Existence and universality of the blow-up profile for the semilinear wave equation in one space dimension

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Abstract: In this paper, we consider the semilinear wave equation with a power nonlinearity in one space dimension. We exhibit a universal one parameter family of functions which stand for the blow-up profile in selfsimilar variables at a non characteristic point, for general initial data. The proof is done in selfsimilar variables. We first characterize all the solutions of the associated stationary problem, as a one parameter family. Then, we use energy arguments coupled with dispersive estimates to show that the solution approaches this family in the energy norm, in the non characteristic case, and to a finite decoupled sum of such a solution in the characteristic case. Finally, in the case where this sum is reduced to one element, which is the case for non characteristic points, we use modulation theory coupled with a nonlinear argument to show the exponential convergence (in the selfsimilar time variable) of the various parameters and conclude the proof. This step provides us with a result of independent interest: the trapping of the solution in selfsimilar variables near the set of stationary solutions, valid also for non characteristic points. The proof of these results is based on a new analysis in the selfsimilar variable.

AMS Classification: 35L05, 35L67

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1 Introduction

1.1 The problem and known results

We consider the following one dimensional semilinear wave equation

$$\begin{cases} \partial^2_{tt} u = \partial^2_{xx} u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{1}$$

where $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}$ with $\|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} = \sup_{a\in\mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$ and $\|v\|^2_{\mathrm{H}^1_{\mathrm{loc,u}}} = \|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\nabla v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}}$.
The Cauchy problem for equation (1) in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$ follows from the finite speed of propagation and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$. See for instance Ginibre, Soffer and Velo [7], Ginibre and Velo [8], Lindblad and Sogge [12] (for the local in time wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$). The existence of blow-up solutions for equation (1) is a consequence of the finite speed of propagation and ODE techniques (see for example Levine [11] and Antonini and Merle [4]). More blow-up results can be found in Caffarelli and Friedman [5], Alinhac [1] and [2], Kichenassamy and Littman [9], [10] and Shatah and Struwe [21]). Note that an important part of the literature on blow-up in the wave framework is devoted to quasilinear wave equations (where the nonlinearity occurs in the diffusion term). Such equations may develop "geometric" blow-up (see Alinhac [1], [2], [3]).

Most of the previous literature considered blow-up for the wave equation from the point of view of prediction. Indeed, most of the papers gave sufficient conditions to have blow-up or constructed special solutions with a prescribed behavior (see [9] and [10] for example). As we did in our earlier work [18], [17] and [19], we adopt in this paper a different point of view and aim at describing the blow-up behavior for *any* blow-up solution. More precisely, this paper is dedicated to the blow-up profile in selfsimilar variables.

If u is a blow-up solution of (1), we define (see for example Alinhac [1]) a continuous curve Γ as the graph of a function $x \to T(x)$ such that u cannot be extended beyond the set

$$D_u = \{(x,t) \mid t < T(x)\}. \tag{2}$$

The set D_u is called the maximal influence domain of u. From the finite speed of propagation, T is a 1-Lipschitz function. Let $\bar{T}$ be the infimum of $T(x)$ for all $x \in \mathbb{R}$. The time $\bar{T}$ and the surface Γ are called (respectively) the blow-up time and the blow-up surface of u.
Let us first introduce the following non degeneracy condition for Γ. If we introduce for all $x \in \mathbb{R}^N$, $t \le T(x)$ and $\delta > 0$, the cone

$$\mathcal{C}_{x,t,\delta} = \{(\xi,\tau) \neq (x,t) \mid 0 \le \tau \le t - \delta|\xi - x|\}, \tag{3}$$

then our non degeneracy condition is the following: x_0 is a non characteristic point if

$$\exists \delta_0 = \delta_0(x_0) \in (0,1) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0,T(x_0),\delta_0}. \tag{4}$$

It is an open problem to tell whether condition (4) holds for all space-time blow-up points.
Let us recall our result about the blow-up rate (valid also in higher dimensions under the condition

$$N \ge 2 \text{ and } 1 < p \le p_c \equiv 1 + \frac{4}{N-1}) : \tag{5}$$

Given some (x_0, T_0) such that $0 < T_0 \le T(x_0)$, we introduce the following self-similar change of variables:

$$w_{x_0,T_0}(y,s) = (T_0 - t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t). \tag{6}$$

If $T_0 = T(x_0)$, then we simply write w_{x_0} instead of $w_{x_0,T(x_0)}$. This change of variables transforms the backward light cone with vertex (x_0, T_0) into the infinite cylinder $(y, s) \in B \times [-\log T_0, +\infty)$ where $B = B(0, 1)$. The function w_{x_0,T_0} (we write w for simplicity) satisfies the following equation for all $y \in B$ and $s \geq -\log T_0$:

$$\partial^2_{ss} w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y\partial^2_{ys} w \tag{7}$$

$$\text{where } \mathcal{L}w = \frac{1}{\rho} \partial_y \left(\rho(1-y^2)\partial_y w\right) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \tag{8}$$

This equation will be studied in the space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \;\middle|\; \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + \left(q_1'\right)^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}, \tag{9}$$

which is the energy space for w. Note that $\mathcal{H} = \mathcal{H}_0 \times L^2_\rho$ where

$$\mathcal{H}_0 = \{ r \in H^1_{loc}(-1,1) \mid \|r\|^2_{\mathcal{H}_0} \equiv \int_{-1}^{1} \left(r'^2 (1-y^2) + r^2 \right) \rho dy < +\infty \}. \tag{10}$$

This is the blow-up bound we obtain in [17] (see also Proposition 2.2 in [18] for a statement):

Uniform bounds on solutions of (7). *If u is a solution of (1) with blow-up surface $\Gamma : \{x \to T(x)\}$ and $x_0 \in \mathbb{R}$, then for all $s \geq -\log T(x_0) + 1$,*
(E1) $E(w_{x_0}(s)) \to E_\infty \geq 0$ as $s \to \infty$.
(E2) There exists $C_0 > 0$ such that for all $s \geq s_0 + 1$, $\int_{-1}^{1} w_{x_0}(y, s)^2 \rho(y) dy \leq C_0$.
(E3) $\int_s^{+\infty} \int_{-1}^{1} \frac{\partial_s w_{x_0}(y, s')^2}{1-y^2} \rho(y) ds' dy \to 0$ as $s \to \infty$.
(E4) There exists $C_0 > 0$ such that for all $s \geq s_0 + 1$,

$$\int_s^{s+1} \int_{-1}^{1} \left\{ \partial_y w^2_{x_0} (1-y^2) + w^2_{x_0} + \partial_s w^2_{x_0} + |w_{x_0}|^{p+1} \right\} (y, s') \rho(y) dy ds' \leq C_0.$$

If in addition x_0 is non characteristic (in the sense (4)), then for all $s \geq -\log T(x_0) + 4$,

$$0 < \epsilon_0(p) \leq \|w_{x_0}(s)\|_{\mathrm{H}^1(-1,1)} + \|\partial_s w_{x_0}(s)\|_{\mathrm{L}^2(-1,1)} \leq K \tag{11}$$

where w_{x_0} is defined in (6) and K depends only on p and on an upper bound on $T(x_0)$, $1/T(x_0)$, $\delta_0(x_0)$ and the initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$.

Remark: Note that the positivity of $E(w_{x_0}(s))$ is the only delicate point in making the analysis of [17] work for characteristic points. See Appendix A.

A natural question then is to know if $w_{x_0}(y, s)$ has a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$).
In the context of Hamiltonian systems, this question is delicate, and there is no natural reason for such a convergence, since equation (1) is time reversible. See Martel and Merle [13] for the case of the L^2 critical Korteweg de Vries equation, and Merle and Raphaël [14]

for the case of the L^2 critical nonlinear Schrödinger equation.
For the case of the heat equation

$$\partial_t u = \Delta u + |u|^{p-1}u \tag{12}$$

where $u : (x,t) \in \Omega \times [0,T) \to \mathbb{R}$ and $\Omega = \mathbb{R}^N$ or Ω is a bounded domain of $\mathbb{R}^N$, $p > 1$ and $(N-2)p < N+2$, the structure in selfsimilar variables is similar to that of the wave equation (1). However, the blow-up time T is unique for equation (12). It is the time when the solution leaves the Cauchy space. What we call the blow-up set then is the set of all $x_0 \in \Omega$ such that $u(x,t)$ does not remain bounded as (x,t) approaches (x_0,T). Unlike the wave equation case, the blow-up set is a subset of $\mathbb{R}^N$ and not $\mathbb{R}^{N+1}$. As in (7), we can define a $w(y,s)$ in selfsimilar variables. We know from Giga and Kohn [6] that this $w(y,s)$ approaches a universal function (actually a constant), which turns to be the unique non zero stationary solution (up to a sign change) in the selfsimilar variable. Note that in the heat equation case, the set of stationary solutions is made of three isolated solutions.

This paper is organized around two main results. We present each of them in a subsection.

1.2 Convergence to the set of stationary solutions

We first classify all $\mathcal{H}_0$ stationary solutions of (7) in one dimension. More precisely, we prove the following proposition in Subsection 2.3:

Proposition 1 (Classification of all stationary solutions of (7) in one dimension) *(i) Consider $w \in \mathcal{H}_0$ a stationary solution of (7). Then, either $w \equiv 0$ or there exist $d \in (-1,1)$ and $\omega = \pm 1$ such that $w(y) = \omega\kappa(d,y)$ where*

$$\forall (d,y) \in (-1,1)^2, \;\; \kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \;\textit{and}\; \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}. \tag{13}$$

(ii) It holds that

$$E(0) = 0 \;\textit{and}\; \forall d \in (-1,1), \;\; E(\kappa(d,\cdot)) = E(-\kappa(d,\cdot)) = E(\kappa_0) > 0 \tag{14}$$

where

$$E(w(s)) = \int_{-1}^{1} \left(\frac{1}{2}\left(\partial_s w\right)^2 + \frac{1}{2}\left(\partial_y w\right)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1}\right) \rho dy. \tag{15}$$

Remark: Note that the set of stationary solutions consists of 3 connected components, one of them is the null singleton, and the two others are symmetric with respect to each other, and depend on one parameter. In the proof, we use the fact that $N = 1$. In higher dimensions, we are unable to classify all stationary solutions of (7) in $\mathcal{H}_0$. Of course, we already know that $\pm\kappa(d,\omega.y)$ is an $\mathcal{H}_0$ stationary solution of (7) for any $|d| < 1$ and $\omega \in \mathbb{R}^N$ with $|\omega| = 1$, but we are unable to say whether there are others or not. This missing information prevents us from extending our results to higher dimensions. Note that $H^1 \subset \mathcal{H}_0$. Thus, the result holds in H^1 as well.

Remark: The functional $E(w(s))$ defined in (15) is a Lyapunov functional for equation (7). Indeed, we know from Antonini and Merle [4] that if $w(y,s)$ is a solution to (7) defined for all $(y,s) \in \mathbb{R} \times [s_1, s_2]$, then,

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1}\int_{s_1}^{s_2}\int_{-1}^{1}(\partial_s w(y,s))^2 \frac{\rho(y)}{1-y^2} dyds. \tag{16}$$

Then, we consider $x_0 \in \mathbb{R}$ and show that $w_{x_0}(y,s)$ defined in (6) approaches a non null connected component of the stationary solutions' set in the non characteristic case, strongly in the $H^1 \times L^2(-1,1)$ norm, and in the characteristic case, a decoupled sum of stationary solutions. More precisely, we prove the following:

Theorem 2 (Strong convergence related to the set of stationary solutions) *Consider u a solution of (1) with blow-up curve $\Gamma : \{x \to T(x)\}$.*
(A) Non characteristic case: *If $x_0 \in \mathbb{R}$ is non characteristic (in the sense (4)), then, there exists $\omega^*(x_0) \in \{-1, 1\}$ such that:*
(A.i) $\inf_{|d|<1} \|w_{x_0}(\cdot, s) - \omega^*(x_0)\kappa(d,\cdot)\|_{H^1(-1,1)} + \|\partial_s w_{x_0}\|_{L^2(-1,1)} \to 0$ *as* $s \to \infty$.
(A.ii) $E(w_{x_0}(s)) \to E(\kappa_0)$ *as* $s \to \infty$.
(B) Characteristic case: *If $x_0 \in \mathbb{R}$ is characteristic, then, there exist $k(x_0) \in \mathbb{N}$, $\omega_i^* = \pm 1$ and continuous $d_i(s) = \tanh \zeta_i(s) \in (-1,1)$ for $i = 1, ..., k$ such that:*

(B.i) $$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k(x_0)} \omega_i^* \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty.$$

(B.ii) $|\zeta_i(s) - \zeta_j(s)| \to \infty$ *as* $s \to \infty$ *for* $i \neq j$.
(B.iii) $E(w_{x_0}(s)) \to k(x_0)E(\kappa_0)$ *as* $s \to \infty$.

Remark: When $k(x_0) = 0$, the sum in (B.i) has to be understood as 0.
A natural question now in the non characteristic case is to see whether $w_{x_0}(s)$ converges to some $\kappa(d_\infty(x_0))$ as $s \to \infty$ for a given $d_\infty(x_0) \in (-1,1)$ (in fact, with the method we use to answer this question, we treat also the characteristic case when $k(x_0) = 1$). This question will be addressed in the next subsection.

1.3 Trapping near the set of non zero stationary solutions

In this part, we work in the space $\mathcal{H}$ defined in (9), which is a natural choice (the energy space in w). We consider $w \in C([s^*, \infty), \mathcal{H})$ a solution to equation (7), where w may be equal to w_{x_0} defined in (6) from u, a blow-up solution to equation (1), with no restriction on x_0. In particular, x_0 may or may not be a characteristic point.
In the following, we show that if $w(s^*)$ is close enough to some non zero stationary solution and satisfies an energy barrier, then $w(s)$ converges to a neighboring stationary solution as $s \to \infty$. More precisely, we have the following:

Theorem 3 (Trapping near the set of non zero stationary solutions of (7)) *There exist positive ϵ_0, μ_0 and C_0 such that if $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution of equation (7) such that*

$$\forall s \geq s^*, \quad E(w(s)) \geq E(\kappa_0), \tag{17}$$

and

$$\left\| \left(\begin{array}{c} w(s^*) \\ \partial_s w(s^*) \end{array} \right) - \omega^* \left(\begin{array}{c} \kappa(d^*, \cdot) \\ 0 \end{array} \right) \right\|_{\mathcal{H}} \le \epsilon^* \tag{18}$$

for some $d^* \in (-1,1)$, $\omega^* = \pm 1$ and $\epsilon^* \in (0, \epsilon_0]$, where $\mathcal{H}$ and its norm are defined in (9) and $\kappa(d,y)$ in (13), then there exists $d_\infty \in (-1,1)$ such that

$$|d_\infty - d^*| \le C_0 \epsilon^* (1 - d^{*2})$$

and for all $s \ge s^*$:

$$\left\| \left(\begin{array}{c} w(s) \\ \partial_s w(s) \end{array} \right) - \omega^* \left(\begin{array}{c} \kappa(d_\infty, \cdot) \\ 0 \end{array} \right) \right\|_{\mathcal{H}} \le C_0 \epsilon^* e^{-\mu_0 (s - s^*)}. \tag{19}$$

Remark: If $w = w_{x_0}$ where x_0 is some non characteristic point of u, a blow-up solution to (1), one sees from (A.ii) of Theorem 2 and the monotonicity of the Lyapunov functional $E(w)$ that condition (17) is already satisfied and can be dropped down from the statement of Theorem 3. More generally, when x_0 is characteristic and

$$k(x_0) = 1, \tag{20}$$

we see from part B in Theorem 2 that conditions (17) and (18) hold for s_0 large. In [20], we will see from Theorem 2 that (20) cannot occur with x_0 characteristic.
Remark: The condition (17) is necessary. Indeed, if the solution converges to some $\kappa(d_\infty, \cdot)$, then we see from the monotonicity of the functional $E(w(s))$ that

$$\forall s \ge s_0, \;\; E(w(s)) \ge \lim_{s \to \infty} E(w(s)) = E(\kappa(d_\infty, \cdot)).$$

Using (14), we see that (17) follows. In particular, the following function

$$w^*(y,s) = (1 + e^s)^{-\frac{2}{p-1}} \kappa \left(d, \frac{y}{1 + e^s} \right) = \kappa_0 \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + e^s + dy)^{\frac{2}{p-1}}}$$

which is a particular solution to (7) (use (31) below) is a heteroclinic orbit connecting $\kappa(d, \cdot)$ as $s \to -\infty$ to 0 as $s \to \infty$ and satisfies $E(w^*(s)) < E(\kappa_0)$ for any $s \in \mathbb{R}$.
Remark: Note that ϵ_0 is independent of d^* in this theorem. This remarkable fact is very important in the characteristic case, as we show in a forthcoming paper [20]. One could think of using the Lorentz transform to reduce the analysis to the case $d^* = 0$, which would give a uniform ϵ_0. This doesn't work, because the Lorentz transform mixes time and space. In our proof, we work uniformly in $|d^*| < 1$ in the space $\mathcal{H}$ (9) which is well adapted to the measure of the distance between two solutions to equation (7), including in the characteristic case, and leads to exponential estimates.

Now, if $w = w_{x_0}$ where x_0 is non characteristic, then Theorems 2 and 3 apply (use (16) to derive (17) from (A.ii) in Theorem 2), and we obtain the convergence of w_{x_0} to some non zero stationary solution in the norm of $\mathcal{H}$. Using the uniform estimates (11), we directly get the following result:

Corollary 4 (Blow-up profile near a non characteristic point) *If u a solution of (1) with blow-up curve $\Gamma : \{x \to T(x)\}$ and $x_0 \in \mathbb{R}$ is non characteristic (in the sense (4)), then there exist $d_\infty(x_0) \in (-1,1)$, $|\omega^*(x_0)| = 1$ and $s^*(x_0) \ge -\log T(x_0)$ such that for all $s \ge s^*(x_0)$, (19) holds with $\epsilon^* = \epsilon_0$, where C_0 and ϵ_0 are given in Theorem 3. Moreover,*

$$\|w_{x_0}(s) - \omega^*(x_0)\kappa(d_\infty(x_0), y)\|_{H^1(-1,1)} + \|\partial_s w_{x_0}(s)\|_{L^2(-1,1)} \to 0 \text{ as } s \to \infty.$$

Remark: The sign $\omega^*(x_0)$ is given by Theorem 2. From condition (18) in Theorem 3, the time $s^*(x_0)$ is completely explicit and characterized by the fact that

$$s^*(x_0) = \inf_{s \ge -\log T(x_0)} \inf_{|d|<1} \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \omega^*(x_0) \begin{pmatrix} \kappa(d,\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \le \epsilon_0.$$

Remark: Theorem 3 and Corollary 4 are a fundamental step towards new blow-up results by the authors in a new paper [20]. We prove there that the set of non characteristic points I_0 is open and that $\forall x \in I_0$, $T'(x) = d_\infty(x)$ defined in (19). This gives a geometrical interpretation for $d_\infty(x)$ as the slope of the blow-up curve. For the moment, we are unable to prove this theorem in higher dimensions. The main difficulty comes from the fact that we are unable to classify all H^1 stationary solutions of (7) in higher dimensions, even in the radially symmetric case. Nevertheless, we hope to carry this program in higher dimensions with the same approach, avoiding the lack of information on the stationary solutions by using some extra arguments.

This paper is organized as follows. In section 2, we give some basic properties of equation (7) and prove Proposition 1 which characterizes the set of stationary solutions. In Section 3, we use energy methods to prove Theorem 2. Then, in Section 4, we study the linearized operator of equation (7) around a non zero stationary solution. That study is far from being trivial, since this linearized operator is not self-adjoint. Finally, Section 5 is devoted to the proof of Theorem 3 (note that Corollary 4 is a direct consequence of Theorems 2 and 3). The proof of Theorem 3 is the most delicate part in the proof, because of the non self-adjoint character of the linear operator, and because every non zero stationary solution of (7) is non isolated. This difficulty will be overcome by using similar concepts to those used for the Korteweg de Vries equation (Martel and Merle [13]) and the nonlinear Schrödinger equation (Merle and Raphaël [14]). See section 5 for more details.

2 Preliminaries

This section is divided in 3 subsections.
- In Subsection 2.1, we give some dispersive estimates of equation (7).
- In Subsection 2.2, we give some properties of the Lorentz transform which keeps equation (1) invariant.
- In Subsection 2.3, we prove Proposition 1 which characterizes the set of stationary solutions.

2.1 Dispersive and spectral properties for equation (7)

We first recall from [4] the following result which gives the boundedness for E and its variation:

Proposition 2.1 (Boundedness of the Lyapunov functional for equation (7)) *(i) Consider $w(y,s)$ a solution to (7) defined for all $(y,s) \in (-1,1) \times [-\log T, +\infty)$ such that $(w, \partial_s w)(-\log T) \in H^1 \times L^2(-1,1)$. For all $s \geq -\log T$, we have*

$$0 \leq E(w(s)) \leq E(w(-\log T))$$

and

$$\int_{-\log T}^{\infty} \int_{-1}^{1} (\partial_s w(y,s))^2 \frac{\rho(y)}{1-y^2} dy ds \leq \frac{p-1}{4} E(w(-\log T)).$$

Remark: Note that with this proposition, the analysis of [17] extends immediately to the case where $w = w_{x_0}$ with x_0 characteristic, and the estimates (E1)-(E4) of page 3 are fully justified.
Proof of Proposition 2.1: See Antonini and Merle [4] and Appendix A. ■

In the following, we give Hardy-Sobolev identities in the space $\mathcal{H}_0$ (10):

Lemma 2.2 (A Hardy-Sobolev type identity) *For all $h \in \mathcal{H}_0$, it holds that*

$$\left(\int_{-1}^{1} h(y)^2 \frac{\rho(y)}{1-y^2} dy \right)^{1/2} \leq C \|h\|_{\mathcal{H}_0}, \tag{21}$$

$$\|h\|_{L^{p+1}_\rho} \leq C \|h\|_{\mathcal{H}_0}, \tag{22}$$

$$\|h(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty(-1,1)} \leq C \|h\|_{\mathcal{H}_0}. \tag{23}$$

Proof of (21): Let us recall from [17] the following Hardy type inequality

$$\int_{-1}^{1} h(y)^2 y^2 \frac{\rho(y)}{1-y^2} dy \leq C \int_{-1}^{1} h(y)^2 \rho(y) + C \int_{-1}^{1} \left(h'(y)\right)^2 (1-y^2)\rho(y) = C\|h\|^2_{\mathcal{H}_0}$$

(see the appendix in [17] for a proof). Using the fact that $\frac{\rho(y)}{1-y^2} = \rho + y^2 \frac{\rho(y)}{1-y^2}$, we get (21).
Proof of (22) and (23): Let us use the following change of variables

$$\xi = \frac{1}{2} \log \left(\frac{1+y}{1-y} \right) \text{ (that is } y = \tanh \xi \text{) and } \bar{h}(\xi) = h(y)(1-y^2)^{\frac{1}{p-1}}.$$

Then,

$$\int_{-1}^{1} h(y)^{p+1} \rho(y) dy = \int_{-1}^{1} \bar{h}(\xi)^{p+1} \frac{dy}{1-y^2} = \int_{\mathbb{R}} \bar{h}(\xi)^{p+1} d\xi \leq C_0 \left(\int_{-1}^{1} \left(\bar{h}^2 + \bar{h}_\xi^2\right) d\xi \right)^{\frac{p+1}{2}},$$

$$\|h(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty(-1,1)} = \|\bar{h}\|_{L^\infty(\mathbb{R})} \leq C_0 \left(\int_{-1}^{1} \left(\bar{h}^2 + \bar{h}_\xi^2\right) d\xi \right)^{\frac{1}{2}}.$$

Note from (21) that

$$\int_{\mathbb{R}} \bar{h}(\xi)^2 d\xi = \int_{-1}^{1} h(y)^2 \rho(y) d\xi = \int_{-1}^{1} \frac{h(y)^2 \rho(y)}{1-y^2} dy \leq C_0 \|h\|^2_{\mathcal{H}_0}, \quad (24)$$

$$\int_{\mathbb{R}} \bar{h}_\xi(\xi)^2 d\xi \leq C_0 \left(\int_{-1}^{1} h_y(y)^2 (1-y^2) \rho(y) dy + \int_{-1}^{1} \frac{h(y)^2 \rho(y)}{1-y^2} dy \right) \leq C_0 \|h\|^2_{\mathcal{H}_0}, \quad (25)$$

which concludes the proof of (22) and (23) and Lemma 2.2. ■

The Legendre operator

$$\mathcal{L} w = \frac{1}{\rho} \partial_y \left(\rho (1-y^2) \partial_y w \right) \text{ where } \rho(y) = (1-y^2)^{\frac{2}{p-1}},$$

involved in the expression of equation (7) has the following properties:

Proposition 2.3 (Properties of the operator $\mathcal{L}$ (8)) *The operator $\mathcal{L}$ is self-adjoint in L^2_ρ. For each $n \in \mathbb{N}$, there exists a polynomial h_n of degree n such that*

$$\mathcal{L} h_n = \gamma_n h_n \text{ where } \gamma_n = -n \left(n + \frac{p+3}{p-1} \right). \quad (26)$$

The family $\{h_n \mid n \in \mathbb{N}\}$ is orthonormal and spans the whole space L^2_ρ. When $n = 0$ and $n = 1$, the eigenfunctions are $h_0 = c_0$ and $h_1 = c_1 y$ for some positive c_0 and c_1, and

$$\mathcal{L} c_0 = 0, \quad \mathcal{L} c_1 y = -\frac{2(p+1)}{p-1} c_1 y. \quad (27)$$

Proof: The proof is straightforward and classical. One can show that for some positive c_n, $h_n = \frac{c_n}{\rho} \frac{d^n}{dy^n} \left(\rho (1-y^2)^n \right)$. ■

We claim the following:

Lemma 2.4 *Consider $u \in L^2_\rho$ such that $\mathcal{L} u \in L^2_\rho$ and*

$$\int_{-1}^{1} u(y) \rho(y) dy = \int_{-1}^{1} u(y) y \rho(y) dy = 0. \quad (28)$$

Then, $\int_{-1}^{1} u \mathcal{L} u \rho dy \leq \gamma_2 \int u^2 \rho dy$ where $\gamma_2 = -2\frac{(3p+1)}{p-1}$.

Proof: From (28) and (27), we have

$$\tilde{u}_0 = \tilde{u}_1 = 0 \text{ where } \tilde{u}_n = \int_{-1}^{1} u h_n \rho dy. \quad (29)$$

Therefore, using (26), we write $u = \sum_{n=2}^{\infty} \tilde{u}_n h_n$ and $\mathcal{L} u = \sum_{n=2}^{\infty} \gamma_n \tilde{u}_n h_n$. Using the orthogonality of the polynomials h_k and the fact that $\gamma_n \leq \gamma_2$ for all $n \geq 2$, we write

$$\int_{-1}^{1} u \mathcal{L} u \rho dy = \sum_{n=2}^{\infty} \gamma_n \tilde{u}_n^2 \leq \gamma_2 \sum_{n=2}^{\infty} \tilde{u}_n^2 = \gamma_2 \int_{-1}^{1} u^2 \rho dy.$$

This concludes the proof of Lemma 2.4. ■

2.2 Invariance of equation (7)

In this section, we consider $u(x,t)$ a solution of (1) defined in the cone

$$\{(\xi,\tau)\ \mid\ t_1 \le \tau < t_0 - |\xi - x_0|\} \tag{30}$$

for some $t_1 < t_0$ and $x_0 \in \mathbb{R}$. Using the transformation (7), we see that $w = w_{x_0,t_0}$ is a solution of (7) defined for all $|y| < 1$ and $s \in [-\log(t_0 - t_1), +\infty)$. Equation (1) is invariant under translations in time and space, scaling and the Lorentz transformation. Through the selfsimilar transformation (7), this provides us with 4 invariant transformations for equation (7). More precisely, the following transformations of $w(y,s)$ are also solutions to (7):

- For any $a \in \mathbb{R}$, the function $w_1(y,s)$ defined for all $s \in [-\log(t_0 - t_1), +\infty)$ and $y \in (-ae^s - 1, -ae^s + 1)$ by

$$w_1(y,s) = w(y + ae^s, s).$$

- For any $b \le t_0 - t_1$, the function $w_2(y,s)$ defined for all $s \ge -\log(t_0 - t_1 - b)$ and $|y| < 1 + be^s$ by

$$w_2(y,s) = (1+be^s)^{-\frac{2}{p-1}} w\left(\frac{y}{1+be^s}, s - \log(1+be^s)\right). \tag{31}$$

- For any $c \in \mathbb{R}$, the function $w_3(y,s)$ defined for all $|y| < 1$ and $s \in [-\log(t_0 - t_1) - c, +\infty)$ by

$$w_3(y,s) = w(y, s + c).$$

- The transposition in selfsimilar variables of the Lorentz transform which will be given in this section.

Let us recall the invariance of equation (1) under the Lorentz transform:

Lemma 2.5 (Invariance of equation (1) under the Lorentz transform)
(i) Consider $u(x,t)$ a solution of equation (1) defined in the cone (30). For any $d \in (-1,1)$, the function $U \equiv Z_d(u)$ defined by

$$U(x',t') = u(x,t) \text{ where } x' = \frac{x + dt}{\sqrt{1-d^2}} \text{ and } t' = \frac{t + dx}{\sqrt{1-d^2}}$$

is also a solution of (1) defined in the set

$$\{(x',t') \mid t_1\sqrt{1-d^2} + dx' \le t' < t_0' - |x' - x_0'|\} \text{ where } x_0' = \frac{x_0 + dt_0}{\sqrt{1-d^2}} \text{ and } t_0' = \frac{t_0 + dx_0}{\sqrt{1-d^2}}.$$

(ii) For all d_1 and d_2 in $(-1,1)$, we have $Z_{d_1} \circ Z_{d_2} = Z_{d_1 * d_2}$ where

$$d_1 * d_2 = \frac{d_1 + d_2}{1 + d_1 d_2}, \tag{32}$$

Remark: From (ii) of this proposition, we deduce that $Z_d \circ Z_{-d} = Z_0 = \mathrm{Id}$ for all $d \in (-1,1)$.

