Blow-up profile for the Complex Ginzburg-Landau equation

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Abstract

We construct a solution to the complex Ginzburg-Landau equation, which blows up in finite time T only at one blow-up point. We also give a sharp description of its blow-up profile. The proof relies on the reduction of the problem to a finite dimensional one, and the use of index theory to conclude. Two major difficulties arise in the proof: the linearized operator around the profile is not self-adjoint and it has a second neutral mode. In the last section, the interpretation of the parameters of the finite dimensional problem in terms of the blow-up time and the blow-up point gives the stability of the constructed solution with respect to perturbations in the initial data.

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1 Introduction and statement

The complex Ginzburg-Landau equation appears in various physical situations. In particular it appears in the theory of superconductivity, the description of several instabilities in fluid dynamics (in particular the plane Poiseuille flow). It can be seen as a generic amplitude equation near the onset of instabilities that lead to chaotic dynamics in fluid mechanical systems, as well as in the theory of phase transitions and superconductivity. We refer to Popp *et al.* [21] and the references therein for the physical background.

We are concerned in this paper with the following Ginzburg-Landau equation:

$$\begin{aligned} u_t &= (1+i\beta)\Delta u + (1+i\delta)|u|^{p-1}u - \gamma u \\ u(.,0) &= u_0 \in L^\infty(\mathbb{R}^N,\mathbb{C}), \end{aligned} \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{C}$, $p > 1$ and the constants β, δ and γ are real. The Cauchy problem for equation (1) can be solved in a variety of spaces using semi-group

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theory as in the case of the heat equation (see for instance [5, 8, 9] for existence results and [19] for some uniqueness results). In particular for an initial data $u_0 \in L^\infty(\mathbb{R}^N, \mathbb{C})$, there exists a time $T_1 > 0$ and a unique solution $u \in C([0, T_1]; L^\infty(\mathbb{R}^N, \mathbb{C}))$ to (1). This solution can be prolonged to $[0, \infty)$ if there is no blow-up.

An extensive literature is devoted to the study of the "twin" equation

$$u_t = (1 + i\beta)\Delta u - (1 - i\delta)|u|^{p-1}u - \gamma u \tag{2}$$

with the negative sign in front of the nonlinear term (global existence of weak solution [6, 10], existence of traveling wave solutions (see Tang [22] for example) are available in this case). The major difference between the two equations can be seen from the associated ODEs

$$u' = (1 + i\delta)|u|^{p-1}u \qquad \text{and} \quad u' = -(1 - i\delta)|u|^{p-1}u. \tag{3}$$

We see that the ODE associated to equation (1) exhibits finite-time blow-up.

To our knowledge, the question of the existence of a blow-up solution for equation (1) remained open so far. In this paper, we will prove the existence of such solutions under some conditions on β and δ. Classical methods based on energy-type estimates (Levine [11] and Ball [1]) break down. However, there have been some strong evidence that blow-up can occur. In their paper, Popp *et al.* [21] give such an evidence through a formal study and numerical simulations.

We also point out that (2) may have blow-up in the focusing case, namely $\beta\delta > 0$. In [20] (see also [4]), Plechac and Sverak give some evidence for the existence of a radial solution which blows up in a self-similar way. Their argument is based on matching a numerical solution in an inner region with an analytical solution in an outer region. As explained in [20], addressing the blow-up question for the complex Ginzburg-Landau equation is important not only for the equation itself , but also because the Ginzburg-Landau equation has common fundamental features with the Navier-Stokes equation, making it a lab model for the study of the singular behavior for Navier-Stokes.

In this paper, we justify the formal method of Popp *et al.* [21] and construct a solution $u(x, t)$ of (1) that blows up in some finite time T, in the sense that

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty,$$

with the same profile and the same range of parameters (see section 3.3 in [21] where the case $p = 3$ is treated, but the analysis extends naturally to other values of $p > 1$). More precisely, this is our existence result:

Theorem 1 (Existence of a blow-up solution for equation (1)) *Consider* $(\beta, \delta) \in \mathbb{R}^2$ *such that*

$$p - \delta^2 - \beta\delta(p + 1) > 0. \tag{4}$$

Then, equation (1) has a solution $u(x, t)$ *which blows up in finite time* $T > 0$ *only at the origin. Moreover,*
(i) for all $t \in [0, T)$,

$$\begin{aligned} &\left\| (T - t)^{\frac{1+i\delta}{p-1}} |\log(T - t)|^{-i\mu} u(x, t) - \left(p - 1 + \frac{b|x|^2}{(T - t)|\log(T - t)|} \right)^{-\frac{1+i\delta}{p-1}} \right\|_{L^\infty(\mathbb{R}^N)} \\ \leq \quad & \frac{C_0}{1 + \sqrt{|\log(T - t)|}} \end{aligned} \tag{5}$$

where

$$b = \frac{(p-1)^2}{4\,(p-\delta^2-\beta\delta-\beta\delta p)} > 0 \text{ and } \mu = -\frac{2b\beta}{(p-1)^2}(1+\delta^2); \tag{6}$$

(ii) for all $x \neq 0$, $u(x,t) \to u^*(x) \in C^2(\mathbb{R}^N\backslash\{0\})$ *and*

$$u^*(x) \sim |2\log|x||^{i\mu}\left[\frac{b}{2}\frac{|x|^2}{|\log|x||}\right]^{-\frac{1+i\delta}{p-1}} \quad \text{as } x \to 0. \tag{7}$$

Remarks:
1) Note that no smallness assumptions are made on β or δ. We just assume (4). In particular, when $\beta = 0$, the result is valid for all $\delta \in (-\sqrt{p}, \sqrt{p})$. This answers a conjecture by Zaag [23] where the result was proved only for small δ.
2) Note that the norm as well as the phase blow up in (7).
3) The result holds with the same proof when the reaction term in (1) is replaced by $f(u)$ which is equivalent to $(1+i\delta)|u|^{p-1}u$ as $|u| \to \infty$. For simplicity, we prove the result when the nonlinear term is exactly $(1+i\delta)|u|^{p-1}u$, that is when $\gamma = 0$ in (1).
4) We will only give the proof when $N = 1$. Indeed, the computation of the eigenfunctions of $\mathcal{L}_{\beta,\delta}$ (defined in (22)) and the projection of (21) on the eigenspaces become much more complicated when $N \geq 2$. Besides, the ideas are exactly the same.
5) The blow-up in Theorem 1 can be seen as a blow-up based on the heat equation. This is different from the blow-up solutions proposed by [20] for equation (2) where the blow-up mechanism is based on the focusing Schrödinger part and is self-similar.
6) Note that the constructed solution satisfies

$$\forall t \in [0,T), \quad \|u(t)\|_{L^\infty} \leq Cv(t) \text{ where } v(t) = (p-1)^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}}$$

is the the solution of

$$v' = v^p \text{ with } \lim_{t\to T} v(t) = +\infty.$$

In the heat equation blow-up literature, such a solution is called of "Type 1" (see Matano and Merle [12]).

Our proof uses some ideas developed by Merle and Zaag in [14] and Bricmont and Kupiainen in [3] for the semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u.$$

In [23], Zaag adapted that method to the case of the following complex-valued equation, where no gradient structure exists:

$$u_t = \Delta u + (1+i\delta)|u|^{p-1}u$$

and where δ is small enough. One may think that the method used in [14], [3] and [23] should work the same for (1) perhaps with some technical complications. This is not the case, since the fact that $\beta \neq 0$ breaks any symmetry in the problem and makes the diffusion operator associated to (1) not self-adjoint. Moreover, note from (4) that we will allow β and δ to take large values as long as the constant b defined in (6) is positive. In other words, the method we present here is not based on a simple perturbation of the semilinear heat equation as in [23].

More precisely, the proof relies on the understanding of the dynamics of the selfsimilar version of (1) (see equation (11) below) around the profile (5). More precisely, we proceed in 2 steps:

- In Step 1, we reduce the question to a two-dimensional problem : We show that it is enough to control a two-dimensional variable in order to control the solution (which is infinite dimensional) near the profile. Two difficulties arise here: The linearized operator is not self-adjoint and there is a new zero eigenvalue coming from the complex character of the solution (modulation is needed to control this new neutral mode).
- In Step 2, we proceed by contradiction to solve the two-dimensional problem and conclude using index theory.

As in [14] and [23], it is possible to make the interpretation of the two-dimensional variable in terms of the blow-up time and the blow-up point. This allows us to derive the stability of the profile (5) in Theorem 1 with respect to perturbations in the initial data. More precisely, we have the following:

Theorem 2 (Stability of the solution constructed in Theorem 1) *Let us denote by $\hat{u}(x,t)$ the solution constructed in Theorem 1 and by $\hat{T}$ its blow-up time. Then, there exists a neighborhood $\mathcal{V}_0$ of $\hat{u}(x,0)$ in L^∞ such that for any $u_0 \in \mathcal{V}_0$, equation (1) has a unique solution $u(x,t)$ with initial data u_0, and $u(x,t)$ blows up in finite time $T(u_0)$ at one single blow-up point $a(u_0)$. Moreover, estimate (5) is satisfied by $u(x-a,t)$ and*

$$T(u_0) \to \hat{T}, \;\; a(u_0) \to 0 \text{ as } \quad u_0 \to \hat{u}_0 \text{ in } L^\infty(\mathbb{R}^N, \mathbb{C}).$$

Remarks:
1) This stability result is more general than stated. Indeed, from our proof, we can show that not only is our solution stable, but also any solution which is trapped in some shrinking neighborhood of the profile in the selfsimilar variables. See section 6 for a precise statement. Even though we don't show that any solution satisfying (5) is trapped in this shrinking neighborhood, we believe that this gap is small in comparison with the the stability proof inside this trap.
2) This stability result gives a physical legitimacy to the solution we construct. In particular, we suspect that a numerical simulation of equation (1) should lead to the profile (5).
3) Following remark 6 after Theorem 1, we note that all the solutions $u(x,t)$ with initial data $u_0 \in \mathcal{V}_0$ are all of Type 1, since they all satisfy (5).

