the number of balls of radius $\frac{1-\delta_0}{2}$ that cover $B(0,1)$. In addition,

$$|(T^*(x_i)-t)-(T^*(x^*)-t)| = |T^*(x_i)-T^*(x^*)| = \delta_0\,||x_i-x_0|-|x^*-x_0||$$

$$\le \quad \delta_0|x_i-x^*| \le \delta_0(T^*(x^*)-t),$$

hence

$$(1-\delta_0)(T^*(x^*)-t) \le T^*(x_i)-t \le (1+\delta_0)(T^*(x^*)-t).$$

It follows then that

$$\begin{aligned}
&(T^*(x^*)-t)^\eta \int_{B(x^*,T^*(x^*)-t)} |f(\xi,t)|^q d\xi \\
\le\ & \sum_{i=1}^{k(\delta_0)} (T^*(x^*)-t)^\eta \int_{B(x_i,\frac{1-\delta_0}{2}(T^*(x^*)-t))} |f(\xi,t)|^q d\xi \\
\le\ & \sum_{i=1}^{k(\delta_0)} \frac{1}{(1-\delta_0)^\eta}(T^*(x_i)-t)^\eta \int_{B(x_i,\frac{T^*(x_i)-t}{2})} |f(\xi,t)|^q d\xi \\
\le\ & \frac{k(\delta_0)}{(1-\delta_0)^\eta} \sup_{|x-x_0|\le\frac{T_0-t}{\delta_0}} (T^*(x)-t)^\eta \int_{B(x,\frac{T^*(x)-t}{2})} |f(\xi,t)|^q d\xi,
\end{aligned}$$

where we used in the last line the fact that $x_i \in B(x^*, T^*(x^*)-t) \subset B(x_0, \frac{T_0-t}{\delta_0})$ by (24). This yields (iii) and concludes the proof of Proposition 3.3. ■

3.3 L^r estimate of w in the whole unit ball

We now claim from Proposition 3.3:

Proposition 3.4 (L^r estimate of w in the whole unit ball) *For all $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$ and $s \ge -\log T_0 + 1$ such that $\mathcal{D}_{x_0,T_0,T_0-e^{-s},\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$, we have*

$$\int_B |w_{x_0,T_0}(y,s)|^2 dy + \int_B |w_{x_0,T_0}(y,s)|^r dy \le C(\delta_0)\Phi(T_0), \tag{26}$$

where $r = \frac{p+3}{2}$ if $N \ge 2$ and $r = p+1$ if $N = 1$ and Φ is defined in (16).

Proof: We only prove the L^r estimate since the L^2 estimate follows after by Hölder's inequality. Consider $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$ and $s \ge -\log T_0 + 1$ such that $\mathcal{D}_{x_0,T_0,T_0-e^{-s},\delta_0} \subset D_u$. From (7), estimating $w_{x_0,T_0}(s)$ is equivalent to estimating $u(t)$ where $t = T_0 - e^{-s}$. Note then that

$$t \in [T_0(1-1/e), T_0)\,. \tag{27}$$

Using Proposition 3.3 with

$$\eta = \frac{2r}{p-1} - N > 0,\ q = r \text{ and } f = u,$$

and the self-similar change of variables (7), we write

$$\begin{aligned}&\int_B \left|w_{x_0,T^*(x_0)}\left(y,-\log(T^*(x_0)-t)\right)\right|^r dy\\ \leq\ & C(\delta_0)\sup_{\{x\ |\ |x-x_0|\leq \frac{T_0-t}{\delta_0}\}}\int_{B_{1/2}}\left|w_{x,T^*(x)}\left(y,-\log(T^*(x)-t)\right)\right|^r dy \qquad (28)\end{aligned}$$

where $T^*(x)$ is defined (21).
Take x such that $|x-x_0|\leq \frac{T_0-t}{\delta_0}$. From (27), (22) and (23), we can apply Corollary 2.3 and get

$$\int_{B_{1/2}}\left|w_{x,T^*(x)}\left(y,-\log(T^*(x)-t)\right)\right|^r dy \leq C\Phi(T^*(x))\leq C(\delta_0)\Phi(T_0). \qquad (29)$$

Using (28) and (29), we get the conclusion of Proposition 3.4. ■

3.4 A local Gagliardo-Nirenberg argument

We claim the following:

Proposition 3.5 (Uniform control of the $\mathrm{H}^1(B)$ norm of $w_{x_0,T_0}(s)$)
For all $x_0\in\mathbb{R}^N$, $T_0\leq T(x_0)$ and $s\geq -\log T_0+1$ such that $\mathcal{D}_{x_0,T_0,T_0-e^{-s},\delta_0}\subset D_u$ for some $\delta_0\in(0,1)$, we have

$$\int_B |\nabla w_{x_0,T_0}(y,s)|^2 dy \leq C(\delta_0)\Phi(T_0).$$

We first introduce the following Gagliardo-Nirenberg estimate:

Lemma 3.6 (Local control of the space L^{p+1} norm by the H^1 norm) *For all $x_0\in\mathbb{R}^N$, $T_0\leq T(x_0)$ and $s\geq -\log T_0+1$ such that $\mathcal{D}_{x_0,T_0,T_0-e^{-s},\delta_0}\subset D_u$ for some $\delta_0\in(0,1)$, we have*

$$\int_B |w_{x_0,T_0}(y,s)|^{p+1}dy \leq C(\delta_0)\Phi(T_0)\left(1+\int_B|\nabla w_{x_0,T_0}(y,s)|^2dy\right)^{\beta}, \qquad (30)$$

where $\beta=\beta(p,N)\in[0,1)$.