Proof: Everything is straightforward, except may be for the composition identity. Consider then $d_1, d_2 \in (-1,1)$ and define

$$U = Z_{d_1} u \text{ by } U(x',t') = u(x,t) \text{ where } x' = \frac{x + d_1 t}{\sqrt{1-d_1^2}} \text{ and } t' = \frac{t + d_1 x}{\sqrt{1-d_1^2}},$$

$$\text{and} \qquad \mathcal{U} = Z_{d_2} U \text{ by } \mathcal{U}(x'',t'') = U(x',t') \text{ where } x'' = \frac{x' + d_2 t'}{\sqrt{1-d_2^2}} \text{ and } t'' = \frac{t' + d_2 x'}{\sqrt{1-d_2^2}}.$$

Then,

$$x'' = \frac{x' + d_2 t'}{\sqrt{1-d_2^2}} = \frac{x + d_1 t + d_2(t + d_1 x)}{\sqrt{(1-d_2^2)(1-d_1^2)}} = \frac{x + t\frac{d_1+d_2}{1+d_1d_2}}{\sqrt{\frac{(1-d_2^2)(1-d_1^2)}{(1+d_2d_1)^2}}} = \frac{x + t(d_1 * d_2)}{\sqrt{1-(d_1 * d_2)^2}}$$

since $\frac{(1-d_2^2)(1-d_1^2)}{(1+d_2d_1)^2} = 1 - \left(\frac{d_1+d_2}{1+d_2d_1}\right)^2$.
Similarly, we have $t'' = (t + x(d_1 * d_2)) / \sqrt{1-(d_1 * d_2)^2}$. Since $\mathcal{U}(x'',t'') = U(x',t') = u(x,t)$, this implies that $Z_{d_1} \circ Z_{d_2} = Z_{d_1 * d_2}$. ■
Through the selfsimilar transformation (6), the Lorentz transform provides a one dimensional group which keeps invariant equation (7). More precisely,

Lemma 2.6 (The Lorentz transform in similarity variables) *Consider $w(y,s)$ a solution of equation (1) defined for all $|y| < 1$ and $s \in (s_0, s_1)$ for some s_0 and s_1 in $\mathbb{R}$, and introduce for any $d \in (-1,1)$, the function $W \equiv \mathcal{T}_d(w)$ defined by*

$$W(Y,S) = \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dY)^{\frac{2}{p-1}}} w(y,s) \text{ where } y = \frac{Y+d}{1+dY} \text{ and } s = S - \log \frac{1+dY}{\sqrt{1-d^2}}. \tag{33}$$

Then $W(Y,S) = \mathcal{T}_d(w)$ is also a solution of (7) defined for all $|Y| < 1$ and $S \in \left(s_0 + \frac{1}{2}\log\frac{1+|d|}{1-|d|}, s_1 - \frac{1}{2}\log\frac{1+|d|}{1-|d|}\right)$.

Remark: From (ii) in Lemma 2.5, we have $\mathcal{T}_{d_1} \circ \mathcal{T}_{d_2} = \mathcal{T}_{d_1 * d_2}$ and $\mathcal{T}_d \circ \mathcal{T}_{-d} = \mathcal{T}_0 = \mathrm{Id}$ where the law $*$ is defined in (32).
Remark: If $w(y)$ is a stationary solution of (7), then the function $W(Y) = \mathcal{T}_d(w)$ depends only on Y and is also a stationary solution of (7).
Proof: Note that the domain of definition of $W(Y,S)$ follows directly from (33). Remains to check that it is a solution to (7).
Let us define $\tilde{W}(\tilde{Y},\tilde{S})$ by

$$\tilde{W}(\tilde{Y},\tilde{S}) = (t_0 - t')^{\frac{2}{p-1}} U(x',t'), \quad \tilde{Y} = \frac{x'-x_0}{t_0-t'} \text{ and } \tilde{S} = -\log(t_0 - t'), \tag{34}$$

$$\text{where} \qquad x_0 = \frac{d}{\sqrt{1-d^2}}, \quad t_0 = \frac{1}{\sqrt{1-d^2}}, \tag{35}$$

$$U(x',t') = u(x,t), \quad x' = \frac{x+dt}{\sqrt{1-d^2}}, \quad t' = \frac{t+dx}{\sqrt{1-d^2}}, \tag{36}$$

$$u(x,t) = (1-t)^{-\frac{2}{p-1}} w(y,s), \quad y = \frac{x}{1-t} \text{ and } s = -\log(1-t). \tag{37}$$

Using the selfsimilar transformation (6), the Lorentz transform (36) and then again (6), we see that u and U are solutions to (1), and then $\tilde{W}(\tilde{Y}, \tilde{S})$ is a solution to (7). In the following, we will prove that $\tilde{W} = W$, $\tilde{Y} = Y$ and $\tilde{S} = S$, which will conclude the proof. Using (37) and (34), we write

$$x = ye^{-s}, \quad t = 1 - e^{-s}, \quad x' = x_0 + \tilde{Y}e^{-\tilde{S}}, \quad t' = t_0 - e^{-\tilde{S}},$$
$$\tilde{W}(\tilde{Y}, \tilde{S}) = e^{-\frac{2\tilde{S}}{p-1}} U(x', t') \text{ and } w(y, s) = e^{-\frac{2s}{p-1}} u(x, t).$$

Using the Lorentz transform (36), we write

$$\tilde{W}(\tilde{Y}, \tilde{S}) = e^{2\frac{s-\tilde{S}}{p-1}} w(y, s), \quad \tilde{Y}e^{-\tilde{S}} + x_0 = \frac{ye^{-s} + d(1 - e^{-s})}{\sqrt{1 - d^2}}, \quad t_0 - e^{-\tilde{S}} = \frac{1 - e^{-s} + dye^{-s}}{\sqrt{1 - d^2}}. \tag{38}$$

Using (35), this gives

$$\tilde{S} = s - \log \frac{1 - dy}{\sqrt{1 - d^2}}, \quad \tilde{Y} = \frac{y - d}{1 - dy} \text{ and } \tilde{W}(\tilde{Y}, \tilde{S}) = \frac{(1 - dy)^{\frac{2}{p-1}}}{(1 - d^2)^{\frac{1}{p-1}}} w(y, s). \tag{39}$$

Therefore,

$$(1 - dy)(1 + d\tilde{Y}) = 1 - d^2, \quad y = \frac{\tilde{Y} + d}{1 + d\tilde{Y}} \text{ and } \frac{1 - dy}{\sqrt{1 - d^2}} = \frac{\sqrt{1 - d^2}}{1 + d\tilde{Y}}.$$

Thus, using (33) and (39), we see that $\tilde{W} = W$, $\tilde{Y} = Y$ and $\tilde{S} = S$. Since $\tilde{W}(\tilde{Y}, \tilde{S})$ is a solution to (7), the same holds for $W(Y, S)$. This concludes the proof of Lemma 2.6. ■

For further purpose, we need to understand precisely the effect of the transformation $\mathcal{T}_d$ defined in (33) on the operator $\mathcal{L}w$ which appears in (7) (regardless of the fact that w is a solution of (7) or not). In (i) of the following Lemma, we transform all the terms (linear and nonlinear) of equation (7). In (ii), we show that in fact, the linearized operator of equation (7) around the constant solution κ_0 (13) transforms into the linearized operator of the same equation around $\kappa(d, y)$, the transformation of κ_0 by the Lorentz transformation in similarity variables. More precisely, we claim the following:

Lemma 2.7 (Transformations of the linearized operator of (7) around κ_0) *Consider a general $w(y, s)$ not necessarily a solution to (7) and $W = \mathcal{T}_d w$ defined in (33). Then, it holds that:*

(i) **(Nonlinear version)**

$$\begin{aligned} &\partial^2_{ss} w - \left(\mathcal{L}w - \tfrac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \tfrac{p+3}{p-1} \partial_s w - 2y \partial^2_{y,s} w\right) \\ = \frac{(1 + dY)^{\frac{2p}{p-1}}}{(1 - d^2)^{\frac{p}{p-1}}} &\left[\partial^2_{SS} W - \left(\mathcal{L}W - \tfrac{2(p+1)}{(p-1)^2} W + |W|^{p-1} W - \tfrac{p+3}{p-1} \partial_S W - 2Y \partial^2_{Y,S} W\right)\right]. \end{aligned} \tag{40}$$

(ii) **(The linearized operator around κ_0)**

$$\begin{aligned} &\partial^2_{ss} w - \left(\mathcal{L}w + \frac{2(p+1)}{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial^2_{y,s} w\right) \\ = \frac{(1 + dY)^{\frac{2p}{p-1}}}{(1 - d^2)^{\frac{p}{p-1}}} &\left(\partial^2_{SS} W - \left(\mathcal{L}W + \psi(d, Y) W - \frac{p+3}{p-1} \partial_S W - 2Y \partial^2_{Y,S} W\right)\right) \end{aligned}$$

$$\textit{where } \psi(d,Y) = p\kappa(d,Y)^{p-1} - \frac{2(p+1)}{(p-1)^2} = \frac{2(p+1)}{(p-1)^2}\left(p\frac{(1-d^2)}{(1+dY)^2} - 1\right). \quad (41)$$

Remark: If we consider $w(y,s) = w(y)$, then it holds for $W = \mathcal{T}_d w$ that

$$\mathcal{L}w(y) + \frac{2(p+1)}{p-1}w(y) = \frac{(1+dY)^{\frac{2p}{p-1}}}{(1-d^2)^{\frac{p}{p-1}}}\left(\mathcal{L}W(Y) + \psi(d,Y)W(Y)\right) \quad (42)$$

where $W \equiv \mathcal{T}_d w$ is given in (33).

Proof of Lemma 2.7:
(i) Using (37), (36) and (34), we write

$$\begin{aligned}
&\partial^2_{ss}w - \left(\mathcal{L}w - \tfrac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w - \tfrac{p+3}{p-1}\partial_s w - 2y\partial^2_{y,s}w\right)\\
= (1-t)^{\frac{2p}{p-1}} \quad &\left(\partial^2_{tt}u - \partial^2_{xx}u - |u|^{p-1}u\right)\\
= (1-t)^{\frac{2p}{p-1}} \quad &\left(\partial^2_{t't'}U - \partial^2_{x'x'}U - |U|^{p-1}U\right)\\
= \left(\tfrac{1-t}{t_0-t'}\right)^{\frac{2p}{p-1}} \quad &\left[\partial^2_{SS}W - \left(\mathcal{L}W - \tfrac{2(p+1)}{(p-1)^2}W + |W|^{p-1}W - \tfrac{p+3}{p-1}\partial_S W - 2Y\partial^2_{Y,S}W\right)\right].
\end{aligned} \quad (43)$$

Using (36), we see that $t = (t' - dx')/\sqrt{1-d^2}$. Therefore, using (37) and (35), we write

$$\frac{1-t}{t_0-t'} = \frac{1 - \frac{t'-dx'}{\sqrt{1-d^2}}}{t_0 - t'} = \frac{t_0 - t' + d(x'-x_0)}{(t_0-t')\sqrt{1-d^2}} = \frac{1+dY}{\sqrt{1-d^2}}.$$

Using (43), this concludes the proof of (i) of Lemma 2.7.
(ii) Using (33), we write

$$p\frac{2(p+1)}{(p-1)^2}w - |w|^{p-1}w = \frac{(1+dY)^{\frac{2p}{p-1}}}{(1-d^2)^{\frac{p}{p-1}}}\left(p\frac{2(p+1)}{(p-1)^2}W\frac{(1-d^2)}{(1+dY)^2} - |W|^{p-1}W\right),$$

which shows the same factor as in (40). Subtracting this from (40), we get the conclusion of Lemma 2.7. ■

In the following, we show that the transformation defined in (33) is continuous from $\mathcal{H}_0$ to $\mathcal{H}_0$ defined in (10).

Lemma 2.8 (Continuity of $\mathcal{T}_d$ in $\mathcal{H}_0$) *There exists $C_0 > 0$ such that for all $d \in (-1,1)$ and $w \in H_0$, we have*

$$\frac{1}{C_0}\|w\|_{\mathcal{H}_0} \le \|T_d(w)\|_{\mathcal{H}_0} \le C_0\|w\|_{\mathcal{H}_0}. \quad (44)$$

Proof: We only prove the second inequality of (44), since the first one follows by applying the second one to $T_{-d}(w)$ and using the fact that $\mathcal{T}_d \circ \mathcal{T}_{-d} = \mathrm{Id}$ (see the remark following Lemma 2.6).
If we consider $W = \mathcal{T}_d w$ defined in (33), then we see that

$$\partial_Y W(Y) = -\frac{2d}{p-1}\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dY)^{\frac{2}{p-1}+1}}w(y) + \frac{(1-d^2)^{\frac{1}{p-1}+1}}{(1+dY)^{\frac{2}{p-1}+2}}\partial_y w(y) \text{ where } y = \frac{Y+d}{1+dY}.$$

Using (10) and (33), we write

$$\begin{aligned}
\|W\|_{\mathcal{H}_0}^2 &\le C\int_{-1}^{1}\frac{(1-d^2)^{\frac{2}{p-1}}}{(1+dY)^{\frac{4}{p-1}}}w\left(\frac{Y+d}{1+dY}\right)^2(1-Y^2)^{\frac{2}{p-1}}dY,\\
&+ C\int_{-1}^{1}\frac{(1-d^2)^{\frac{2}{p-1}}}{(1+dY)^{\frac{4}{p-1}+2}}w\left(\frac{Y+d}{1+dY}\right)^2(1-Y^2)^{\frac{2}{p-1}+1}dY,\\
&+ C\int_{-1}^{1}\frac{(1-d^2)^{\frac{2}{p-1}+2}}{(1+dY)^{\frac{4}{p-1}+4}}\left(\partial_y w\left(\frac{Y+d}{1+dY}\right)\right)^2(1-Y^2)^{\frac{2}{p-1}+1}dY.
\end{aligned}$$

Performing the change of variables $y=\frac{Y+d}{1+dY}$, we get

$$\begin{aligned}
\|W\|_{\mathcal{H}_0}^2 &\le C\int_{-1}^{1}(1-y^2)^{\frac{2}{p-1}}w(y)^2\frac{1-d^2}{(1-dy)^2}dy + C\int_{-1}^{1}(1-y^2)^{\frac{2}{p-1}+1}w(y)^2\frac{1}{(1-dy)^2}dy,\\
&+ C\int_{-1}^{1}(1-y^2)^{\frac{2}{p-1}+1}\left(\partial_y w(y)\right)^2dy. \qquad (45)
\end{aligned}$$

Using the fact that

$$\forall (d,y)\in(-1,1)^2,\ \ |y+d|+|1-d^2|+(1-y^2)\le C(1+dy), \qquad (46)$$

and (21), we see that

$$\|W\|_{\mathcal{H}_0}^2 \le \int_{-1}^{1}(1-y^2)^{\frac{2}{p-1}-1}w(y)^2dy + C\|w\|_{\mathcal{H}_0}^2 \le C\|w\|_{\mathcal{H}_0}^2$$

and the conclusion follows. ■

2.3 Characterization of the stationary solutions in self-similar variables

In this section, we prove Proposition 1 which characterizes all $\mathcal{H}_0$ solutions of

$$\frac{1}{\rho}\left(\rho(1-y^2)w'\right)' - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w = 0, \qquad (47)$$

the stationary version of (7). Note that since 0 and $\pm\kappa_0$ are trivial solutions to equation (7), we see from Lemma 2.6 that $\pm\mathcal{T}_d\kappa_0 = \pm\kappa(d,y)$ are also stationary solutions to (7). Let us introduce the set

$$S \equiv \{0,\ \kappa(d,\cdot),\ -\kappa(d,\cdot)\ \mid\ |d|<1\}. \qquad (48)$$

Now, we prove Proposition 1 which states that there are no more solutions of (47) in $\mathcal{H}_0$ other than the set S. We first prove (ii) since it is shorter and then prove (i).

(ii) Since we clearly have from the definition of $E(\kappa(d,\cdot))$ (15), $E(0)=0$, we only compute $E(\pm\kappa(d,\cdot))$. Since $\kappa(d,y)$ is a solution to equation (47), we multiply the equation by $\kappa(d,y)\rho(y)$ and integrate it with respect to $y\in(-1,1)$ to get

$$-\int_{-1}^{1}|\partial_y\kappa(d,y)|^2(1-y^2)\rho(y) - \frac{2(p+1)}{(p-1)^2}\int_{-1}^{1}\kappa(d,y)^2\rho(y)dy + \int_{-1}^{1}\kappa(d,y)^{p+1}\rho(y)dy = 0.$$

Therefore, we see from (15) that $E(\kappa(d,\cdot)) = \frac{p-1}{2(p+1)}\int_{-1}^1 \kappa(d,y)^{p+1}\rho(y)dy$. Making the change of variables $Y = \frac{y+d}{1+dy}$, we see that

$$\begin{aligned} E(\kappa(d,\cdot)) = \tfrac{p-1}{2(p+1)}\int_{-1}^1 \kappa(d,y)^{p+1}\rho(y)dy = \frac{p-1}{2(p+1)}\kappa_0^{p+1}\int_{-1}^1 \rho(Y)dY &= E(\kappa_0) > 0, \\ \tfrac{1}{2}\int_{-1}^1 |\partial_y\kappa(d,y)|^2(1-y^2)\rho(y) + \frac{(p+1)}{(p-1)^2}\int_{-1}^1 \kappa(d,y)^2\rho(y)dy &= \tfrac{p+1}{p-1}E(\kappa_0). \end{aligned} \tag{49}$$

Thus, (14) follows.

(i) Consider $w \in \mathcal{H}_0$ a non zero solution of (47). Let us prove that there is some $d \in (-1,1)$ such that $w = \pm\kappa(d,\cdot)$. For this purpose, consider

$$\xi = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right) \text{ (that is } y = \tanh\xi) \text{ and } \bar{w}(\xi) = w(y)(1-y^2)^{\frac{1}{p-1}}. \tag{50}$$

Remark first from (24) and (25) that $\bar{w} \in H^1(\mathbb{R})$. Let us prove that if $w \not\equiv 0$ is solution to (47) is equivalent to $\bar{w} \not\equiv 0$ is a solution to

$$\bar{w}_{\xi\xi} + |\bar{w}|^{p-1}\bar{w} - \frac{4}{(p-1)^2}\bar{w} = 0. \tag{51}$$

Indeed, we have

$$\begin{aligned} \bar{w}_\xi &= -\frac{2y}{p-1}(1-y^2)^{\frac{1}{p-1}}w + w_y(1-y^2)^{\frac{1}{p-1}+1}, \\ \bar{w}_{\xi\xi} &= \left[-\frac{2}{p-1}y(1-y^2)^{\frac{1}{p-1}}\right]_y (1-y^2)w - \frac{2}{p-1}y(1-y^2)^{\frac{1}{p-1}+1}w_y \\ &-\frac{2yp}{p-1}(1-y^2)^{\frac{1}{p-1}+1}w_y + w_{yy}(1-y^2)^{\frac{1}{p-1}}(1-y^2)^2 \\ &= \left(\left[-\frac{2(1-y^2)}{p-1} + \frac{4y^2}{(p-1)^2}\right]w - \frac{2(p+1)}{p-1}yw_y(1-y^2) + w_{yy}(1-y^2)^2\right)(1-y^2)^{\frac{1}{p-1}}. \end{aligned}$$

Thus,

$$\begin{aligned} &\bar{w}_{\xi\xi} - \frac{4}{(p-1)^2}\bar{w} + |\bar{w}|^{p-1}\bar{w} \\ = \;&(1-y^2)^{1+\frac{1}{p-1}}\left[-2\frac{(p+1)}{(p-1)^2}w - \frac{2(p+1)}{p-1}yw_y + w_{yy}(1-y^2) + |w|^{p-1}w\right] \end{aligned}$$

which proves the equivalence.
It is classical that all non zero solutions of (51) in $H^1(\mathbb{R})$ are

$$\bar{w}(\xi) = \pm\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi+\xi_0)} \text{ for } \xi_0 \in \mathbb{R}. \tag{52}$$

Thus, for $d = \tanh \xi_0 \in (-1,1)$ and $y = \tanh \xi$, we write.

$$\bar{w}(\xi) = \pm\kappa_0 \left[1 - \tanh(\xi+\xi_0)^2\right]^{\frac{1}{p-1}} = \pm\kappa_0 \left[1 - \left(\frac{\tanh\xi + \tanh\xi_0}{1+\tanh\xi\tanh\xi_0}\right)^2\right]^{\frac{1}{p-1}}$$

$$= \pm\kappa_0 \left[1 - \left(\frac{y+d}{1+dy}\right)^2\right]^{\frac{1}{p-1}} = \pm\kappa_0 \left[\frac{(1-d^2)(1-y^2)}{(1+dy)^2}\right]^{\frac{1}{p-1}} = \pm\kappa(d,y)(1-y^2)^{\frac{1}{p-1}} \quad (53)$$

This means by (50) that $w(y) = \pm\kappa(d,y)$, which concludes the proof of Proposition 1. ■

3 Energy estimates and convergence to the set of stationary solutions

This section is devoted to the proof of Theorem 2. In a pedagogical approach, we treat the non characteristic case first, and then the general case. Indeed, in this first case, we will replace the use of an averaging property of the equation (useful in the general case) by the use of the finite speed of propagation.

3.1 The non characteristic case

We prove part A of Theorem 2 in this section. Note first that using the continuity of the Lyapunov functional $E(w)$ (15) in the space $H^1 \times L^2(-1,1)$ and (14), (A.ii) directly follows from (A.i). Thus, we only prove (A.i). Consider $x_0 \in \mathbb{R}$ a non characteristic point and introduce

$$w = w_{x_0} = w_{x_0,T(x_0)}.$$

From (11) (proved in [18]), the Sobolev injection and Proposition 2.1, we have the following bounds:

Lemma 3.1 (Boundedness of $w(s)$ [18]) *There exists $K > 0$ such that for all $s \geq -\log \frac{T(x_0)}{4}$,*

$$0 < \epsilon_0(p) \leq \|w(s)\|_{\mathrm{H}^1(-1,1)} + \|\partial_s w(s)\|_{\mathrm{L}^2(-1,1)} \leq K, \quad (54)$$

$$\|w(s)\|_{L^\infty(-1,1)} \leq K, \quad (55)$$

and

$$\int_{-\log T(x_0)}^{\infty} \int_{-1}^{1} (\partial_s w(y,s))^2 \frac{\rho(y)}{1-y^2} dy ds \leq K. \quad (56)$$

We will show that there exists $\omega(x_0) \in \{-1,1\}$ such that

$$\inf_{|d|<1} \|w(\cdot,s) - \omega(x_0)\kappa(d,\cdot)\|_{H^1(-1,1)} + \|\partial_s w\|_{L^2(-1,1)} \to 0 \text{ as } s \to \infty. \quad (57)$$

It is a remarkable fact for a dispersive equation that a solution converges *strongly* to a stationary solution (as in the case of a dissipative equation). We first have the following reduction:

Proposition 3.2 *In order to prove (57), it is enough to prove that*

$$\inf_{\tilde{w}\in S} \|w(s) - \tilde{w}\|_{H^1(-1,1)} + \|\partial_s w\|_{L^2(-1,1)} \to 0 \text{ as } s \to \infty, \tag{58}$$

where S (48) is the set of all $\mathcal{H}_0$ stationary solutions to (7).

Proof: From Proposition 1 and (48), we know that $S = S_1 \cup S_2 \cup \{0\}$ where $S_1 = \{\kappa(d,\cdot) \mid |d| < 1\}$ and $S_2 = \{-\kappa(d,\cdot) \mid |d| < 1\}$. From the Sobolev injection, positivity and (13), we have

$$\begin{aligned} \text{for } i = 1,\ 2,\ \ d_{H^1(-1,1)}(S_i, 0) \geq C d_{L^\infty(-1,1)}(S_1, 0) &\geq C \inf_{|d|<1} \|\kappa(d,\cdot)\|_{L^\infty(-1,1)} \geq C_0 > 0. \\ d_{H^1(-1,1)}(S_1, S_2) \geq C d_{L^\infty(-1,1)}(S_1, S_2) &\geq C d_{L^\infty(-1,1)}(S_1, 0) \geq C_0 > 0. \end{aligned}$$

Since $(w(s), \partial_s w(s))$ is continuous as a function of s in $H^1 \times L^2(-1,1)$ and its norm is bounded from below by (54), we see that (58) implies (57). This concludes the proof of Proposition 3.2. ■

We now prove (58), which by Proposition 3.2, will conclude the proof of (57) and Part A of Theorem 2. We proceed by contradiction and assume that there exist $\epsilon_0 > 0$ and a sequence $s_n \to \infty$ such that

$$\inf_{\tilde{w}\in S} \|w(s_n) - \tilde{w}\|_{H^1(-1,1)} + \|\partial_s w(s_n)\|_{L^2(-1,1)} \geq \epsilon_0 > 0. \tag{59}$$

We proceed in 2 steps:

- In Step 1, we show that $w(s_n)$ converges in $L^\infty(-1,1)$ to some $w^* \in S$. This step will be a consequence of the existence of the Lyapunov functional E (15) and compactness related to the uniform bounds we have in (54).

- In Step 2, using the space-time localization of the original energy for the function $u(t)$, we find an estimate on $w(s_n)$ which contradicts (59). This step is remarkable, in the setting of Hamiltonian systems (for example, this fact is false for L^2 critical NLS and L^2 critical KdV; see [14] and [13]).

Step 1: Convergence of $w(s_n)$ to a stationary solution in $L^\infty(-1,1)$
From (54), we there is a subsequence (still denoted by s_n) and $w^* \in H^1(-1,1)$ such that

$$\|w(s_n) - w^*\|_{L^\infty(-1,1)} \to 0 \text{ as } n \to \infty.$$

We have the following:

Lemma 3.3
(i) For any $M > 0$, we have

$$w(y, s_n + s) - w^*(y) \to 0 \text{ as } n \to \infty, \text{ uniformly for } |y| < 1 \text{ and } |s| < M.$$

(ii) We have $w^* \in S$.

Proof:

(i) From (54) and (56), we have for all $M>0$,

$$\begin{aligned}
&\int_{|y|<1-\frac{1}{M}} |w(y,s_n+s)-w^*(y)|^2 dy\\
\le\ & \int_{|y|<1-\frac{1}{M}} |w(y,s_n)-w^*(y)|^2 dy + C_0\int_{s_n-M}^{s_n+M}\left(\int_{|y|<1-\frac{1}{M}} (\partial_s w(s_n+s',y))^2 dy\right)^{1/2} ds'\\
\le\ & \int_{|y|<1-\frac{1}{M}} |w(y,s_n)-w^*(y)|^2 dy + C(M)\left(\int_{s_n-M}^{s_n+M}\int_{-1}^{1} (\partial_s w(s_n+s',y))^2 \rho dy\right)^{1/2} \to 0
\end{aligned}$$

as $n\to\infty$. From the fact that $\|v\|^2_{L^\infty(|y|<1-\frac{1}{M})} \le C(M)\|v\|_{L^2(|y|<1-\frac{1}{2M})}\|\bar{w}\|_{H^1(|y|<1-\frac{1}{2M})}$, we see that $w(y,s_n+s)-w^*(y)\to 0$ as $n\to\infty$, uniformly for $|y|<1-\frac{1}{M}$ and $|s|<M$. Since from (54), we have $\|w(y,s_n+s)-w^*(y)\|_{C^{\frac{1}{2}}(-1,1)}\le C_0$, (i) follows.