Some authors write the complex Ginzburg-Landau in the nonlinear Schrödinger equation style. Following that choice, we have this corollary:

Corollary 3 (Blow-up solutions in the NLS style) *Theorems 1 and 2 yield stable blow-up solutions for:*
(i) the following perturbed defocusing Schrödinger equation with $\nu > 0$ and $\nu' > 0$

$$u_t = (\nu' + i)\Delta u + (\nu - i)|u|^{p-1}u \text{ if } p - \frac{1}{\nu^2} + \frac{p+1}{\nu\nu'} > 0, \tag{8}$$

(ii) the focusing Schrödinger equation with $\nu > 0$ and $\nu' > 0$

$$u_t = (\nu' + i)\Delta u + (\nu + i)|u|^{p-1}u \text{ if } p - \frac{1}{\nu^2} - \frac{p+1}{\nu\nu'} > 0. \tag{9}$$

In both cases, the solutions blow up only at one point and the blow-up profile is given by (5) after appropriate scaling.

To prove the corollary, we notice that the condition (4) becomes $p - \frac{\delta^2}{\nu^2} - \frac{\beta}{\nu'}\frac{\delta}{\nu}(p+1) > 0$ for the equation $u_t = (\nu' + i\beta)\Delta u + (\nu + i\delta)|u|^{p-1}u$ with $\nu > 0$ and $\nu' > 0$. Note in particular that for the defocusing Schrödinger equation

$$u_t = i\Delta u - i|u|^{p-1}u,$$

a small perturbation of the type $\nu(\Delta u + |u|^{p-1}u)$ yields blow up solutions of heat equation type with ν very small. However, for the focusing Schrödinger, we need a big enough perturbation to get a blow-up solution of the heat equation type.

Following our results, many problems remain open, like determining sufficient and/or necessary conditions that yield Type 1 blow-up. There is also the question of taking an *arbitrary* blow-up solution of (1) and determining its blow-up behavior. If such a solution is of Type 1, we expect that the classification of all Type 1 solutions of (1) defined for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ for some $T > 0$ would play an important role. As a matter of fact, in the case of the pure heat equation

$$\beta = \delta = 0 \text{ and } (N-2)p < N+2,$$

such a classification was done (in the form of a Liouville Theorem) by Merle and Zaag in [15] and [16] (see also Nouaili [17]), yielding uniform estimates for blow-up solutions. The existence of a Lyapunov functional in this case was essential in the proof. When

$$\beta = 0 \text{ and } \delta \neq 0,$$

there is no Lyapunov functional, and the method of [15] and [16] breaks down. Nevertheless, Nouaili and Zaag obtained a classification in [18] for small δ and not so large solutions.

We proceed in 6 sections to prove Theorems 1 and 2. In Section 2, we use a formal argument to derive the profile in (5). Since the formal argument cannot be justified, we adopt a different strategy and make a new formulation of the problem in Section 3. Then, we give the proof of Theorem 1 in Section 4 (section 5 is devoted to the proof of a technical lemma which is central in the step of reduction to a finite dimensional problem). Finally, we prove Theorem 2 in Section 6.

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2 A formal approach

Given an arbitrary $T > 0$, we introduce the following self-similar transformation of equation (1)

$$w(y,s) = (T-t)^{\frac{1+i\delta}{p-1}} u(x,t), \quad y = \frac{x}{\sqrt{T-t}}, \quad s = -\log(T-t). \tag{10}$$

If $u(x,t)$ satisfies (1) for all $(x,t) \in \mathbb{R}^N \times [0,T)$ (with $\gamma = 0$), then $w(y,s)$ satisfies for all $(y,s) \in \mathbb{R}^N \times [-\log T, +\infty)$ the following equation

$$\frac{\partial w}{\partial s} = (1+i\beta)\Delta w - \frac{1}{2} y.\nabla w - \frac{1+i\delta}{p-1} w + (1+i\delta)|w|^{p-1}w. \tag{11}$$

Thus, constructing a solution $u(x,t)$ for equation (1) that blows up at T like $(T-t)^{-\frac{1+i\delta}{p-1}}$ reduces to constructing a global solution $w(y,s)$ for equation (11) such that

$$0 < \epsilon_0 < \limsup_{s\to\infty} \|w(s)\|_{L^\infty(\mathbb{R}^N)} < \frac{1}{\epsilon_0}.$$

Let us try to find a solution for equation (11) of the form

$$w(y,s) = e^{i\mu \log s} \sum_{j=0}^{\infty} \frac{1}{s^j} w_j(\frac{y}{\sqrt{s}}). \tag{12}$$

Let us denote $z = \frac{y}{\sqrt{s}}$. By looking at the leading order, we find that w_0 should satisfy

$$-\frac{1}{2} z.\nabla w_0 - \frac{1+i\delta}{p-1} w_0 + (1+i\delta)|w_0|^{p-1} w_0 = 0. \tag{13}$$

Hence, modulo a phase, there exists $b > 0$ such that

$$w_0(z) = \kappa \left(1 + \frac{b}{p-1}\frac{|y|^2}{s}\right)^{-\frac{1+i\delta}{p-1}} \text{ where } \kappa = (p-1)^{-\frac{1}{p-1}}. \tag{14}$$

At the order $\frac{1}{s}$, we get

$$F(z) \equiv -i\mu w_0 + \frac{1}{2} z.\nabla w_0 + (1+i\beta)\Delta w_0 - \frac{1}{2} z.\nabla w_1 - \frac{1+i\delta}{p-1} w_1 + \\ +(1+i\delta)\left[\frac{p+1}{2}|w_0|^{p-1}w_1 + \frac{p-1}{2}|w_0|^{p-3}w_0^2\overline{w_1}\right] = 0.$$

Computing $F(z=0)$, we get

$$-i\mu\kappa + (1+i\beta)\Delta w_0(0) + (1+i\delta)\Re w_1(0) = 0.$$

Taking the real part and the imaginary part, we deduce that

$$\Re w_1(0) = \frac{2\kappa b}{(p-1)^2}(1-\delta\beta) \text{ and } \mu = -\frac{2b\beta}{(p-1)^2}(1+\delta^2).$$

Expanding w_1 in powers of z, namely $w_1(z) = w_1(0) + \gamma z + \alpha z^2 + O(z^3)$ and expanding $F(z)$ in powers of z, we get that

$$\gamma = 0$$

by looking at the term of order z and that

$$i\mu\kappa\frac{(1+i\delta)b}{(p-1)^2} - \kappa\frac{(1+i\delta)b}{(p-1)^2} + (1+i\beta)6\kappa\frac{1+i\delta}{(p-1)}\left(\frac{1+i\delta}{(p-1)}+1\right)\frac{b^2}{(p-1)^2} + \\ +(1+i\delta)\Re\alpha - \alpha + (1+i\delta)\frac{b}{2(p-1)^2}\left[(p+1)w_1(0) + (p-1+2i\delta)\overline{w_1}(0)\right] = 0$$

by looking at the term of order z^2. Taking the real part, we see that α and $\Im w_1(0)$ disappear and we get an equation only involving b, μ and $\Re w_1(0)$, namely

$$-\frac{\mu\kappa\delta b}{(p-1)^2} - \frac{\kappa b}{(p-1)^2} + \left(p-(p+1)\beta\delta-\delta^2\right)\frac{6\kappa b^2}{(p-1)^4} + \frac{b}{(p-1)^2}[\delta^2-p]\Re w_1(0) = 0. \quad (15)$$

Writing μ and $\Re w_1(0)$ in terms of b, we deduce that

$$b = \frac{(p-1)^2}{4\left(p-\delta^2-\beta\delta-\beta\delta p\right)}.$$

3 Formulation of the problem

The preceding calculation is purely formal. We know of no proof that the expansion (12) can be continued to all orders (see Berger and Kohn [2]). However, the formal expansion in (12) provides us with the profile of the function $\left(w(y,s) = e^{i\mu\log s}\left(w_0\left(\frac{y}{\sqrt{s}}\right) + ...\right)\right)$. Our idea for the actual proof is then to linearize equation (11) around that profile and prove that the linearized equation as well as the nonlinear equation have a solution that goes to zero as $s \to \infty$. Let us introduce $q(y,s)$ and $\theta(s)$ such that

$$\begin{aligned} &w(y,s) = e^{i(\mu\log s+\theta(s))}\left(\varphi(y,s)+q(y,s)\right)\\ \text{where}\quad &\varphi(y,s) = \varphi_0\left(\frac{y}{\sqrt{s}}\right) + \frac{a}{s}(1+i\delta) \equiv \kappa^{-i\delta}\left(p-1+b\frac{|y|^2}{s}\right)^{-\frac{1+i\delta}{p-1}} + \frac{a}{s}(1+i\delta),\\ &a = \frac{2\kappa b}{(p-1)^2}(1-\delta\beta), \end{aligned} \quad (16)$$

κ is defined in (14) and b and μ are defined in (6) (in order to guarantee that only one couple (q,θ) satisfies (16), an additional constraint is needed; see (51) below.