Proof: When $N=1$, the conclusion follows from Proposition 3.4. When $N\geq 2$, we have the following Gagliardo-Nirenberg inequality:

$$\int_B |w|^{p+1}\leq \left(\int_B |w|^{\frac{p+3}{2}}\right)^{\eta}\left(\int_B|\nabla w|^2+|w|^2\right)^{\beta} \qquad (31)$$

where

$$\eta=\frac{1-(p+1)/2^*}{1-(p+3)/(2.2^*)},\quad \beta=\frac{(p-1)/4}{1-(p+3)/(2.2^*)}\ \text{and}\ \frac{1}{2^*}=\frac{N-2}{2N}.$$

Since $p<p_c$, it follows from the definition of p_c (2) that

$$\beta(p,N)<\beta(p_c,N)=\frac{(p_c-1)/4}{1-(N-2)(p_c+3)/(4N)}=\frac{1}{N-1}\frac{1}{[1-\frac{(N-2)4N}{4N(N-1)}]}=1.$$

Thus, the conclusion follows from (31) and Proposition 3.4. ■

Proof of Proposition 3.5: Fix $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$ and $s \ge -\log T_0 + 1$ such that $\mathcal{D}_{x_0,T_0,T_0-e^{-s},\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$. From (7), estimating $w_{x_0,T_0}(s)$ is equivalent to estimating $u(t)$ where $t = T_0 - e^{-s}$. Note that

$$t \in [T_0(1-1/e), T_0)\,. \tag{32}$$

Using Proposition 3.3 with $\eta = \frac{2(p+1)}{p-1} - N$ (which is positive), $q = 2$ and $f = \nabla u$, and the self-similar change of variables (7), we write on the one hand

$$\begin{aligned}
&\sup_{\{x \mid |x-x_0| \le \frac{T_0-t}{\delta_0}\}} \int_B \left|\nabla w_{x,T^*(x)}\left(y, -\log(T^*(x)-t)\right)\right|^2 dy \\
\le\quad & C(\delta_0) \sup_{\{x \mid |x-x_0| \le \frac{T_0-t}{\delta_0}\}} \int_{B_{1/2}} \left|\nabla w_{x,T^*(x)}\left(y, -\log(T^*(x)-t)\right)\right|^2 dy
\end{aligned} \tag{33}$$

where $T^*(x)$ is defined (21).
On the other hand, for any x such that $|x - x_0| \le \frac{T_0-t}{\delta_0}$, we write from the definition (11) and boundedness (15) of the Lyapunov function E, (23) and the Cauchy-Schwarz inequality:

$$\begin{aligned}
\int_{B_{1/2}} & \left|\nabla w_{x,T^*(x)}\left(y, -\log(T^*(x)-t)\right)\right|^2 dy \\
\le\quad & \int_B \left|\nabla w_{x,T^*(x)}\left(y, -\log(T^*(x)-t)\right)\right|^2 (1-|y|^2)^{\alpha+1} dy \\
\le\quad & \Phi(T^*(x)) + C \int_B |w_{x,T^*(x)}(y, -\log(T^*(x)-t))|^{p+1} dy.
\end{aligned}$$

From (32), (22) and (23), we can use the control of the L^{p+1} by the H^1 norm of Lemma 3.6 to obtain

$$\begin{aligned}
&\sup_{\{x \mid |x-x_0| \le \frac{T_0-t}{\delta_0}\}} \int_{B_{1/2}} \left|\nabla w_{x,T^*(x)}\left(y, -\log(T^*(x)-t)\right)\right|^2 dy \\
\le\quad & C(\delta_0)\Phi(T_0) \left(1 + \sup_{|x-x_0| \le \frac{T_0-t}{\delta_0}} \int_B \left|\nabla w_{x,T^*(x)}\left(y, -\log(T^*(x)-t)\right)\right|^2 dy\right)^\beta
\end{aligned} \tag{34}$$

where $\beta \in [0,1)$. From (33) and (34) and the fact that $\beta < 1$, we see that

$$\sup_{|x-x_0| \le \frac{T_0-t}{\delta_0}} \int_B \left|\nabla w_{x,T^*(x)}\left(y, -\log(T^*(x)-t)\right)\right|^2 dy \le C(\delta_0)\Phi(T_0).$$

In particular, for $x = x_0$, we have $T^*(x_0) = T_0$ and $-\log(T^*(x_0) - t) = s$, which yields the conclusion of Proposition 3.5. ■

3.5 Estimate of $\partial_s w$ in $L^2(B)$

We claim the following:

Proposition 3.7 (Uniform control of the $\mathrm{L}^2(B)$ norm of $\partial_s w_{x_0,T_0}(s)$) *For all $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$ and $s \ge -\log T_0 + 1$ such that $\mathcal{D}_{x_0,T_0,T_0-e^{-s},\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$, we have*

$$\int_B |\partial_s w_{x_0,T_0}(y,s)|^2 dy \le C(\delta_0)\Phi(T_0).$$

Proof: Let $t = T_0 - e^{-s}$. We first claim the following:

Claim 1 *For all x such that $|x-x_0| \le (T_0-t)/\delta_0$, the following holds for $\rho = \frac{1}{2}$ and $\rho = 1$:*

$$\begin{aligned}\left| \int_{B(0,\rho)} \left(\partial_s w_{x,T^*(x)}(y,s)\right)^2 dy \right. &- \left. (T^*(x)-t)^{\frac{2(p+1)}{p-1}-N} \int_{B(x,\rho(T^*(x)-t))} \left(\partial_t u(\xi,t)\right)^2 d\xi \right| \\ &\le C(\delta_0)\phi(T_0),\end{aligned}$$

where $T^*(x)$ is defined in (21).

Proof: Consider x such that $|x-x_0| \le (T_0-t)/\delta_0$. Using the self-similar transformation (7), we see that

$$(T^*(x)-t)^{\frac{p+1}{p-1}}\partial_t u(\xi,t) = -\frac{2}{p-1} w_{x,T^*(x)}(y,s) + \partial_s w_{x,T^*(x)}(y,s) + y.\nabla w_{x,T^*(x)}(y,s) \quad (35)$$

where $\xi = x + ye^{-s}$ and $t = T^*(x) - e^{-s}$. Using Lemma 3.2, we see that $\mathcal{D}_{x,T^*(x),t,\delta_0} \subset D_u$. Therefore, using (22) and (23), we can apply Propositions 3.4 and 3.5 and get

$$\int_B |\nabla w_{x,T^*(x)}(y,s)|^2 dy + \int_B |w_{x,T^*(x)}(y,s)|^2 dy \le C(\delta_0)\Phi(T_0).$$

Using (35), this concludes the proof of Claim 1. ∎

Using Proposition 3.3 with $\eta = \frac{2(p+1)}{p-1} - N$ (which is positive), $q = 2$ and $f = \partial_t u$, we write

$$\begin{aligned}&(T_0-t)^{\frac{2(p+1)}{p-1}-N} \int_{B(x_0,T_0-t)} |\partial_t u(\xi,t)|^2 d\xi \\ \le\ & C(\delta_0) \sup_{\{x \mid |x-x_0| \le \frac{T_0-t}{\delta_0}\}} (T^*(x)-t)^{\frac{2(p+1)}{p-1}-N} \int_{B(x,\frac{T^*(x)-t}{2})} |\partial_t u(\xi,t)|^2 d\xi.\end{aligned}$$

Using Claim 1, this yields

$$\int_B |\partial_s w_{x_0,T_0}(y,s)|^2 dy \le C(\delta_0)\Phi(T_0)\left(1 + \sup_{\{x \mid |x-x_0| \le \frac{T_0-t}{\delta_0}\}} \int_{B(0,\frac{1}{2})} |\partial_s w_{x,T^*(x)}(y,s)|^2 dy\right). \quad (36)$$

From the expression and the boundedness of E (see (12) and (15)), Propositions 3.4 and 3.5, and Lemma 3.6 (which we can apply thanks to Lemma 3.2), we obtain for all x such that $|x-x_0| \le (T_0-t)/\delta_0$,

$$\begin{aligned}
&\int_{B_{1/2}} |\partial_s w_{x,T^*(x)}(y,s)|^2 dy \le C\int_B |\partial_s w_{x,T^*(x)}(y,s)|^2 \rho(y) dy \\
\le\ & 2CE(w_{x,T^*(x)}(-\log T^*(x))) \\
+\ & 2C\int_B \left(-\frac{(p+1)}{(p-1)^2}|w_{x,T^*(x)}(y,s)|^2 + \frac{1}{p+1}|w_{x,T^*(x)}(y,s)|^{p+1}\right)\rho dy \qquad (37)\\
-\ & C\int_B \left(|\nabla w_{x,T^*(x)}(y,s)|^2 - (y.\nabla w_{x,T^*(x)}(y,s))^2\right)\rho dy \le C\Phi(T^*(x)) \le C(\delta_0)\Phi(T_0)
\end{aligned}$$

(here, we just use (23)).

Using (37) and (36), this concludes the proof of Proposition 3.7. ■

3.6 Conclusion of the proof of the main theorems

We conclude the proof of Theorem 1 and Theorem 2' in this subsection. Please note that Theorem 2 and Corollary 3 follow from Theorem 2' by the self-similar transformation (7).