(ii) Here, we use the fact that $w(y,s)$ is a weak solution of (7), i.e. for any C^∞ function $\varphi(y,s)$ compactly supported in $(-1,1)\times(s_1,\infty)$ and some $s_1\in\mathbb{R}$,

$$\begin{aligned}
I &= \int\left(\mathcal{L}(\varphi)w - 2\frac{(p+1)}{(p-1)^2}w\varphi + |w|^{p-1}w\varphi\right)\varphi\rho dyds\\
&+ \int \partial_s w\left\{\partial_s\varphi - \frac{p+3}{p-1}\varphi + \frac{1}{\rho}\partial_y(2y\rho\varphi)\right\}\rho dyds = 0 \qquad (60)
\end{aligned}$$

(see below for a proof of this fact).
For $\varphi_1(y)\in C^\infty$ compactly supported in $[-1+\frac{1}{M},1-\frac{1}{M}]$, consider $\varphi(y,s)=\varphi_1(y)\varphi_2(s-s_n)$ where $\varphi_2\in C^\infty$, $\operatorname{supp}\varphi_2\in[-2,2]$ and $\int_{\mathbb{R}}\varphi_2 = 1$ and apply (60).
Since $\int_{s_n-2}^{s_n+2}\int_{|y|<1-\frac{1}{M}}(\partial_s w(y,s'))^2 dyds' \to 0$ from (56), we use (i) of this lemma and the Cauchy-Schwartz inequality to get as $n\to\infty$:

$$\int_{-1}^{1}\left[w^*\mathcal{L}\varphi_1 + \left(-\frac{2(p+1)}{(p-1)^2}w^* + |w^*|^{p-1}w^*\right)\varphi_1\right]\rho dy = 0. \qquad (61)$$

Since $w^*\in H^1(-1,1)$, we obtain from classical elliptic regularity theory that $w^*\in C^2(-1,1)$, therefore, w^* satisfies equation (47), which is the conclusion of (ii) of Lemma 3.3. Remains to prove (60).
Proof of (60): Let us remark from the definition of w given in (6) that

$$\partial^2_{ss}w - \left(\mathcal{L}w - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w - \frac{p+3}{p-1}\partial_s w - 2y\partial^2_{y,s}w\right) = \frac{(\partial^2_{tt}u - \partial^2_{xx}u - |u|^{p-1}u)}{(T-t)^{-\frac{2p}{p-1}}},$$

and thus for all C^∞ function $\varphi(y,s)$ compactly supported in $(-1,1)\times(s_1,\infty)$, for some $s_1\in\mathbb{R}$, we have

$$I = \int_{\mathcal{C}}\left(u\partial^2_{tt}\psi - u\partial^2_{x,x}\psi - |u|^{p-1}u\psi\right)dxdt \qquad (62)$$

where $\mathcal{C} = \{(x,t) \mid T - e^{s_1} < t < T,\ |x - x_0| < T - t\}$ and $\psi(x,t)$ is C^∞ compactly supported in $\mathcal{C}$ and defined by $\psi(x,t) = \varphi(y,s)e^{-\frac{2s}{p-1}}\rho(y)$, where $y = \frac{x-x_0}{T-t}$ and $s = -\log(T-t)$.
The Duhamel representation for u (where $u_0 \in H^1_{loc}$ and $u_1 \in L^2_{loc}$):

$$u(x,t) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_1 + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} |u|^{p-1}u(z,\tau)dzd\tau \quad (63)$$

yields that u is also a weak solution of (1), hence, $I = 0$. Let us briefly recall the proof of this fact. Making the change of variables

$$\tilde{u}(\xi,\eta) = u(x,t),\ \tilde{\psi}(\xi,\eta) = \psi(x,t) \text{ with } \xi = x+t \text{ and } \eta = x-t,$$

we write

$$\begin{aligned} I &= \frac{1}{2}\int \left(-4\tilde{u}(\xi,\eta)\partial^2_{\xi\eta}\tilde{\psi}(\xi,\eta) - |\tilde{u}|^{p-1}\tilde{u}(\xi,\eta)\tilde{\psi}(\xi,\eta)\right) d\xi d\eta,\\ \tilde{u}(\xi,\eta) &= \frac{1}{2}\left(u_0(\xi) + u_0(\eta)\right) + \frac{1}{2}\int_\eta^\xi u_1 + \frac{1}{2}\int_0^{\frac{\xi-\eta}{2}} \int_{\eta+\tau}^{\xi-\tau} |u|^{p-1}u(z,\tau)dzd\tau. \end{aligned}$$

Integrating by parts and using Fubini's identity, we get

$$\begin{aligned} &-4\int \tilde{u}(\xi,\eta)\partial^2_{\xi\eta}\tilde{\psi}(\xi,\eta)d\xi d\eta = 2\int\left(\partial_\xi \int_0^{\frac{\xi-\eta}{2}}\int_{\eta+\tau}^{\xi-\tau} |u|^{p-1}u(z,\tau)dzd\tau\right)\partial_\eta\tilde{\psi}(\xi,\eta)d\xi d\eta\\ &= 2\int\left(\int_0^{\frac{\xi-\eta}{2}} |u|^{p-1}u(\xi-\tau,\tau)d\tau\right)\partial_\eta\tilde{\psi}(\xi,\eta)d\xi d\eta = \int |\tilde{u}|^{p-1}\tilde{u}(\xi,\tau)\tilde{\psi}(\xi,\eta)d\xi d\eta. \end{aligned}$$

Hence, $I = 0$ and (60) is proved. This concludes the proof of Lemma 3.3. ■

Step 2: H^1 control through the localization in the u variable

The following lemma allows us to conclude the proof of Theorem 2 in the non characteristic case:

Lemma 3.4 *For n large, we have*

$$\|w(s_n) - w^*\|_{H^1(-1,1)} + \|\partial_s w(s_n)\|_{L^2(-1,1)} \le \frac{\epsilon_0}{2}$$

where ϵ_0 is defined in (59).

Indeed, taking n large, we have from this lemma a contradiction with (59), hence, (58) holds and by Proposition 3.2, (57) holds and so does Theorem 2 in the non characteristic case.

Proof of Lemma 3.4: We claim it as a consequence of the localization of the energy in the u variable (finite speed of propagation) and the scaling factor coming from the self-similar transformation (6).

For $B = B(\epsilon_0) > 0$ to be chosen later large enough, consider

$$W_n(y,s) = w(y, s + s_n - B). \quad (64)$$

From (54) and the previous step, we know that for all $n \in \mathbb{N}$

- W_n and w^* are solutions to equation (7),
- for all $s \geq 0$, $\|W_n(s)\|_{H^1(-1,1)} + \|\partial_s W_n(s)\|_{L^2(-1,1)} + \|w^*\|_{H^1(-1,1)} \leq C$,
- $$\sup_{s\in[0,B]} \|W_n(s) - w^*\|_{L^\infty(-1,1)} \leq \epsilon_n \to 0. \tag{65}$$

Introducing u_n and u defined as in the selfsimilar transformation (7) by

$$u_n(\xi,\tau) = (1-\tau)^{-\frac{2}{p-1}} W_n\left(\frac{\xi}{1-\tau}, -\log(1-\tau)\right),\ u^*(\xi,\tau) = (1-\tau)^{-\frac{2}{p-1}} w^*\left(\frac{\xi}{1-\tau}\right), \tag{66}$$

we see that

- u_n and u^* are solutions of (1) defined in $\{(\xi,\tau) \mid 0 \leq \tau < 1 \text{ and } |\xi| < 1-\tau\}$,
- $\|u_n(0)\|_{H^1(-1,1)} + \|\partial_\tau u_n(0)\|_{L^2(-1,1)} + \|u^*(0)\|_{H^1(-1,1)} \leq C_0$ (note that C_0 is independent from B),
- $\sup_{\tau\in[0,\tau_B]} \|u_n(\tau) - u^*(\tau)\|_{L^\infty(|\xi|<1-\tau)} = C(B)\epsilon_n \to 0$ where $\tau_B = 1 - e^{-B}$.

Consider for $\tau \in [0,\tau_B]$, $v_n(\tau) = u_n(\tau) - u^*(\tau)$. We have:

- $(\partial^2_{\tau\tau} - \partial^2_{\xi\xi})v_n = f_n$ where $\sup_{\tau\in[0,\tau_B]} \|f_n(\tau)\|_{L^\infty(|\xi|<1-\tau)} = C(B)\epsilon_n \to 0$ as $n \to \infty$,
- there is $C_0 > 0$ such that for all n, $I(0) \leq C_0$ where
$$I(\tau) = \int_{|\xi|<1-\tau} \left((\partial_\xi v_n(\xi,\tau))^2 + (\partial_\tau v_n(\xi,\tau))^2\right) d\xi.$$

Let us prove that for n large, $I(\tau_B) \leq 2C_0$. Indeed, we have by a direct computation, for all $\tau \in [0,\tau_B]$,

$$I'(\tau) \leq 2\int_{|\xi|<1-\tau} f_n \partial_\tau v_n(\xi,\tau) d\xi \leq C(B)\epsilon_n\sqrt{I(\tau)},$$

which leads by integration in time for ϵ_n small enough, to $I(\tau_B) \leq 2C_0$.

Note that we have from (66),

$$\begin{aligned} \partial_\xi u_n(\xi,\tau) &= (1-\tau)^{-\frac{2}{p-1}-1}\partial_y W_n\left(\tfrac{\xi}{1-\tau}, -\log(1-\tau)\right) \\ \partial_\tau u_n(\xi,\tau) &= (1-\tau)^{-\frac{2}{p-1}-1}\left(\partial_\tau W_n + y.\partial_y W_n + \tfrac{2}{p-1}W_n\right)\left(\tfrac{\xi}{1-\tau}, -\log(1-\tau)\right), \end{aligned} \tag{67}$$

and the same holds for u^*. Using (66) and (67), we obtain

$$\|\partial_y W_n(B) - \partial_y w^*(B)\|_{L^2(-1,1)} \leq e^{-\frac{2B}{p-1}-\frac{B}{2}}\|\partial_\xi v_n(\tau_B)\|_{L^2(|\xi|<\tau_B)} \leq C_0' e^{-\frac{2B}{p-1}-\frac{B}{2}}, \tag{68}$$

where C_0' is independent from B, and similarly, using (65)

$$\begin{aligned} \|\partial_s W_n(B)\|_{L^2(-1,1)} &\leq e^{-\frac{2B}{p-1}-\frac{B}{2}}\left(\|\partial_\tau v_n(\tau_B)\|_{L^2(|\xi|<\tau_B)} + \|\partial_\xi v_n(\tau_B)\|_{L^2(|\xi|<\tau_B)}\right) \\ &+ \frac{2}{p-1}\|W_n(B) - w^*\|_{L^\infty(-1,1)} \\ &\leq C_0' e^{-\frac{2B}{p-1}} + C\epsilon_n. \end{aligned} \tag{69}$$

Therefore, since $W_n(y,B) = w(y,s_n)$ by (64), we have from (65), (68) and (69)

$$\|w(s_n) - w^*\|_{H^1(-1,1)} + \|\partial_s w(s_n)\|_{L^2(-1,1)} \le C_0' e^{-\frac{2B}{p-1}} + C(B)\epsilon_n.$$

Taking $B = B(\epsilon_0)$ and n large enough, we get to the conclusion of Lemma 3.4. ∎

3.2 The characteristic case

Let us now consider x_0 a characteristic point and introduce $s_0 = -\log T(x_0)$. The known facts are limited in the non characteristic case. Nevertheless, thanks to Appendix A, Section 2 of [17] applies and we know for $w = w_{x_0}$ that (E1)-(E4) cited in page 3 hold. Note that the proof we present works of course in the non characteristic case.
We proceed in two Parts:

- In Part 1, we show that all the terms in the Lyapunov functional are bounded (Proposition 3.5), and then, we prove a local convergence result under a non vanishing condition (Proposition 3.8).
- In Part 2, we conclude the proof of Theorem 2 in the characteristic case.

Part 1: Local convergence under a non vanishing condition

Improving (E1)-(E4), we now claim that each term of the Lyapunov functional $E(w)$ is bounded separately:

Proposition 3.5 (Boundedness of each term of $E(w)$ and convergence)
(i) There is a $C_0 > 0$ such that for all $s \ge s_0 + 3$,

$$\int_{-1}^{1} \left(\partial_y w(s)^2(1-y^2) + w(s)^2 + \partial_s w(s)^2 + |w(s)|^{p+1}\right)\rho \le C_0.$$

(ii) $\displaystyle \frac{1}{2}\int_{-1}^{1} \partial_y w(s)^2(1-y^2)\rho + \frac{p+1}{(p-1)^2}\int_{-1}^{1} w(s)^2\rho + \frac{1}{2}\int_{-1}^{1}\partial_s w(s)^2\rho \to \frac{p+1}{p-1}E_\infty$ *as* $s \to \infty$.

(iii) $\displaystyle \frac{1}{p+1}\int_{-1}^{1} |w(y,s)|^{p+1}\rho \to \frac{2}{p-1}E_\infty$ *as* $s \to \infty$.

Remark: Part (i) of this proposition gives a different proof of the result of [18] when $k = 1$. However, the dependence of the bound on initial data is less clear here. Note that in the characteristic case, our new estimate is stronger than that of [18]. In addition, the energy partition we obtain in (ii) and (iii) is the same as for a stationary solution (see (49)).

Let us first establish two preliminary lemmas:

Lemma 3.6 *There is a $C_0 > 0$ such that for all $s \ge s_0 + \frac{3}{2}$,*

$$\int_{-1}^{1} \frac{w(y,s)^2}{1-y^2}\rho(y)dy \le C_0.$$

Proof: From the Hardy-Sobolev estimate (21) and (E4), we obtain

$$\forall s \geq s_0 + \frac{3}{2},\quad \int_{s-\frac{1}{2}}^{s+\frac{1}{2}} \int \frac{w(y,s')^2}{1-y^2}\rho(y)ds'dy \leq C_0 \tag{70}$$

for some $C_0 > 0$. Thus, there is $s_1(s) \in [s-\frac{1}{2}, s]$ such that $\displaystyle\int \frac{w(y,s_1)^2}{1-y^2}\rho(y)dy \leq 2C_0$. We then have from (E3) and (70)

$$\begin{aligned}\int_{-1}^{1} \frac{w(y,s)^2}{1-y^2}\rho(y)dy &= \int_{-1}^{1} \frac{w(y,s_1)^2}{1-y^2}\rho(y)dy + 2\int_{s_1}^{s}\int_{-1}^{1} \frac{w\partial_s w(y,s')}{1-y^2}\rho(y)dyds',\\ \leq\;& 2C_0 + \left(\int_{s_1}^{s}\int_{-1}^{1} \frac{w^2(y,s')}{1-y^2}\rho(y)ds' + \int_{s_1}^{s}\int_{-1}^{1} \frac{\partial_s w^2(y,s')}{1-y^2}\rho(y)ds'\right) \leq C_0'\end{aligned}$$

and the conclusion of Lemma 3.6 follows. ■

We now have from the proof of (E4) given in [17] a refinement of these estimates:

Lemma 3.7 *There are $s_1(s)$ and $s_2(s)$ defined for $s \geq s_0 + 1$ such that:*
(i) $|s_1(s) - s| + |s_2(s) - s| \to 0$ as $s \to \infty$.
(ii) $\displaystyle\int_{s_1(s)}^{s_2(s)+1}\int_{-1}^{1} \frac{|w(y,s)|^{p+1}}{p+1}\rho \to \frac{2}{p-1}E_\infty$ *and*

$$\int_{s_1(s)}^{s_2(s)+1}\int_{-1}^{1}\left\{\frac{1}{2}\partial_y w(y,s)^2(1-y^2)\rho + \frac{1}{2}\partial_s w(y,s)^2\rho + \frac{p+1}{(p-1)^2}w(y,s)^2\rho\right\} \to \frac{p+1}{p-1}E_\infty$$

as $s \to \infty$.

Proof: Remark from [17] (identity (11) page 1152) that we have for all $s_1 \geq s_0$ and $s_2 \geq s_0 + 1$,

$$\begin{aligned}\frac{p-1}{2(p+1)}\int_{s_1}^{s_2+1}\int_{-1}^{1}|w(y,s)|^{p+1}\rho &= \int_{s_1}^{s_2+1} E(w(s))ds + \frac{1}{2}\left[\int_{-1}^{1} w\partial_s w\rho\right]_{s_1}^{s_2+1}\\ + \int_{s_1}^{s_2+1}\int_{-1}^{1}&\left\{-\partial_s w(y,s)^2\rho - \partial_s w y\partial_y w\rho - \partial_s w w y \partial_y\rho + \frac{5-p}{2(p-1)}w\partial_s w\rho\right\}.\end{aligned}$$

Then, using (E3), we claim that for $s \geq s_0 + 1$, there are $s_1(s)$ and $s_2(s)$ such that (i) in Lemma 3.7 holds, $\displaystyle\int_{-1}^{1}(\partial_s w(s_1(s)))^2\frac{\rho}{1-y^2} \to 0$ and $\displaystyle\int_{-1}^{1}(\partial_s w(s_2(s)+1))^2\frac{\rho}{1-y^2} \to 0$ as $s \to \infty$. Indeed, if $\displaystyle\eta(s) = \int_{s}^{s+1}\int_{-1}^{1}(\partial_s w(s'))^2\frac{\rho}{1-y^2}ds'$, then (E3) implies that $\eta(s) \to 0$ as $s \to \infty$. Therefore, considering $s_1(s) \in [s, s+\sqrt{\eta(s)}]$ such that

$$\begin{aligned}\int_{-1}^{1}(\partial_s w(s_1(s)))^2\frac{\rho}{1-y^2} &= \frac{1}{\sqrt{\eta(s)}}\int_{s}^{s+\sqrt{\eta(s)}}\int_{-1}^{1}(\partial_s w(s'))^2\frac{\rho}{1-y^2}ds'\\ &\leq \frac{1}{\sqrt{\eta(s)}}\int_{s}^{s+1}\int_{-1}^{1}(\partial_s w(s'))^2\frac{\rho}{1-y^2}ds' = \frac{\eta(s)}{\sqrt{\eta(s)}} \to 0,\end{aligned}$$

we conclude for $s_1(s)$. Taking $s_2(s) = s_1(s+1) - 1$ closes the proof.

Now, using (E2), (E3) and (E4), we see that $\left[\int_{-1}^{1} w\partial_s w\rho\right]_{s_1}^{s_2+1} \to 0$ and

$$\int_{s_1}^{s_2+1}\int_{-1}^{1}\left\{-\partial_s w(y,s)^2\rho - \partial_s wy\partial_y w\rho - \partial_s wwy\partial_y\rho + \frac{5-p}{2(p-1)}w\partial_s w\rho\right\} \to 0$$

as $s \to \infty$.

Since $E(w(s)) \to E_\infty$ and $|s_1(s)-s|+|s_2(s)-s| \to 0$ as $s \to \infty$, we get $\int_{s_1}^{s_2+1} E(w(s))ds \to E_\infty$, and the conclusion follows for $\int_{s_1}^{s_2+1} |w(y,s)|^{p+1}\rho$. Using the definition of $E(w(s))$ (15), we conclude the proof of Lemma 3.7. ■

Let us prove Proposition 3.5 now.

Proof of Proposition 3.5: We proceed by a priori estimates. Using $E(w(s))$, it is enough to prove that $\int_{-1}^{1}\left(\frac{1}{2}\partial_y w(y,s)^2(1-y^2) + \frac{1}{2}\partial_s w(y,s)^2 + \frac{p+1}{(p-1)^2}w(y,s)^2\right)\rho(y)dy$ is bounded and converges to $\frac{p+1}{p-1}E_\infty$ as $s \to \infty$.

We have from Lemma 3.7 and (E3) that for all $\epsilon_0 \in (0, \frac{p+1}{(p-1)^2})$, there is $s_{\epsilon_0} \geq s_0 + 5$ such that for all $s \geq s_{\epsilon_0}$, we have:

$$\left|\int_{s_1(s-2)}^{s_2(s-2)+1}\int_{-1}^{1}\left(\frac{1}{2}\partial_y w(y,s')^2(1-y^2) + \frac{1}{2}\partial_s w^2 + \frac{p+1}{(p-1)^2}w^2\right)\rho(y)ds' - \frac{p+1}{p-1}E_\infty\right| \leq \frac{\epsilon_0}{2}, \quad (71)$$

$$\int_{s_2(s-2)}^{s}\int_{-1}^{1}\frac{\partial_s w(y,s')^2}{1-y^2}\rho ds'dy \leq \delta_0(\epsilon_0), \quad (72)$$

and $|s_1(s)-s| + |s_2(s)-s| \leq \delta_0(\epsilon_0)$, where small δ_0 will be fixed later dependent of ϵ_0.

We now claim for all $s \geq s_{\epsilon_0}$,

$$\left|\int_{-1}^{1}\frac{1}{2}\partial_y w(y,s)^2(1-y^2)\rho + \frac{1}{2}\partial_s w(y,s)^2\rho + \frac{p+1}{(p-1)^2}w^2\rho - \frac{p+1}{p-1}E_\infty\right| \leq \epsilon_0, \quad (73)$$

which concludes the proof of Proposition 3.5.

Proof of (73): From (71), we know that for all $s \geq s_{\epsilon_0}$, there is $s_3(s) \in [s_1(s-2), s_2(s-2)+1]$ such that,

$$\left|(1+s_2-s_1)\int_{-1}^{1}\left[\frac{\partial_y w(s_3)^2}{2}(1-y^2) + \frac{\partial_s w(s_3)^2}{2} + \frac{p+1}{(p-1)^2}w(s_3)^2\right]\rho - \frac{p+1}{p-1}E_\infty\right| \leq \frac{\epsilon_0}{2},$$

therefore,

$$\left|\int_{-1}^{1}\left[\frac{\partial_y w(s_3)^2}{2}(1-y^2) + \frac{\partial_s w(s_3)^2}{2} + \frac{p+1}{(p-1)^2}w(s_3)^2\right]\rho - \frac{p+1}{p-1}E_\infty\right| \leq \frac{\epsilon_0}{2} + C_0\delta_0, \quad (74)$$

where $s_3 \in [s-3, s-\frac{1}{2}]$.
If we impose that $C_0\delta_0 < \frac{\epsilon_0}{2}$, then (73) holds for s_3. Let us prove (73) for all $s' \in [s_3, s]$ if ϵ_0 is small enough and δ_0 is small enough in terms of ϵ_0.
By contradiction, assume that (73) holds for all $s' \in [s_3, s_4]$ and that for $s' = s_4$, we have equality in (73), where $s_4 \in [s_3, s]$. Then, from (23) and (73), we have for all $s' \in [s_3, s_4]$, $\|w(s')(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty} \le C_0(E_\infty+1)$. Thus, using the derivative of the Lyapunov functional (16) and Lemma 3.6, we have for all $s' \in [s_3, s_4]$,

$$\begin{aligned}
&\left|\frac{d}{ds}\left[\int_{-1}^{1}\left(\frac{1}{2}\partial_y w(y,s')^2(1-y^2)+\frac{1}{2}\partial_s w^2+\frac{p+1}{(p-1)^2}w^2\right)\rho\right]\right| \\
= &\left|-\frac{4}{p-1}\int_{-1}^{1}\frac{\partial_s w^2}{1-y^2}\rho+\int_{-1}^{1}\partial_s w|w|^{p-1}w\rho\right| \\
\le &\ C_0\int_{-1}^{1}\frac{\partial_s w^2}{1-y^2}\rho+C_0\int_{-1}^{1}\frac{|\partial_s w||w|}{1-y^2}\rho \le C_0\int_{-1}^{1}\frac{\partial_s w^2}{1-y^2}\rho+C_0\left(\int_{-1}^{1}\frac{\partial_s w^2}{1-y^2}\rho\right)^{1/2}.
\end{aligned}$$

Integrating in time between s_3 and s_4, we obtain from (74) and (72),

$$\epsilon_0 \le \frac{\epsilon_0}{2}+C_0\delta_0+C_0\delta_0+C_0\int_{s_3}^{s_4}\left(\int_{-1}^{1}\frac{\partial_s w^2}{1-y^2}\rho\right)^{1/2} \le \frac{\epsilon_0}{2}+C_0(\delta_0+\delta_0^{1/2}).$$

Therefore, we obtain a contradiction by taking $\delta_0 = \epsilon_0^4$ and ϵ_0 small enough. Thus, (73) is proved. This concludes the proof of Proposition 3.5. ∎

Note in addition that from Proposition 3.5 and (23), there is $C_0 > 0$ such that

$$\forall s \ge s_0+3 \text{ and } y \in (-1,1),\quad |w(y,s)(1-y^2)^{\frac{1}{p-1}}| \le C_0. \tag{75}$$

From the dispersion property of the flow (16), we are able to prove that any recurrent nonlinear object in the dynamics as $s \to \infty$ is a stationary solution. Considering the space variable ξ which allows us to write easily decoupling properties:

$$\xi = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right) \in \mathbb{R} \text{ (i.e. } y=\tanh\xi\text{) and } \bar{w}(\xi,s) = w(y,s)(1-y^2)^{\frac{1}{p-1}}, \tag{76}$$

we have the following:

Proposition 3.8 (Local convergence under a non vanishing condition) *Consider a sequence (y_n, s_n) and $\epsilon_0 > 0$ such that $s_n \to \infty$ and $|w(y_n,s_n)|(1-y_n^2)^{\frac{1}{p-1}} \ge \epsilon_0$. Then, there is $\xi_0 \in \mathbb{R}$ and $\omega_0 = \pm 1$ such that up to a subsequence:*

$$(i)\quad \left|\bar{w}(\xi+\xi_n, s+s_n) - \omega_0\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi-\xi_0)}\right| \to 0 \tag{77}$$

as $n \to \infty$, uniformly on compact sets of $|\xi|+|s|$ where $\xi_n = \frac{1}{2}\log\left(\frac{1+y_n}{1-y_n}\right)$.

$$(ii)\quad \forall M>0,\ \int_{\{y||\xi-\xi_n|<M\}}|w(y,s_n)-\omega_0\kappa(d_n,y)|^{p+1}\rho dy \to 0 \tag{78}$$

as $n \to \infty$, where

$$d_n = \tanh\tilde{\xi}_n \text{ and } \tilde{\xi}_n \text{ is such that } \xi_n+\tilde{\xi}_n = -\xi_0. \tag{79}$$

Remark: We have for all $n \in \mathbb{N}$, $1/C \le \frac{1-y_n^2}{1-d_n^2} \le C$.
Proof of Proposition 3.8: Arguing as for (24) and (25), we see from (76), Proposition 3.5, Lemma 3.6 and (E3), that there is $C_0 > 0$ such that

$$\forall s \ge s_0 + 3, \ \ \|\bar{w}\|_{H^1(\mathbb{R})} \le C_0, \tag{80}$$

$$\int_{s^*}^{\infty}\int_{\mathbb{R}} \partial_s \bar{w}^2 ds d\xi \le C_0. \tag{81}$$

Recall from (52) that the corresponding set of stationary solutions in $H^1(\mathbb{R})$ in the $\bar{w}$ variable (to the stationary solution in $\mathcal{H}_0$ in the w variable) is

$$\pm\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi-\xi_0)} \text{ where } \xi_0 \in \mathbb{R}. \tag{82}$$

Proposition 3.8 reduces then to prove that up to a subsequence (also denoted by s_n) and for some $\bar{w}^* \not\equiv 0$, a stationary solution (that is a solution of equation (51)), we have

$$|\bar{w}(\xi+\xi_n, s+s_n) - \bar{w}^*(\xi)| \to 0 \text{ as } n \to \infty \tag{83}$$

uniformly on compact sets $|\xi| + |s| \le M$.
Indeed, if (83) holds, then (i) of Proposition 3.8 follows from the fact that a non zero stationary solution $\bar{w}^*$ is given by (82).
As for (ii) of Proposition 3.8, remark from (75) and (79) that

$$\begin{aligned}
&\int_{\{y||\xi-\xi_n|<M\}} |w(y,s_n) - \omega_0\kappa(d_n,y)|^{p+1}\rho dy \\
\le\ & C_0 \int_{\{y||\xi-\xi_n|<M\}} \frac{|w(y,s_n)(1-y^2)^{\frac{1}{p-1}} - \omega_0\kappa(d_n,y)(1-y^2)^{\frac{1}{p-1}}|^2}{1-y^2} dy \\
\le\ & C_0 \int_{|\xi-\xi_n|<M} \left|\bar{w}(\xi,s_n) - \omega_0 \frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi+\tilde{\xi}_n)}\right|^2 d\xi \\
\le\ & C_0 \int_{|\xi|<M} \left|\bar{w}(\xi_n+\xi,s_n) - \omega_0 \frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi-\xi_0)}\right|^2 d\xi \to 0
\end{aligned}$$

as $n \to \infty$ using (77). Thus, we just need to prove (83).