Note that $\varphi_0(z) = w_0(z)$ defined in (14) has been exhibited in the formal approach and satisfies equation (13), which makes $\varphi(y,s)$ an approximate solution of (11). If w satisfies equation (11), then q satisfies the following equation

$$\frac{\partial q}{\partial s} = \mathcal{L}_\beta q - \frac{(1+i\delta)}{p-1}q + L(q,\theta',y,s) + R^*(\theta',y,s) \quad (17)$$

where

$$\begin{aligned} &\mathcal{L}_\beta q = (1+i\beta)\Delta q - \frac{1}{2}y.\nabla q,\\ &L(q,\theta',y,s) = (1+i\delta)\left\{|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi\right\} - i\left(\frac{\mu}{s}+\theta'(s)\right)q,\\ &R^*(\theta',y,s) = R(y,s) - i\left(\frac{\mu}{s}+\theta'(s)\right)\varphi \text{ with}\\ &R(y,s) = -\frac{\partial\varphi}{\partial s} + (1+i\beta)\Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{(1+i\delta)}{p-1}\varphi + (1+i\delta)|\varphi|^{p-1}\varphi. \end{aligned} \quad (18)$$

Our aim is to find $\theta \in C^1([-\log T, \infty), \mathbb{R})$ such that equation (17) has a solution $q(y,s)$ defined for all $(y,s) \in \mathbb{R}^N \times [-\log T, \infty)$ such that

$$\|q(s)\|_{L^\infty} \to 0 \text{ as } s \to \infty.$$

From (16), one sees that the variable $z = \frac{y}{\sqrt{s}}$ plays a fundamental role. Thus, we will consider the dynamics for $|z| > K$ and $|z| < 2K$ separately for some $K > 0$ to be fixed large.

3.1 The outer region where $|y| > K\sqrt{s}$

Let us consider a non-increasing cut-off function $\chi_0 \in C^\infty(\mathbb{R}^+, [0,1])$ such that $\chi_0(\xi) = 1$ for $\xi < 1$ and $\chi_0(\xi) = 0$ for $\xi > 2$ and introduce

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K\sqrt{s}}\right) \tag{19}$$

where K will be fixed large. Let us define

$$q_e(y,s) = e^{\frac{i\delta}{p-1}s} q(y,s)\left(1 - \chi(y,s)\right). \tag{20}$$

q_e is the part of $q(y,s)$ for $|y| > K\sqrt{s}$. As we will explain in subsection 5.3, the linear operator of the equation satisfied by q_e is negative, which makes it easy to control $\|q_e(s)\|_{L^\infty(\mathbb{R})}$. This is not the case for the part of $q(y,s)$ for $|y| < 2K\sqrt{s}$, where the linear operator has two positive eigenvalues, a zero eigenvalue in addition to infinitely many negative ones. Therefore, we have to expand q with respect to these eigenvalues in order to control $\|q(s)\|_{L^\infty(|y|<2K\sqrt{s})}$. This requires more work than for q_e. The following subsection is dedicated to that purpose. From now on, K will be a fixed constant which is chosen such that $\|\varphi(s')\|_{L^\infty\left(|y|\geq K\sqrt{s'}\right)}$ is small enough, namely $\|\varphi_0(z)\|^{p-1}_{L^\infty(|z|\geq K)} \leq \frac{1}{4C(p-1)}$ (see subsection 5.3 for more details). We point out for instance that K goes to infinity when b goes to zero.

3.2 The inner region where $|y| < 2K\sqrt{s}$

If we linearize the term $L(q, \theta', y, s)$ in equation (17), then we can write (17) as

$$\frac{\partial q}{\partial s} = \mathcal{L}_{\beta,\delta} q - i\left(\frac{\mu}{s} + \theta'(s)\right) q + V_1 q + V_2 \bar{q} + B(q,y,s) + R^*(\theta', y, s) \tag{21}$$

where

$$\mathcal{L}_{\beta,\delta} q = \mathcal{L}_\beta q + (1+i\delta)\Re q,$$

$$V_1(y,s) = \tfrac{p+1}{2}(1+i\delta)\left(|\varphi|^{p-1} - \tfrac{1}{p-1}\right), \quad V_2(y,s) = \tfrac{(p-1)}{2}(1+i\delta)\left(|\varphi|^{p-3}\varphi^2 - \tfrac{1}{p-1}\right),$$

$$B(q,y,s) = (1+i\delta)\left\{|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi - |\varphi|^{p-1}q - \tfrac{(p-1)}{2}|\varphi|^{p-3}\varphi\left(\varphi\bar{q} + \bar{\varphi}q\right)\right\} \tag{22}$$

with $\mathcal{L}_\beta$ and $R^*(\theta', y, s)$ defined in (18).
Note that the term $B(q, y, s)$ is built to be quadratic in the inner region $|y| \leq 2K\sqrt{s}$. Indeed, we have for all $K > 1$ and $s \geq 1$,

$$\sup_{|y|\leq 2K\sqrt{s}} |B(q, y, s)| \leq C(K)|q|^2. \tag{23}$$

Note also that $R(y, s)$ measures the defect of $\varphi(y, s)$ from being an exact solution of (11). However, since $\varphi(y, s)$ is an approximate solution of (11), one easily derives from (13) the fact that

$$\|R(s)\|_{L^\infty} \leq \frac{C}{s}. \tag{24}$$

Therefore, if $\theta'(s)$ goes to zero as $s \to \infty$, we expect the term $R^*(\theta', y, s)$ to be small, since (18) and (24) yield

$$|R^*(\theta', y, s)| \leq \frac{C}{s} + |\theta'(s)|. \tag{25}$$

Therefore, since we would like to make q go to zero as $s \to \infty$, the dynamics of equation (21) are influenced by the asymptotic limit of its linear term,

$$\mathcal{L}_{\beta,\delta} q + V_1 q + V_2 \bar{q},$$

as $s \to \infty$. In the sense of distributions (see the definitions of V_1 and V_2 (22) and φ (16)) this limit is $\mathcal{L}_{\beta,\delta} q$. We would like to find a basis where $\mathcal{L}_{\beta,\delta}$ is diagonal or at least in Jordan blocks' form. In order to do so, we first deal with $\mathcal{L}_\beta$.

3.3 Spectral properties of $\mathcal{L}_\beta$

Here, we take $N \geq 1$ and starting from subsection 3.4, we will restrict to $N = 1$. We consider the Hilbert space $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$ which is the set of all $f \in L^2_{loc}(\mathbb{R}^N, \mathbb{C})$ such that

$$\int_{\mathbb{R}^N} |f(y)|^2 |\rho_\beta(y)| dy < +\infty,$$

where

$$\rho_\beta(y) = \frac{e^{-\frac{|y|^2}{4(1+i\beta)}}}{(4\pi(1+i\beta))^{N/2}} \text{ and } |\rho_\beta(y)| = \frac{e^{-\frac{|y|^2}{4(1+\beta^2)}}}{\left(4\pi\sqrt{1+\beta^2}\right)^{N/2}}. \tag{26}$$

We can diagonalize $\mathcal{L}_\beta$ in $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$. Indeed, we can write it in divergence form

$$\mathcal{L}_\beta q = \frac{1}{\rho_\beta} \operatorname{div}(\rho_\beta \nabla q).$$

We notice that $\mathcal{L}_\beta$ is formally "self-adjoint" with respect to the weight ρ_β. Indeed, for any v and w in $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$ satisfying $\mathcal{L}_\beta v$ and $\mathcal{L}_\beta w$ in $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$, it holds that

$$\int v \mathcal{L}_\beta w \rho_\beta dy = \int w \mathcal{L}_\beta v \rho_\beta dy. \tag{27}$$

If we introduce for each $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$ the polynomial

$$f_\alpha(y) = c_\alpha \Pi_{i=1}^N H_{\alpha_i}\left(\frac{y_i}{2\sqrt{1+i\beta}}\right) \tag{28}$$

where H_n is the standard one dimensional Hermite polynomial and $c_\alpha \in \mathbb{C}$ is chosen so that the term of highest degree in f_α is $\Pi_{i=1}^N y_i^{\alpha_i}$, then we get a family of eigenfunctions of $\mathcal{L}_\beta$, "orthogonal" with respect to the weight ρ_β, in the sense that for any different α and ζ in $\mathbb{N}^N$,

$$\begin{aligned} \mathcal{L}_\beta f_\alpha &= -\frac{|\alpha|}{2} f_\alpha, && (29)\\ \int_{\mathbb{R}^N} f_\alpha(y) f_\zeta(y)\rho_\beta(y)dy &= 0. && (30)\end{aligned}$$

Moreover, the family f_α is a basis for $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$ considered as a $\mathbb{C}$ vector space. All the facts about the operator $\mathcal{L}_\beta$ and the family f_α can be found in Appendix A.

3.4 Spectral properties of $\mathcal{L}_{\beta,\delta}$

In the sequel, we will assume that $N = 1$. Now, with the explicit basis diagonalizing $\mathcal{L}_\beta$, we are able to write $\mathcal{L}_{\beta,\delta}$ in a Jordan blocks' form. More precisely,

Lemma 3.1 (Jordan blocks' decomposition of $\mathcal{L}_{\beta,\delta}$) *For all $n \in \mathbb{N}$, there exist two polynomials*

$$\begin{aligned} h_n &= if_n + \sum_{j=0}^{n-1} d_{j,n} f_j \ \textit{where}\ d_{j,n} \in \mathbb{C} && (31)\\ \textit{and}\ \tilde{h}_n &= f_n + \delta i f_n + \sum_{j=0}^{n-1} \tilde{d}_{j,n} f_j \ \textit{where}\ \tilde{d}_{j,n} \in \mathbb{C} && (32)\end{aligned}$$

of degree n such that

$$\mathcal{L}_{\beta,\delta} h_n = -\frac{n}{2} h_n, \ \ \mathcal{L}_{\beta,\delta}\tilde{h}_n = (1 - \frac{n}{2})\tilde{h}_n + c_n h_{n-2} \tag{33}$$

with $c_n \in \mathbb{C}$ (and we take $h_k \equiv 0$ for $k < 0$). The term of highest degree of h_n (resp. $\tilde{h}_n$) is iy^n (resp. $(1 + i\delta)y^n$).