Proof Theorem 1:
Fix some $x_0 \in \mathbb{R}^N$ and $t \in [\frac{3}{4}T(x_0), T(x_0))$. Introduce

$$T_0(x_0,t) = (T(x_0)+t)/2 \text{ and } s = -\log(T_0(x_0,t)-t).$$

Note that

$$\frac{3}{4}T(x_0) \le t \le T_0(x_0,t) \le T(x_0) \text{ and } T_0(x_0,t) - t = \frac{T(x_0)-t}{2}. \qquad (38)$$

By definition of the domain of u, we know that the backward light cone $\mathcal{C}_{x_0,T(x_0),1} \subset D_u$. We claim that we can apply Propositions 3.4, 3.5 and 3.7 to $w_{x_0,T_0(x_0,t)}$. Indeed, on the one hand, since $\mathcal{D}_{x_0,T_0(x_0,t),t,\frac{1}{2}} \subset \mathcal{C}_{x_0,T(x_0),1}$, it follows that $\mathcal{D}_{x_0,T_0(x_0,t),t,\frac{1}{2}} \subset D_u$. On the other hand, since $\frac{3}{4} \ge 1-1/e$, we use (38) to derive that $t \ge T_0(x_0,t)(1-1/e)$ and $s \ge -\log T_0(x_0,t)+1$. Thus, we can apply Propositions 3.4, 3.5, 3.7 and using (38), we get

$$\begin{aligned}
\int_B \left(|w_{x_0,T_0(x_0,t)}(y,s)|^2 + |\partial_s w_{x_0,T_0(x_0,t)}(y,s)|^2 + |\nabla w_{x_0,T_0(x_0,t)}(y,s)|^2\right) dy &\le \Phi(T_0(x_0,t)) \\
&\le \Phi(T(x_0)).
\end{aligned}$$

Going back to the $u(x,t)$ formulation (use (7), (35) and (38)), we have

$$\begin{aligned}
&(T(x_0)-t)^{\frac{2}{p-1}-N}\int_{B\left(x_0,\frac{T(x_0)-t}{2}\right)} |u(\xi,t)|^2 d\xi \\
+\ & (T(x_0)-t)^{\frac{2(p+1)}{p-1}-N}\int_{B\left(x_0,\frac{T(x_0)-t}{2}\right)} \left(|\nabla u(\xi,t)|^2 + |\partial_t u(\xi,t)|^2\right) d\xi \le \Phi(T(x_0)).
\end{aligned}$$

This yields the conclusion of Theorem 1. ■

Proof of Theorem 2':
Please note that the upper estimate in Theorem 2' could be derived directly from Theorem 1 and a simple covering argument. However, we prove the theorem directly.

Consider some non characteristic point x_0 and $s \geq -\log \frac{T(x_0)}{4}$. Let $\delta_0 = \delta_0(x_0) < 1$, the constant introduced in (6).
Since $s \geq -\log T(x_0) + 1$ and $\mathcal{D}_{x_0,T(x_0),T(x_0)-e^{-s},\delta_0} \subset \mathcal{C}_{x_0,T(x_0),\delta_0}$, we can apply Lemma 3.1 to get the lower bound, Proposition 3.4 for the L^2 norm of $w_{x_0,T(x_0)}$, Proposition 3.5 for the L^2 norm of the gradient and Proposition 3.7 for the L^2 norm of the time derivative.
This concludes the proof of Theorem 2'. ■

4 The critical case

We take $p = p_c$ defined in (2) in this section and prove Theorems 1 and 2' (note in particular that $N \geq 2$). As in the subcritical case, Theorem 2 and Corollary 3 follow from Theorem 2' by (7). The result follows in a straightforward way if one relies on these already given arguments:
- the energy-type estimates of the critical case presented in [17].
- the covering technique of the previous section.

Therefore, we just sketch the proof in the following and emphasize only one delicate averaging technique, necessary to overcome the degeneracy in the dissipation of the function E (11). Only Step 2 below is rather different from [17]. It is explained with some details in the appendix.

Step 1: Bounds on the Lyapunov functional and its dissipation
We recall the following from [17]:

Proposition 4.1 *Consider w a solution of (7) defined for all $(y,s) \in B \times [s_0, \infty)$ for some $s_0 \in \mathbb{R}$ and $E(w)$ the functional defined by (11) (note that $\rho \equiv 1$).*
i) **(E is a Lyapunov functional)**: *For all $s_2 \geq s_1 \geq s_0$,*

$$E(w(s_2)) - E(w(s_1)) = -\int_{s_1}^{s_2} \int_{\partial B} \partial_s w(\sigma, s)^2 d\sigma ds.$$

ii) **(Bounds on E and its dissipation)** *For all $s \geq s_0$, $s_2 \geq s_1 \geq s_0$,*

$$\begin{aligned} 0 \leq E(w(s)) &\leq E(w(s_0)), \\ \int_{s_1}^{s_2} \int_{\partial B} \partial_s w(\sigma, s)^2 d\sigma ds &\leq E(w(s_0)). \end{aligned} \tag{39}$$

Proof: See Lemma 2.1 and Corollary 2.3 in [17]. ■

Comparing the previous Proposition and Proposition 2.1 which is the analogous in the subcritical case, we see the main difference between the two cases: the energy dissipation (which is a dispersion estimate) degenerates to the boundary of the unit ball in the critical case. Therefore, the following step (Step 2) is specific to the critical case. We make in it averages of estimate (39) where $w = w_{x_0,T_0}$ in order to get an estimate supported in the whole unit ball.