Proof of (83): The proof is similar to the one of Lemma 3.3. From (80), there is a subsequence s_n and $\bar{w}^* \in H^1(\mathbb{R})$ such that

$$\bar{w}(\xi+\xi_n, s_n) \to \bar{w}^*(\xi) \text{ in } C(|\xi| < M) \text{ for all } M > 0. \tag{84}$$

Remark from (76) and the hypotheses of Preposition 3.8 that

$$|\bar{w}(\xi_n, s_n)| \ge \epsilon_0, \text{ thus } |\bar{w}^*(0)| \ge \epsilon_0 \text{ and } \bar{w}^* \not\equiv 0. \tag{85}$$

Moreover, (80) and (81) give for all $M > 0$ and $|s| < M$

$$\begin{aligned}
&\int_{|\xi|<M} |\bar{w}(\xi_n+\xi, s_n+s) - \bar{w}^*(\xi)|^2 d\xi \\
\leq\ & \int_{|\xi|<M} |\bar{w}(\xi_n+\xi, s_n) - \bar{w}^*(\xi)|^2 d\xi + C_0 \int_{s_n-M}^{s_n+M} \left(\int_{|\xi|<M} (\partial_s \bar{w}(\xi_n+\xi, s'))^2 d\xi \right)^{1/2} ds' \\
\leq\ & \int_{|\xi|<M} |\bar{w}(\xi_n+\xi, s_n) - \bar{w}^*(\xi)|^2 d\xi + C_0\sqrt{M} \left(\int_{s_n-M}^{s_n+M} \int_{\mathbb{R}} (\partial_s \bar{w}(\xi, s'))^2 d\xi ds' \right)^{1/2} \to 0
\end{aligned}$$

as $n \to \infty$, and from the fact that

$$\|\bar{w}\|^2_{L^\infty(|\xi|<M)} \leq C_0 \|\bar{w}\|_{L^2(|\xi|<M+1)} \|\bar{w}\|_{H^1(|\xi|<M+1)},$$

we have (83) with $\bar{w}^*$ defined in (84). Remains to prove that $\bar{w}^*(\xi)$ corresponds to a stationary solution. Let us remark from similar computations to page 15 that

$$\begin{aligned}
&(1-y^2)^{\frac{1}{p-1}+1} \left[-\partial^2_{ss} w - \frac{p+3}{p-1}\partial_s w - 2y\partial^2_{y,s} w + \mathcal{L}w - 2\frac{(p+1)}{(p-1)^2} w + |w|^{p-1}w \right] \\
=\ & -\frac{\partial^2_{ss}\bar{w}}{\cosh^2\xi} + \left(\tanh^2\xi - \frac{p+3}{p-1} \right) \partial_s \bar{w} - 2\tanh\xi \partial^2_{\xi s}\bar{w} + \bar{w}_{\xi\xi} - \frac{4}{(p-1)^2}\bar{w} + |\bar{w}|^{p-1}\bar{w}
\end{aligned}$$

and thus for all $\bar{\varphi}(\xi, s)$ C^∞ with compact support included in $\{s \geq s^*\}$,

$$\begin{aligned}
&\int \left(\frac{1}{\cosh^2\xi} \partial_s \bar{\varphi} d\xi ds - \left(\frac{p+3}{p-1} - \tanh^2\xi \right) \bar{\varphi} + \partial_\xi \left(2\bar{\varphi}\tanh^2\xi \right) \right) \partial_s \bar{w} d\xi ds \\
+\ & \int \left(\bar{w}\bar{\varphi}_{\xi\xi} - \frac{4}{(p-1)^2}\bar{w}\bar{\varphi} + |\bar{w}|^{p-1}\bar{w}\bar{\varphi} \right) d\xi ds \\
=\ & \int \left\{ \partial_s\varphi - \frac{p+3}{p-1}\varphi + \frac{1}{\rho}\partial_y(2y\rho\varphi) \right\} \partial_s w \rho dy ds \\
+\ & \int \left(\mathcal{L}(\varphi) w - 2\frac{(p+1)}{(p-1)^2} w\varphi + |w|^{p-1} w\varphi \right) \varphi\rho dy ds = 0
\end{aligned} \tag{86}$$

with $\varphi(y, s) = \bar{\varphi}(\xi, s)$. The fact that the latter expression is zero follows from the same computations to the non characteristic case (see Step 1 in subsection 3.1).
Consider now an arbitrary $\bar{\varphi}_1(\xi)$ C^∞ compactly supported. Apply identity (86) with

$$\bar{\varphi}(\xi, s) = \bar{\varphi}_1(\xi - \xi_n)\bar{\varphi}_2(s - s_n)$$

where $\bar{\varphi}_2 \in C_c^\infty$, $\operatorname{supp}\bar{\varphi}_2 \in [-2, 2]$ and $\int_{\mathbb{R}} \bar{\varphi}_2 = 1$.
Since we know from (81) that $\displaystyle\int_{s_n-2}^{s_n+2} \int \partial_s \bar{w}^2 \to \infty$ as $n \to \infty$, we use (83) and the Cauchy-Schwartz inequality to get as $n \to \infty$

$$\int \bar{w}^* \partial^2_{\xi\xi}\bar{\varphi}_1 + \int \left(|\bar{w}^*|^{p-1}\bar{w}^* - \frac{4}{(p-1)^2}\bar{w}^* \right) \bar{\varphi}_1 = 0.$$

From the fact that $\bar{w}^* \in H^1(\mathbb{R})$ and classical elliptic regularity theory, we have $\bar{w}^* \in H^3$. Therefore, $\bar{w}^* \in C^2(\mathbb{R})$ and $\bar{w}^*$ satisfies

$$\partial^2_{\xi\xi}\bar{w}^* + |\bar{w}^*|^{p-1}\bar{w}^* - \frac{4}{(p-1)^2}\bar{w}^* = 0 \text{ for } \xi \in \mathbb{R},$$

which concludes the proof of Proposition 3.8. ■

Part 2: Conclusion of the proof of Theorem 2 in the non characteristic case

From (E1), we know that $E_\infty \geq 0$. If $E_\infty = 0$, then from Proposition 3.5, we have $\|w(s)\|_{\mathcal{H}} \to 0$ as $s \to \infty$ and the conclusion is valid with $k = 0$. Assume from now on that

$$E_\infty > 0.$$

Step 1: Localization of the energy packets

Remark first from Proposition 3.5, Lemma 3.6 and (23), that there is $C_0 > 0$ and $s_1 \geq s_0 + 3$ such that for all $s \geq s_1$,

$$\int_{-1}^1 \frac{w(s)^2}{1-y^2}\rho + \int_{-1}^1 \frac{\partial_s w(y,s)^2}{1-y^2}\rho + \|w(s)(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty} \leq C_0 \text{ and } \int_{-1}^1 |w(s)|^{p+1}\rho \geq \frac{1}{C_0}.$$

Therefore, $\frac{1}{C_0} \leq \int |w|^{p+1}\rho \leq \int \frac{w^2}{1-y^2}\rho \|w(1-y^2)^{\frac{1}{p-1}}\|^{p-1}_{L^\infty} \leq C_0\|w(1-y^2)^{\frac{1}{p-1}}\|^{p-1}_{L^\infty}$, hence, there exists $\epsilon_0 \in (0, \frac{\kappa_0}{4})$ such that for all $s \geq s_1$,

$$\|w(s)(1-y^2)^{\frac{1}{p-1}}\|^{p-1}_{L^\infty} \geq 2\epsilon_0. \tag{87}$$

In particular, if we define

$$\tilde{A}(s) = \{\xi \mid |\bar{w}(\xi,s)| \geq \epsilon_0\} \text{ and } A(s) = \{\xi \mid d(\xi, \tilde{A}(s)) < 1\},$$

then, for all $s \geq s_1^*$, $\tilde{A}(s) \neq \emptyset$ and $A(s) \neq \emptyset$. We now have the following:

Lemma 3.9 *There is $k \in \mathbb{N}^*$, s_2 and $\mu_0 > 0$ such that for all $s \geq s_2$,*
(i) $A(s) = \cup_{i=1}^k (\xi_i(s) - \mu_i(s), \xi_i(s) + \mu_i(s))$ where $\xi_i(s)$ is a continuous function of s,

$$|\xi_i(s) - \xi_j(s)| \to \infty \text{ for } i \neq j \text{ and } \mu_i(s) \to \mu_0 \tag{88}$$

as $s \to \infty$.
(ii)

$$\left|\bar{w}(\xi + \xi_i(s), s) - \omega_i \frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi)}\right| \to 0$$

uniformly on compact sets of $|\xi|$, where $\omega_i = \pm 1$.
(iii) For all $\epsilon > 0$, there exist $M_\epsilon > 0$ and $s_\epsilon \geq s_2$ such that if $s \geq s_\epsilon$ and $\inf_{i=1,..,k} |\xi - \xi_i(s)| > M_\epsilon$, then $|\bar{w}(\xi, s)| \leq \epsilon$.

Proof:
(i)-(ii) Note first that $A(s)$ is an open set of $\mathbb{R}$, that is a disjoint union of open intervals. Let $k(s) \in \bar{\mathbb{N}}$ be the number of connected components of $A(s)$. Let us show that for s large enough,

$$k(s) \le 2\frac{E_\infty}{E(\kappa_0)} + 1. \tag{89}$$

Let us assume by contradiction that for some $m > 2\frac{E_\infty}{E(\kappa_0)}+1$, there are $s_n \to \infty$, $\xi_{1,n} < ... < \xi_{m,n}$ in $\mathbb{R}$ and positive $\mu_{1,n}$,...,$\mu_{k,n}$ such that $(\xi_{i,n}-\mu_{i,n}, \xi_{i,n}+\mu_{i,n})$ are disjoint and $A(s_n) \supset \cup_{i=1}^k(\xi_{i,n}-\mu_{i,n}, \xi_{i,n}+\mu_{i,n})$. By definition of $A(s_n)$, there exist $\xi'_{i,n} \in (\xi_{i,n}-1, \xi_{i,n}+1)\cap\tilde{A}(s_n)$ such that $|\bar{w}(\xi'_{i,n}, s_n)| \ge \epsilon_0$. Therefore, it follows from Proposition 3.5 that up to a subsequence and for all $i = 1, ..., m$,

$$|\bar{w}(\xi + \xi_{i,n}, s_n + s) - \omega_i \frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi - x_i)}| \to 0 \text{ uniformly for } |\xi| + |s| \le M \tag{90}$$

for some $x_i \in \mathbb{R}$ and $\omega_i = \pm 1$. Moreover, since $(\xi_{i,n} - \mu_{i,n}, \xi_{i,n} + \mu_{i,n})$ is a connected component of $A(s)$ with center $\xi_{i,n}$, we use (90) and the fact that

$$\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi)} > \epsilon_0 \text{ iff } -\mu'_0 \le \xi \le \mu'_0 \text{ for some } \mu'_0 = \mu'_0(\epsilon_0) > 0$$

to derive that for all $i = 1, ..., m$:

- $x_i = 0$,
- $\mu_{i,n} \to \mu'_0 + 1$ (use the fact that for any $\delta > 0$ and n large enough, we have $\tilde{A}(s_n) \cap (\xi_{i,n} - 2(\mu'_0 + 1), \xi_{i,n} + 2(\mu'_0 + 1)) \subset (\xi_{i,n} - (\mu'_0 + \delta), \xi_{i,n} + (\mu'_0 + \delta)))$,
- $|\xi_{i,n} - \xi_{j,n}| \to \infty$ as $n \to \infty$, for $i \ne j$.

Making the change of variables $y = \tanh \xi$, we see from (49) that $\int_{\mathbb{R}} \left|\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi)}\right|^{p+1} d\xi = \kappa_0^{p+1}\int_{-1}^1 \rho(y)dy = \frac{2(p+1)}{p-1}E(\kappa_0)$. Fix then $M > 0$ such that $\int_{|\xi|>M} \left|\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi)}\right|^{p+1} d\xi < \frac{1}{100}\frac{2}{p-1}E(\kappa_0)$, and, from Proposition 3.5 and (90), take $n \ge n_0(M)$ such that the intervals $(\xi_{i,n} - M, \xi_{i,n} + M)$ are disjoint for $i = 1, ..., m$ and

$$\begin{aligned}
&\frac{2(p+1)}{p-1}\left(E_\infty + \frac{1}{100}E(\kappa_0)\right) \ge \int |w(y, s_n)|^{p+1}\rho dy = \int |\bar{w}(\xi, s_n)|^{p+1} d\xi \\
\ge\ & \sum_{i=1}^m \int_{|\xi-\xi_{i,n}|\le M} |\bar{w}(\xi, s_n)|^{p+1} d\xi \ge \sum_{i=1}^m \left(\int_{\mathbb{R}} \left|\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}\xi}\right|^{p+1} d\xi - \frac{2}{100}\frac{2(p+1)}{p-1}E(\kappa_0)\right) \\
=\ & m\frac{2(p+1)}{p-1}E(\kappa_0)(1 - \frac{2}{100}),
\end{aligned}$$

hence, $m \le \frac{100}{98}\frac{E_\infty}{E(\kappa_0)} + \frac{1}{98}$, which is a contradiction. Thus, (89) holds.

Let us show now that $k(s)$ is constant for s large, that is, $k(s) = k \in \mathbb{N}^*$. We proceed by contradiction and consider $s_n \to \infty$ and $\delta_n \in (-1,1)$ such that $k(s_n+\delta_n) < k(s_n) = m$. Making the same construction for s_n as we did for the previous proof, defining in particular $\xi_{1,n} < ... < \xi_{m,n}$ in $\mathbb{R}$, we see that (90) holds with $x_i = 0$. Applying (90) with $s = \delta_n \in (-1,1)$, we see that $A(s_n+\delta_n)$ has at least m connected components inherited from those of $A(s_n)$ (here we use the fact that $\epsilon_0 < \frac{\kappa_0}{4}$). Contradiction. Thus, $k(s) = k \in \mathbb{N}^*$ for $s \geq s_2$ for some s_2 large enough.

We are now able to define for all $s \geq s_2$, $\xi_1(s) < ... < \xi_k(s)$, $\mu_i(s)$ such that (88) holds. Note that (90) now writes

$$|\bar{w}(\xi+\xi_i(s), s+\sigma) - \omega_i(s)\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi)}| \to 0 \text{ as } s\to\infty \text{ uniformly for } |\xi|+|\sigma| \leq M$$

for some $\omega_i(s) = \pm 1$. In particular, $\xi_i(s)$ is a continuous function of s and $\omega_i(s)$ is constant for s large. This concludes the proof of (i) and (ii) of Lemma 3.9.

(iii) This estimate follows by contradiction considering some $\epsilon_1 \in (0, \frac{\kappa_0}{4})$ and (ξ_n, s_n) such that $s_n \to \infty$, $\min_{i=1,...,k} |\xi_n - \xi_i(s_n)| \to \infty$ and $|\bar{w}(\xi_n, s_n)| \geq \epsilon_1$. Applying Proposition 3.8 and the fact that $\epsilon_1 \leq \frac{\kappa_0}{4}$, we see that $\mathrm{dist}(\xi_n, A(s_n)) \leq M_1(\epsilon_1)$, which is a contradiction. This concludes the proof of Lemma 3.9. ∎

Using the fact that

$$|\xi_i(s) - \xi_j(s)| \to \infty \text{ as } s \to \infty \text{ for } i \neq j, \tag{91}$$

we have the following:

Claim 3.10 *If*

$$d_i(s) = -\tanh \xi_i(s), \tag{92}$$

then we have as $s \to \infty$*:*

$$\int_{-1}^{1}\left|\sum_{i=1}^{k}\omega_i\kappa(d_i(s),y)\right|^{p+1}\rho - \left(\sum_{i=1}^{k}\int_{-1}^{1}\kappa(d_i(s),y)^{p+1}\rho\right) \to 0,$$

$$\int_{-1}^{1}\left(\sum_{i=1}^{k}\omega_i\partial_y\kappa(d_i(s),y)\right)^2(1-y^2)\rho - \left(\sum_{i=1}^{k}\int_{-1}^{1}(\partial_y\kappa(d_i(s),y))^2(1-y^2)\rho\right) \to 0,$$

$$\int_{-1}^{1}\left(\sum_{i=1}^{k}\omega_i\kappa(d_i(s),y)\right)^2\rho - \left(\sum_{i=1}^{k}\int_{-1}^{1}\kappa(d_i(s),y)^2\rho\right) \to 0,$$

$$\int_{-1}^{1}\left(\sum_{i=1}^{k}\omega_i\kappa(d_i(s),y)\right)\left(\sum_{i=1}^{k}\omega_i\kappa(d_i(s),y)^p\right)\rho - \int_{-1}^{1}\sum_{i=1}^{k}\kappa(d_i(s),y)^{p+1}\rho \to 0.$$

Proof: We only prove the first inequality since the two others follow in the same way. Since $\kappa(d_i(s),y)$ becomes $\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi-\xi_i(s))}$ by the transformation (76), we use the linear character of (76) to get

$$\int_{-1}^{1}\left|\sum_{i=1}^{k}\omega_i\kappa(d_i(s),y)\right|^{p+1}\rho = \int_{\mathbb{R}}\left|\sum_{i=1}^{k}\omega_i\frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi-\xi_i(s))}\right|^{p+1}d\xi.$$

Since we know from (91) that

$$\int_{\mathbb{R}} \left(\left| \sum_{i=1}^{k} \omega_i \frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi - \xi_i(s))} \right|^{p+1} - \sum_{i=1}^{k} \left| \frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi - \xi_i(s))} \right|^{p+1} \right) d\xi \to 0$$

as $s \to \infty$, we just use again (76) to conclude the proof of Claim 3.10. ■

Step 2: Conclusion of the proof

We want to prove that

$$q(y,s) \equiv w(y,s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y) \to 0$$

in the energy norm. Using Step 1, we first prove the convergence in L_ρ^{p+1}. From (iii) in Proposition 3.5, this implies the quantization of E_∞. Then, using the weak convergence of $q(s)$ to 0 is the energy space and the convergence of the norm in (ii) of Proposition 3.5, we prove the strong convergence.

Let us prove now the following:

Claim 3.11 (Convergence in L_ρ^{p+1}) *As* $s \to \infty$,

$$\int |w(s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)|^{p+1} \rho \to 0 \ \textit{and} \int |w(s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)|^2 \rho \to 0. \quad (93)$$

Proof: Remark first that the Hölder inequality and the L^{p+1} estimate imply the L^2 estimate. Let us then prove the L^{p+1} estimate.
For all $\epsilon > 0$, there are from (iii) of Lemma 3.9 $M_\epsilon > 0$ and s_ϵ such that if $s \geq s_\epsilon$ and $\forall i = 1,.,k$, $|\xi - \xi_i(s)| \geq M_\epsilon$, then

$$\begin{aligned} |w(y,s)|(1-y^2)^{\frac{1}{p-1}} &\leq \frac{\epsilon}{2}, \\ \left| \sum_{i=1}^{k} \kappa(d_i(s), y) \right| (1-y^2)^{\frac{1}{p-1}} = \left| \sum_{i=1}^{k} \frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi - \xi_i(s))} \right| &\leq \frac{\epsilon}{2}, \\ |w(y,s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)|(1-y^2)^{\frac{1}{p-1}} &\leq \epsilon \end{aligned} \quad (94)$$

where $y = \tanh \xi$. Therefore, for $s \geq s_\epsilon$,

$$\begin{aligned} \int |w(s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s))|^{p+1} \rho &\leq \int_{\{y \,|\forall i,\, |\xi - \xi_i(s)| \geq M_\epsilon\}} |w(y,s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)|^{p+1} \rho \\ + & \sum_{i=1}^{k} \int_{\{y \,||\xi - \xi_i(s)| < M_\epsilon\}} |w(y,s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)|^{p+1} \rho, \\ \leq & \ \epsilon^{p-1} \int \frac{|w(y,s) - \sum_{i=1}^{k} \omega_i \kappa(d_i(s), y)|^2}{1-y^2} \rho \\ + & \sum_{i=1}^{k} \int_{\{y \,||\xi - \xi_i(s)| < M_\epsilon\}} |w(y,s) - \omega_i \kappa(d_i(s), y)|^{p+1} \rho + o(1) \end{aligned}$$

(from (94) and the fact that $|\xi_i(s)-\xi_j(s)|\to\infty$ as $s\to\infty$ for $i\neq j$). Therefore, for s large,

$$\begin{aligned}\int |w(s)-\sum_{i=1}^k \omega_i\kappa(d_i(s))|^{p+1}\rho &\le C\epsilon^{p-1}\left(\|w(s)\|^2_{\mathcal{H}_0}+\sum_{i=1}^k\|\kappa(d_i(s))\|^2_{\mathcal{H}_0}\right)+o(1)\\ &\le C_0\epsilon^{p-1}+o(1)\le 2C_0\epsilon^{p-1}\end{aligned}$$

(from (23) and (ii) in Lemma 3.9). Letting $\epsilon\to 0$ allows us to conclude.

As a consequence, we have the following energy constraint:

Corollary 3.12 (Quantization of the limit of $E(w(s))$) *It holds that $E_\infty = kE(\kappa_0)$, where $k\in\mathbb{N}^*$ was introduced in Lemma 3.9.*

Indeed, on one hand, we have from Proposition 3.5

$$\int_{-1}^1 |w(s)|^{p+1}\rho \to \frac{2(p+1)}{p-1}E_\infty \text{ as } s\to\infty.$$

On the other hand, from Claims 3.11 and 3.10, and (49), we have

$$\begin{aligned}\lim_{s\to\infty}\int_{-1}^1|w(s)|^{p+1}\rho &= \lim_{s\to\infty}\int_{-1}^1\left(\sum_{i=1}^k\kappa(d_i(s),y)\right)^{p+1}\rho=\lim_{s\to\infty}\sum_{i=1}^k\int_{-1}^1\kappa(d_i(s),y)^{p+1}\rho\\ &= \lim_{s\to\infty}\sum_{i=1}^k\int_{-1}^1\kappa_0^{p+1}\rho = k\int_{-1}^1\kappa_0^{p+1}\rho=\frac{2(p+1)}{p-1}kE(\kappa_0),\end{aligned}$$

and the corollary follows. ■

We now have the following:

Claim 3.13 *If we define $I(s)$ by*

$$\int_{-1}^1\left(\frac12|\partial_y w-\sum_{i=1}^k\omega_i\partial_y\kappa(d_i(s))|^2(1-y^2)+\frac{p+1}{(p-1)^2}|w-\sum_{i=1}^k\omega_i\kappa(d_i(s))|^2+\frac12(\partial_s w)^2\right)\rho,$$

then we have $I(s)\to 0$ as $s\to\infty$.

Proof: Note first that

$$I(s)=\frac12\int\partial_y w(y,s)^2(1-y^2)\rho+\frac{p+1}{(p-1)^2}\int w^2\rho+\frac12\int\partial_s w(y,s)^2\rho+J(s)+K(s)\quad(95)$$

where

$$\begin{aligned}J(s) &= \frac12\int\left(\sum_{i=1}^k\omega_i\partial_y\kappa(d_i(s),y)\right)^2(1-y^2)\rho+\frac{(p+1)}{(p-1)^2}\int\left(\sum_{i=1}^k\omega_i\kappa(d_i(s),y)\right)^2\rho,\\ K(s) &= -\int\partial_y w\partial_y\left(\sum_{i=1}^k\omega_i\kappa(d_i(s),y)\right)(1-y^2)\rho-\frac{2(p+1)}{(p-1)^2}\int w\left(\sum_{i=1}^k\omega_i\kappa(d_i(s))\right)\rho.\end{aligned}$$

Using Claim 3.10 and (49), we see that

$$
\begin{aligned}
J(s) &= \sum_{i=1}^{k}\int\left(\frac{1}{2}(\partial_y\kappa(d_i(s),y))^2(1-y^2)\rho+\frac{(p+1)}{(p-1)^2}(\kappa(d_i(s),y))^2\right)\rho+o(1)\\
&= k\frac{(p+1)}{p-1}E(\kappa_0)+o(1). \qquad (96)
\end{aligned}
$$

We claim

$$
K(s)\to -2\frac{k(p+1)}{p-1}E(\kappa_0)\text{ as } s\to\infty. \qquad (97)
$$

Indeed, from integration by parts and the fact that $\kappa(d_i(s),\cdot)$ is a solution of (47), we have

$$
\begin{aligned}
K(s) &= \int w(s)\left[\sum_{i=1}^{k}\left(\frac{1}{\rho}\partial_y\left(\omega_i\partial_y\kappa(d_i(s),y)(1-y^2)\rho\right)-\frac{2(p+1)}{(p-1)^2}\omega_i\kappa(d_i(s),y)\right)\right]\rho\\
&= -\int w(s)\left[\sum_{i=1}^{k}\omega_i\kappa(d_i(s),y)^p\right]\rho.
\end{aligned}
$$

Therefore, from (49), Hölder's inequality and Claims 3.11 and 3.10, we write

$$
K(s)=-\int\left[\sum_{i=1}^{k}\omega_i\kappa(d_i(s),y)\right]\left[\sum_{i=1}^{k}\omega_i\kappa(d_i(s),y)^p\right]\rho+o(1)
$$

$$
= -\sum_{i=1}^{k}\int\kappa(d_i(s),y)^{p+1}\rho+o(1)=-k\int\kappa_0^{p+1}\rho+o(1)=-2k\frac{(p+1)}{p-1}E(\kappa_0)+o(1),
$$

which concludes the proof of (97).
Using (95), Proposition 3.5, (96) and (97), we write

$$
I(s)\to\frac{p+1}{p-1}E_\infty+k\frac{(p+1)}{p-1}E(\kappa_0)-2k\frac{(p+1)}{p-1}E(\kappa_0)=\frac{p+1}{p-1}(E_\infty-kE(\kappa_0))=0
$$

by Claim 3.12, which proves Claim 3.13. ■

Claim 3.13 together with Corollary 3.12 conclude the proof of Theorem 2 in the characteristic case (use Lemma 3.9 and (92) for the continuity of $d_i(s)$; use (88) and (92) to derive estimate (B.ii)). ■

4 The linearized operator around a non zero stationary solution

In this section, we study the properties of the linearized operator of equation (7) around the stationary solution $\kappa(d,y)$ (13).
If we introduce $q=(q_1,q_2)=\begin{pmatrix}q_1\\ q_2\end{pmatrix}$ for all $s\in[s_0,\infty)$ by

$$
\begin{pmatrix}w(y,s)\\ \partial_s w(y,s)\end{pmatrix}=\begin{pmatrix}\kappa(d,y)\\ 0\end{pmatrix}+\begin{pmatrix}q_1(y,s)\\ q_2(y,s)\end{pmatrix}, \qquad (98)
$$

then we see from equation (7) that q satisfies the following equation for all $s \geq s_0$ (for the proof in a more general case, see the proof of (ii) of Proposition 5.1 below) :

$$\frac{\partial}{\partial s}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = L_d \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix} \tag{99}$$

where

$$\begin{aligned} L_d \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} q_2 \\ \mathcal{L}q_1 + \psi(d,y)q_1 - \frac{p+3}{p-1}q_2 - 2yq_2' \end{pmatrix}, \\ f_d(q_1) &= |\kappa(d,\cdot) + q_1|^{p-1}(\kappa(d,\cdot)+q_1) - \kappa(d,\cdot)^p - p\kappa(d,\cdot)^{p-1}q_1, \end{aligned} \tag{100}$$

$\mathcal{L}$, $\psi(d,\cdot)$ and $\kappa(d,\cdot)$ are defined respectively in (8), (41) and (13). In this section, we study the linear operator L_d in the energy space $\mathcal{H}$ defined in (9). Note from (9) that we have

$$\|q\|_{\mathcal{H}} = [\phi(q,q)]^{1/2} < +\infty \tag{101}$$

where the inner product ϕ is defined by

$$\phi(q,r) = \phi\left(\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\right) = \int_{-1}^{1} \left(q_1 r_1 + q_1' r_1'(1-y^2) + q_2 r_2\right)\rho dy. \tag{102}$$

Using integration by parts and the definition of $\mathcal{L}$ (8), we have the following identity

$$\phi(q,r) = \int_{-1}^{1} \left(q_1\left(-\mathcal{L}r_1 + r_1\right) + q_2 r_2\right)\rho dy. \tag{103}$$

One of the major difficulties in the proof of the convergence in Theorem 3 comes from the fact that the linear operator L_d is not self-adjoint. In particular, standard spectral theory does not apply. Nevertheless, using a modified version of Proposition 2.3, one can directly show that

$$\lambda_n = 1 - n \text{ and } \mu_n = -2\frac{(p+1)}{p-1} - n, \quad n \in \mathbb{N},$$

are eigenvalues of L_d and that the corresponding eigenfunctions are polynomials of degree n that span the whole space $\mathcal{H}$. Note that L_d has one positive direction ($\lambda = 1$) and one null direction ($\lambda = 0$), and the rest of the spectrum is negative ($\lambda \leq -1$). Then, one can expand the solution q according to the positive, null and negative part of the spectrum. The general strategy is to obtain properties of L_d with the hope to extend them to the nonlinear equation (99). From the Hamiltonian structure of the original equation or the non self-adjoint character of L_d, few examples are known in the literature where this strategy works. Indeed, the problem we are looking to is related to the so called existence and asymptotic stability of blow-up profile in the energy space (for L^2 critical generalized KdV, see Martel and Merle [13] and for L^2 critical NLS equation, see Merle and Raphaël [14]). In this section,

- We first show that $\lambda = 1$ and $\lambda = 0$ are eigenvalues of L_d and compute explicitly the corresponding eigenfunctions (Lemma 4.2).

- Then, we compute explicitly eigenfunctions of L_d^* (the adjoint of L_d with respect to the inner product ϕ) for $\lambda = 1$ and $\lambda = 0$, which will give projections on the corresponding eigenspace of L_d.

- Finally, subtracting from the solution the projections on eigenspaces of $\lambda = 1$ and $\lambda = 0$, we obtain the projection on the negative part of the spectrum. However, to control that part, no spectral theory will be used, because of the weakness and the technical character of such an approach in the Hamiltonian context. Instead, we use a different approach based on the nonlinear equation (99) and its dispersive relation. See similar results in the context of KdV and NLS equations in the references.

4.1 The conjugate operator L_d^*

In the following, we compute L_d^*.

Lemma 4.1 (The conjugate operator of L_d with respect to the inner product ϕ) *For any $|d| < 1$, the operator L_d^* conjugate of L_d with respect to ϕ is given by*

$$L_d^* \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} R_d(r_2) \\ -\mathcal{L}r_1 + r_1 + \frac{p+3}{p-1} r_2 + 2yr_2' - \frac{8}{(p-1)} \frac{r_2}{(1-y^2)} \end{pmatrix} \tag{104}$$

for any $(r_1, r_2) \in (\mathcal{D}(\mathcal{L}))^2$, where $r = R_d(r_2)$ is the unique solution of

$$-\mathcal{L}r + r = \mathcal{L}r_2 + \psi(d, y) r_2. \tag{105}$$

Remark: The domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ defined in (8) is the set of all $r \in L^2_\rho$ such that $\mathcal{L}r \in L^2_\rho$.