Remark: We notice that $c_0 = c_1 = 0$, which means that $\tilde{h}_0$ and $\tilde{h}_1$ are also eigenfunctions for $\mathcal{L}_{\beta,\delta}$. On the contrary, $c_2 \neq 0$. When $n \geq 3$, we expect that $c_n \neq 0$, however, we didn't try to prove it and actually this fact is not necessary for our proof. When $n \geq 2$, we also notice that Span $\{h_{n-2}, \tilde{h}_n\}$ is the characteristic space associated to the eigenvalue $1 - n/2$.

Proof of Lemma 3.1: Let us first remark that in the basis $(if_n, f_n)_{n\in\mathbb{N}}$, the operator $\mathcal{L}_{\beta,\delta}$ is an infinite upper triangular matrix. Indeed, since the term of highest degree in f_n is y^n, which is real, we can expand

$$\Re f_n = f_n + \sum_{j=0}^{n-1} a_{j,n} f_j \text{ and } \Re(if_n) = \sum_{j=0}^{n-1} b_{j,n} f_j$$

where $a_{j,n}$ and $b_{j,n}$ are complex. Therefore, by definition (22) of $\mathcal{L}_{\beta,\delta}$, we have

$$\begin{aligned}
\mathcal{L}_{\beta,\delta}(if_n) &= \mathcal{L}_{\beta}(if_n) + (1+i\delta)\Re(if_n) \\
&= -\frac{n}{2}if_n + (1+i\delta)\sum_{j=0}^{n-1} b_{j,n}f_j \\
&= -\frac{n}{2}if_n + \sum_{j=0}^{n-1} \Re((1+i\delta)b_{j,n})f_j + \Im((1+i\delta)b_{j,n})f_j
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
\mathcal{L}_{\beta,\delta}f_n &= \mathcal{L}_{\beta}f_n + (1+i\delta)\Re f_n \\
&= -\frac{n}{2}f_n + (1+i\delta)\left(f_n + \sum_{j=0}^{n-1} a_{j,n}f_j\right) \\
&= \left(1-\frac{n}{2}\right)f_n + \delta i f_n + \sum_{j=0}^{n-1} \Re((1+i\delta)a_{j,n})f_j + \Im((1+i\delta)a_{j,n})if_j.
\end{aligned} \tag{35}$$

Thus, $\mathcal{L}_{\beta,\delta}$ is an infinite upper triangular matrix in the basis $(if_n, f_n)_{n\in\mathbb{N}}$, and its diagonal is

$$0,\ 1,\ -\frac{1}{2},\ \frac{1}{2},\ -1,\ 0,\ \ldots -\frac{n}{2},\ 1-\frac{n}{2},\ \ldots.$$

We first consider $\lambda = 0$ or $\lambda = 1$. Looking for eigenfunctions for $\mathcal{L}_{\beta,\delta}$ as a linear combination of the two first basis elements (if_0 and f_0) yields $\tilde{h}_0(y) \equiv (1+i\delta)$ and $\tilde{h}_1(y) \equiv (1+i\delta)y$ respectively.
Taking $\lambda = -\frac{n}{2}$, we remark that it occurs first in the diagonal of $\mathcal{L}_{\beta,\delta}$ at the position of the column of $\mathcal{L}_{\beta,\delta}(if_n)$ (see (34)) and then at the column of $\mathcal{L}_{\beta,\delta}(f_{n+2})$ (see (35) and note that $1-\frac{n+2}{2} = -\frac{n}{2}$).
Looking for an eigenfunction for $\lambda = -\frac{n}{2}$ as a combination of if_n and the preceding vectors in the basis yields the existence of h_n of the form (31) such that $\mathcal{L}_{\beta,\delta}h_n = -\frac{n}{2}h_n$.
Taking advantage of the second occurrence of $\lambda = -\frac{n}{2}$ as $1-\frac{n+2}{2}$ (in the column of $\mathcal{L}_{\beta,\delta}(f_{n+2})$), we look for an eigenfunction as a combination of f_{n+2} and the preceding vectors in the basis (if_{n+1}, if_j and f_j for $0 \le j \le n$). If this is possible, then there is some $\tilde{h}_{n+2}$ of the form (32), that is

$$\tilde{h}_{n+2} = f_{n+2} + \delta i f_{n+2} + \sum_{j=0}^{n+1} \tilde{d}_{j,n+2}f_j \text{ where } \tilde{d}_{j,n+2} \in \mathbb{C}, \tag{36}$$

such that $\mathcal{L}_{\beta,\delta}\tilde{h}_{n+2} = -\frac{n}{2}\tilde{h}_{n+2}$. If this is not possible, then we can define $\tilde{h}_{n+2}$ of the form (36) (hence of the form (32)) such that $\mathcal{L}_{\beta,\delta}\tilde{h}_{n+2} = -\frac{n}{2}\tilde{h}_{n+2} + c_{n+2}h_n$ (Jordan blocks' decomposition).
Since the term of highest degree of f_j is y^j, (31) and (32) yield the degree and the term of highest degree of h_n and $\tilde{h}_n$. This concludes the proof of Lemma 3.1. ■

For the small values of n, we have the following expressions for h_n and $\tilde{h}_n$:

Lemma 3.2 (The basis vectors of degree less or equal to 4) *We have*

$$h_0(y) = \; i, \qquad \tilde{h}_0(y) = \; (1+i\delta)$$

$$h_1(y) = \; iy, \qquad \tilde{h}_1(y) = \; (1+i\delta)y$$

$$h_2(y) = \; iy^2 + \beta - i(2+\delta\beta), \qquad \tilde{h}_2(y) = \; (1+i\delta)\left(y^2 - 2 + 2\delta\beta\right),$$

$$\begin{aligned} h_4(y) =& \; iy^4 + y^2(6\beta + id_{4,2}) + c_{4,0} + id_{4,0} \\ \tilde{h}_4(y) =& \; (1+i\delta)y^4 + y^2\left(12(\beta\delta - 1) + i\tilde{d}_{4,2}\right) + \tilde{c}_{4,0} + i\tilde{d}_{4,0}. \end{aligned}$$

Moreover, $\mathcal{L}_{\beta,\delta}\tilde{h}_0 = \tilde{h}_0$, $\mathcal{L}_{\beta,\delta}\tilde{h}_1 = \frac{1}{2}\tilde{h}_1$ *and* $\mathcal{L}_{\beta,\delta}\tilde{h}_2 = 2\beta(1+\delta^2)h_0 = 2i\beta(1+\delta^2)$.

Proof: The proof is straightforward though a bit lengthy. One has to start from (31) and (32) and identify the missing coefficients thanks to (33). ∎

We also have the following corollary for Lemma 3.1:

Corollary 3.3 (Basis for the set of polynomials) *The family* $(h_n, \tilde{h}_n)_{n\in\mathbb{N}}$ *is a basis of* $\mathbb{C}[X]$*, the* $\mathbb{R}$ *vector space of complex polynomials.*

3.5 Decomposition of q

For the sake of controlling q in the region $|y| < 2K\sqrt{s}$, we will expand the unknown function q (and not just χq where χ is defined in (19)) with respect to the family f_n and then with respect to the family h_n. We start by writing

$$q(y,s) = \sum_{n\le M} Q_n(s) f_n(y) + q_-(y,s) \tag{37}$$

where f_n is the eigenvalue of $\mathcal{L}_\beta$ defined in (28), $Q_n(s) \in \mathbb{C}$, q_- satisfies

$$\int q_-(y,s) f_n(y)\rho_\beta(y)dy = 0 \text{ for all } n \le M$$

and M is a fixed even integer satisfying

$$M \ge 4(\sqrt{1+\delta^2} + 1 + 2 \max_{i=1,2,\ y\in\mathbb{R},\ s\ge 1} |V_i(y,s)|), \tag{38}$$

with V_i defined in (22). From (30), we have

$$Q_n(s) = \frac{\int q(y,s) f_n(y)\rho_\beta(y)dy}{\int f_n(y)^2 \rho_\beta(y)dy} \equiv F_n(q(s)). \tag{39}$$