Step 2: A bound on the time average of the L^2 norm of $\partial_s w$ in the whole unit ball

From now on, we consider u a solution of (1) with blow-up surface $\Gamma : \{t = T(x)\}$. We work in the variable $w_{x_0,T_0}(y,s)$ defined in (7). The energy dissipation of the Lyapunov functional (7) degenerates to the boundary of the unit ball in the critical case. As in [17], we use an averaging technique in order to get from Proposition 4.1 an estimate supported in the whole unit ball. In addition, we get a non concentration result.

Proposition 4.2 (An averaging technique) *For all $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$ and $s_2 \ge s_1 \ge -\log T_0 + 1$ such that $s_2 - s_1 \le 10$ and $\mathcal{D}_{x_0,T_0,T_0-e^{-s_1},\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$, it holds that*

$$\int_{s_1}^{s_2} \int_B (\partial_s w_{x_0,T_0}(z,s) - \lambda(s,s_1) w_{x_0,T_0}(z,s))^2 \, ds dz \le C(\delta_0)\Phi(T_0) \tag{40}$$

and **(non concentration property)** *for all $b \in \mathbb{R}^N$ and $r_0 \in (0,1)$ such that $B(b,r_0) \subset B(0,1/\delta_0)$,*

$$\int_{s_1}^{s_1+\sqrt{r_0}} \int_{B(b,r_0)} (\partial_s w_{x_0,T_0}(z,s) - \lambda(s,s_1) w_{x_0,T_0}(z,s))^2 \, ds dz \le C(\delta_0)\Phi(T_0) r_0 \tag{41}$$

where $0 \le \lambda(s,s_1) \le C(\delta_0)$.

Proof: One has just to adapt the proofs of Propositions 2.4 and 3.1 in [17] to the geometric context. This is only technical, however somehow delicate. We give a sketch of the proof in Appendix A for the reader's convenience. ■

Step 3: Dispersion estimates in self-similar variables

Using the first identity in Proposition 4.2, one has to do as in subsection 2.3 in [17] to obtain the following equivalent of Proposition 2.2:

Proposition 4.3 *For all $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$ and $s \ge -\log T_0 + 1$ such that $\mathcal{D}_{x_0,T_0,T_0-e^{-s},\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$, it holds that*

$$\begin{aligned} \int_s^{s+1} \int_B \left(\partial_s w_{x_0,T_0}^2 + |\nabla w_{x_0,T_0}|^2 + |w_{x_0,T_0}|^{p_c+1}\right) dy ds' &\le C(\delta_0)\Phi(T_0), \\ \int_B |w_{x_0,T_0}(y,s)|^2 dy + \int_B |w_{x_0,T_0}(y,s)|^{\frac{p_c+3}{2}} dy &\le C(\delta_0)\Phi(T_0), \end{aligned}$$

where C depends on δ_0 as well as N, and an upper bound on the norm of initial data for u in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$.

Remark: Since $\rho \equiv 1$ in the critical case (see (9)), we already have estimates with respect to the Lebesgue measure in the whole unit ball, so we don't need to truncate the domain from $B(0,1)$ to $B_{1/2}$ as we did between Proposition 2.2 and Corollary 2.3 in the subcritical case. This way, Proposition 4.3 is in the same time an equivalent of Proposition 2.2 and

Corollary 2.3.
Proof: If we introduce

$$W_{x_0,T_0}(y,s,s_1) = \exp\left(-\int_{s_1}^{s} \lambda(s',s_1)ds'\right) w_{x_0,T_0}(y,s),$$

then the dispersion estimate of Proposition 4.2 reads as follows:

$$\int_{s_1}^{s_2}\int_B \left(\partial_s W_{x_0,T_0}(z,s)\right)^2 dsdz \le C(\delta_0)\Phi(T_0),$$

which is equivalent to what we got in [17] for w_{x_0,T_0}. Thus, the interpolation estimates made for $w_{x_0,T_0}(y,s)$ in the course of the proofs or Proposition 2.5 and 2.6 in [17] hold here for $W_{x_0,T_0}(y,s,s_1)$ which yields the conclusion of Proposition 4.3.

Another way to make the adaptation of [17] to the present context is to do as for the proof of Propositions 2.5 and 2.6 in [17], with systematically putting $\partial_s w_{x_0,T_0}(z,s) - \lambda(s,s_1)w_{x_0,T_0}(z,s)$ instead of $\partial_s w_{x_0,T_0}(z,s)$. ■

Using these estimates and the non concentration property of Proposition 4.2, we obtain the following non concentration result for the $L^{\frac{p_c+3}{2}}$ norm:

Proposition 4.4 (Non concentration of the $L^{\frac{p_c+3}{2}}$ norm) *For all $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$ and $s \ge -\log T_0 + 1$ such that $\mathcal{D}_{x_0,T_0,T_0-e^{-s},\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$, for all $b \in \mathbb{R}^N$ and $r_0 \in (0,1)$ such that $B(b,r_0) \subset B(0,1/\delta_0)$, it holds that*

$$\int_{B(b,r_0)} |w_{x_0,T_0}(z,s)|^{\frac{p_c+3}{2}} dsdz \le C(\delta_0)\Phi(T_0)\sqrt{r_0}.$$

Proof: See Proposition 3.1 in [17]. ■

Step 4: Conclusion of the proof

The $L^2(B)$ estimate on w_{x_0,T_0}: It has been obtained in Proposition 4.3.