Proof of Lemma 4.1: By definition of L_d^*, we have for all $q = (q_1, q_2)$ and $r = (r_1, r_2)$ in $\mathcal{H}$,

$$\phi\left(L_d(q), r\right) = \phi\left(q, L_d^*(r)\right). \tag{106}$$

Using (100) and (103), we write for arbitrary (q_1, q_2) and (r_1, r_2) in $\mathcal{H}$,

$$\begin{aligned} \phi\left(L_d(q), r\right) &= \phi\left(\begin{pmatrix} q_2 \\ \mathcal{L}q_1 + \psi(d,y)q_1 - \frac{p+3}{p-1}q_2 - 2yq_2' \end{pmatrix}, \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\right) \\ &= \int_{-1}^{1} \left(q_2\left(-\mathcal{L}r_1 + r_1\right) + \left(\mathcal{L}q_1 + \psi(d,y)q_1 - \frac{p+3}{p-1}q_2 - 2yq_2'\right) r_2\right) \rho dy. \end{aligned}$$

Integrating by parts, we write

$$\begin{aligned} -2\int_{-1}^{1} yq_2' r_2 \rho dy &= 2\int_{-1}^{1} q_2 \left(r_2\rho + yr_2'\rho + yr_2\rho'\right) dy \\ &= 2\int_{-1}^{1} q_2 \left(r_2\rho + yr_2'\rho - yr_2 \frac{4}{(p-1)} \frac{y\rho}{(1-y^2)}\right) dy \\ &= \int_{-1}^{1} q_2 \left(2\frac{p+3}{p-1} r_2 + 2yr_2' - \frac{8r_2}{(p-1)(1-y^2)}\right) \rho dy. \end{aligned} \tag{107}$$

Therefore, since $\mathcal{L}$ is self-adjoint, we get

$$\begin{aligned}\phi\left(L_d(q), r\right) &= \int_{-1}^{1} q_1\left(\mathcal{L}r_2 + \psi(d,y)r_2\right)\rho \\ &+ \int_{-1}^{1} q_2\left(-\mathcal{L}r_1 + r_1 + \frac{p+3}{p-1}r_2 + 2yr_2' - \frac{8}{(p-1)}\frac{r_2}{(1-y^2)}\right)\rho. \quad (108)\end{aligned}$$

Now, we define $R_d : L^2_\rho(-1,1) \to L^2_\rho(-1,1)$ by (105). Note that R_d is well defined, whenever r_2 and $\mathcal{L}r_2$ are in L^2_ρ (or $r_2 \in \mathcal{D}(\mathcal{L})$), since $\mathcal{H}_0$ equipped with the inner product

$$< u, v >_{\mathcal{H}_0} = \int_{-1}^{1}\left(u'(y)v'(y)(1-y^2) + u(y)v(y)\right)\rho(y)dy = \int_{-1}^{1}(-\mathcal{L}u(y) + u(y))v(y)\rho(y)dy \quad (109)$$

is a Hilbert space. Using (105), (108) and (103), we see that

$$\begin{aligned}\phi\left(L_d(q), r\right) &= \int_{-1}^{1} q_1\left(-\mathcal{L}R_d(r_2) + R_d(r_2)\right)\rho \\ &+ \int_{-1}^{1} q_2\left(-\mathcal{L}r_1 + r_1 + \frac{p+3}{p-1}r_2 + 2yr_2' - \frac{8}{(p-1)}\frac{r_2}{(1-y^2)}\right)\rho \\ &= \phi\left(\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \begin{pmatrix} R_d(r_2) \\ -\mathcal{L}r_1 + r_1 + \frac{p+3}{p-1}r_2 + 2yr_2' - \frac{8}{(p-1)}\frac{r_2}{(1-y^2)} \end{pmatrix}\right).\end{aligned}$$

Using the characterization of L_d^* by (106), we get (104). This concludes the proof of Lemma 4.1. ∎

4.2 Nonnegative directions of L_d

Let us now find nonnegative directions of L_d. We claim the following:

Lemma 4.2 (Nonnegative eigenvalues and eigenfunctions for L_d)
(i) For all $|d| < 1$, $\lambda = 1$ and $\lambda = 0$ are eigenvalues of the linear operator L_d and the corresponding eigenfunctions are respectively

$$F_1^d(y) = (1-d^2)^{\frac{p}{p-1}}\begin{pmatrix}(1+dy)^{-\frac{2}{p-1}-1} \\ (1+dy)^{-\frac{2}{p-1}-1}\end{pmatrix} \text{ and } F_0^d(y) = (1-d^2)^{\frac{1}{p-1}}\begin{pmatrix}\frac{y+d}{(1+dy)^{\frac{2}{p-1}+1}} \\ 0\end{pmatrix}. \quad (110)$$

(ii) Moreover, it holds for some $C_0 > 0$ and any $\lambda \in \{0,1\}$ that

$$\forall |d| < 1,\ \frac{1}{C_0} \le \|F_\lambda^d\|_{\mathcal{H}} \le C_0 \text{ and } \|\partial_d F_\lambda^d\|_{\mathcal{H}} \le \frac{C_0}{1-d^2}. \quad (111)$$

Proof :
(i) Since we know Proposition 1 and (31) that for any $(b,d) \in (-1,1)^2$, the function

$$G_{b,d}(y,s) = \kappa_0(1-d^2)^{\frac{1}{p-1}}\begin{pmatrix}(1+be^s+dy)^{-\frac{2}{p-1}} \\ -\frac{2be^s}{p-1}(1+be^s+dy)^{-\frac{2}{p-1}-1}\end{pmatrix} \quad (112)$$

is a particular solution to equation (7) put in the following vectorial form

$$\frac{\partial}{\partial s}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ \mathcal{L}w_1 - \frac{2(p+1)}{(p-1)^2}w_1 + |w_1|^{p-1}w_1 - \frac{p+3}{p-1}w_2 - 2y\partial_y w_2 \end{pmatrix}, \tag{113}$$

it follows that $\partial_b G_{0,d}$ and $\partial_d G_{0,d}$ are particular solutions to the linearized equation around $G_{0,d} = \kappa(d,\cdot)$, which is precisely $\partial_s(w_1,w_2) = L_d(w_1,w_2)$ by definition of L_d (100). Since we have from (112)

$$\begin{aligned} \partial_b G_{0,d}(y,s) &= -\frac{2\kappa_0 e^s}{p-1}(1-d^2)^{\frac{1}{p-1}}\begin{pmatrix} (1+dy)^{-\frac{2}{p-1}-1} \\ (1+dy)^{-\frac{2}{p-1}-1} \end{pmatrix}, \\ \partial_d G_{0,d}(y,s) &= \begin{pmatrix} \partial_d\kappa(d,y) \\ 0 \end{pmatrix} = -\frac{2\kappa_0(1-d^2)^{\frac{1}{p-1}-1}}{p-1}\begin{pmatrix} (y+d)(1+dy)^{-\frac{2}{p-1}-1} \\ 0 \end{pmatrix} \end{aligned} \tag{114}$$

this concludes the proof of (i).
(ii) We first give the following claim:

Claim 4.3 *Consider for some $\alpha > -1$ and $\beta \in \mathbb{R}$ the following integral*

$$I(d) = \int_{-1}^{1} \frac{(1-y^2)^\alpha}{(1+dy)^\beta}dy.$$

Then, there exists $K(\alpha,\beta) > 0$ such that the following holds for all $d \in (-1,1)$,
(i) if $\alpha+1-\beta > 0$, then $\frac{1}{K} \le I(d) \le K$,
(ii) if $\alpha+1-\beta = 0$, then $\frac{1}{K} \le I(d)/|\log(1-d^2)| \le K$,
(iii) if $\alpha+1-\beta < 0$, then $\frac{1}{K} \le I(d)(1-d^2)^{-(\alpha+1)+\beta} \le K$.

Proof: Since $I(d)$ is continuous, positive and even, it is enough to show the desired estimate as $d \to -1$. Note first that (i) follows from the Lebesgue theorem. For (ii) and (iii), we perform the following change of variables $y = 1 + \frac{d+1}{d}z$ and write

$$I(d) = \frac{(1+d)^{\alpha+1-\beta}}{(-d)^{\alpha+1}}\int_0^{\frac{-2d}{d+1}} (2 + \frac{d+1}{d}z)^\alpha \frac{z^\alpha}{(1+z)^\beta}dz. \tag{115}$$

In the case (iii), we just use the Lebesgue theorem to see that $I(d)(1+d)^{-(\alpha+1)+\beta} \to 2^\alpha \int_0^\infty \frac{z^\alpha}{(1+z)^\beta}dz$. In the case (ii), note that the integral in (115) behaves like $2^\alpha|\log\left(\frac{-2d}{d+1}\right)|$ to get the result and conclude the proof of Claim 4.3. ■

Using (46) together with the definition of F^d_λ (110) and straightforward computations, we see that for $\lambda = 1$ or 0, $i = 1$ or 2 and $|d| < 1$,

$$\begin{aligned} |F^d_{\lambda,i}(y)| &\le C\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}}, & |\partial_y F^d_{\lambda,i}(y)| &\le C\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}+1}} \\ |\partial_d F^d_{\lambda,i}(y)| &\le C\frac{(1-d^2)^{\frac{1}{p-1}-1}}{(1+dy)^{\frac{2}{p-1}}}, & |\partial^2_{d,y} F^d_{\lambda,i}(y)| &\le C\frac{(1-d^2)^{\frac{1}{p-1}-1}}{(1+dy)^{\frac{2}{p-1}+1}}. \end{aligned}$$

Using this and Claim 4.3, we see that the upper bounds in (111) holds. For the lower bounds, we just write from Lemma 2.8

$$\|F_\lambda^d\|_{\mathcal{H}} \geq \|F_{\lambda,1}^d\|_{\mathcal{H}_0} \geq \frac{1}{C_0}\|\mathcal{T}_d(F_{\lambda,1}^d)\|_{\mathcal{H}_0}.$$

Since we see from (110) that $\mathcal{T}_d(F_{1,1}^d) = 1 - dy$ and $\mathcal{T}_d(F_{0,1}^d) = y$, we get the lower bounds in (111) holds. This concludes the proof of Lemma 4.2. ■

4.3 Nonnegative directions of L_d^* and corresponding projections for L_d

Let us now find the eigenfunctions of L_d^* associated to the eigenvalues $\lambda = 1$ and $\lambda = 0$.

Lemma 4.4 (Eigenfunctions of L_d^* associated with the eigenvalues $\lambda = 1$ and $\lambda = 0$)
(i) **(Existence)** *For all $|d| < 1$ and $\lambda \in \{0, 1\}$, there exists $W_\lambda^d \in \mathcal{H}$ continuous in terms of d such that $L_d^*\left(W_\lambda^d\right) = \lambda W_\lambda^d$ where*

$$W_{1,2}^d(y) = c_1(d)\frac{1-y^2}{(1+dy)^{\frac{2}{p-1}+1}}, \quad W_{0,2}^d(y) = c_0(d)\frac{y+d}{(1+dy)^{\frac{2}{p-1}+1}}, \tag{116}$$

$W_{\lambda,1}^d$ is uniquely determined by

$$-\mathcal{L}r + r = \left(\lambda - \frac{p+3}{p-1}\right) r_2 - 2yr_2' + \frac{8}{p-1}\frac{r_2}{1-y^2} \tag{117}$$

with $r_2 = W_{\lambda,2}^d$ and the C^1 function $c_\lambda(d) > 0$ fixed by the relation

$$\phi(W_\lambda^d, F_\lambda^d) = 1. \tag{118}$$

(ii) **(Orthogonality)** *For all $|d| < 1$ and $\lambda \in \{0, 1\}$, we have $\phi(W_\lambda^d, F_{1-\lambda}^d) = 0$.*
(iii) **(Normalization)** *There exists $C_0 > 0$ such that for $\lambda = 1$ or 0 and $|d| < 1$,*

$$\|W_\lambda^d\|_{\mathcal{H}} \leq C_0 \text{ and } \|\partial_d W_\lambda^d\|_{\mathcal{H}} \leq \frac{C_0}{1-d^2}. \tag{119}$$

Proof:
(ii) This is a standard orthogonality relation between eigenfunctions of L_d and L_d^* for different eigenvalues.
(i) We restrict ourselves to the proof of existence of $(W_{\lambda,1}^d, W_{\lambda,2}^d)$ such that (116) and (117) hold with $c_\lambda(d) = 1$. Indeed:
- The fact that $W_\lambda^d \in \mathcal{H}$ will follow from (iii).
- The condition (118) follows directly from (116) and (117) as we show now:
Using (103) and (117), we write

$$\begin{aligned}
\phi\left(W_\lambda^d, F_\lambda^d\right) &= \int_{-1}^1 \left(\left(-\mathcal{L}W_{\lambda,1}^d + W_{\lambda,1}^d\right) F_{\lambda,1}^d + W_{\lambda,2}^d F_{\lambda,2}^d\right)\rho dy \\
&= \int_{-1}^1 \left(\left(\lambda - \frac{p+3}{p-1}\right) W_{\lambda,2}^d - 2y{W_{\lambda,2}^d}' + \frac{8}{p-1}\frac{W_{\lambda,2}^d}{(1-y^2)}\right) F_{\lambda,1}^d \rho dy \qquad (120) \\
&+ \int_{-1}^1 W_{\lambda,2}^d F_{\lambda,2}^d \rho dy
\end{aligned}$$

When $\lambda = 1$, we use (107), Lemma 4.2 (in particular the fact that $F^d_{1,1} = F^d_{1,2}$) and (116) to write

$$\begin{aligned}\phi\left(W_1^d, F_1^d\right) &= \int_{-1}^{1} W^d_{1,2}\left(\frac{3p+1}{p-1}F^d_{1,1} + 2yF^{d\ \prime}_{1,1}\right)\rho dy\\ &= c_1(d)(1-d^2)^{\frac{p}{p-1}}\int_{-1}^{1}\frac{1-y^2}{(1+dy)^{\frac{2}{p-1}+1}}\left(\frac{1+dy+2(p+1)/(p-1)}{(1+dy)^{\frac{2}{p-1}+2}}\right)\rho dy\end{aligned}$$

which shows the integral of a positive function on $(-1,1)$. Therefore, one can fix $c_1(d)$ such that $\phi\left(W_1^d, F_1^d\right) = 1$. Using Claim 4.3, we see that for $\lambda = 1$, the following holds:

$$0 < c_\lambda(d) \le C(1-d^2)^{\frac{1}{p-1}} \text{ and } |c'_\lambda(d)| \le C(1-d^2)^{\frac{1}{p-1}-1}. \tag{121}$$

When $\lambda = 0$, we use (120), Lemma 4.2 and (116) (note in particular that $W^d_{0,2}(y) = \frac{c_0(d)}{(1-d^2)^{\frac{1}{p-1}}}F^d_{0,1}(y)$) to write

$$\begin{aligned}&\phi\left(W_0^d, F_0^d\right)\\ =\ & \frac{c_0(d)}{(1-d^2)^{\frac{1}{p-1}}}\left[\int_{-1}^{1}\left(-\frac{p+3}{p-1} + \frac{8}{(p-1)(1-y^2)}\right)\left(F^d_{0,1}\right)^2\rho dy + \int_{-1}^{1}F^{d\ 2}_{0,1}(y\rho)'dy\right]\\ =\ & \frac{c_0(d)}{(1-d^2)^{\frac{1}{p-1}}}\int_{-1}^{1}\left(-\frac{p+3}{p-1} + \frac{8}{(p-1)(1-y^2)} + 1 - \frac{4y^2}{(p-1)(1-y^2)}\right)F^{d\ 2}_{0,1}\rho dy\\ =\ & c_0(d)(1-d^2)^{\frac{1}{p-1}}\frac{4}{p-1}\int_{-1}^{1}\frac{(y+d)^2}{(1+dy)^{\frac{4}{p-1}+2}}\frac{\rho}{1-y^2}dy\end{aligned}$$

showing a positive integral. Therefore, one can fix $c_0(d)$ such that $\phi\left(W_0^d, F_0^d\right) = 1$. Using Claim 4.3, we see that (121) holds.
We now start the proof of the existence of $(W^d_{\lambda,1}, W^d_{\lambda,2})$ satisfying (116) and (117). The following claim allows us to conclude:

Claim 4.5
(i) For any $r_2 \in \mathcal{H}_0$, the equation (117) has a unique solution $r \in \mathcal{H}_0$ (10) such that

$$\|r\|_{\mathcal{H}_0} \le C\|r_2\|_{\mathcal{H}_0}. \tag{122}$$

(ii) For any $|d| < 1$, $\lambda \in \mathbb{R}$ and $r \in \mathcal{H}_0$, we have the following equivalence:
$L^*_d(r) = \lambda r$ if and only if the function $e^{-\lambda s}r_2(y)$ is a solution to the equation

$$\partial^2_{ss}w = \mathcal{L}w + \psi(d,y)w - \frac{p+3}{p-1}\partial_s w - 2y\partial^2_{y,s}w + \frac{8}{p-1}\frac{\partial_s w}{1-y^2} \tag{123}$$

and r_1 is a solution to (117).

Indeed, let us first use this claim to conclude the proof of (i) of Lemma 4.4. We first consider the case $d = 0$.

Case $d = 0$. One can check by hand that $e^{-s}(1-y^2)$ and y are solutions to (123) (one may use (27) when $\lambda = 0$). Therefore, from Claim 4.5, the function $(W^0_{\lambda,1}, W^0_{\lambda,2})$ where

$W^0_{1,2}(y) = 1 - y^2$, $W^0_{0,2}(y) = y$ and $W^d_{\lambda,1}$ is the unique solution of (117) with $r_2 = W^d_{\lambda,2}$ is an eigenfunction of L^*_d corresponding to the eigenvalue λ.

Case $d \neq 0$. From the case $d = 0$, consider $(q_1, q_2) \in \mathcal{H}$ where

$$q_2(y) = 1 - y^2 \text{ (respectively } q_2(y) = y) \tag{124}$$

an eigenfunction of L^*_0 corresponding to the eigenvalue $\lambda = 1$ (respectively $\lambda = 0$). If we introduce

$$w(y, s) = e^{-\lambda s} q_2(y), \tag{125}$$

then we see from (ii) of Claim 4.5 that w is a solution to equation (123) with $d = 0$. If we introduce $W(Y, S) = \mathcal{T}_d w$ defined by (33), then we see from Lemma 2.7 and the fact that $\frac{\partial_s w}{1-y^2} = \frac{(1+dY)^{\frac{2p}{p-1}}}{(1-d^2)^{\frac{p}{p-1}}} \frac{\partial_S W}{1-Y^2}$ that $W(Y, S)$ satisfies equation (123) too. Since by (125), (33) and (124), we see that

$$W(Y, S) = \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dY)^{\frac{2}{p-1}}} w\left(\frac{Y+d}{1+dY}, S - \log \frac{1+dY}{\sqrt{1-d^2}}\right)$$

$$= \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dY)^{\frac{2}{p-1}}} e^{-\lambda(S - \log \frac{1+dY}{\sqrt{1-d^2}})} q_2\left(\frac{Y+d}{1+dY}\right) = e^{-\lambda S} \frac{(1-d^2)^{\frac{1}{p-1} - \frac{\lambda}{2}}}{(1+dY)^{\frac{2}{p-1} - \lambda}} q_2\left(\frac{Y+d}{1+dY}\right),$$

which is of the form $e^{-\lambda S} Q_2(Y)$ with $Q_2(Y) = q_2\left(\frac{y+d}{1+yd}\right)(1+dy)^{-\frac{2}{p-1}+\lambda}$ with

$$Q^d_2(y) = (1-d^2)\frac{1-y^2}{(1+dy)^{\frac{2}{p-1}+1}} \text{ (respectively } Q^d_2(y) = \frac{y+d}{(1+dy)^{\frac{2}{p-1}+1}}),$$

using (ii) of Claim 4.5, we see that (Q^d_1, Q^d_2) where $Q^d_1(y)$ is uniquely determined by the equation (117) with $r_2 = Q^d_2$ is an eigenvalue of L^*_d for the eigenvalue λ. Remains to prove Claim 4.5 to conclude the proof of (i) of Lemma 4.4.

Proof of Claim 4.5: Note first that (ii) is classical and straightforward from the expression of L^*_d (104).
(i) If $r_2 \in \mathcal{H}_0$ and

$$f \equiv \left(\lambda - \frac{p+3}{p-1}\right) r_2 - 2yr'_2 + \frac{8}{(p-1)} \frac{r_2}{(1-y^2)}, \tag{126}$$

then we write from the Cauchy-Schwartz inequality and the Hardy estimate (21) for all $h \in \mathcal{H}_0$,

$$\begin{aligned} \left|\int_{-1}^1 fh\rho\right| &\leq C\|r_2\|_{L^2_\rho}\|h\|_{L^2_\rho} + C\left(\|r'_2\sqrt{1-y^2}\|_{L^2_\rho} + \|\frac{r_2}{\sqrt{1-y^2}}\|_{L^2_\rho}\right)\|\frac{h}{\sqrt{1-y^2}}\|_{L^2_\rho} \\ &\leq C\|r_2\|_{\mathcal{H}_0}\|h\|_{\mathcal{H}_0}. \end{aligned}$$

Therefore, the linear form $h \to \int_{-1}^1 f(y)h(y)\rho(y)dy$ is in the dual of $\mathcal{H}_0$ and $\|f\|_{\mathcal{H}'_0} \leq C\|r_2\|_{\mathcal{H}_0}$. Since $\mathcal{H}_0$ equipped with the inner product defined in (109) is a Hilbert space, there is a unique $r \in H_0$ such that

$$\forall h \in \mathcal{H}_0,\ < r, h >_{\mathcal{H}_0} = \int_{-1}^1 f(y)h(y)\rho(y)dy \text{ and } \|r\|_{\mathcal{H}_0} \leq \|f\|_{\mathcal{H}'_0} \leq C\|r_2\|_{\mathcal{H}_0}. \tag{127}$$

Using (109), we see that r is the unique solution of equation (117) and (122) follows from (127). This concludes the proof of Claim 4.5. ■

(iii) **(Normalization)** Since $W^d_{\lambda,1}$ and $\partial_d W^d_{\lambda,1}$ are solutions to equation (117) respectively with $r_2 = W^d_{\lambda,2}$ and $r_2 = \partial_d W^d_{\lambda,2}$, we see from (i) in Claim 4.5 that for $\lambda = 1$ or 0 and $|d| < 1$,

$$\|W^d_\lambda\|_{\mathcal{H}} \le C_0 \|W^d_{\lambda,2}\|_{\mathcal{H}_0} \text{ and } \|\partial_d W^d_\lambda\|_{\mathcal{H}} \le C_0 \|\partial_d W^d_{\lambda,2}\|_{\mathcal{H}_0}. \tag{128}$$

Using (46) together with the definition of $W^d_{l,2}$, (121) and straightforward computations, we see that for $\lambda = 1$ or 0 and $|d| < 1$,

$$\begin{array}{rclrcl} |W^d_{\lambda,2}(y)| &\le& C\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}}, & |\partial_y W^d_{\lambda,2}(y)| &\le& C\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}+1}} \\ |\partial_d W^d_{\lambda,2}(y)| &\le& C\frac{(1-d^2)^{\frac{1}{p-1}-1}}{(1+dy)^{\frac{2}{p-1}}}, & |\partial^2_{d,y} W^d_{\lambda,2}(y)| &\le& C\frac{(1-d^2)^{\frac{1}{p-1}-1}}{(1+dy)^{\frac{2}{p-1}+1}}. \end{array}$$

Since we have from this, Claim 4.3 and the definition of the norm in $\mathcal{H}_0$, $\|W^d_{\lambda,2}\|_{\mathcal{H}_0} + (1-d^2)\|\partial_d W^d_{\lambda,2}\|_{\mathcal{H}_0} \le C_0$, we see that (119) follows from (128). This concludes the proof of Lemma 4.4. ■

4.4 Expansion of q with respect to the eigenspaces of L_d

In the following, we expand any $q \in \mathcal{H}$ with respect to the eigenspaces of L_d partially computed in Lemma 4.2. We claim the following:

Definition 4.6 (Expansion of q with respect to the eigenspaces of L_d) *Consider $q \in \mathcal{H}$ and introduce for $\lambda = 1$ and $\lambda = 0$*

$$\pi^d_\lambda(q) = \phi\left(W^d_\lambda, q\right) \tag{129}$$

where W^d_λ is the eigenfunction of L^*_d computed in Lemma 4.4, and $\pi^d_-(q) = q_-$ defined by

$$q = \pi^d_1(q) F^d_1(y) + \pi^d_0(q) F^d_0(y) + \pi^d_-(q). \tag{130}$$

Applying the operator π^d_λ to (130), we write

$$\pi^d_\lambda(q) = \pi^d_\lambda(q)\pi^d_\lambda(F^d_\lambda) + \pi^d_{1-\lambda}(q)\pi^d_\lambda(F^d_{1-\lambda}) + \pi^d_\lambda\left(\pi^d_-(q)\right).$$

Since

$$\pi^d_\lambda(F^d_\mu) = \delta_{\lambda,\mu} \tag{131}$$

by (118) and (ii) of Lemma 4.4, this yields

$$\phi(F^d_\lambda, q_-) = \pi^d_\lambda\left(\pi^d_-(q)\right) = 0. \tag{132}$$

Therefore, we have

$$\pi^d_-(q) \in \mathcal{H}^d_- \equiv \left\{ r \in \mathcal{H} \ \mid \ \pi^d_1(r) = \pi^d_0(r) = 0 \right\}. \tag{133}$$

Remark: Note that if $q \in \mathcal{H}_-^d$, then $\pi_-^d(q) = q$ (just use (130) and (133)) and $L_d q \in H_-^d$. Indeed, using the definition of π_λ^d (129), (106) and Lemma 4.4, we write $\pi_\lambda^d(L_d q) = \phi\left(W_\lambda^d, L_d q\right) = \phi\left(L_d^* W_\lambda^d, q\right) = \phi\left(\lambda W_\lambda^d, q\right) = \lambda \pi_\lambda^d(q) = 0$. Moreover $\pi_-^d(F_\lambda^d) = 0$ for $\lambda = 0$ or 1 (just use (130) with $q = F_\lambda^d$ and (131)).

Remark: Note that $\pi_\lambda^d(q)$ is the projection of q on the eigenfunction of L_d associated to λ, and that $\pi_-^d(q)$ is the negative part of q.

4.5 Equivalent norms on $\mathcal{H}$ and $\mathcal{H}_-^d$ adapted to the dispersive structure

For the proof of the main theorem, we will need to prove in some sense dispersive estimates on $q_- = \pi_-^d(q)$ when q is a solution to (99). In order to achieve this, we need to manipulate a function of q_- (equivalent to the norm $\|q_-\|_{\mathcal{H}} = \phi(q_-, q_-)^{1/2}$ in $\mathcal{H}_-^d$) which will capture the dispersive character of the equation (99). Such a quantity will be

$$\begin{aligned}
\varphi_d(q, r) &= \int_{-1}^{1} \left(-\psi(d, y) q_1 r_1 + q_1' r_1' (1 - y^2) + q_2 r_2\right) \rho dy && (134)\\
&= \int_{-1}^{1} \left(-q_1 \left(\mathcal{L} r_1 + \psi(d, y) r_1\right) + q_2 r_2\right) \rho dy && (135)
\end{aligned}$$

where $\psi(d, y)$ is defined in (100). This bilinear form is in fact the second variation of $E(w(s))$ defined in (141) around $\kappa(d, y)$ (13), the stationary solution of (7), and can be seen as the energy norm in $\mathcal{H}_-^d$ (space where it will be definite positive). More precisely, we have the following:

Proposition 4.7 (Equivalence in $\mathcal{H}_-^d$ of the $\mathcal{H}$ norm and the energy norm) *There exists $C_0 > 0$ such that for all $|d| < 1$, the following holds:*
(i) **(Equivalence of norms in $\mathcal{H}_-^d$)** *For all $q_- \in \mathcal{H}_-^d$,*

$$\frac{1}{C_0} \|q_-\|_{\mathcal{H}}^2 \le \varphi_d(q_-, q_-) \le C_0 \|q_-\|_{\mathcal{H}}^2 .$$

(ii) **(Equivalence of norms in $\mathcal{H}$)** *For all $q \in \mathcal{H}$,*

$$\frac{1}{C_0} \|q\|_{\mathcal{H}} \le \left(\left|\pi_1^d(q)\right| + \left|\pi_0^d(q)\right| + \sqrt{\varphi_d(q_-, q_-)}\right) \le C_0 \|q\|_{\mathcal{H}}$$

where φ_d is given in (134) and q is expanded as in (130).

Remark: Note that φ_d is not positive in $\mathcal{H}$ (for example, $\varphi_d((1,0),(1,0)) = -\int \psi \rho dy < 0$). In particular, its quadratic form cannot be considered as a norm in $\mathcal{H}$. However, we will show that it is definite positive on the space $\mathcal{H}_-^d$, uniformly for $|d| < 1$, which gives the control of the norm by φ_d (independent of d). A remarkable fact, is that the constant C_0 is independent of d. In the following, we reduce the proof of Proposition 4.7 to the proof of the fact that the following approximation of φ_d defined for $\epsilon > 0$ is nonnegative:

$$\begin{aligned}
\varphi_{d,\epsilon}(q, r) &= \varphi_d(q, r) - \epsilon \int_{-1}^{1} (q_1' r_1' (1 - y^2) + \frac{2(p+1)}{(p-1)^2} q_1 r_1 + q_2 r_2) \rho dy && (136)\\
&= \int_{-1}^{1} q_1 \left(-(1-\epsilon)\mathcal{L} r_1 + \left(-(1-\epsilon)\psi(d, y) - \epsilon \frac{2p(p+1)}{(p-1)^2} \frac{(1-d^2)}{(1+dy)^2}\right) r_1\right) \rho dy && (137)\\
&+ (1-\epsilon) \int_{-1}^{1} q_2 r_2 \rho dy.
\end{aligned}$$

We claim that the following lemma directly implies Proposition 4.7:

Lemma 4.8 (Reduction of the proof of Proposition 4.7) *There exists* $\epsilon_0 \in (0,1)$ *such that for all* $|d| < 1$ *and* $q_- \in \mathcal{H}^d_-$, $\varphi_{d,\epsilon_0}(q_-, q_-) \geq 0$ *where* φ_{d,ϵ_0} *is defined in (136).*

Remark: One could choose other approximations of φ_d, but our choice (136) is particularly well adapted for the proof, as it gives a simple form after the Lorentz transform in similarity variables given in Lemma 2.6. See the proof of Lemma 4.10 below.

Indeed, let us first assume Lemma 4.8 and prove Proposition 4.7.

Lemma 4.8 implies Proposition 4.7:

Proof of (i): For the upper bound, just note that since we easily have

$$\frac{(1-d^2)(1-y^2)}{(1+dy)^2} \leq 1, \text{ hence } |\psi(d,y)| \leq \frac{C}{1-y^2}$$

we see from the definitions of φ_d (134) and the Hardy-Sobolev estimate (21) that for any $|d| < 1$ and q and r in $\mathcal{H}$,

$$|\varphi_d(q,r)| \leq \|q\|_{\mathcal{H}}\|r\|_{\mathcal{H}} + C\left\|\frac{q_1}{\sqrt{1-y^2}}\right\|_{L^2_\rho}\left\|\frac{r_1}{\sqrt{1-y^2}}\right\|_{L^2_\rho} \leq C_0\|q\|_{\mathcal{H}}\|r\|_{\mathcal{H}}. \tag{138}$$

For the lower bound, fix $\epsilon = \epsilon_0$ defined in Lemma 4.8, take $|d| < 1$, $q_- \in \mathcal{H}^d_-$ and write

$$0 \leq \varphi_{d,\epsilon_0}(q_-, q_-) = \varphi_d(q_-, q_-) - \epsilon_0 \int_{-1}^{1} \left(q_{-,1}{}'^2(1-y^2) + q_{-,2}^2 + \frac{2(p+1)}{(p-1)^2}q_{-,1}^2\right)\rho dy. \tag{139}$$

Therefore,

$$\varphi_d(q_-, q_-) \geq \alpha_0\epsilon_0 \int_{-1}^{1} \left(q_{-,1}'^2(1-y^2) + q_{-,1}^2 + q_{-,2}^2\right)\rho dy = \alpha_0\epsilon_0\|q_-\|^2_{\mathcal{H}}$$

for some positive α_0 which is the conclusion of (i).

Proof of (ii): Using the definition of ϕ (102) and (130), we write

$$\|q\|^2_{\mathcal{H}} = \phi(q,q) = \left(\pi^d_1(q)\right)^2 \|F^d_1\|^2_{\mathcal{H}} + \left(\pi^d_0(q)\right)^2 \|F^d_0\|^2_{\mathcal{H}} + \|q_-\|^2_{\mathcal{H}}.$$

Using (111), we get the following equivalence of norms:

$$\frac{1}{C}\|q\|_{\mathcal{H}} \leq \sum_{\lambda=0}^{1}\left|\pi^d_\lambda(q)\right| + \|q_-\|_{\mathcal{H}} \leq C\|q\|_{\mathcal{H}}. \tag{140}$$

Since $q_- \in \mathcal{H}^d_-$ by (133), we can use (i) to conclude. This concludes the proof of Proposition assuming Lemma 4.8. ■

Let us now prove Lemma 4.8.