The function $q_-(y,s)$ can be seen as the projection of $q(y,s)$ onto the subset of the spectrum of $\mathcal{L}_\beta$ which is smaller than $(1-M)/2$. We will call it the infinite dimensional part of q and we will denote it $q_- = P_{-,M}(q)$. We also introduce $P_{+,M} = Id - P_{-,M}$. Notice that $P_{-,M}$ and $P_{+,M}$ are projections. In the sequel, we will denote $P_- = P_{-,M}$ and $P_+ = P_{+,M}$. The complementary part $q_+ = q - q_-$ will be called the finite dimensional part of q. We will expand it with respect to the Jordan decomposition of $\mathcal{L}_{\beta,\delta}$ (see Lemma 3.1) and write therefore

$$q_+(y,s) = \sum_{n\le M} Q_n(s)f_n(y) = \sum_{n\le M} q_n(s)h_n(y) + \tilde{q}_n(s)\tilde{h}_n(y) \tag{40}$$

where $q_n(s), \tilde{q}_n(s) \in \mathbb{R}$. Finally, we notice that for all s, we have

$$\int q_-(y,s)q_+(y,s)\rho_\beta(y)dy = 0.$$

Our purpose is to project equation (21) in order to write an equation for q_n and $\tilde{q}_n$. For that, we need to write down the expression of $q_n(s)$ and $\tilde{q}_n(s)$ in terms of $Q_n(s)$. Since the matrix of $(h_n, \tilde{h}_n)_{n\le M}$ in the basis (if_n, f_n) is upper triangular (see (31) and (32)), the same holds for its inverse. Thus, we derive from (40)

$$\begin{aligned} q_n(s) = &\quad \Im Q_n(s) - \delta \Re Q_n(s) &+ \sum_{j=n+1}^{M} A_{j,n}\Im Q_j(s) + B_{j,n}\Re Q_j(s) \equiv P_{n,M}(q(s)) \\ \tilde{q}_n(s) = &\qquad\qquad\qquad \Re Q_n(s) &+ \sum_{j=n+1}^{M} \tilde{A}_{j,n}\Im Q_j(s) + \tilde{B}_{j,n}\Re Q_j(s) \equiv \tilde{P}_{n,M}(q(s)) \end{aligned} \tag{41}$$

where all the constants are real. Note that the coefficients of $\Im Q_n$ and $\Re Q_n$ in the expression of q_n and $\tilde{q}_n$ are explicit. This comes from the fact that the same holds for the coefficients of if_n and f_n in the expression of h_n and $\tilde{h}_n$ (see (31) and (32)).
Note that the projectors $P_{n,M}(q)$ and $\tilde{P}_{n,M}(q)$ are well-defined thanks to (39). We will project equation (21) on the different modes h_n and $\tilde{h}_n$. Note that from (37) and (40), it holds that

$$q(y,s) = \left(\sum_{n\le M} q_n(s)h_n(y) + \tilde{q}_n(s)\tilde{h}_n(y)\right) + q_-(y,s). \tag{42}$$

We should keep in mind that this decomposition is unique.

4 Proof of the existence result

This section is devoted to the proof of the existence result (Theorem 1). We proceed in 4 steps, each of them making a separate subsection.

- In the first subsection, we define a shrinking set $V_A(s)$ and translate our goal of making $q(s)$ go to 0 in $L^\infty(\mathbb{R})$ in terms of belonging to $V_A(s)$. We also exhibit a two parameter initial data family for equation (21) whose coordinates are very small (with respect to the requirements of $V_A(s)$), except the two first $\tilde{q}_0$ and $\tilde{q}_1$.

- In the second subsection, we solve the local in time Cauchy problem for equation (21) coupled with some orthogonality condition.

- In the third subsection, using the spectral properties of equation (21), we reduce our goal from the control of $q(s)$ (an infinite dimensional variable) in $V_A(s)$ to the control of its two first components $(\tilde{q}_0(s), \tilde{q}_1(s))$ (a two-dimensional variable) in $[-\frac{A}{s^2}, \frac{A}{s^2}]^2$.

- In the fourth subsection, we solve the finite dimensional problem using index theory and conclude the proof of Theorem 1.

We could work in the set of even functions to construct a blow-up solution. However, since we need to prove the stability of the constructed solution in the set of all functions with no evenness assumption, we have to handle general functions.

4.1 Definition of a shrinking set $V_A(s)$ and preparation of initial data

Let us first introduce the following:

Proposition 4.1 (A set shrinking to zero) *For all $K > 1$, $A \geq 1$ and $s \geq e$, we define $V_A(s)$ as the set of all $r \in L^\infty(\mathbb{R})$ such that*

$$\begin{array}{rclrcl} \|r_e\|_{L^\infty(\mathbb{R})} & \leq & \dfrac{A^{M+2}}{\sqrt{s}}, & \left\|\frac{r_-(y)}{(1+|y|)^{M+1}}\right\|_{L^\infty(\mathbb{R})} & \leq & \dfrac{A^{M+1}}{s^{\frac{M+2}{2}}}, \\ |r_j|, |\tilde{r}_j| & \leq & \dfrac{A^j}{s^{\frac{j+1}{2}}} \text{ for all } 3 \leq j \leq M, & |\tilde{r}_0|, |\tilde{r}_1| & \leq & \dfrac{A}{s^2}, \\ |\tilde{r}_2| & \leq & \frac{A^5 \log s}{s^2}, & |r_1| & \leq & \dfrac{A^4}{s^2}, \\ |r_2| & \leq & \dfrac{A^2}{s^2}, & |r_0| & \leq & \dfrac{1}{s^2}, \end{array} \tag{43}$$

where r_e is defined as in (20), r_-, r_n and $\tilde{r}_n$ are defined in (42). Then, we have for all $s \geq 1$ and $r \in V_A(s)$,
(i) $\|r\|_{L^\infty(|y|<2K\sqrt{s})} \leq C(K)\frac{A^{M+1}}{\sqrt{s}}$ and $\|r\|_{L^\infty(\mathbb{R})} \leq \frac{C(K)A^{M+2}}{\sqrt{s}}$.
(ii) for all $y \in \mathbb{R}$, $|r(y)| \leq CA^{M+1}\frac{\log s}{s^2}(1+|y|^{M+1})$.

Proof: Take $r \in V_A(s)$ and $y \in \mathbb{R}$.
(i) If $|y| \geq 2K\sqrt{s}$, then we have from the definition of r_e (20), $|r(y)| = |r_e(y)| \leq \frac{A^{M+2}}{\sqrt{s}}$.
Now, if $|y| < 2K\sqrt{s}$, since we have for all $0 \leq j \leq M$, $|r_j| + |\tilde{r}_j| \leq C\frac{A^{M+1}}{s^{\frac{j+1}{2}}}$ from (43) (use the fact that $M \geq 4$), we write from (42)

$$\begin{aligned} |r(y)| &\leq \left(\sum_{j\leq M} |r_j||h_j(y)| + |\tilde{r}_j||\tilde{h}_j(y)|\right) + |r_-(y)| \qquad (44) \\ &\leq C\sum_{j\leq M} \frac{A^{M+1}}{s^{\frac{j+1}{2}}}(1+|y|)^j + \frac{A^{M+1}}{s^{\frac{M+2}{2}}}(1+|y|)^{M+1} \\ &\leq C\sum_{j\leq M} \frac{A^{M+1}}{s^{\frac{j+1}{2}}}(1+K\sqrt{s})^j + \frac{A^{M+1}}{s^{\frac{M+2}{2}}}(1+K\sqrt{s})^{M+1} \leq C\frac{(KA)^{M+1}}{\sqrt{s}}, \end{aligned}$$

which gives (i).
(ii) Just use (44) together with the fact that for all $0 \leq j \leq M$, $|r_j| + |\tilde{r}_j| \leq CA^{M+1}\frac{\log s}{s^2}$ from (43). This ends the proof of Proposition 4.1. ■

Initial data (at time $s = s_0 \equiv -\log T$) for the equation (21) will depend on two real parameters d_0 and d_1 (besides s_0) as given in the following proposition:

Proposition 4.2 (Decomposition of initial data on the different components) *For all $A \geq 1$, there exists $T_1(A) \in (0, 1/e)$ such that for all $T \leq T_1$:*

(i) $P_{0,M}\left(i\chi(2y,s_0)\right) \neq 0$ *and the following function is well defined:*

$$\begin{array}{rcl}\psi_{s_0,d_0,d_1}(y) &=& \frac{A}{s_0^2}\left(d_0(1+i\delta)+d_1(1+i\delta)y+d_2 i\right)\chi(2y,s_0) \text{ where } s_0=-\log T,\\ d_2(s_0,d_0,d_1) &=& -\dfrac{d_0P_{0,M}((1+i\delta)\chi(2y,s_0))+d_1P_{0,M}((1+i\delta)y\chi(2y,s_0))}{P_{0,M}(i\chi(2y,s_0))}\end{array} \quad (45)$$

and χ is defined in (19).

(ii) There exists a quadrilateral $\mathcal{D}_T \subset [-2,2]^2$ such that the mapping $(d_0,d_1) \to (\tilde\psi_0,\tilde\psi_1)$ (where ψ stands for ψ_{s_0,d_0,d_1}) is linear, one to one from $\mathcal{D}_T$ onto $[-\frac{A}{s_0^2},\frac{A}{s_0^2}]^2$ and maps $\partial\mathcal{D}_T$ onto $\partial[-\frac{A}{s_0^2},\frac{A}{s_0^2}]^2$. Moreover, it is of degree 1 on the boundary.

(iii) For all $(d_0,d_1)\in\mathcal{D}_T$, $\psi_e\equiv 0$, $\psi_0=0$, $|\psi_i|+|\tilde\psi_j|\le CAe^{-\gamma s_0}$ for some $\gamma>0$ and for all $1\le i\le M$ and $2\le j\le M$. Moreover, $\left\|\frac{\psi_-(y)}{(1+|y|)^{M+1}}\right\|_{L^\infty(\mathbb{R})}\le C\dfrac{A}{s_0^{\frac{M}{2}+2}}$.

(iv) For all $(d_0,d_1)\in\mathcal{D}_T$, $\psi_{s_0,d_0,d_1}\in V_A(s_0)$ with strict inequalities except for $(\tilde\psi_0,\tilde\psi_1)$.

Remark: In some sense, ψ_{s_0,d_0,d_1} is reduced to its components on $\tilde h_0$ and $\tilde h_1$. In N dimensions, one has to take $d_0\in\mathbb{R}$ and $d_1\in\mathbb{R}^N$, because the finite dimensional problem we reduce to is in $\mathbb{R}^{N+1}$.