The $L^2(B)$ estimate on $\nabla w_{x_0,T_0}$: With the non concentration result of Proposition 4.4, we use the critical Gagliardo-Nirenberg estimate of [17] in balls of radius r_0 small enough and the covering argument of Proposition 3.3. Then, we prove Proposition 3.5 in the critical case (see [17] for details), which yields the L^2 norm bound on the gradient.

The $L^2(B)$ estimate on $\partial_s w_{x_0,T_0}$: The statement and the proof of Proposition 3.7 presented in the subcritical case hold here as well. This yields the bound on the L^2 norm of the time derivative.

The lower bound: The statement and the proof of Lemma 3.1 hold in the critical case as well.

In conclusion, we see that Lemma 3.1, Propositions 3.4, 3.5 and 3.7 hold in the critical case. Hence, as in the subcritical case, Theorem 1 and 2' follow. Theorem 2 and Corollary 3 follow from Theorem 2' by the self-similar transformation (7), which concludes the proof in the critical case.

Remains to give some details about Step 2 in the appendix.

A Appendix: Averaging technique for the critical case energy dispersion

We sketch the proof of Proposition 4.2 here. The proof is the same as the proof of Propositions 2.3 and 3.1 in [17], except some minor technical adaptations. In both cases, we make averages of the estimate in (39) where $w = w_{x_0,T_0}$. The main difference is geometrical:
- in [17], the domain for $u(x,t)$ is the whole strip $\mathbb{R}^N \times [0,\bar{T}]$ and T_0 is always equal to $\bar{T}$. Thus, we performed averages of (39) where $w = w_{x_0,\bar{T}}$ and x_0 lives in a sphere;
- in the context of this paper, the function $u(x,t)$ is defined in a cone and not in a strip. We perform averagess of (39) where $w = w_{x_0,T_0}$, $T_0 < T(x_0) < \bar{T}$ and x_0 lives in a sphere constrained by the fact that (x_0,T_0) is in the cone.

For the reader's convenience, we give the adaptations necessary for the proof of (40) and refer him to the proof of Proposition 3.1 in [17] for the proof of (41).
Let us first introduce for all $\sigma \in \mathbb{R}^N$, $z \in \mathbb{R}^N$ and $\delta \le 0$ the averaging kernel,

$$P(\sigma,z,\delta) = 1 - \frac{z.\sigma}{(1-\delta)|z|^2 + 1/N}.$$

Proceeding as in Appendix A in [17], we have the following:

Claim 2

$$\int_{\partial B(z(1-\delta),1)} P(\sigma,z,\delta)(\sigma+\delta z)d\sigma = 0, \tag{42}$$

$$\frac{\displaystyle\int_{\partial B(z(1-\delta),1)} P(\sigma,z,\delta)^2 d\sigma}{\left(\displaystyle\int_{\partial B(z(1-\delta),1)} P(\sigma,z,\delta)d\sigma\right)^2} = \frac{N|z|^2+1}{(1-N\delta|z|^2)|\partial B|}. \tag{43}$$

We now start the proof of (40). Consider $x_0 \in \mathbb{R}^N$, $T_0 \le T(x_0)$, $s_1 \ge -\log T_0 + 1$ and $\delta_0 \in (0,1)$ such that $\mathcal{D}_{x_0,T_0,t_1,\delta_0} \subset D_u$ for some $\delta_0 \in (0,1)$ where $t_1 = T_0 - e^{-s_1}$ and introduce

$$T_1 = T_0 - \frac{2\delta_0}{1+\delta_0}e^{-s_1} \in [t_1, T_0) \subset [T_0(1-1/e), T_0). \tag{44}$$

We only prove the result when

$$s_2 = s_1 - \log\left(\frac{1+3\delta_0}{2(1+\delta_0)}\right) \in [s_1, s_1+\log 2] \subset [s_1, -\log(T_0-T_1)). \tag{45}$$

The proof for any $s_2 \ge s_1$ follows then by a simple iteration.