Proof of Lemma 4.8: We proceed in 3 parts:

- In Part 1, we find a subspace of $\mathcal{H}$ of codimension 2 where $\varphi_{d,\epsilon}$ is nonnegative.
- In Part 2, we find a plane in $\mathcal{H}$, where $\varphi_{d,\epsilon}$ is negative and which is orthogonal to $\mathcal{H}_-^d$ with respect to $\varphi_{d,\epsilon}$.
- In Part 3, we proceed by contradiction and prove that $\varphi_{d,\epsilon}$ is nonnegative on $\mathcal{H}_-^d$.

Part 1 : $\varphi_{d,\epsilon}$ is nonnegative on a subspace of codimension 2
We claim the following:

Lemma 4.9 ($\varphi_{d,\epsilon}$ is nonnegative on a subspace of codimension 2) *There exists $\epsilon_1 > 0$ such that for all $|d| < 1$ and $\epsilon \in (0, \epsilon_1]$, $\varphi_{d,\epsilon}$ is nonnegative on the subspace*

$$E_2 = \left\{ q \in \mathcal{H} \quad | \quad \int_{-1}^{1} \mathcal{T}_{-d}(q_1)\rho(y)dy = \int_{-1}^{1} \mathcal{T}_{-d}(q_1)y\rho(y)dy = 0 \right\} \tag{141}$$

where $\mathcal{T}_{-d}$ is defined in (33).

Proof: Define from (26) $\epsilon_1 = \min\left(1, \frac{\gamma_1-\gamma_2}{\frac{2(p+1)}{(p-1)^2}-\gamma_2}\right) > 0$ and fix $\epsilon \in (0, \epsilon_1]$. We consider $(u_1, u_2) \in E_2$, and write from (137)

$$\begin{aligned} \varphi_{d,\epsilon}(u,u) &= \textstyle\int_{-1}^{1} u_1 \left(-(1-\epsilon)\mathcal{L}u_1 + \left[-(1-\epsilon)\psi(d,y) - \epsilon\frac{2p(p+1)}{(p-1)^2}\frac{(1-d^2)}{(1+dy)^2}\right]u_1\right)\rho(y)dy \\ &+ (1-\epsilon)\textstyle\int u_2^2\rho(y)dy. \end{aligned} \tag{142}$$

If $U_1 = \mathcal{T}_{-d}u_1$, then $u_1 = \mathcal{T}_d U_1$ and we have from (33) and (42),

$$\begin{aligned} u_1(y) &= \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}}U_1(z) \text{ with } z = \frac{y+d}{1+dy}, \\ \mathcal{L}u_1(y) + \psi(d,y)u_1(y) &= \frac{(1-d^2)^{\frac{1}{p-1}+1}}{(1+dy)^{\frac{2}{p-1}+2}}\left(\mathcal{L}U_1(z) + \frac{2(p+1)}{p-1}U_1(z)\right), \\ \rho(y)dy &= \frac{(1+dy)^{\frac{2(p+1)}{p-1}}}{(1-d^2)^{\frac{p+1}{p-1}}}\rho(z)dz, \\ 0 &= \int U_1(z)\rho(z)dz = \int U_1(z)z\rho(z)dz. \end{aligned} \tag{143}$$

Therefore, we see from (142) and Lemma 2.4 (use (143)) that

$$\begin{aligned} \varphi_{d,\epsilon}(u,u) &= \int_{-1}^{1} U_1\left(-(1-\epsilon)\mathcal{L}U_1 - \left(\frac{2(p+1)}{p-1} + \frac{2(p+1)}{(p-1)^2}\epsilon\right)U_1\right)\rho(z)dz \\ &+ (1-\epsilon)\int_{-1}^{1} u_2^2\rho(y)dy. \\ &\geq \left(-(1-\epsilon)\gamma_2 + (\gamma_1 - \frac{2(p+1)}{(p-1)^2}\epsilon)\right)\int_{-1}^{1} U_1^2\rho dy + (1-\epsilon)\int_{-1}^{1} u_2^2\rho(y)dy \geq 0 \end{aligned}$$

since $\epsilon \leq \epsilon_1$ hence $-(1-\epsilon)\gamma_2 + (\gamma_1 - \frac{2(p+1)}{(p-1)^2}\epsilon) \geq 0$ and $1-\epsilon \geq 0$. This concludes the proof of Lemma 4.9. ■

Part 2 : $\varphi_{d,\epsilon}$ is negative on a plane orthogonal to $\mathcal{H}_-^d$

We need to find $V_0^{d,\epsilon}$ and $V_1^{d,\epsilon}$ linearly independent in $\mathcal{H}$ such that $\varphi_{d,\epsilon}(V_\lambda^{d,\epsilon}, r) = 0$ for any $r \in \mathcal{H}_-^d$. Since we know from the definition of $\mathcal{H}_-^d$ (133) that

$$\forall r \in \mathcal{H}_-^d, \;\; \phi(W_1^d, r) = \pi_1^d(r) = 0 \text{ and } \phi(W_0^d, r) = \pi_0^d(r) = 0,$$

a convenient way to conclude is to find $V_1^{d,\epsilon}$ and $V_0^{d,\epsilon}$ such that

$$\forall q \in \mathcal{H}, \;\; \phi(W_1^d, q) = \varphi_{d,\epsilon}(V_1^{d,\epsilon}, q) \text{ and } \phi(W_0^d, q) = \varphi_{d,\epsilon}(V_0^{d,\epsilon}, q). \tag{144}$$

Then, we will show that $\varphi_{d,\epsilon}$ is negative on the plane spanned by $V_1^{d,\epsilon}$ and $V_0^{d,\epsilon}$. Consider $\epsilon > 0$ going to zero and take $|d| < 1$. We claim the following:

Lemma 4.10 *There exists $\epsilon_2 > 0$ such that for all $\epsilon \in (0, \epsilon_2]$ and $|d| < 1$:*
(i) There exist continuous functions $V_\lambda^{d,\epsilon}$ for $\lambda \in \{0, 1\}$ such that (144) holds.
(ii) Moreover, it holds that

$$\sup_{|d|<1} \left\| V_1^{d,\epsilon}(y) - \begin{pmatrix} -W_{1,2}^d(y) \\ W_{1,2}^d(y) \end{pmatrix} - \alpha_1(d) F_0^d(y) \right\|_{\mathcal{H}_0} + \left\| \epsilon V_0^{d,\epsilon}(y) + \alpha_2 F_0^d(y) \right\|_{\mathcal{H}_0} \to 0 \tag{145}$$

as $\epsilon \to 0^+$ where $\alpha_1(d)$ is continuous, $\alpha_2 > 0$, $W_{1,2}^d$ and F_0^d are defined in (116) and (110).
(iii) The bilinear form $\varphi_{d,\epsilon}$ is negative on the plane of $\mathcal{H}$ spanned by $V_0^{d,\epsilon}$ and $V_1^{d,\epsilon}$.

Remark: Note that in this lemma, we find explicit solutions for $V_\lambda^{d,\epsilon}$ which was not the case for KdV and NLS (see [13] and [14]).

Proof of Lemma 4.10: We proceed in 3 steps:
- In Step 1, we find a PDE satisfies by $V_\lambda^{d,\epsilon}$ and transform it with the Lorentz transform in similarity variables defined in (33).
- In Step 2, we solve the transformed PDE and find the asymptotic behavior of $V_\lambda^{d,\epsilon}$ as $\epsilon \to 0$, uniformly in $|d| < 1$, which gives (i) and (ii).
- In Step 3, we use that asymptotic behavior to show that $\varphi_{d,\epsilon}$ is negative on the plane spanned by $V_1^{d,\epsilon}$ and $V_0^{d,\epsilon}$, which gives (iii).

Step 1: Reduction to the solution of some PDE

(i) From the definitions of $\varphi_{d,\epsilon}$ (137) and ϕ (103), we see that in order to satisfy (144), it is enough to take

$$V_{\lambda,2}^{d,\epsilon} = W_{\lambda,2}^d / (1 - \epsilon) \tag{146}$$

and to prove the existence of $V_{\lambda,1}^{d,\epsilon}$ solution to

$$-(1-\epsilon)\mathcal{L}V_{\lambda,1}^{d,\epsilon} + \left(-(1-\epsilon)\psi(d,y) - \epsilon \frac{2p(p+1)}{(p-1)^2} \frac{(1-d^2)}{(1+dy)^2} \right) V_{\lambda,1}^{d,\epsilon} = -\mathcal{L}W_{\lambda,1}^d + W_{\lambda,1}^d. \tag{147}$$

In the following, we use use the Lorentz transform (33) and transform this equation to make it ready to solve using the spectral properties of $\mathcal{L}$ stated in Proposition 2.3. More precisely, we have the following:

Claim 4.11 (Reduction to an explicitly solvable PDE) *Consider $V_{\lambda,1}^{d,\epsilon}$ and introduce $\tilde v_{\lambda,1}^{d,\epsilon}$ defined by*

$$\tilde v_{\lambda,1}^{d,\epsilon} = \mathcal{T}_{-d} V_{\lambda,1}^{d,\epsilon} \tag{148}$$

where $\mathcal{T}_{-d}$ is defined in (33). Then,
(i) $V_{\lambda,1}^{d,\epsilon}$ is a solution to (147) if and only if $\tilde v_{\lambda,1}^{d,\epsilon}$ is a solution to the equation

$$(1-\epsilon)\mathcal{L}\tilde v_{\lambda,1}^{d,\epsilon}(z) + \left(-\gamma_1 + \frac{2(p+1)}{(p-1)^2}\epsilon\right)\tilde v_{\lambda,1}^{d,\epsilon}(z) = f_\lambda^d \equiv \frac{1-d^2}{(1-dz)^2}\mathcal{T}_{-d}\left(\mathcal{L}W_{\lambda,1}^d - W_{\lambda,1}^d\right) \tag{149}$$

and $\gamma_1 = -\frac{2(p+1)}{p-1}$ is defined in (26).
(ii) The linear form $h \to \int_{-1}^1 f_\lambda^d h\rho$ is continuous on $\mathcal{H}_0$ and for some $C_0 > 0$, we have

$$\forall d \in (-1,1),\ \ \|f_\lambda^d\|_{\mathcal{H}_0'} \le C_0\|W_\lambda^d\|_{\mathcal{H}} \le C_0^2.$$

Proof:
(i) Using (33) and Lemma 2.6, we see that

$$\begin{aligned} V_{\lambda,1}^{d,\epsilon}(y) &= \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} v_{\lambda,1}^{d,\epsilon}(z) \text{ with } z = \frac{y+d}{1+dy}, \\ \mathcal{L}V_{\lambda,1}^{d,\epsilon}(y) + \psi(d,y)V_{\lambda,1}^{d,\epsilon}(y) &= \frac{(1-d^2)^{\frac{1}{p-1}+1}}{(1+dy)^{\frac{2}{p-1}+2}}\left(\mathcal{L}v_{\lambda,1}^{d,\epsilon}(z) + \frac{2(p+1)}{p-1}v_{\lambda,1}^{d,\epsilon}(z)\right). \end{aligned} \tag{150}$$

Since $\frac{(1-dz)^2}{1-d^2} = \frac{1-d^2}{(1+dy)^2}$ and $\gamma_1 = -\frac{2(p+1)}{p-1}$ (see (26)), we see that equations (147) and (149) are equivalent.

(ii) Note from (33) that for all V_1 and V_2 in L_ρ^2,

$$\int_{-1}^1 V_1(Y)V_2(Y)\rho(Y)dY = \int_{-1}^1 \frac{1-d^2}{(1-dy)^2}v_1(y)v_2(y)\rho(y)dy \tag{151}$$

where $v_i = \mathcal{T}_{-d}V_i$. Therefore, using (149) and (151), we have for any $h \in \mathcal{H}_0$,

$$\int_{-1}^1 f_\lambda^d(z)h(z)\rho(z)dy = \int_{-1}^1 (\mathcal{L}W_{\lambda,1}^d - W_{\lambda,1}^d)H\rho = -\int_{-1}^1 (\partial_y W_{\lambda,1}^d \partial_y H(1-y^2) + W_{\lambda,1}^d H)\rho$$

where $H = \mathcal{T}_d h$. Therefore, using the continuity of $\mathcal{T}_d$ in $\mathcal{H}_0$ (see Lemma 2.8) and the bound on $\|W_\lambda^d\|_{\mathcal{H}}$ (119), we see that

$$\left|\int_{-1}^1 f_\lambda^d(z)h(z)\rho(z)dy\right| \le \|W_{\lambda,1}^d\|_{\mathcal{H}_0}\|H\|_{\mathcal{H}_0} \le C_0\|W_\lambda^d\|_{\mathcal{H}}\|h\|_{\mathcal{H}_0} \le C_0^2\|h\|_{\mathcal{H}_0}.$$

which closes the proof of Claim 4.11.

Step 2: Solution of equation (149) and asymptotic behavior as $\epsilon \to 0$
We prove (i) and (ii) of Lemma 4.10 in this step.

Proof of (i): Note first that since

$$\mathcal{T}_d(z) = F^d_{0,1} \tag{152}$$

by definition of $\mathcal{T}_d$ (33) and F^d_0 (110), we have from the definition of f^d_λ (149), (151), the expression of ϕ (103) and Lemma 4.4: for all $|d|<1$,

$$\int_{-1}^{1} f^d_\lambda(z) z\rho(z)dz = \int_{-1}^{1}\left(\mathcal{L}W^d_{\lambda,1}(y) - W^d_{\lambda,1}(y)\right) F^d_{0,1}(y)\rho(y)dy = -\delta_{\lambda,0}. \tag{153}$$

We have the following claim which follows directly from Proposition 2.3:

Claim 4.12 (Solution of equation (149)) *Consider*

$$f = \sum_{n=0}^{\infty} \tilde{f}_n h_n(y) \in \mathcal{H}'_0$$

where h_n are the eigenfunctions of $\mathcal{L}$ defined in Proposition 2.3. Then, for any $\epsilon \in (0,\frac{1}{2})$, the following equation

$$(1-\epsilon)\mathcal{L}v + (-\gamma_1 + \frac{2(p+1)}{(p-1)^2}\epsilon)v = f \tag{154}$$

has a unique solution in $\mathcal{H}_0$ given by

$$v = \sum_{n=0}^{\infty} \frac{\tilde{f}_n}{\gamma_n - \gamma_1 + \epsilon(\frac{2(p+1)}{(p-1)^2} - \gamma_n)} h_n \tag{155}$$

where $\gamma_n \le 0$ are the eigenvalues of $\mathcal{L}$ introduced in Proposition 2.3.

From this claim and (ii) in Claim 4.11, we see that for all $\epsilon \in (0,\frac{1}{2})$, $|d|<1$ and $\lambda = 1$ or $\lambda = 0$, equation (149) has a solution $\tilde{v}^{d,\epsilon}_{\lambda,1}$. Using (i) in Claim 4.11, we see that equation (147) has a solution $V^{d,\epsilon}_{\lambda,1}$ given by (148), which closes the proof of (i) of Lemma 4.10.

Proof of (ii):
When $\lambda = 1$, we see from (153), (29) and (27) that $\left(\tilde{f^d_1}\right)_1 = \int_{-1}^{1} f^d_1(z) z\rho(z)dz = 0$. Therefore, we see from Claim 4.12 and the definition of f^d_1 (149) that for ϵ small enough,

$$\sup_{|d|<1} \left\|\tilde{v}^{d,\epsilon}_{1,1} - v^*\right\|_{H_0} \le C\epsilon \|f^{d,\epsilon}_1\|_{H'_0} \le C_0\epsilon \text{ where } v^*(z) = \sum_{n\neq 1} \frac{\left(\tilde{f^d_1}\right)_n}{\gamma_n - \gamma_1} h_n(z)$$

is the unique solution of

$$\mathcal{L}v(z) - \gamma_1 v(z) = f^d_1(z) \text{ with } \int_{-1}^{1} v(z) z\rho(z)dz = 0. \tag{156}$$

Therefore, we see from (148) and Lemma 2.8 that for ϵ small enough,

$$\sup_{|d|<1} \left\|V^{d,\epsilon}_{1,1} - V^*\right\|_{\mathcal{H}_0} \le C_0\epsilon, \tag{157}$$

where $V^* = \mathcal{T}_d v^*$ is the unique solution of

$$\mathcal{L}V(y) + \psi(d,y)V(y) = \mathcal{L}W_{1,1}^d - W_{1,1}^d \text{ with } \int_{-1}^1 V(y)F_{0,1}^d(y)\frac{\rho(y)}{(1+dy)^2}dy = 0$$

(note that this equation is the version of (147) with $\epsilon = 0$ and use (151) together with (152) to get the orthogonality condition). Since

$$-\mathcal{L}W_{1,1}^d + W_{1,1}^d = \mathcal{L}W_{1,2}^d + \psi(d,y)W_{1,2}^d \text{ and } \mathcal{L}F_{0,1}^d + \psi(d,y)F_{0,1}^d = 0$$

(use the fact that $L_d^*\left(W_1^d\right) = W_1^d$ and $L_d(F_0^d) = 0$ from Lemmas 4.4 and 4.2), we see from uniqueness that $V^*(y) = -W_{1,2}^d(y) + \alpha_1(d)F_{0,1}^d(y)$ where

$$\alpha_1(d) = \int_{-1}^1 W_{1,2}^d(y)F_{0,1}^d(y)\frac{\rho(y)}{(1+dy)^2}dy / \int_{-1}^1 F_{0,1}^d(y)^2\frac{\rho(y)}{(1+dy)^2}dy$$

is continuous. Thus, the first identity in (145) follows from (157), (146) and (116).
When $\lambda = 0$, we see from (153), (29) and (27) that $\left(\tilde{f_0^d}\right)_1 = \int_{-1}^1 f_0^d(z)z\rho(z)dz = -1$. Therefore, since $h_1(y) = c_1 y$ by (27), we see from Claim 4.12 and (ii) in Claim 4.11 that for ϵ small enough,

$$\left\|\tilde{v}_{0,1}^{d,\epsilon}(z) + \frac{\alpha_2}{\epsilon}z\right\|_{\mathcal{H}_0} \le C\|f_0^d\|_{\mathcal{H}_0'} \le C_0 \text{ where } \alpha_2 = \frac{1}{\left(\frac{2(p+1)}{(p-1)^2} - \gamma_1\right)\int_{-1}^1 y^2\rho(y)dy} > 0 \quad (158)$$

(note from (26) that $\gamma_1 = -\frac{2(p+1)}{p-1} < 0$). Since the estimate for $V_{\lambda,2}^{d,\epsilon}$ follows from (146) and (116), we see that (145) follows from (158), (148) and (152). This closes the proof of (i) and (ii) in Lemma 4.10.

Step 3: Sign of $\varphi_{d,\epsilon}$ on the plane spanned by $V_1^{d,\epsilon}$ and $V_0^{d,\epsilon}$

Proof of (iii): We finish the proof of Lemma 4.10 here, by proving that $\varphi_{d,\epsilon}$ is negative on the plane of $\mathcal{H}$ spanned by $V_1^{d,\epsilon}$ and $V_0^{d,\epsilon}$. It is enough to find ϵ_4 such that for all $0 < \epsilon \le \epsilon_4$ and $|d| < 1$,

$$\varphi_{d,\epsilon}(V_0^{d,\epsilon}, V_0^{d,\epsilon}) < 0 \text{ and } \begin{vmatrix} \varphi_{d,\epsilon}(V_1^{d,\epsilon}, V_1^{d,\epsilon}) & \varphi_{d,\epsilon}(V_1^{d,\epsilon}, V_0^{d,\epsilon}) \\ \varphi_{d,\epsilon}(V_1^{d,\epsilon}, V_0^{d,\epsilon}) & \varphi_{d,\epsilon}(V_0^{d,\epsilon}, V_0^{d,\epsilon}) \end{vmatrix} > 0. \quad (159)$$

In the following, we will estimate $\varphi_{d,\epsilon}(V_\lambda^{d,\epsilon}, V_\mu^{d,\epsilon})$ as $\epsilon \to 0^+$, uniformly for $|d| < 1$, using the asymptotic behavior of $V_\lambda^{d,\epsilon}$ given in (145).
- First, using (144) and the expression of ϕ (103), we write $\varphi_{d,\epsilon}(V_\lambda^{d,\epsilon}, V_\mu^{d,\epsilon}) = \phi(V_\lambda^{d,\epsilon}, W_\mu^d)$ for $\lambda, \mu \in \{0,1\}$. Since $\phi(F_\lambda^d, W_\mu^d) = \delta_{\lambda,\mu}$ by Lemma 4.4, taking $\lambda = 0$ and $\mu \in \{0,1\}$, we see from (145) and the continuity of ϕ in $\mathcal{H}$ that

$$\sup_{|d|\le d_0}\left|\epsilon\varphi_{d,\epsilon}(V_0^{d,\epsilon}, V_\mu^{d,\epsilon}) + \alpha_2\delta_{0,\mu}\right| \to 0 \text{ as } \epsilon \to 0. \quad (160)$$

Now, taking $\lambda = \mu = 1$, we see from (145) that

$$\sup_{|d|\le d_0}\left|\varphi_{d,\epsilon}(V_1^{d,\epsilon}, V_1^{d,\epsilon}) - \phi\left(W_1^d, \begin{pmatrix} -W_{1,2}^d \\ W_{1,2}^d \end{pmatrix}\right)\right| \to 0 \text{ as } \epsilon \to 0. \quad (161)$$

Using (103) again together with (117), we write

$$\phi\left(W_1^d, \left(\begin{array}{c} -W_{1,2}^d \\ W_{1,2}^d \end{array}\right)\right) = \int_{-1}^1 W_{1,2}^d(y)\left(\mathcal{L}W_{1,1}^d(y) - W_{1,1}^d(y) + W_{1,2}^d(y)\right)\rho(y)dy$$

$$= \int_{-1}^1 W_{1,2}^d(y)\left(\frac{p+3}{p-1}W_{1,2}^d(y) + 2yW_{1,2}^{d}{}'(y) - \frac{8}{p-1}\frac{W_{1,2}^d(y)}{1-y^2}\right)\rho(y)dy$$

$$= \int_{-1}^1 \frac{W_{1,2}^d(y)^2}{p-1}\left(p+3-\frac{8}{1-y^2}\right)\rho(y)dy - \int_{-1}^1 W_{1,2}^d(y)^2(y\rho(y))'dy$$

$$= -\frac{4}{p-1}\int_{-1}^1 W_{1,2}^d(y)^2\frac{\rho(y)}{1-y^2}dy. \tag{162}$$

Using (160), (161) and (162), we see that

$$\varphi_{d,\epsilon}(V_0^{d,\epsilon}, V_0^{d,\epsilon}) \sim -\frac{\alpha_2}{\epsilon} \text{ and} \tag{163}$$

$$\left|\begin{array}{cc} \varphi_{d,\epsilon}(V_1^{d,\epsilon}, V_1^{d,\epsilon}) & \varphi_{d,\epsilon}(V_1^{d,\epsilon}, V_0^{d,\epsilon}) \\ \varphi_{d,\epsilon}(V_1^{d,\epsilon}, V_0^{d,\epsilon}) & \varphi_{d,\epsilon}(V_0^{d,\epsilon}, V_0^{d,\epsilon}) \end{array}\right| \sim \frac{4\alpha_2}{\epsilon(p-1)}\int_{-1}^1 W_{1,2}^d(y)^2\frac{\rho(y)}{1-y^2}dy$$

as $\epsilon \to 0$ uniformly for $|d| < 1$. Hence, since $\alpha_2 > 0$, (159) follows for ϵ small and positive and $|d| < 1$, which implies that $\varphi_{d,\epsilon}$ is negative in the plane spanned by $V_0^{d,\epsilon}$ and $V_1^{d,\epsilon}$. This concludes the proof of Lemma 4.10. ■

Part 3: End of the proof of Lemma 4.8:

From Lemmas 4.9 and 4.10, we define $\epsilon_0 = \min(\epsilon_1, \epsilon_2) \in (0, 1)$. We will now prove by contradiction that φ_{d,ϵ_0} is negative on $\mathcal{H}_-^d$ for all $|d| < 1$.

From Lemma 4.10 and (144), for all $|d| < 1$ and $\epsilon \in (0, \epsilon_0]$, we write the definition of $\mathcal{H}_-^d$ (133) as follows:

$$\mathcal{H}_-^d = \left\{r \in \mathcal{H} \mid \varphi_{d,\epsilon}\left(V_\lambda^{d,\epsilon}, r\right) = 0 \text{ for all } \lambda \in \{0,1\}\right\}. \tag{164}$$

We proceed by contradiction and assume that

$$\text{there is } r \in \mathcal{H}_-^d \text{ such that } \varphi_{d,\epsilon}(r,r) < 0. \tag{165}$$

Since the determinant in (163) is not zero, we see from (164) that $r \notin \text{span}\left(V_1^{d,\epsilon}, V_0^{d,\epsilon}\right)$. Therefore, the vector subspace

$$E_1 = \text{span}\left(V_1^{d,\epsilon}, V_0^{d,\epsilon}, r\right)$$

is of dimension 3. Hence, since the subspace E_2 (141) is of codimension 2, there exists a non zero $u \in E_1 \cap E_2$.

On the one hand, since $u \in E_2$, we have from Lemma 4.9 that

$$\varphi_{d,\epsilon}(u,u) \geq 0. \tag{166}$$

On the other hand, since $\varphi_{d,\epsilon}$ is negative on E_1 by (iii) of Lemma 4.10, we must have from (164) and (165),

$$\varphi_{d,\epsilon}(u,u) < 0.$$

This contradicts (166). Thus, (165) does not hold, and $\varphi_{d,\epsilon}$ is nonnegative on $\mathcal{H}_-^d$. This concludes the proof of Lemma 4.8 and Proposition 4.7. ■

5 Trapping near the set of stationary solutions

We prove Theorem 3 in this section. Note that in this section, we work in the space $\mathcal{H}$, which is a natural choice. Indeed, if $(w, \partial_s w) \in \mathcal{H}$, then the Lyapunov functional $E(w)$ (15) is well defined, thanks to the Hardy-Sobolev inequality of Lemma 2.2.
We proceed in 3 steps, each of them making a separate subsection.

- In subsection 5.1, assuming that (18) holds for some $s^* \in \mathbb{R}$, $d^* \in (-1,1)$, $\omega^* = \pm 1$ and $\epsilon^* > 0$ small enough and independent of d^*, we use modulation theory to introduce a parameter $d(s)$ adapted to the linearized operator of equation (7) around the stationary solution $\kappa(d, \cdot)$ (see section 4).
- In subsection 5.2, under the a priori estimate that $\|(w(s), \partial_s w(s)) - (\kappa(d(s), \cdot), 0)\|_{\mathcal{H}}$ is small, we project the linearized equation of (7) around $\kappa(d(s), \cdot)$ and derive from the energy barrier (17) the smallness of the unstable direction with respect to the stable.
- In subsection 5.3, we use the two first steps and prove Theorem 3 by showing the convergence of $(w(s), \partial_s w(s))$ to some $\kappa(d_\infty, \cdot)$ as $s \to \infty$ in the norm of $\mathcal{H}$.

5.1 Modulation theory

In this section, we use modulation theory and introduce a parameter $d(s)$ adapted to the dispersive property of the equation (7) whenever (18) holds. We claim the following:

Proposition 5.1 (Modulation of w with respect to $\kappa(d, \cdot)$)
There exists $\epsilon_1 > 0$ and $K_1 > 0$ such that if $(w, \partial_s w) \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution to equation (7) which satisfies (18) for some $|d^*| < 1$, $\omega^* = \pm 1$ and $\epsilon^* \le \epsilon_1$, then the following is true:
(i) **(Choice of the modulation parameter)** *There exists $d(s) \in C^1([s^*, \infty), (-1,1))$ such that for all $s \in [s^*, \infty)$,*

$$\pi_0^{d(s)}(q(s)) = 0 \tag{167}$$

where π_0^d is defined in (129), $q = (q_1, q_2)$ is defined for all $s \in [s_0, \infty)$ by

$$\begin{pmatrix} w(y,s) \\ \partial_s w(y,s) \end{pmatrix} = \begin{pmatrix} \kappa(d(s), y) \\ 0 \end{pmatrix} + \begin{pmatrix} q_1(y,s) \\ q_2(y,s) \end{pmatrix}. \tag{168}$$

Moreover,

$$\left|\log\left(\frac{1+d(s^*)}{1-d(s^*)}\right) - \log\left(\frac{1+d^*}{1-d^*}\right)\right| + \|q(s^*)\|_{\mathcal{H}} \le K_1 \epsilon^*. \tag{169}$$

(ii) **(Equation on q)** *For all $s \in [s^*, \infty)$:*

$$\frac{\partial}{\partial s}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = L_{d(s)}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f_{d(s)}(q_1) \end{pmatrix} - d'(s)\begin{pmatrix} \partial_d \kappa(d,y) \\ 0 \end{pmatrix} \tag{170}$$

$$\begin{aligned} \text{where } L_d\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} q_2 \\ \mathcal{L}q_1 + \psi(d,y)q_1 - \frac{p+3}{p-1}q_2 - 2yq_2' \end{pmatrix}, \\ f_d(q_1) &= |\kappa(d,\cdot) + q_1|^{p-1}(\kappa(d,\cdot)+q_1) - \kappa(d,\cdot)^p - p\kappa(d,\cdot)^{p-1}q_1, \end{aligned} \tag{171}$$

$\mathcal{L}$, $\psi(d, \cdot)$ and $\kappa(d, \cdot)$ are defined respectively in (8) and (41) and (13).

Remark: We recall from (129) that π_0^d is the projection on F_0^d (110), the null eigenspace of L_d span by $(\partial_d\kappa(d,y),0)$ by (110) and (114). In particular, the modulation term (i.e. containing $d'(s)$) in (170) is proportional to F_0^d.

Proof of Proposition 5.1:
Up to replacing $w(y,s)$ by $-w(y,s)$, we can assume that $\omega^*=1$ in (18).
(i) In (18), we see that there is a parameter $d^*\in(-1,1)$ which makes the distance between the solution $(w(s^*),\partial_s w(s^*))$ and a particular element of the family of stationary solutions $\{(\kappa(d,y),0)\mid |d|<1\}$ small. Now, we would like to sharpen the decomposition and find for all $s\in[s^*,\sigma^*]$ for some $\sigma^*>s^*$ a different parameter $d(s)$ close to d^* which not only makes the difference between $(w(s),\partial_s w(s))$ and $\kappa(d(s),\cdot)$ small, but also satisfies the orthogonality condition (167).