Proof of Proposition 4.2: For simplicity, we write ψ instead of ψ_{s_0,d_0,d_1}. Note first from Proposition 4.1 that (iv) follows from (ii) and (iii) by taking $s_0=-\log T$ large enough (that is, T small enough). Thus, we only prove (i), (ii) and (iii). Consider some $K\ge 1$, $A\ge 1$ and $T\le 1/e$. Note that $s_0=-\log T\ge 1$.

(i) The proof of (i) is a direct consequence of (iii) of the following claim:

Lemma 4.3 *There exist $\gamma=\frac{1}{32(1+\beta^2)}>0$ and $T_2<1/e$ such that for all $K\ge 1$ and all $T\le T_2$, if g is given by $(1+i\delta)\chi(2y,s_0)$, $(1+i\delta)y\chi(2y,s_0)$ or $i\chi(2y,s_0)$, then $\left\|\frac{g_-(y)}{(1+|y|)^{M+1}}\right\|_{L^\infty(\mathbb{R})}\le\dfrac{C}{s_0^{\frac{M}{2}}}$ and all g_i and $\tilde g_i$ for $0\le i\le M$ are less than $Ce^{-\gamma s_0}$, except:*
(i) $|\tilde g_0-1|\le Ce^{-\gamma s_0}$ when $g=(1+i\delta)\chi(2y,s_0)$,
(ii) $|\tilde g_1-1|\le Ce^{-\gamma s_0}$ when $g=(1+i\delta)y\chi(2y,s_0)$,
(iii) $|g_0-1|\le Ce^{-\gamma s_0}$ when $g=i\chi(2y,s_0)$.

Proof: In all cases, we write

$$g(y)=p(y)+r(y) \text{ where } p(y)=(1+i\delta),\ (1+i\delta)y \text{ or } i \text{ and } r(y)=p(y)(\chi(2y,s_0)-1). \quad (46)$$

From the uniqueness of the decomposition (42) and Lemma 3.2, we see that $p_-\equiv 0$ and all p_i and $\tilde p_i$ are zero except

$$\tilde p_0=1 \text{ when } p(y)=1+i\delta,\ \tilde p_1=1 \text{ when } p(y)=(1+i\delta)y \text{ and } p_0=1 \text{ when } p(y)=i. \quad (47)$$

Considering the cases $2|y|<K\sqrt{s}$ and $2|y|>K\sqrt{s}$, we have by definition of χ (19),

$$\begin{array}{rcl}1-\chi(2y,s) &\le& \left(\frac{2|y|}{K\sqrt{s_0}}\right)^M,\\ |\rho_\beta(y)(1-\chi(2y,s))| &\le& \sqrt{|\rho_\beta(y)|}\sqrt{|\rho_\beta\left(\frac{K}{2}\sqrt{s_0}\right)|}\le Ce^{-\frac{K^2s_0}{32(1+\beta^2)}}\sqrt{|\rho_\beta(y)|}.\end{array}$$

Therefore, from (46), (39) and (41), we see that

$$\begin{aligned}
&|r(y)| \le C(1+|y|)\left(\frac{2|y|}{K\sqrt{s_0}}\right)^M \le C\frac{(1+|y|)^{M+1}}{s_0^{\frac{M}{2}}},\\
&|R_j|+|r_j|+|\tilde r_j| \le Ce^{-\frac{K^2 s_0}{32(1+\beta^2)}} \text{ for all } j\le M.
\end{aligned} \tag{48}$$

Hence, using (48), (37) and the fact that $|f_j(y)| \le C(1+|y|)^M$ for all $j \le M$, we get also

$$|r_-(y)| \le C\frac{(1+|y|)^{M+1}}{s_0^{\frac{M}{2}}}.$$

Using (46) and the estimates for $p(y)$ stated in (47) and before, this concludes the proof of Lemma 4.3 and (i) of Proposition 4.2. ■

(ii) From (45), we see that

$$\begin{pmatrix} \tilde\psi_0 \\ \tilde\psi_1 \end{pmatrix} = G\begin{pmatrix} d_0 \\ d_1 \end{pmatrix} \text{ where } G=(g_{i,j})_{0\le i,j\le 1}. \tag{49}$$

Using first Lemma 4.3, we see from (45) that

$$|d_2| \le C(|d_0|+|d_1|)e^{-\gamma s_0} \tag{50}$$

for T small enough. Using again Lemma 4.3, we see that $\frac{s_0^2}{A}G \to \mathrm{Id}$ as $s_0\to\infty$ (for fixed K and A), which concludes the proof of (ii).

(iii) Since $\mathrm{supp}(\psi)\subset B(0,K\sqrt{s_0})$ by (45) and (19), we see from (20) that $\psi_e\equiv 0$.
By definition of ψ (45), we see that $\psi_0 = P_{0,M}(\psi) = d_0P_{0,M}((1+i\delta)\chi(2y,s_0))+d_1P_{0,M}((1+i\delta)y\chi(2y,s_0)) + d_2P_{0,M}(i\chi(2y,s_0))$ which is zero by definition of d_2 (45).
Using the fact that $|d_i|\le 2$ and the bound on d_2 (50), we see that the estimates on ψ_i, $\tilde\psi_j$ and ψ_- in (iii) follow from (45) and Lemma 4.3. This concludes the proof of Proposition 4.2. ■

4.2 Local in time solution for the problem (21)-(51)

In the following, we find a local in time solution for equation (21) coupled with the condition

$$P_{0,M}(q(s)) = 0. \tag{51}$$

Proposition 4.4 (Local in time solution for problem (21)-(51) with initial data (45)) *For all $A\ge 1$, there exists $T_3(A)\in(0,1/e)$ such that for all $T\le T_3$, the following holds:*
For all $(d_0,d_1)\in\mathcal{D}_T$, there exists $s_{max} > s_0 = -\log T$ such that the problem (21)-(51) with initial data at $s=s_0$,

$$(q(s_0),\theta(s_0)) = (\psi_{s_0,d_0,d_1},0)$$

where ψ_{s_0,d_0,d_1} is given by (45), has a unique solution satisfying $q(s)\in V_{A+1}(s)$ for all $s\in[s_0,s_{max})$.

Proof: From the solution of the local in time Cauchy problem for equation (1) in $L^\infty(\mathbb{R})$, there exists $s_1 > s_0$ such that equation (11) with initial data (at $s = s_0$) $\varphi(y, s_0) + \psi_{s_0,d_0,d_1}(y)$, where $\varphi(y, s)$ is given by (16) has a unique solution $w(s) \in C([s_0, s_1), L^\infty(\mathbb{R}))$. Now, we have to find a unique $(q(s), \theta(s))$ such that

$$w(y, s) = e^{i(\mu \log s + \theta(s))}(\varphi(y, s) + q(y, s)) \tag{52}$$

and (51) is satisfied. Since $f_0 = 1$ from the convention after (28) and $\int_{\mathbb{R}} \rho_\beta(y)dy = 1$ (see Appendix A), we use (41) and (39) to write condition (51) as

$$P_{0,M}(q) = \Im(\int q(y, s)\rho_\beta(y)dy) - \delta\Re(\int q(y, s)\rho_\beta(y)dy) = \Im((1-i\delta)\int q(y, s)\rho_\beta(y)dy) = 0,$$

or using (52),

$$F(s, \theta(s)) \equiv \Im((1 - i\delta)\int (e^{-i(\mu \log s+\theta(s))}w(y, s) - \varphi(y, s))\rho_\beta(y)dy) = 0.$$

Note that

$$\frac{\partial F}{\partial \theta}(s, \theta) = -\Re((1 - i\delta)\int (e^{-i(\mu \log s+\theta(s))}w(y, s))\rho_\beta(y)dy).$$

Moreover, from (iii) in Proposition 4.2, $F(s_0, 0) = P_{0,M}(\psi_{s_0,d_0,d_1}) = 0$ and $\frac{\partial F}{\partial \theta}(s_0, 0) = -\Re((1 - i\delta)\int(\varphi(y, s_0) + \psi_{s_0,d_0,d_1}(y))\rho_\beta(y)dy) = -\kappa + O(\frac{1}{\sqrt{s_0}})$ as $s_0 \to \infty$ (for fixed K and A). Therefore, if T is small enough in terms of A, then $\frac{\partial F}{\partial \theta}(s_0, 0) \neq 0$, and from the implicit function theorem, there exists $s_2 \in (s_0, s_1)$ and $\theta \in C^1([s_0, s_2), \mathbb{R})$ such that $F(s, \theta(s)) = 0$ for all $s \in [s_0, s_2))$. Defining $q(., s) \in L^\infty(\mathbb{R})$ by (52) gives a unique solution of problem (21)-(51) for all $s \in [s_0, s_2)$. Now, since we have from (iv) of Proposition 4.2, $q(s_0) \in V_A(s_0) \underset{\neq}{\subset} V_{A+1}(s_0)$, there exists $s_3 \in (s_0, s_2)$ such that for all $s \in [s_0, s_3)$, $q(s) \in V_{A+1}(s)$. This concludes the proof of Proposition 4.4. ■

4.3 Reduction to a finite dimensional problem

In the following, we reduce the problem to a finite dimensional one:

Proposition 4.5 (Control of $q(s)$ in $V_A(s)$ by $(\tilde q_0(s), \tilde q_1(s))$) *There exist $A_4 \geq 1$ such that for all $A \geq A_4$, there exists $T_4(A) \in (0, 1/e)$ such that for all $T \leq T_4$, the following holds:*
If (q, θ) is a solution of (21)-(51) with initial data at $s = s_0 = -\log T$ given by (45) with $(d_0, d_1) \in \mathcal{D}_T$, and $q(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ with $q(s_1) \in \partial V_A(s_1)$ for some $s_1 \geq s_0$, then:

(i) **(Smallness of the modulation parameter)** *For all $s \in [s_0, s_1]$, $|\theta'(s)| \leq C\frac{A^5 \log s}{s^2}$.*

(ii) $(\tilde q_0(s_1), \tilde q_1(s_1)) \in \partial[-\frac{A}{s_1^2}, \frac{A}{s_1^2}]^2$.