Using the self-similar change of variables (7), we see that w_{x_0,T_0} is well defined in $B(0,1/\delta_0)\times[s_1,s_2]$ and that for all $(z,s) \in B(0,1/\delta_0)\times[s_1,s_2]$ and $x_1 \in \mathbb{R}^N$ such that $x_1 \in \mathbb{R}^N$ and $T_1 \le T(x_1)$,

$$\begin{aligned} e^{\frac{2s'}{p-1}} w_{x_1,T_1}(y,s') &= e^{\frac{2s}{p-1}} w_{x_0,T_0}(z,s) = u(x,t), \\ \partial_s w_{x_1,T_1}(y,s') &= (1-\delta)^{-\frac{p+1}{p-1}}\left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + (\sigma+\delta z).\nabla w_{x_0,T_0}(z,s)\right. \\ &+ \left.\partial_s w_{x_0,T_0}(z,s)\right] \end{aligned} \tag{46}$$

where $t(s)$, $s'(s)$, $\delta(s)$, $x(z,s)$, $\sigma(x,s)$ and $y(z,s,x_1)$ are uniquely determined by the relations (we omit the dependence on x_0, T_0, s_1 and δ_0 which are permanently fixed in the proof):

$$T_0 - e^{-s} = t = T_1 - e^{-s'}, \qquad \delta = (T_1 - T_0)e^{s'}, \tag{47}$$

$$x_0 + ze^{-s} = x = x_1 + ye^{-s'} \quad \text{and} \quad \sigma = (x_1 - x_0)e^{s'}. \tag{48}$$

Note from (44) and (45) that

$$\forall s \in [s_1, s_2], \quad -\frac{4\delta_0}{1-\delta_0} \le \delta \le -\frac{2\delta_0}{1-\delta_0}. \tag{49}$$

Remark: Let us just remark that in [17], we worked in the strip $\mathbb{R}^N \times [0, \bar{T})$ and we took

$$T_0 = T_1 = \bar{T}. \tag{50}$$

Our aim then was to perform a well chosen average on (46) to get rid of both the w_{x_0,T_0} and the $\nabla w_{x_0,T_0}$ terms in the right-hand side of the second line in (46) so that applying (39) to the left-hand side gives us an estimate on w_{x_0,T_0} in the whole unit ball. In the context of this paper, estimate (50) is never satisfied (see the beginning of the appendix) and when we adapt the method of [17], we only get rid of the $\nabla w_{x_0,T_0}$ term in the right-hand side of the second line in (46), hence, we get an $L^2(B \times [s_1, s_2])$ estimate on the sum $\partial_s w_{x_0,T_0}(z,s) - \lambda(s,s_1) w_{x_0,T_0}(z,s)$ and not just on $\partial_s w_{x_0,T_0}(z,s)$ as in [17].

Applying Proposition 4.1 to w_{x_1,T_1} where $x_1 \in \mathbb{R}^N$ and $T_1 \le T(x_1)$ and using (15), we see that for all $s \ge -\log T_0$ and $s_2' \ge s_1' \ge -\log T_1$,

$$\begin{aligned} 0 \le E(w_{x_1,T_1}(s)) &\le C\Phi(T_1), \\ \int_{s_1'}^{s_2'} \int_{\partial B} \partial_s w_{x_1,T_1}(\sigma,s)^2 d\sigma ds &\le C\Phi(T_1). \end{aligned} \tag{51}$$

Fix some $z \in B(0, 1/\delta_0)$ and $s \in [s_1, s_2]$. In the following, We bound $\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)$ (where $\delta = \delta(s) \le 0$ is defined in (47)) by an average on the sphere $\partial B(x, e^{-s'})$.

Lemma A.1 *For all $z \in B(0, 1/\delta_0)$ and $s \in [s_1, s_2]$,*

$$\left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right]^2 \le C(\delta_0) e^{(N-1)s'} \int_{\partial B(x,e^{-s'})} \left(\partial_s w_{x_1,T_1}(y,s')\right)^2 dx_1 \tag{52}$$

where $s'(s)$, $x(z,s)$ and $y(z,s,x_1)$ are uniquely determined form z, s and $x_1 \in \partial B(x, e^{-s'})$ by (47) and (48).

Proof: Using (42), we have,

$$\begin{aligned} &\int_{\partial B(z(1-\delta),1)} P(\sigma,z,\delta) \left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + (\sigma + \delta z) .\nabla w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right] d\sigma \\ = &\left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right] \int_{\partial B(z(1-\delta),1)} P(\sigma,z,\delta) d\sigma. \end{aligned}$$

Therefore, using the Cauchy-Schwarz inequality and (43), we write

$$
\begin{aligned}
& \left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right]^2 \\
= & \left(\int_{\partial B(z(1-\delta),1)} P(\sigma,z,\delta) d\sigma\right)^{-2} \left(\int_{\partial B(z(1-\delta),1)} P(\sigma,z,\delta)\left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s)\right.\right. \\
+ & \left.\left.(\sigma+\delta z).\nabla w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right] d\sigma\right)^2 \\
\leq & \; C \int_{\partial B(z(1-\delta),1)} \left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + (\sigma+\delta z).\nabla w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right]^2 d\sigma.
\end{aligned}
$$

If we change σ by x_1 according to (48), then we write

$$
\begin{aligned}
& \left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right]^2 \\
\leq & \; C e^{(N-1)s'} \int_{\partial B(x_0+z(1-\delta)e^{-s'}, e^{-s'})} \left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s)\right. \\
+ & \left.(\sigma+\delta z).\nabla w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right]^2 dx_1.
\end{aligned}
$$