From (129), we see that condition (167) becomes $\Phi\left((w(s),\partial_s w(s)),d\right)=0$ where $\Phi\in C(\mathcal{H}\times(-1,1),\mathbb{R})$ is defined by

$$\Phi(v,d)=\phi\left(v-(\kappa(d,\cdot),0),W_0^d\right) \tag{172}$$

and ϕ and W_0^d are given in (103) and Lemma 4.4. The implicit function theorem allows us to conclude. Indeed,
- Note first that we have

$$\Phi((\kappa(d^*,\cdot),0),d^*)=0. \tag{173}$$

- Then, we compute from (172), the expressions of $\partial_d\kappa(d,y)$ (114) and F_0^d (110) and the orthogonality relation (118),

$$\begin{aligned} D_v\Phi(v,d)(u) &= \phi(u,W_0^d) \text{ for all } u\in\mathcal{H},\\ \partial_d\Phi(v,d) &= -\phi\left((\partial_d\kappa(d,\cdot),0),W_0^d\right)+\phi\left(v-(\kappa(d,\cdot),0),\partial_d W_0^d\right),\\ &= \frac{2\kappa_0}{(p-1)(1-d^2)}+\phi\left(v-(\kappa(d,\cdot),0),\partial_d W_0^d\right). \end{aligned}$$

Using the continuity of ϕ in $\mathcal{H}$, the bound (119), and the fact that

$$\forall d_1,d_2\in(-1,1),\ \ \|\kappa(d_1,\cdot)-\kappa(d_2,\cdot)\|_{\mathcal{H}_0}\le C_0|\theta_1-\theta_2| \text{ where } \theta_i=\frac{1}{2}\log\left(\frac{1+d_i}{1-d_i}\right) \tag{174}$$

(see below for the proof of (174)), we see that if

$$\left|\log\left(\frac{1+d}{1-d}\right)-\log\left(\frac{1+d^*}{1-d^*}\right)\right|+\|v-(\kappa(d^*,\cdot),0)\|_{\mathcal{H}_0}\le\epsilon_1$$

for some $\epsilon_1>0$ small enough independent of d^*, then we have

$$\|D_v\Phi(v,d)\|\le C_0 \text{ and } 0<\frac{1}{C_0(1-d^2)}\le\partial_d\Phi(v,d)\le\frac{C_0}{1-d^2}. \tag{175}$$

Now, if we introduce $\Psi\in C(\mathcal{H}\times\mathbb{R},\mathbb{R})$ defined by

$$\Psi(v,\theta)=\Phi(v,d) \text{ where } d=\tanh\theta,$$

then, since $\theta = \frac{1}{2}\log\left(\frac{1+d}{1-d}\right)$ and $\tanh'(\theta) = 1 - \tanh(\theta)^2$, we see from (173) and (175) that the implicit function theorem applies to Ψ and we get the existence of $d(s)$ for all $s \in [s^*, \sigma^*)$ for some $\sigma^* \leq \infty$. Assume by contradiction that $\sigma^* < +\infty$. Applying the implicit function theorem around $(v,d) = ((w(s_n), \partial_s w(s_n)), d(s_n))$ where $s_n = \sigma^* - \frac{1}{n}$, and the uniform continuity of $(w(s), \partial_s w(s))$ from $[\sigma_* - \eta_0, \sigma_* + \eta_0]$ to $\mathcal{H}$ for some $\eta_0 > 0$, we see that for n large enough, we can define $d(s)$ for all $s \in [s_n, s_n + \epsilon_0]$ for some $\epsilon_0 > 0$ independent of n. Therefore, for n large enough, $d(s)$ exists beyond σ^*, which is a contradiction. Thus, $\sigma^* = \infty$ and (i) is proved.
Remains to prove (174).

Proof of (174):
Case $d_1 = 0$: Since $\kappa(d_2, \cdot) = \mathcal{T}_{d_2}\kappa_0$ by (33),we see from Lemma 2.8 that for all $d_2 \in (-1,1)$, $\|\kappa(d_2,\cdot)\|_{\mathcal{H}_0} \leq \|\kappa_0\|_{\mathcal{H}_0} \leq C$. Therefore, $\|\kappa(d_2,\cdot) - \kappa_0\|_{\mathcal{H}_0}$ is a bounded C^1 function of $\theta_2 = \frac{1}{2}\log\left(\frac{1+d}{1-d}\right)_2$ which is zero when d_2 is zero. This directly implies (174).
Case $d_1 \neq 0$: Using the remark after Lemma 2.6, we see that $\kappa(d_2,\cdot) - \kappa(d_1,\cdot) = \mathcal{T}_{d_1}(\kappa(d_2 * (-d_1)) - \kappa_0)$. Using the continuity estimate of $\mathcal{T}_{d_1}$ in $\mathcal{H}_0$ (see Lemma 2.8) and the case $d_1 = 0$, we see that

$$\|\kappa(d_1,\cdot) - \kappa(d_2,\cdot)\|_{\mathcal{H}_0} \leq C_0 \|\kappa(d_2 * (-d_1), \cdot) - \kappa_0\|_{\mathcal{H}_0} \leq C_0|\tilde{\theta}|$$

where $\tilde{\theta} = \frac{1}{2}\log\left(\frac{1+(d_2*(-d_1))}{1-(d_2*(-d_1))}\right)$, or equivalently, $\tanh\tilde{\theta} = d_2 * (-d_1)$. Since we have from (32)

$$d_2 * (-d_1) = \frac{d_2 - d_1}{1 - d_2 d_1} = \frac{\tanh\theta_2 - \tanh\theta_1}{1 - \tanh\theta_1 \tanh\theta_2} = \tanh(\theta_2 - \theta_1),$$

we see that $\tilde{\theta} = \theta_2 - \theta_1$, which concludes the proof of (174) and (i) of Proposition 5.1.

(ii) is a direct consequence of the equation (7) satisfied by w put in vectorial form:

$$\partial_s w = v \tag{176}$$
$$\partial_s v = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w - \frac{p+3}{p-1}v - 2y\partial_y v \tag{177}$$

and the fact that $(\kappa(d,\cdot), 0)$ is a stationary solution of (176)-(177), that is $\kappa(d,\cdot)$ is a solution of

$$\mathcal{L}\kappa(d,\cdot) - \frac{2(p+1)}{(p-1)^2}\kappa(d,\cdot) + |\kappa(d,\cdot)|^{p-1}\kappa(d,\cdot) = 0 \tag{178}$$

(see Proposition 1).
Indeed, since we have from (168), the definition of $\mathcal{L}$ (8) and $f_d(q_1)$ (171)

$$\begin{aligned} w(y,s) &= q_1(y,s) + \kappa(d(s),y), \\ \mathcal{L}w(y,s) &= \mathcal{L}q_1(y,s) + \mathcal{L}\kappa(d(s),y), \\ |w|^{p-1}w(y,s) &= f_d(q_1) + \kappa(d(s),y)^p + p\kappa(d,y)^{p-1}q_1(y,s), \end{aligned}$$

and from (176) and (168) $v = \partial_s w = q_2$, we see that equation (170) follows immediately from (176)-(178). This concludes the proof of Proposition 5.1. ■

5.2 Projection on the eigenspaces of the operator L_d

Given $s \geq s^*$ and following the previous section, we make in this subsection the following a priori estimate:

$$\|q(s)\|_{\mathcal{H}} \leq \epsilon \tag{179}$$

for some $\epsilon > 0$. From (167), we will expand q according to the spectrum of the linear operator L_d as in (130):

$$q(y,s) = \alpha_1(s) F_1^{d(s)}(y) + q_-(y,s) \tag{180}$$

where

$$\alpha_1(s) = \pi_1^{d(s)}(q), \;\; \alpha_0(s) = \pi_0^{d(s)}(q) = 0, \;\; \alpha_-(s) = \sqrt{\varphi_d\left(q_-, q_-\right)} \tag{181}$$

and

$$q_- = \begin{pmatrix} q_{-,1} \\ q_{-,2} \end{pmatrix} = \pi_-^d(q) = \pi_-^d \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{182}$$

From (180) and Proposition 4.7, we see that for all $s \geq s_0$,

$$\begin{aligned} \tfrac{1}{C_0}\alpha_-(s) \leq \quad & \|q_-(s)\|_{\mathcal{H}} && \leq C_0 \alpha_-(s), \\ \tfrac{1}{C_0}\left(|\alpha_1(s)| + \alpha_-(s)\right) \leq \quad & \|q(s)\|_{\mathcal{H}} && \leq C_0\left(|\alpha_1(s)| + \alpha_-(s)\right) \end{aligned} \tag{183}$$

for some $C_0 > 0$. In the following proposition, we derive from (170) differential inequalities satisfied by $\alpha_1(s)$, $\alpha_-(s)$ and $d(s)$:

Proposition 5.2 *There exists $\epsilon_2 > 0$ such that if w a solution to equation (7) satisfying (167) and (179) at some time s for some $\epsilon \leq \epsilon_2$, where q is defined in (168), then:*
(i) **(Control of the modulation parameter)**

$$|d'| \leq C_0(1-d^2)({\alpha_1}^2 + {\alpha_-}^2). \tag{184}$$

(ii) **(Projection of equation (170) on the different eigenspaces of L_d)**

$$\left|{\alpha_1}' - \alpha_1\right| \leq C_0\left({\alpha_1}^2 + {\alpha_-}^2\right), \tag{185}$$

$$\left(R_- + \frac{1}{2}{\alpha_-}^2\right)' \leq -\frac{4}{p-1}\int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} dy + C_0\left({\alpha_1}^2 + {\alpha_-}^2\right)^{3/2} \tag{186}$$

for some $R_-(s)$ satisfying

$$|R_-(s)| \leq C_0({\alpha_1}^2 + {\alpha_-}^2)^{\frac{1+\bar{p}}{2}} \text{ where } \bar{p} = \min(p,2) > 1. \tag{187}$$

(iii) **(Additional relation)**

$$\frac{d}{ds}\int_{-1}^{1} q_1 q_2 \rho \leq -\frac{4}{5}{\alpha_-}^2 + C_0 \int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} + C_0 {\alpha_1}^2. \tag{188}$$

(iv) **(Energy barrier)** *If moreover (17) holds, then*

$$\alpha_1(s) \leq C_0 \alpha_-(s). \tag{189}$$

Remark: Here, (186) and (188) are coming from the relations we use in [17] to bound uniformly $(w(s), \partial_s w(s))$ in $H^1 \times L^2(-1,1)$. Identities (186) and (188) together will be fundamental to control the dynamics of the infinite dimensional part q_- of the solution, and allow us thus to overcome the difficulty coming from the non self-adjoint character of the linear operator L_d. Such a use of conservation laws to control the dynamics is in the same spirit as the case of NLS (Viriel identity and the mass ejection law; see Merle and Raphaël [15] and [16]).

Proof of Proposition 5.2: Before the proof, let us give the following nonlinear estimate:

Claim 5.3 (Nonlinear estimates) *For all* $y \in (-1,1)$,

$$\begin{aligned} |f_{d(s)}(q_1(y,s))| &\leq \text{mM}\left(\kappa(d(s),y)^{p-2}|q_1(y,s)|^2, C_0|q_1(y,s)|^p\right), && (190)\\ |\mathcal{F}_{d(s)}(q_1(y,s))| &\leq \text{mM}\left(\kappa(d(s),y)^{p-2}|q_1(y,s)|^3, C_0|q_1(y,s)|^{p+1}\right) && (191)\end{aligned}$$

where $\text{mM} = \min$ *if* $1 < p < 2$ *and* $\text{mM} = \max$ *if* $p \geq 2$, *and*

$$\mathcal{F}_d(q_1) = \int_0^{q_1} f_d(q')dq' = \frac{|\kappa(d,\cdot)+q_1|^{p+1}}{p+1} - \frac{\kappa(d,\cdot)^{p+1}}{p+1} - \kappa(d,\cdot)^p q_1 - \frac{p}{2}\kappa(d,\cdot)^{p-1}q_1^2. \quad (192)$$

Proof: Introducing $\xi = q_1/\kappa(d(s),y)$ and considering the cases where $|\xi| < 1$ and $|\xi| \geq 1$, we directly get (i). Since (ii) follows from (i) by integration, this concludes the proof of Claim 5.3. ■

(i)-(ii) We proceed in 2 steps:

- In Step 1, we project equation (170) with the projector π_λ^d (129) for $\lambda = 0$ and $\lambda = 1$ and derive the smallness condition on d' (184) and the equation satisfied by α_1 (185).

- In Step 2, we write an equation satisfied by $(q_{-,1}, q_{-,2})$ which is the difficult part in this non self-adjoint framework. We claim that (186) follows from the existence of the Lyapunov functional $E(w)$ (15) for equation (7). Here, the Lyapunov functional structure will be revealed by the quadratic form φ_d (134).

Step 1: Projection of equation (170) on the modes $\lambda = 0$ and $\lambda = 1$

Projecting equation (170) with the projector π_λ^d (129) for $\lambda = 0$ and $\lambda = 1$, we write

$$\pi_\lambda^d(\partial_s q) = \pi_\lambda^d(L_d q) + \pi_\lambda^d\begin{pmatrix} 0 \\ f_d(q_1)\end{pmatrix} - d'(s)\pi_\lambda^d\begin{pmatrix}\partial_d\kappa(d,y) \\ 0\end{pmatrix}. \quad (193)$$

- Since $\alpha_\lambda(s) = \pi_\lambda^d(q) = \phi(W_\lambda^d, q)$ by (181) and the definition of π_λ^d (129), we write

$$\alpha'_\lambda(s) = \pi_\lambda^d(\partial_s q) + d'(s)\phi(\partial_d W_\lambda^d, q)$$

Using (119) and (183), we get

$$\left|\pi_\lambda^d(\partial_s q) - \alpha'_\lambda(s)\right| \leq \frac{C_0}{1-d^2}|d'|(|\alpha_1| + \alpha_-). \quad (194)$$

- Using (i) of Lemma 4.4, the definition of π_λ^d (129), the duality relation (106) and (181), we write

$$\pi_\lambda^d(L_d(q)) = \phi\left(W_\lambda^d, L_d(q)\right) = \phi\left(L_d^*\left(W_\lambda^d\right), q\right) = \lambda\phi\left(W_\lambda^d, q\right) = \lambda\pi_\lambda^d(q) = \lambda\alpha_\lambda(s). \quad (195)$$

- Using (46), the definition of $W^d_{\lambda,2}$ (116) and (121), we have

$$\forall (d,y)\in(-1,1)^2,\ \ |W^d_{\lambda,2}(y)|\le C\kappa(d,y) \tag{196}$$

Therefore, using the definitions of π^d_λ (129) and ϕ (102), and Claim 5.3, we see that

$$\left|\pi^d_\lambda\begin{pmatrix}0\\ f_d(q_1)\end{pmatrix}\right|\le C\int_{-1}^1\kappa(d,y)\,|f_d(q_1)|\,\rho(y)dy \tag{197}$$

$$\begin{aligned}&\le\ C_0\int_{-1}^1\kappa(d,y)^{p-1}q_1(y,s)^2\rho dy+C_0\delta_{\{p\ge2\}}\int_{-1}^1\kappa(d,y)|q_1(y,s)|^p\rho dy\\ &\le\ C_0\|q_1\|^2_{L^{p+1}_\rho}\|\kappa(d,y)\|^{p-1}_{L^{p+1}_\rho}+C_0\delta_{\{p\ge2\}}\|q_1\|^p_{L^{p+1}_\rho}\|\kappa(d,y)\|_{L^{p+1}_\rho}\end{aligned} \tag{198}$$

where $\delta_{\{p\ge2\}}$ is 0 if $1<p<2$ and 1 otherwise. Therefore, using (49), (197), (198), Lemma 2.2, (179) and (183), we get

$$\left|\pi^d_\lambda\begin{pmatrix}0\\ f_d(q_1)\end{pmatrix}\right|\le C\int_{-1}^1\kappa(d,y)\,|f_d(q_1)|\,\rho(y)dy.\le C_0\left(\alpha_1(s)^2+\alpha_-(s)^2\right). \tag{199}$$

- Using (114), (110) (131) and (121), we write

$$\pi^d_\lambda\begin{pmatrix}\partial_d\kappa(d,y)\\ 0\end{pmatrix}=-\frac{2\kappa_0}{(p-1)(1-d^2)}\pi^d_\lambda\left(F^d_0\right)=-\frac{2\kappa_0}{(p-1)(1-d^2)}\delta_{\lambda,0}. \tag{200}$$

- Using (193), (194), (195), (199), (200) and the fact that $\alpha_0\equiv\alpha_0'\equiv0$ by (181), we get for $\lambda=0,1$:

$$\begin{aligned}\frac{2\kappa_0}{(p-1)(1-d^2)}|d'|\ &\le\ \frac{C_0}{1-d^2}|d'|(|\alpha_1|+\alpha_-)+C_0\left({\alpha_1}^2+{\alpha_-}^2\right),\\ \left|\alpha_1'(s)-\alpha_1(s)\right|\ &\le\ \frac{C_0}{1-d^2}|d'|(|\alpha_1|+\alpha_-)+C_0\left({\alpha_1}^2+{\alpha_-}^2\right).\end{aligned}$$

Using the smallness condition (179) and (183), we obtain (184) and (185) for ϵ small enough.

Step 2: Differential inequality on α_-

In the following Lemma, we project equation (170) on the negative modes, which gives a partial differential inequality satisfied by q_-:

Claim 5.4 (Preliminary estimates) *There exists $\epsilon_3>0$ such that if $\epsilon\le\epsilon_3$ in the hypotheses of Proposition 5.2, then*

$$\left\|\partial_s q_- - L_d(q_-)-\pi^d_-\begin{pmatrix}0\\ f_d(q_1)\end{pmatrix}\right\|_{\mathcal H}\ \le\ C_0({\alpha_1}^2+{\alpha_-}^2)^{3/2}, \tag{201}$$

$$\left|\varphi_d\left(q_-,\pi^d_-\begin{pmatrix}0\\ f_d(q_1)\end{pmatrix}\right)-\int_{-1}^1 q_2 f_d(q_1)\rho dy\right|\ \le\ C_0\left({\alpha_1}^2+{\alpha_-}^2\right)^{3/2}, \tag{202}$$

$$\left|\int_{-1}^1 q_2 f_d(q_1)\rho dy-\frac{d}{ds}\int_{-1}^1\mathcal F_d(q_1)\rho dy\right|\ \le\ C_0\left({\alpha_1}^2+{\alpha_-}^2\right)^2 \tag{203}$$

where $\mathcal F_d(q_1)$ is defined in (192).

Remark: Note that the term in (203) cannot be controlled directly and has to be seen as a time derivative.

Assuming now Claim 5.4, we are able to conclude the proof of the differential inequality (186) satisfied by α_-.

Proof of (186) assuming Claim 5.4:
In fact, the whole proof is based on the fact that the derivative of α_-^2 is related to the quadratic form $\varphi_d(q_-, L_d(q_-))$ defined in (134), which inherits the properties of the Lyapunov functional defined in (15) (and give an almost self-adjoint behavior).
Note from the definition we took for α_- (181) that

$$\alpha_-(s)^2 = \varphi_d(q_-(s), q_-(s))$$

Using the definition (134) of φ_d, we have by differentiation

$$\alpha_-{}'.\alpha_- = \varphi_d(q_-, \partial_s q_-) - \frac{1}{2}d'(s)\int_{-1}^{1} \partial_d\psi(d,y)q_{-,1}^2\rho. \tag{204}$$

Using the Hölder inequality, the Hardy-Sobolev estimate of Lemma 2.2 and (183), we write

$$\left|\int_{-1}^{1} \partial_d\psi(d,y)q_{-,1}^2\rho\right| \le \|\partial_d\psi(d,y)\|_{L_\rho^{\frac{p+1}{p-1}}}\|q_{-,1}\|^2_{L_\rho^{p+1}} \le C_0\|\partial_d\psi(d,y)\|_{L_\rho^{\frac{p+1}{p-1}}}\alpha_-(s)^2. \tag{205}$$

Since $|\partial_d\psi(d,y)| \le C/(1+dy)^2$ for all $(d,y)\in(-1,1)^2$ by (41), using Claim 4.3, we see that $\|\partial_d\psi(d,y)\|_{L_\rho^{\frac{p+1}{p-1}}} \le C/(1-d^2)$. Therefore, using (204), (205), and the bound (184) on $|d'(s)|$, we get

$$\left|\alpha_-{}'.\alpha_- - \varphi_d(q_-, \partial_s q_-)\right| \le C_0|d'|\frac{\alpha_-^2}{1-d^2} \le C_0({\alpha_1}^2 + {\alpha_-}^2)^2. \tag{206}$$

From (206), the continuity of φ_d (138), Claim 5.4, (183), we write

$$\begin{aligned}
&\left|\alpha_-{}'\alpha_- - \varphi_d\left(q_-, L_d(q_-)\right) - \frac{d}{ds}\int_{-1}^{1}\mathcal{F}_d(q_1)\rho dy\right| \\
\le\ & C_0({\alpha_1}^2+{\alpha_-}^2)^{3/2} + \left|\varphi_d\left(q_-, \partial_s q_- - L_d(q_-) - \pi_-^d\begin{pmatrix} 0 \\ f_d(q_1)\end{pmatrix}\right)\right| \\
\le\ & C_0({\alpha_1}^2+{\alpha_-}^2)^{3/2} + \|q_-\|_{\mathcal{H}}({\alpha_1}^2+{\alpha_-}^2)^{3/2} \le C_0({\alpha_1}^2+{\alpha_-}^2)^{3/2}.
\end{aligned} \tag{207}$$

On the one hand, using the expressions of L_d (171) and φ_d (135), we have

$$\begin{aligned}
\varphi_d\left(q_-, L_d(q_-)\right) &= \varphi_d\left(\begin{pmatrix} q_{-,2} \\ \mathcal{L}q_{-,1} + \psi(d,y)q_{-,1} - \frac{p+3}{p-1}q_{-,2} - 2yq_{-,2}'\end{pmatrix}, \begin{pmatrix} q_{-,1} \\ q_{-,2}\end{pmatrix}\right) \\
&= -\int_{-1}^{1} q_{-,2}\left(\mathcal{L}q_{-,1} + \psi(d,y)q_{-,1}\right)\rho dy \\
&+ \int_{-1}^{1}\left(\mathcal{L}q_{-,1} + \psi(d,y)q_{-,1} - \frac{p+3}{p-1}q_{-,2} - 2yq_{-,2}'\right)q_{-,2}\rho(y)dy \\
&= -\frac{p+3}{p-1}\int_{-1}^{1} q_{-,2}^2\rho dy - \int_{-1}^{1} y\left(q_{-,2}^2\right)'\rho dy = -\frac{p+3}{p-1}\int_{-1}^{1} q_{-,2}^2\rho dy + \int q_{-,2}^2\left(\rho - y\rho'\right)dy \\
&= -\frac{4}{p-1}\left[\int_{-1}^{1} q_{-,2}^2\rho dy + \int_{-1}^{1} q_{-,2}^2\frac{y^2\rho}{1-y^2}dy\right] = -\frac{4}{p-1}\int_{-1}^{1} q_{-,2}^2\frac{\rho}{1-y^2}dy.
\end{aligned} \tag{208}$$

Using (207) and (208), we see that estimate (186) holds with

$$R_-(s) = -\int_{-1}^{1} \mathcal{F}_d(q_1)\rho dy. \tag{209}$$

Using (ii) of Claim 5.3, Lemma 2.2 and condition (179) (considering first the case $p \geq 2$ and then the case $1 < p < 2$), we see that (187) holds. Remains to prove Claim 5.4 to conclude the proof of (i)-(ii) of Proposition 5.2.

Proof of Claim 5.4:
Proof of (201): We first project equation (170) using the negative projector π_-^d introduced in Definition 4.6:

$$\pi_-^d(\partial_s q) = \pi_-^d(L_d q) + \pi_-^d \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix} - d'(s)\pi_-^d \begin{pmatrix} \partial_d \kappa(d,y) \\ 0 \end{pmatrix}. \tag{210}$$

- We will use the notation (182) here. Differentiating (180) and using the expansion (130) with $\partial_s q$, we write

$$\begin{aligned} \partial_s q(y,s) &= \alpha_1'(s)F_1^d(y) + \alpha_1(s)d'(s)\partial_d F_1^d(y) + \partial_s q_-(y,s), & (211)\\ \partial_s q(y,s) &= \pi_1^d(\partial_s q)\, F_1^d(y) + \pi_0^d(\partial_s q)\, F_0^d(y) + \pi_-^d(\partial_s q)\,. & (212)\end{aligned}$$

Making the difference between (211) and (212) and using (111), we get

$$\left\|\pi_-^d(\partial_s q) - \partial_s q_-(y,s)\right\|_{\mathcal{H}} \leq C_0\left(\left|\pi_1^d(\partial_s q) - \alpha_1'(s)\right| + \left|\pi_0^d(\partial_s q)\right| + \frac{|\alpha_1 d'(s)|}{1-d^2}\right).$$

Using (194), (167) and (184), we obtain

$$\left\|\pi_-^d(\partial_s q) - \partial_s q_-(y,s)\right\|_{\mathcal{H}} \leq C_0(\alpha_1^2 + {\alpha_-}^2)^{\frac{3}{2}}. \tag{213}$$

- Applying the operator L_d to (180) and using the fact that $L_d F_1^d = F_1^d$ (see Lemma 4.2), we obtain

$$L_d q = \alpha_1(s)F_1^d + L_d(q_-)\,. \tag{214}$$

Since $\pi_-^d(F_1^d) = 0$ and $\pi_-^d(L_d(q_-)) = L_d(q_-)$ (see the remark after Definition 4.6 and note in particular that $L_d(q_-) \in \mathcal{H}_-^d$ because $q_- \in \mathcal{H}_-^d$), we get from (214)

$$\pi_-^d(L_d(q)) = L_d(q_-). \tag{215}$$

- Using (114), (110) and the remark after Definition 4.6, we write

$$\pi_-^d \begin{pmatrix} \partial_d \kappa(d,y) \\ 0 \end{pmatrix} = -\frac{2\kappa_0}{p-1}(1-d^2)^{-1}\pi_-^d\left(F_0^d\right) = 0. \tag{216}$$

Using (210), (213), (215) and (216), we write

$$\left\|\partial_s q_- - L_d(q_-) - \pi_-^d \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix}\right\|_{\mathcal{H}} \leq C_0({\alpha_1}^2 + {\alpha_-}^2)^{3/2}.$$

This concludes the proof of (201).
Proof of (202): Recall from (180) and (130) that we have

$$\begin{aligned} q(y,s) &= \alpha_1(s)F_1^d(y) + q_-(y,s), & (217)\\ \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix} &= \beta_1(s)F_1^d(y) + \beta_0(s)F_0^d(y) + \pi_-^d \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix} & (218) \end{aligned}$$

where $\beta_\lambda(s) = \pi_\lambda^d \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix}$. Note from the definition (134) and the bilinearity of φ_d, the bound on the norm of $\mathcal{F}_\lambda^d$ (111), (138) and (183) that

$$\begin{aligned} &\left|\varphi_d\left(q_-, \pi_-^d \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix}\right) - \int_{-1}^1 q_2 f_d(q_1)\rho dy\right| \\ = &\left|\varphi_d\left(q_-, \pi_-^d \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix}\right) - \varphi_d\left(q, \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix}\right)\right| \\ \leq\ & C_0\left(|\alpha_1| + |\alpha_-|\right)\left(|\beta_1| + |\beta_0|\right) + |\alpha_1|\left|\varphi_d(F_1^d, \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix})\right| \end{aligned}$$

Since we have from the expression (134) of φ_d, the fact that $|F_{1,2}^d(y)| \leq C\kappa(d,y)$ and (199),

$$\left|\varphi_d\left(F_1^d, \begin{pmatrix} 0 \\ f_d(q_1) \end{pmatrix}\right)\right| = \left|\int_{-1}^1 F_{1,1}^d(y) f_d(q_1)\rho(y)dy\right| \leq C_0\left({\alpha_1}^2 + {\alpha_-}^2\right), \quad (219)$$

$$|\beta_1(s)| + |\beta_0(s)| \leq C_0 \int_{-1}^1 \kappa(d,y)|f_d(q_1)|\rho dy \leq C_0\left({\alpha_1}^2 + {\alpha_-}^2\right), \quad (220)$$

this gives (202).