(iii) **(Transverse crossing)** *There exists $m \in \{0, 1\}$ and $\omega \in \{-1, 1\}$ such that*

$$\omega\tilde q_m(s_1) = \frac{A}{s_1^2} \text{ and } \omega\frac{d\tilde q_m}{ds}(s_1) > 0.$$

Remark: In N dimensions, $\tilde{q}_0 \in \mathbb{R}$ and $\tilde{q}_1 \in \mathbb{R}^N$. In particular, the finite dimensional problem is of dimension $N+1$. This is why in initial data (45), one has to take $d_0 \in \mathbb{R}$ and $d_1 \in \mathbb{R}^N$.
The idea of the proof of Proposition 4.5 is to project equation (21) on the different components of the decomposition (42). More precisely, we claim that Proposition 4.5 is a consequence of the following:

Proposition 4.6 *There exists $A_5 \geq 1$ such that for all $A \geq A_5$, there exists $s_5(A)$ such that the following holds for all $s_0 \geq s_5$:*
Assume that for all $s \in [\tau, s_1]$ for some $s_1 \geq \tau \geq s_0$, $q(s) \in V_A(s)$ and $q_0(s) = 0$, then the following holds for all $s \in [\tau, s_1]$:
(i) **(Smallness of the modulation parameter)***:*

$$|\theta'(s)| \leq C\frac{A^5 \log s}{s^2}.$$

(ii) **(ODE satisfied by the expanding modes)***: For $m = 0$ and 1, we have*

$$|\tilde{q}_m' - \left(1 - \frac{m}{2}\right)\tilde{q}_m| \leq \frac{C}{s^2}.$$

(iii) **(Control of null and negative modes)***:*

$$\begin{aligned}
|\tilde{q}_2(s)| &\leq \frac{\tau^2}{s^2}|\tilde{q}_2(\tau)| + C\frac{A^4}{\tau s^2}(s-\tau),\\
|q_1(s)| &\leq e^{-\frac{(s-\tau)}{2}}|q_1(\tau)| + \frac{C\,A^3}{s^2},\\
|q_2(s)| &\leq e^{-(s-\tau)}|q_2(\tau)| + \frac{C}{s^2},\\
|q_j(s)| &\leq e^{-\frac{j(s-\tau)}{2}}|q_j(\tau)| + \frac{C\,A^{j-1}}{s^{\frac{j+1}{2}}} \text{ for all } 3 \leq j \leq M,\\
|\tilde{q}_j(s)| &\leq e^{-\frac{(j-2)(s-\tau)}{2}}|\tilde{q}_j(\tau)| + \frac{C\,A^{j-1}}{s^{\frac{j+1}{2}}},\\
\left\|\frac{q_-(s)}{1+|y|^{M+1}}\right\|_{L^\infty} &\leq e^{-\frac{M+1}{4}(s-\tau)}\left\|\frac{q_-(\tau)}{1+|y|^{M+1}}\right\|_{L^\infty} + C\frac{A^M}{s^{\frac{M+2}{2}}},\\
\|q_e(s)\|_{L^\infty} &\leq e^{-\frac{(s-\tau)}{2(p-1)}}\|q_e(\tau)\|_{L^\infty} + \frac{CA^{M+1}}{\sqrt{\tau}}(1+s-\tau).
\end{aligned}$$

The idea of the proof of Proposition 4.6 is to project equations (21) and (17) according to the decomposition (42). However, because of the number of parameters in our problem (p, δ and β) and the coordinates in (42), the computations become too long. That is why a whole section (the next one) is devoted to the proof of Proposition 4.6. Let us now derive Proposition 4.5 from Proposition 4.6.

Proof of Proposition 4.5 assuming Proposition 4.6:
We will take $A_4 \geq A_5$. Hence, we can use the conclusions of Proposition 4.6.

The proof of (i) follows from (i) in Proposition 4.6. Indeed, by choosing T_4 small enough, we can make $s_0 = -\log T$ bigger than $s_5(A)$.

To prove (ii), we notice that from Proposition 4.1 and the fact that $q_0(s) = 0$, it is enough to prove that for all $s \in [s_0, s_1]$,

$$
\begin{array}{rclrcl}
\|q_e\|_{L^\infty(\mathbb{R})} & \leq & \dfrac{A^{M+2}}{2\sqrt{s}}, & \|\frac{q_-(y)}{(1+|y|)^{M+1}}\|_{L^\infty(\mathbb{R})} & \leq & \dfrac{A^{M+1}}{2s^{\frac{M+2}{2}}}, \\
|q_j|, |\tilde{q}_j| & \leq & \dfrac{A^j}{2s^{\frac{j+1}{2}}} \text{ for all } 3 \leq j \leq M, & |\tilde{q}_2| & \leq & \dfrac{A^5 \log s}{s^2} - \dfrac{1}{s^3}, \\
|q_1| & \leq & \dfrac{A^4}{2s^2}, & |q_2| & \leq & \dfrac{A^2}{2s^2}.
\end{array} \tag{53}
$$

Define $\sigma = \log A$ and take $s_0 \geq \sigma$ (that is, $T \leq e^{-\sigma} = 1/A$) so that for all $\tau \geq s_0$ and $s \in [\tau, \tau + \sigma]$, we have

$$
\tau \leq s \leq \tau + \sigma \leq \tau + s_0 \leq 2\tau \text{ hence } \frac{1}{2\tau} \leq \frac{1}{s} \leq \frac{1}{\tau} \leq \frac{2}{s}. \tag{54}
$$

We consider two cases in the proof.

Case 1: $s \leq s_0 + \sigma$.

Note that (54) holds with $\tau = s_0$. Using (iii) of Proposition 4.6 and estimate (iii) of Proposition 4.2 on the initial data $q(., s_0)$ (where we use (54) with $\tau = s_0$), we write

$$
\begin{array}{rcl}
|\tilde{q}_2(s)| & \leq & CAe^{-\gamma\frac{s}{2}} + C\dfrac{A^4}{\frac{s^3}{2}} \log A, \\
|q_1(s)| & \leq & CAe^{-\gamma\frac{s}{2}} + \dfrac{C\,A^3}{s^2}, \\
|q_2(s)| & \leq & CAe^{-\gamma\frac{s}{2}} + \dfrac{C}{s^2}, \\
|q_j(s)| & \leq & CAe^{-\gamma\frac{s}{2}} + \dfrac{C\,A^{j-1}}{s^{\frac{j+1}{2}}} \text{ for all } 3 \leq j \leq M, \\
|\tilde{q}_j(s)| & \leq & CAe^{-\gamma\frac{s}{2}} + \dfrac{C\,A^{j-1}}{s^{\frac{j+1}{2}}} \text{ for all } 3 \leq j \leq M, \\
\left\|\dfrac{q_-(s)}{1+|y|^{M+1}}\right\|_{L^\infty} & \leq & C\dfrac{A}{\left(\frac{s}{2}\right)^{\frac{M}{2}+2}} + C\dfrac{A^M}{s^{\frac{M+2}{2}}}, \\
\|q_e(s)\|_{L^\infty} & \leq & \dfrac{CA^{M+1}}{\sqrt{\frac{s}{2}}}\,(1 + \log A)\,.
\end{array}
$$

Thus, if $A \geq A_6$ and $s_0 \geq s_6(A)$ (that is $T \leq e^{-s_6(A)}$) for some positive A_6 and $s_6(A)$, we see that (53) holds.

Case 2: $s > s_0 + \sigma$.

Let $\tau = s - \sigma > s_0$. Applying (iii) of Proposition 4.6 and using the fact that $q(\tau) \in V_A(\tau)$, we write (we use (54) to bound any function of τ by a function of s, except for the first line which requires a more delicate treatment):

$$\begin{array}{rcl}
|\tilde q_2(s)| & \le & \frac{\tau^2}{s^2}\frac{A^5\log\tau}{\tau^2}+C\frac{A^4}{\tau s^2}\sigma,\\
|q_1(s)| & \le & e^{-\frac{\sigma}{2}}\frac{A^4}{\left(\frac{s}{2}\right)^2}+\frac{C\,A^3}{s^2},\\
|q_2(s)| & \le & e^{-\sigma}\frac{A^2}{\left(\frac{s}{2}\right)^2}+\frac{C}{s^2},\\
|q_j(s)| & \le & e^{-\frac{j\sigma}{2}}\frac{A^j}{\left(\frac{s}{2}\right)^{\frac{j+1}{2}}}+\frac{C\,A^{j-1}}{s^{\frac{j+1}{2}}}\text{ for all }3\le j\le M,\\
|\tilde q_j(s)| & \le & e^{-\frac{(j-2)\sigma}{2}}\frac{A^j}{\left(\frac{s}{2}\right)^{\frac{j+1}{2}}}+\frac{C\,A^{j-1}}{s^{\frac{j+1}{2}}}\text{ for all }3\le j\le M,\\
\left\|\frac{q_-(s)}{1+|y|^{M+1}}\right\|_{L^\infty} & \le & e^{-\frac{M+1}{4}\sigma}\frac{A^{M+1}}{\left(\frac{s}{2}\right)^{\frac{M+2}{2}}}+C\frac{A^M}{s^{\frac{M+2}{2}}},\\
\|q_e(s)\|_{L^\infty} & \le & e^{-\frac{\sigma}{2(p-1)}}\frac{A^{M+2}}{\sqrt{\frac{s}{2}}}+\frac{CA^{M+1}}{\sqrt{\frac{s}{2}}}(1+\sigma)\,.
\end{array}$$

For all the coordinates except $\tilde q_2(s)$, it is clear that if $A\ge A_7$ and $s_0\ge s_7(A)$ for some positive A_7 and $s_7(A)$, then (53) is satisfied (remember that $\sigma=\log A$). For $\tilde q_2(s)$, (53) will be satisfied if

$$A^5\frac{\log(s-\sigma)}{s^2}+C\frac{A^4}{(s-\sigma)s^2}\sigma\le A^5\frac{\log s}{s^2}-\frac{1}{s^3}. \tag{55}$$

Since for fixed A, we have

$$A^5\frac{\log(s-\sigma)}{s^2}+C\frac{A^4}{(s-\sigma)s^2}\sigma=A^5\frac{\log s}{s^2}+(CA^4-A^5)\frac{\sigma}{s^3}+O\left(\frac{1}{s^4}\right)$$

as $s\to\infty$, we see that if $A\ge A_8$ and $s_0\ge s_8(A)$ for some positive A_8 and $s_8(A)$, then (55) is satisfied, hence (53) is satisfied for $\tilde q_2(s)$.