Using (46) and (49), this concludes the proof of Lemma A.1. ■

Now, if we integrate (52) for $(z,s) \in B \times [s_1, s_2]$ and use Fubini's property, then we get

$$
\begin{aligned}
& \int_{s_1}^{s_2} \int_B \left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right]^2 ds dz \\
\leq & \; C(\delta_0) e^{Ns_2} \int_{B\left(x_0, \frac{2e^{-s_1}}{1+\delta_0}\right)} dx_1 \int_{-\log(T_1-t_1)}^{-\log(T_1-t_2)} ds \int_{\partial B} \left(\partial_s w_{x_1,T_1}(\sigma,s)\right)^2 d\sigma \qquad (53)
\end{aligned}
$$

where $t_2 = T_0 - e^{-s_2} \in [t_1, T_1)$ by (45). Note that the right-hand side of (53) is well defined. Indeed, for all $x_1 \in B\left(x_0, \frac{2e^{-s_1}}{1+\delta_0}\right)$, we have from (44), $T_1 = T_0 - \frac{2\delta_0}{1+\delta_0} e^{-s_1} \leq T_0 - \delta_0 |x_1 - x_0|$ and $T_1 \in [t_1, T_0)$. Hence, $(x_1, T_1) \in \mathcal{D}_{x_0,T_0,t_1,\delta_0} \subset D_u$. This means that u is well defined on the backward light cone with vertex (x_1, T_1), hence, (7) implies that w_{x_1,T_1} is well defined for all $(y,s) \in [-\log T_1, +\infty)$.

Using the estimate on the sphere (51), we end-up with

$$
\begin{aligned}
\int_{s_1}^{s_2} \int_B \left[\frac{2\delta}{p-1} w_{x_0,T_0}(z,s) + \partial_s w_{x_0,T_0}(z,s)\right]^2 ds dz \leq & \; C(\delta_0) e^{Ns_2} \left| B\left(x_0, \frac{2e^{-s_1}}{1+\delta_0}\right)\right| \Phi(T_1) \\
\leq & \; C(\delta_0) e^{Ns_2} C e^{-Ns_1} \Phi(T_0) \\
\leq & \; C(\delta_0) \Phi(T_0)
\end{aligned}
$$

where we used (44), (45) and (16). Taking $\lambda(s, s_1) = -\frac{2\delta(s)}{p-1}$ and using (49), this concludes the proof of (40). For the proof of (41), one has to do the same and use the ideas of the proof of Proposition 3.1 in [17]. ■

References

[1] S. Alinhac. *Blowup for nonlinear hyperbolic equations*, volume 17 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston Inc., Boston, MA, 1995.

[2] C. Antonini and F. Merle. Optimal bounds on positive blow-up solutions for a semilinear wave equation. *Internat. Math. Res. Notices*, (21):1141–1167, 2001.

[3] P. Bizón, T. Chmaj, and Z. Tabor. On blowup for semilinear wave equations with a focusing nonlinearity. *Nonlinearity*, (17):2187–2201, 2004.

[4] L. A. Caffarelli and A. Friedman. The blow-up boundary for nonlinear wave equations. *Trans. Amer. Math. Soc.*, 297(1):223–241, 1986.

[5] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.*, 38(3):297–319, 1985.

[6] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36(1):1–40, 1987.

[7] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.

[8] F. John. Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Math.*, 28(1-3):235–268, 1979.

[9] S. Kichenassamy and W. Littman. Blow-up surfaces for nonlinear wave equations. I. *Comm. Partial Differential Equations*, 18(3-4):431–452, 1993.

[10] S. Kichenassamy and W. Littman. Blow-up surfaces for nonlinear wave equations. II. *Comm. Partial Differential Equations*, 18(11):1869–1899, 1993.

[11] H. Lindblad and C. D. Sogge. On existence and scattering with minimal regularity for semilinear wave equations. *J. Funct. Anal.*, 130(2):357–426, 1995.

[12] F. Merle and P. Raphael. Blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Annals of Math.* to appear.

[13] F. Merle and P. Raphael. Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation. *Geom. Funct. Anal.*, 13(3):591–642, 2003.

[14] F. Merle and P. Raphael. On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation. *Invent. Math.*, 156(3):565–672, 2004.

[15] F. Merle and H. Zaag. A Liouville theorem for vector-valued nonlinear heat equations and applications. *Math. Annalen*, 316(1):103–137, 2000.

[16] F. Merle and H. Zaag. Determination of the blow-up rate for the semilinear wave equation. *Amer. J. Math.*, 125:1147–1164, 2003.

[17] F. Merle and H. Zaag. Determination of the blow-up rate for a critical semilinear wave equation. *Math. Annalen*, 2005. to appear.

[18] J. Shatah and M. Struwe. *Geometric wave equations*. New York University Courant Institute of Mathematical Sciences, New York, 1998.

Address:
Université de Cergy Pontoise, Département de mathématiques, 2 avenue Adolphe Chauvin, BP 222, 95302 Cergy Pontoise cedex, France.

e-mail: `merle@math.u-cergy.fr`

École Normale Supérieure, Département de mathématiques et applications, CNRS UMR 8553, 45 rue d'Ulm, 75230 Paris cedex 05, France.

e-mail: `Hatem.Zaag@ens.fr`