Proof of (203): Since $q_2 = \partial_s q_1 + d'\partial_d\kappa(d,y)$ by (168), we use (192) to write

$$\begin{aligned} &\int_{-1}^1 q_2 f_d(q_1)\rho dy = \int_{-1}^1 \partial_s q_1 f_d(q_1)\rho dy + d'(s)\int_{-1}^1 \partial_d\kappa(d,y) f_d(q_1)\rho dy \\ = &\frac{d}{ds}\int_{-1}^1 \mathcal{F}_d(q_1)\rho dy + d'(s)\int_{-1}^1 \left(\partial_d\kappa(d,y) f_d(q_1) - \partial_d\mathcal{F}_d(q_1)\right)\rho dy \\ = &\frac{d}{ds}\int_{-1}^1 \mathcal{F}_d(q_1)\rho + d'(s)\frac{p(p-1)}{2}\int_{-1}^1 \partial_d\kappa(d,y)\kappa(d,y)^{p-2} q_1(y,s)^2\rho dy. \quad (221) \end{aligned}$$

Since we have $\|\partial_d\kappa(d,y)\kappa(d,y)^{p-2}\|_{L_\rho^{\frac{p+1}{p-1}}} \leq C_0/(1-d^2)$, from the definitions of $\partial_d\kappa(d,y)$ (114), F_0^d (110) and Claim 4.3, we use the Hölder inequality and the Hardy-Sobolev inequality of Lemma 2.2 to derive that

$$\left|\int_{-1}^1 \partial_d\kappa(d,y)\kappa(d,y)^{p-2} q_1(y,s)^2\rho dy\right| \leq \frac{C_0}{1-d^2}\|q_1\|_{L_\rho^{p+1}}^2 \leq \frac{C_0}{1-d^2}\|q(s)\|_{\mathcal{H}}^2. \quad (222)$$

Using (183) and (184), we see that (221) and (222) give (203). This concludes the proof of Claim 5.4 as well as (i)-(ii) of Proposition 5.2. ■

(iii) This inequality is a consequence of the coercivity of the quadratic form φ_d on the space $\mathcal{H}_-^d$ stated in Proposition 4.7.
From equation (170) and the definition of L_d (171), we write

$$\begin{aligned}\frac{d}{ds}\int_{-1}^1 q_1q_2\rho &= \int_{-1}^1 q_2\partial_s q_1\rho + \int_{-1}^1 q_1\partial_s q_2\rho\\ &= \int_{-1}^1 q_2^2\rho - d'(s)\int_{-1}^1 q_2\partial_d\kappa(d,y)\rho\\ &+ \int_{-1}^1 q_1\left(\mathcal{L}q_1 + \psi(d,y)q_1 - \frac{p+3}{p-1}q_2 - 2y\partial_y q_2 + f_d(q_1)\right)\rho. \quad (223)\end{aligned}$$

- First, note from (183) that

$$\int_{-1}^1 q_1^2\rho + \int_{-1}^1 (\partial_y q_1)^2(1-y^2)\rho + \int_{-1}^1 q_2^2\rho \le C_0(\alpha_1^2+\alpha_-^2). \quad (224)$$

- Using (180), the Hardy estimate (22) and the bound (111), we write

$$\int_{-1}^1 q_2^2\frac{\rho}{1-y^2} \le 2\int_{-1}^1 q_{-,2}^2\frac{\rho}{1-y^2} + 2{\alpha_1}^2\int_{-1}^1 (F_{1,1}^d)^2\frac{\rho}{1-y^2} \le 2\int_{-1}^1 q_{-,2}^2\frac{\rho}{1-y^2} + C_0{\alpha_1}^2. \quad (225)$$

- From the expression of φ_d (135), (180), the definition of α_- (181), the continuity estimate (138), the bound (111) on F_1^d and (183), we write

$$\begin{aligned}\int_{-1}^1 & q_1\left(\mathcal{L}q_1+\psi(d,y)q_1\right)\rho = -\varphi_d\left(\begin{pmatrix}q_1\\0\end{pmatrix},\begin{pmatrix}q_1\\0\end{pmatrix}\right) = -\varphi_d\left(\begin{pmatrix}q_{-,1}\\0\end{pmatrix},\begin{pmatrix}q_{-,1}\\0\end{pmatrix}\right)\\ - & {\alpha_1}^2\varphi_d\left(\begin{pmatrix}F_{1,1}^d\\0\end{pmatrix},\begin{pmatrix}F_{1,1}^d\\0\end{pmatrix}\right) - \alpha_1\varphi_d\left(\begin{pmatrix}F_{1,1}^d\\0\end{pmatrix},\begin{pmatrix}q_{-,1}\\0\end{pmatrix}\right)\\ \le & -{\alpha_-}^2 + \int_{-1}^1 q_{-,2}^2\frac{\rho}{1-y^2} + C_0({\alpha_1}^2+|\alpha_1|\alpha_-)\\ \le & -\frac{9}{10}\alpha_-(s)^2 + \int_{-1}^1 q_{-,2}^2\frac{\rho}{1-y^2} + C_0{\alpha_1}^2\end{aligned}$$

- Since $\|\partial_d\kappa(d,y)\|_{L^2_\rho} \le C_0/(1-d^2)$ from the definition of $\partial_d\kappa(d,y)$ and Claim 4.3, we use the Cauchy-Schwartz inequality, (184), (224) and (179) to write for ϵ small enough,

$$\begin{aligned}\left|d'(s)\int_{-1}^1 q_2\partial_d\kappa(d,y)\rho dy\right| &\le C|d'(s)|\left(\int_{-1}^1 q_2^2\rho\right)^{1/2}\|\partial_d\kappa(d)\|_{L^2_\rho}\\ &\le C_0({\alpha_1}^2+{\alpha_-}^2)^{3/2} \le \frac{1}{100}({\alpha_1}^2+{\alpha_-}^2). \quad (226)\end{aligned}$$

- Using integration by parts, the fact that $|y\partial_y \rho(y)| \le C\frac{\rho(y)}{1-y^2}$, the Cauchy-Schwartz inequality, the Hardy-Sobolev estimate (21), (224) and (225), we write

$$
\begin{aligned}
& \left| -\frac{p+3}{p-1}\int_{-1}^{1} q_1 q_2 \rho - 2\int_{-1}^{1} q_1 y\partial_y q_2 \rho \right| \\
= & \left| 2\int_{-1}^{1} q_2 \partial_y q_1 y\rho + \left(2-\frac{p+3}{p-1}\right)\int_{-1}^{1} q_2 q_1 \rho + 2\int_{-1}^{1} q_2 q_1 y\partial_y \rho \right|. \\
\le & C\int_{-1}^{1}\left(|q_2||\partial_y q_1|\rho + |q_2||q_1|\frac{\rho}{1-y^2}\right) \\
\le & C\left(\int_{-1}^{1}|q_2|^2\frac{\rho}{1-y^2}\right)^{1/2}\left[\int_{-1}^{1}(\partial_y q_1)^2(1-y^2)\rho + \int_{-1}^{1} q_1^2\frac{\rho}{1-y^2}\right]^{1/2} \\
\le & C_0\left(\alpha_1^2+\alpha_-^2\right)^{1/2}\left(\int_{-1}^{1} q_{-,2}^2\frac{\rho}{1-y^2} + C_0\alpha_1^2\right)^{1/2} \le 100C_0^2\int_{-1}^{1} q_{-,2}^2\frac{\rho}{1-y^2} + C\alpha_1^2 + \frac{\alpha_-^2}{100}.
\end{aligned}
$$

- Using (49), Claim 5.3, the Hölder inequality and Lemma 2.2, (183) and (179), we write for ϵ small enough (note that $\bar{p}+1>2$ and use (179)),

$$
\begin{aligned}
& \int_{-1}^{1} q_1 f_d(q_1)\rho \le C_0\delta_{\{p\ge 2\}}\int_{-1}^{1}\kappa(d,y)^{p-2}|q_1|^3\rho + C_0\int_{-1}^{1}|q_1|^{p+1}\rho \\
\le & C_0\delta_{\{p\ge 2\}}\|\kappa(d,y)\|_{L_\rho^{p+1}}^{p-2}\|q_1\|_{L_\rho^{p+1}}^{3} + C_0\|q_1\|_{L_\rho^{p+1}}^{p+1} \le C_0\|q\|_{\mathcal{H}}^{\bar{p}+1} \le \tfrac{1}{100}({\alpha_1}^2+{\alpha_-}^2).
\end{aligned} \tag{227}
$$

Collecting (223)-(227) concludes the proof of (iii) of Proposition 5.2.

(iv) Using the definition of $q(y,s)$ (168), we can make an expansion of $E(w(s))$ (15) for $q\to 0$ in $\mathcal{H}$ and get after from straightforward computations

$$
E(w(s)) = E(\kappa(d,\cdot)) + \frac{1}{2}\varphi_d(q,q) - \int_{-1}^{1}\mathcal{F}_d(q_1)\rho dy \tag{228}
$$

where φ_d and $\mathcal{F}_d(q_1)$ are defined in (134) and (192). Note in particular that there is no linear term, since $\kappa(d,\cdot)$ is a stationary solution to (7), hence, a critical point of $E(w(s))$. Moreover, as we announced right after (134), the second variation of $E(w(s))$ around $\kappa(d,\cdot)$ is given by φ_d.
Since we have (209), (187), (179) and (183)

$$
\left|\int_{-1}^{1}\mathcal{F}_d(q_1)\rho dy\right| \le C\|q(s)\|_{\mathcal{H}}^{\bar{p}+1} \le C\epsilon^{\bar{p}-1}\left(\alpha_1^2+\alpha_-^2\right) \tag{229}
$$

where $\bar{p}=\min(p,2)$, we claim that the conclusion follows from the fact that

$$
\varphi_d(q,q) \le C_0\alpha_-^2 - C_1{\alpha_1}^2 \tag{230}
$$

for some $C_1>0$. Indeed, from (17), (228), (230) and (229), we see that taking ϵ small enough so that $C\epsilon^{\bar{p}-1}\le\frac{C_1}{4}$, we get

$$
0 \le E(w(s)) - E(\kappa(d,\cdot)) \le \left(\frac{C_0}{2}+\frac{C_1}{4}\right)\alpha_-^2 - \frac{C_1}{4}{\alpha_1}^2,
$$

which yields (189). Remains to prove (230).
Proof of (230): Since $L_d(F_1^d) = F_1^d$ by Lemma 4.2, calculation (208) holds with q_- replaced by F_1^d, and we get from Claim 4.3 for some $C_1 > 0$,

$$\varphi_d(F_1^d, F_1^d) = -\frac{4}{p-1}\int_{-1}^{1}\left(F_{1,1}^d\right)^2 \frac{\rho}{1-y^2}dy \le -2C_1. \tag{231}$$

Since we have from the decomposition (180), the definition of α_- (181), the continuity of φ_d (138), the bound on F_1^d (111), (183) and (231),

$$\begin{aligned}\varphi_d(q,q) &= \varphi_d(q_-,q_-) + 2\alpha_1\varphi_d(F_1^d, q_-) + {\alpha_1}^2\varphi_d(F_1^d, F_1^d) \quad (232)\\ &\le {\alpha_-}^2 + \frac{C_0^2}{C_1}{\alpha_1}^2 + C_1{\alpha_-}^2 - 2C_1{\alpha_1}^2, \quad (233)\end{aligned}$$

this yields (230) and concludes the proof of Proposition 5.2. ■

5.3 Exponential decay of the different components

We prove Theorem 3 in this subsection. Let us first introduce a more adapted notation and rewrite Proposition 5.2.
If we introduce

$$\theta(s) = \frac{1}{2}\log\left(\frac{1+d(s)}{1-d(s)}\right), \quad a(s) = \alpha_1(s)^2 \text{ and } b(s) = \alpha_-(s)^2 + 2R_-(s) \tag{234}$$

(note that $d(s) = \tanh(\theta(s))$), then we see from (187), and (183) that if (179) holds, then $|b - {\alpha_-}^2| \le C_0\epsilon^{\bar{p}-1}({\alpha_1}^2 + {\alpha_-}^2)$, hence

$$\frac{99}{100}{\alpha_-}^2 - \frac{1}{100}a \le b \le \frac{101}{100}{\alpha_-}^2 + \frac{1}{100}a \tag{235}$$

for ϵ small enough. Therefore, using Proposition 5.2, estimate (179), (183) and the fact that $\theta'(s) = \frac{d'(s)}{1-d(s)^2}$, we derive the following:

Corollary 5.5 (Relations between a, b, θ and $\int_{-1}^1 q_1q_2\rho$) *There exist positive ϵ_4, K_4 and K_5 such that if w is a solution to equation (7) such that (167) and (179) hold at some time s for some $\epsilon \le \epsilon_4$, where q is defined in (168), then using the notation (234), we have:*
*(i)***(Size of the solution)**

$$\frac{1}{K_4}(a(s)+b(s)) \le \|q(s)\|_{\mathcal{H}}^2 \le K_4(a(s)+b(s)) \le K_4^2\epsilon^2, \tag{236}$$

$$|\theta'(s)| \le K_4(a(s)+b(s)) \le K_4^2\|q(s)\|_{\mathcal{H}}^2, \tag{237}$$

$$\left|\int_{-1}^1 q_1q_2\rho\right| \le K_4(a(s)+b(s)) \tag{238}$$

and (235) holds.
(ii) **(Equations)**

$$\begin{aligned}\frac{3}{2}a - K_4\epsilon b \le a' &\le \frac{5}{2}a + K_4\epsilon b, &(239)\\ b' &\le -\frac{8}{p-1}\int_{-1}^{1} q_{-,2}^2\frac{\rho}{1-y^2} + K_4\epsilon(a+b), &(240)\\ \frac{d}{ds}\int_{-1}^{1} q_1q_2\rho &\le -\frac{3}{5}b + K_4\int_{-1}^{1} q_{-,2}^2\frac{\rho}{1-y^2} + K_4 a. &(241)\end{aligned}$$

(iii) **(Energy barrier)** *If (17) holds, then*

$$a(s) \le K_5 b(s). \tag{242}$$

At this level, we still don't have exponential decay of a and b. However, with this corollary and the following analysis, we are ready to prove Theorem 3.

Proof of Theorem 3: Consider $w \in C([s^*,\infty),\mathcal{H})$ for some $s^* \in \mathbb{R}$ a solution of equation (7) such that (17) and (18) hold for some $d^* \in (-1,1)$, $\omega^* = \pm 1$ and $\epsilon^* \in (0,\epsilon_0]$. Up to replacing $w(y,s)$ by $-w(y,s)$, we can assume that $\omega^* = 1$ in (18). Consider then $\epsilon = 2K_0K_1\epsilon^*$ where K_1 is given in Proposition 5.1 and K_0 will be fixed later. If

$$\epsilon^* \le \epsilon_1 \text{ and } \epsilon \le \epsilon_4, \tag{243}$$

then we see that Proposition 5.1, Corollary 5.5 and (235) apply respectively with ϵ^* and ϵ. In particular, there is a maximal solution $d(s) \in C^1([s^*,\infty),(-1,1))$ such that (167) holds for all $s \in [s^*,\infty)$ where $q(y,s)$ is defined in (168) and

$$|\theta(s^*) - \theta^*| + \|q(s^*)\|_{\mathcal{H}} \le K_1\epsilon^* \text{ with } \theta^* = \frac{1}{2}\log\left(\frac{1+d^*}{1-d^*}\right). \tag{244}$$

If in addition we have

$$K_0 \ge 1 \text{ hence, } \epsilon \ge 2K_1\epsilon^*, \tag{245}$$

then, we can give two definitions:
- We define first from (244) and (245) $s_1^* \in (s^*,\infty)$ such that for all $s \in [s^*, s_1^*]$,

$$\|q(s)\|_{\mathcal{H}} < \epsilon \tag{246}$$

and if $s_1^* < \infty$, then $\|q(s_1^*)\|_{\mathcal{H}} = \epsilon$.
- Then, we define $s_2^* \in [s^*, s_1^*]$ as the first $s \in [s^*, s_1^*]$ such that

$$a(s) \ge \frac{b(s)}{5K_4} \tag{247}$$

where K_4 is introduced in Corollary 5.5, or $s_2^* = s_1^*$ if (247) is never satisfied on $[s^*, s_1^*]$.
We proceed in 3 steps:
- In Step 1, using (247), we integrate the equations (240)-(241) on the time interval $[s^*, s_2^*]$ and obtain for some positive K_6, μ_6 and $f(s)$

$$\forall s \in [s^*, s_1^*],\ \ \frac{1}{K_6}\|q\|_{\mathcal{H}}^2 \le f \le K_6^2\|q\|_{\mathcal{H}}^2 \text{ and } f' \le -2\mu_6 f.$$

- In Step 2, integrating the equation (239) satisfied by a on the time interval $[s_2^*, s_1^*]$, we obtain some exponential estimate.
- In Step 3, we conclude the proof by showing first that $s_1^* - s_2^* \le \sigma_0$ for some σ_0, then $s_1^* = \infty$. Then, integrating the equation obtained in Step 1, we conclude.

In the 3 steps, we use the notation C_i for an arbitrary constant.

Step 1: Integration of the equations on $[s^*, s_2^*]$
We claim the following:

Claim 5.6 *There exist positive ϵ_6, μ_6, K_6 and $f \in C^1([s^*, s_2^*], \mathbb{R}^+)$ such that if $\epsilon \le \epsilon_6$, then for all $s \in [s^*, s_2^*]$:*
(i)

$$\frac{1}{2} f(s) \le b(s) \le 2f(s) \text{ and } f'(s) \le -2\mu_6 f(s),$$

(ii)

$$\|q(s)\|_{\mathcal{H}} \le K_6 \|q(s^*)\|_{\mathcal{H}} e^{-\mu_6(s-s^*)} \le K_6 K_1 \epsilon^* e^{-\mu_6(s-s^*)}.$$

Proof:
(i) By definition of s_2^*, we see that

$$\forall s \in [s^*, s_2^*], \quad a(s) \le \frac{b(s)}{5K_4} \tag{248}$$

where $a(s)$ and $b(s)$ are defined in (234). Since $[s^*, s_2^*] \subset [s^*, s_1^*]$, the interval where (246) is satisfied, we can apply Corollary 5.5. Therefore, using equations (240) and (241), we write for all $s \in [s^*, s_2^*]$,

$$b'(s) \le -\frac{8}{p-1} \int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} + C_1 \epsilon b(s), \tag{249}$$

$$\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho \le -\frac{2}{5} b(s) + K_4 \int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} \tag{250}$$

for some $C_1 > 0$ and ϵ^* small enough. We claim that

$$f(s) = b(s) + \eta_6 \int q_1 q_2 \rho$$

satisfies the desired property, where $\eta_6 > 0$ will be fixed small independent of ϵ. Using (238), we see that if η_6 is small enough, then we get for all $s \in [s^*, s_2^*]$,

$$\frac{1}{2} b(s) \le f(s) \le 2b(s), \tag{251}$$

and using (248) and the equivalence of norms (236), we obtain for some $C_3 > 0$

$$\frac{1}{C_3} \|q(s)\|_{\mathcal{H}}^2 \le f(s) \le C_3 \|q(s)\|_{\mathcal{H}}^2. \tag{252}$$

Then, using (249), (250) and (251), we have for all $s \in [s^*, s_2^*]$,

$$f'(s) \le -\left(\frac{2}{5}\eta_6 - C_1\epsilon\right) b(s) - \left(\frac{8}{p-1} - K_4\eta_6\right) \int_{-1}^{1} q_{-,2}^2 \frac{\rho}{1-y^2} \le -\frac{\eta_6}{4} b \le -\frac{\eta_6}{8} f(s) \tag{253}$$

if η_6 is small enough independent of ϵ, and ϵ is small enough. Using (251), (244) and (253), this concludes the proof of (i).

(ii) Integrating equation (253), we get for all $s \in [s^*, s_2^*]$, $f(s) \le f(s^*)e^{-\frac{\eta_6}{8}(s-s^*)}$. Using (252), this concludes the proof of Claim 5.6. ■

Step 2: Integration of the equations on $[s_2^*, s_1^*]$
We claim the following:

Claim 5.7 *(i) There exists* $\epsilon_7 > 0$ *such that for all* $\sigma > 0$, *there exists* $K_7(\sigma) > 0$ *such that if* $\epsilon \le \epsilon_7$, *then*

$$\forall s \in [s_2^*, \min(s_2^* + \sigma, s_1^*)], \quad \|q(s)\|_{\mathcal{H}} \le K_7 \|q(s^*)\|_{\mathcal{H}} e^{-\mu_6(s-s^*)} \le K_7 K_1 \epsilon^* e^{-\mu_6(s-s^*)}$$

where μ_6 *has been introduced in Claim 5.6.*
(ii) There exists $\epsilon_8 > 0$ *such that if* $\epsilon \le \epsilon_8$, *then*

$$\forall s \in (s_2^*, s_1^*], \quad b(s) \le a(s) \left(5K_4 e^{-\frac{(s-s_2^*)}{2}} + \frac{1}{4K_5} \right) \tag{254}$$

where K_4 *and* K_5 *have been introduced in Corollary 5.5.*

Proof:
(i) Using equations (239) and (240), we see that for all $s \in [s_2^*, \min(s_2^* + \sigma, s_1^*)]$,

$$(a+b)' \le 3(a+b), \text{ hence } a(s) + b(s) \le e^{3\sigma}(a(s_2^*) + b(s_2^*))$$

for ϵ small enough. Therefore, we see from (236) that $\|q(s)\|_{\mathcal{H}} \le K_4 e^{\frac{3\sigma}{2}} \|q_2(s_2^*)\|_{\mathcal{H}}$. Using (ii) in Claim 5.6 with $s = s_2^*$ gives the conclusion.
(ii) By definition of s_1^*, (246) is satisfied for all $s \in [s_2^*, s_1^*]$, hence, Corollary 5.5 applies and equations (239) and (240) hold.
Let us first prove that

$$\forall s \in (s_2^*, s_1^*], \quad a(s) \ge \frac{b(s)}{5K_4} \tag{255}$$

where K_4 is introduced in Corollary 5.5. We need to assume that $s_2^* < s_1^*$, otherwise the set $(s_2^*, s_1^*]$ is empty. Let $g = a - \frac{b}{5K_4}$ where a and b are defined in (234). From equations (239) and (240), we write for some $C_1 > 0$ and for all $s \in [s_2^*, s_1^*]$,

$$\begin{aligned} a' &\ge \frac{3}{2}a - C_1\epsilon b, \quad b' \le C_1\epsilon(a+b), \qquad (256)\\ g' = (a - \frac{b}{5K_4})' &\ge \frac{3}{2}a - C_1\epsilon b - \frac{C_1}{5K_4}\epsilon(a+b) \ge C_1\epsilon\,(1+5K_4)\,g + a \end{aligned}$$

for ϵ small enough. Since by definition of s_2^*, we have $g(s_2^*) \ge 0$ (remember that $s_2^* < s_1^*$), (255) follows. Using (256) and (255), we obtain for ϵ small enough,

$$\forall s \in (s_2^*, s_1^*], \quad a'(s) \ge \frac{3}{2}a - 5K_4C_1\epsilon a \ge a(s) \text{ hence } a(s) \ge e^{s-s_2^*}a(s_2^*). \tag{257}$$

If $q_2(s_2^*) \equiv 0$, then $w(y, s_2^*) \equiv \kappa(d(s_2^*), y)$ by (168), and from the uniqueness of solutions to equation (7), we have $w(y,s) \equiv \kappa(d(s_2^*), y)$ and $q(y,s) \equiv 0$ for all $s \ge s_2^*$, hence

$a(s) = b(s) = 0$ by (236) and (254) holds trivially.
Now, if $q(s_2^*) \not\equiv 0$, we can define $h = \frac{b}{a}$ for all $s \in (s_2^*, s_1^*]$ and derive from (256) and (257) for all $s \in (s_2^*, s_1^*]$,

$$h' = \frac{b'a - ba'}{a^2} \le \frac{C_1\epsilon(a+b)a - ba}{a^2} \le -\frac{h}{2} + C_1\epsilon$$

for ϵ small enough. Integrating this equation gives

$$b(s) \le a(s)\left(e^{-\frac{(s-s_2^*)}{2}}\frac{b(s_2^*)}{a(s_2^*)} + 2C_1\epsilon\right).$$

Using (255) and taking ϵ small enough gives (254) and concludes the proof of Claim 5.7.■

Step 3: Conclusion of the proof

We use Step 1 and 2 to conclude the proof of Theorem 3 here.
Let us first fix $\sigma_0 > 0$ such that

$$5K_4^{-\frac{\sigma_0}{2}} + \frac{1}{4K_5} \le \frac{1}{2K_5}. \tag{258}$$

where K_4 and K_5 are introduced in Corollary 5.5. Then, we impose the condition

$$\epsilon = 2K_0K_1\epsilon^* \text{ where } K_0 = \max\left(2, K_6, K_7(\sigma_0)\right) \tag{259}$$

and the constants are defined in Proposition 5.1 and Claims 5.6 and 5.7. Finally, we fix

$$\epsilon_0 = \min\left(1, \epsilon_1, \frac{\epsilon_i}{2K_0K_1} \text{ for } i \in \{4,\ 6,\ 7,\ 8\}\right) \tag{260}$$

and the constants are defined in Proposition 5.1, Corollary 5.5, Claims 5.6 and 5.7.
Now , if $\epsilon^* \le \epsilon_0$, then Corollary 5.5 and Steps 1 and 2 apply. We claim that for all $s \in [s^*, s_1^*]$,

$$\|q(s)\|_{\mathcal{H}} \le K_0\|q(s^*)\|_{\mathcal{H}}e^{-\mu_6(s-s^*)} \le K_0K_1\epsilon^*e^{-\mu_6(s-s^*)} = \frac{\epsilon}{2}e^{-\mu_6(s-s^*)}. \tag{261}$$

Indeed, if $s \in [s^*, \min(s_2^* + \sigma_0, s_1^*)]$, then, this comes from (ii) of Claim 5.6 or (i) of Claim 5.7 and the definition of k_0 (259).
Now, if $s_2^* + \sigma_0 < s_1^*$ and $s \in [s_2^* + \sigma_0, s_1^*]$, then we have from (254) and the definition of σ_0, $b(s) \le \frac{a(s)}{2K_5}$ on the one hand. On the other hand, from (iii) in Corollary 5.5, we have $a(s) \le K_5 b(s)$, hence, $a(s) = b(s) = 0$ and from (236), $q(y,s) \equiv 0$, hence (261) is satisfied trivially.
In particular, we have for all $s \in [s^*, s_1^*]$, $\|q(s)\|_{\mathcal{H}} \le \frac{\epsilon}{2}$, hence, by definition of s_1^*, this means that $s_1^* = \infty$. Therefore, from (261) and (237), we have

$$\forall s \ge s^*,\ \ \|q(s)\|_{\mathcal{H}} \le \frac{\epsilon}{2}e^{-\mu_6(s-s^*)} \text{ and } |\theta'(s)| \le K_4^2\frac{\epsilon^2}{4}e^{-2\mu_6(s-s^*)}. \tag{262}$$

Hence, there is $\theta_\infty \in \mathbb{R}$ such that $\theta(s) \to \theta_\infty$ as $s \to \infty$ and

$$\forall s \ge s^*,\ \ |\theta_\infty - \theta(s)| \le C_1{\epsilon^*}^2e^{-2\mu_6(s-s^*)} = C_2\epsilon^2e^{-2\mu_6(s-s^*)} \tag{263}$$

for some positive C_1 and C_2. Taking $s = s^*$ here and using (244), we see that $|\theta_\infty - \theta^*| \le C_0\epsilon^*$. If $d_\infty = \tanh\theta_\infty$, then we see that $|d_\infty - d^*| \le C_3(1 - {d^*}^2)\epsilon^*$.
Using the definition of q (168), (174), (262) and (263), we write

$$\begin{aligned} & \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \begin{pmatrix} \kappa(d_\infty, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \\ \le & \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \begin{pmatrix} \kappa(d(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} + \|\kappa(d(s), \cdot) - \kappa(d_\infty, \cdot)\|_{\mathcal{H}_0} \\ \le & \|q(s)\|_{\mathcal{H}} + C|\theta_\infty - \theta(s)| \le C_4\epsilon^* e^{-\mu_6(s - s^*)}. \end{aligned}$$

This concludes the proof of Theorem 3. ∎

A Positivity of the Lyapunov functional $E(w)$

We prove Proposition 2.1 here. In [17], the proof is given in the "non characteristic" case, that is when $w = w_{x_0}$ defined from some solution $u(x,t)$ to (1) where x_0 is a non characteristic point of u. That proof naturally extends to the case where the set $[-1,1] \times [-\log T, +\infty)$ is in the interior of the domain of definition of w. Let us then focus on the remaining case. Note from (16) and [17] that we only need to prove the positivity of $E(w(s))$.

Let us introduce for all $\sigma \ge 1/(T - \frac{1}{n})$ and $|z| < 1 + \frac{e^\sigma}{n}$,

$$w_n(z, \sigma) = (1 - \frac{e^s}{n})^{\frac{2}{p-1}} w(y, s), \quad y = \frac{z}{1 + \frac{e^\sigma}{n}} \text{ and } s = \sigma - \log(1 + \frac{e^\sigma}{n}). \tag{264}$$

For a given n, since by definition, $w_n(y, s)$ is defined for all $|y| < 2$ for s large, we see that $E(w_n(s)) \to 0$ as $s \to \infty$. Thus, since by hypothesis, we have $(w, \partial_s w)(-\log T) \in H^1 \times L^2(-1, 1)$, we obtain for all $s \in (-\log T + 2, \infty)$ and for all n large enough,

$$0 \le E(w_n(s)) \le E_0. \tag{265}$$

One has to prove in a certain sense that $E(w_n(s_n)) \to E(w(s_0)) = E_0$ where $s_n \to s_0$.
Using [17], we have for all $s \in (-\log T + 1, \infty)$ and $n \in \mathbb{N}$,

$$\int_s^{s+1} \int_{-1}^1 \left((\partial_y w_n)^2(1 - y^2) + |w_n|^{p+1} + (\partial_s w_n)^2 + w_n^2\right)\rho \le C(E_0 + 1).$$

By convergence in energy space, we obtain for all $\delta > 0$ and $s \in (-\log T + 1, \infty)$,

$$\int_s^{s+1} \int_{|y| < 1 - \delta} \left((\partial_y w)^2(1 - y^2) + |w|^{p+1} + w^2 + (\partial_s w)^2\right)\rho \le C(E_0 + 1).$$

Thus,

$$\int_s^{s+1} \int_{|y| < 1} \left((\partial_y w)^2(1 - y^2) + |w|^{p+1} + w^2 + (\partial_s w)^2\right)\rho \le C(E_0 + 1). \tag{266}$$

We have by the Lebesgue theorem,

$$\forall s \in (-\log T + 2, \infty), \quad \int_s^{s+1} E(w_n(\tau))d\tau \to \int_s^{s+1} E(w(\tau))d\tau$$

which proves for all $s \geq -\log T + 2$, $E(w(s)) \geq 0$ from (265). Indeed, for all $s \in (-\log T + 2, \infty)$ and $|y| < 1$ for n large (depending on s),

$$\begin{aligned} & \left((\partial_y w_n)^2(1-z^2) + w_n^2 + |w_n|^{p+1} + (\partial_s w_n)^2\right)(z,\sigma)\rho(z) \\ \leq \; & C_0\left((\partial_y w)^2(1-y^2) + w^2 + |w|^{p+1} + (\partial_s w)^2\right)(y,s)\rho(y) \end{aligned}$$

where (z,σ) and (y,s) are linked by (264), therefore, we have

$$\int_s^{s+1} E(w_n)d\tau_n \to \int_s^{s+1} E(w)d\tau.$$

Using (265) and the monotonicity of $E(w)$ (16), we have the conclusion. ■

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