Conclusion of (ii): If $A\ge\max(A_6,A_7,A_8)$ and $s_0\ge\max(s_6(A),s_7(A),s_8(A))$, then (53) is satisfied. Since we know that $q(s_1)\in\partial V_A(s_1)$, we see from the definition of $V_A(s)$ (Proposition 4.1) that $(\tilde q_1(s_1),\tilde q_2(s_1))\in\partial[-\frac{A}{s_1^2},\frac{A}{s_1^2}]^2$. This concludes the proof of (ii) of Proposition 4.5.

(iii) From (ii), there is $m=0$ or 1 and $\omega=\pm1$ such that $\tilde q_m(s_1)=\omega\frac{A}{s_1^2}$. Using (ii) of Proposition 4.6, we see that

$$\omega\tilde q_m'(s_1)\ge(1-\frac{m}{2})\omega\tilde q_m(s_1)-\frac{C}{s_1^2}\ge\frac{(1-m/2)A-C}{s_1^2}.$$

Taking A large enough gives $\omega\tilde q_m'(s_1)>0$ and concludes the proof of Proposition 4.5 assuming Proposition 4.6 is true. It remains to prove Proposition 4.6 to finish the proof of Proposition 4.5. This will be done later in Section 5. ■

4.4 Proof of the finite dimensional problem and Proof of Theorem 1

We prove Theorem 1 using the previous subsections. We proceed in two parts:

- In Part 1, we solve the finite dimensional problem and prove the existence of $A \geq 1$, $T > 0$, $(d_0, d_1) \in \mathcal{D}_T$ such that problem (21)-(51) with initial data at $s = s_0 = -\log T$, $(\psi_{s_0,d_0,d_1}, 0)$ given by (45) has a solution $(q(s), \theta(s))_{d_0,d_1}$ defined for all $s \in [s_0, \infty)$ such that

$$q(s) \in V_A(s) \text{ and } |\theta'(s)| \leq C\frac{A^5 \log s}{s^2} \text{ for all } s \in [-\log T, +\infty). \tag{56}$$

- In Part 2, we show that the solution constructed in Part 1 provides a blow-up solution of equation (1) which satisfies all the properties stated in Theorem 1, which concludes the proof.

Part 1: Solution of the finite dimensional problem

We take $A = A_4$ and $T = \min(T_1(A), T_3(A), T_4(A))$ so that Propositions 4.2, 4.4 and 4.5 apply. We will find the parameter (d_0, d_1) in the set $\mathcal{D}_T$ defined in Proposition 4.2. We proceed by contradiction and assume from (iv) of Proposition 4.2 that for all $(d_0, d_1) \in \mathcal{D}_T$, there exists $s_*(d_0, d_1) \geq -\log T$ such that $q_{d_0,d_1}(s) \in V_A(s)$ for all $s \in [-\log T, s_*]$ and $q_{d_0,d_1}(s_*) \in \partial V_A(s_*)$. From (ii) of Proposition 4.5, we see that $(\tilde{q}_0(s_*), \tilde{q}_1(s_*)) \in \partial[-\frac{A}{{s_*}^2}, \frac{A}{{s_*}^2}]^2$ and the following function is well defined:

$$\begin{aligned} \Phi \quad &: \mathcal{D}_T \to \partial[-1, 1]^2 \\ &(d_0, d_1) \to \tfrac{s_*^2}{A}\left(\tilde{q}_0, \tilde{q}_1\right)_{d_0,d_1}(s_*). \end{aligned} \tag{57}$$

From (iii) of Proposition 4.5, Φ is continuous. If we manage to prove that Φ is of degree 1 on the boundary, then we have a contradiction from the degree theory. Let us prove that.

Using (ii) and (iv) of Proposition 4.2 and the fact that $q(-\log T) = \psi_{d_0,d_1}$, we see that when (d_0, d_1) is on the boundary of the quadrilateral $\mathcal{D}_T$, $(\tilde{q}_0, \tilde{q}_1)(-\log T) \in \partial[-\frac{A}{(\log T)^2}, \frac{A}{(\log T)^2}]^2$ and $q(-\log T) \in V_A(-\log T)$ with strict inequalities for the other components. Applying the transverse crossing property of Proposition 4.5, we see that $q(s)$ leaves $V_A(s)$ at $s = -\log T$, hence $s_*(d_0, d_1) = -\log T$. Using (ii) of Proposition 4.2, we see that the restriction of Φ to the boundary is of degree 1. A contradiction then follows. Thus, there exists a value $(d_0, d_1) \in \mathcal{D}_T$ such that $\forall s \geq -\log T$, $q_{d_0,d_1}(s) \in V_A(s)$. Using (i) of Proposition 4.5, we get the bound on $\theta'(s)$. This concludes the proof of (56).

Part 2: Proof of Theorem 1

Here, we use the solution of problem (21)-(51) constructed in Part 1 to exhibit a blow-up solution of equation (1) and prove Theorem 1.

(i) Consider $(q(s), \theta(s))$ constructed in Part 1 such that (56) holds. From (56) and Proposition 4.1, we see that $\theta(s) \to \theta_0$ as $s \to \infty$ such that

$$|\theta(s) - \theta_0| \leq CA^5 \int_s^\infty \frac{\log \tau}{\tau^2} d\tau \leq C\frac{A^5 \log s}{s} \text{ and } \|q(s)\|_{L^\infty(\mathbb{R})} \leq \frac{C_0(K, A)}{\sqrt{s}}.$$

Introducing $w(y, s) = e^{i(\mu \log s + \theta(s))}(\varphi(y, s) + q(y, s))$, we see from the beginning of section 3 that w is a solution of equation (11) that satisfies for all $s \geq -\log T$ and $y \in \mathbb{R}$,

$$|w(y, s) - e^{i\theta_0 + i\mu \log s}\varphi(y, s)| \leq C\|q(s)\|_{L^\infty} + C|\theta(s) - \theta_0| \leq \frac{C_0}{\sqrt{s}}.$$

Introducing

$$u(x,t) = e^{-i\theta_0}\kappa^{i\delta}(T-t)^{\frac{1+i\delta}{p-1}} w\left(\frac{s}{\sqrt{T-t}}, -\log(T-t)\right),$$

we see from (10) and the definition of φ (16) that u is a solution of equation (1) defined for all $(x,t) \in \mathbb{R} \times [0,T)$ which satisfies (5).
If $x_0 = 0$, then we see from (5) that $|u(0,t)| \sim \kappa(T-t)^{-\frac{1}{p-1}}$ as $t \to T$. Hence, u blows up at time T at $x_0 = 0$. It remains to prove that when $x_0 \neq 0$, x_0 is not a blow-up point. The following result from Giga and Kohn [7] allows us to conclude:

Proposition 4.7 (Giga and Kohn - No blow-up under some threshold) *For all $C_0 > 0$, there is $\eta_0 > 0$ such that if $v(\xi,\tau)$ solves*

$$|\partial_t v - (1+i\beta)\Delta v| \leq C_0(1+|v|^p)$$

and satisfies

$$|v(\xi,\tau)| \leq \eta_0(T-t)^{-\frac{1}{p-1}}$$

for all $(\xi,\tau) \in B(a,r) \times [T-r^2, T)$ for some $a \in \mathbb{R}$ and $r > 0$, then v does not blow up at (a,T).

Indeed, since we see from (5) and (16) that

$$\sup_{|x-x_0|<\frac{|x_0|}{2}} (T-t)^{\frac{1}{p-1}}|u(x,t)| \leq \left|\varphi_0\left(\frac{|x_0|/2}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| + \frac{C}{\sqrt{|\log(T-t)|}} \to 0$$

as $t \to T$, x_0 is not a blow-up point of u from Proposition 4.7. This concludes the proof of (i) of Theorem 1.
Proof of Proposition 4.7: Although Giga and Kohn give in [7] the proof only when $\beta = 0$, their argument remains valid for other values of β, simply because the semigroup and the fundamental solution generated by $(1+i\beta)\Delta v$ have the same regularizing effect independently from β. ■

(ii) Using the techniques of [13], we derive the existence of a blow-up profile $u^* \in C^2(\mathbb{R}^*)$ such that $u(x,t) \to u^*(x)$ as $t \to T$, uniformly on compact sets of $\mathbb{R}^*$. The profile u^* is singular at the origin. In the following, we would like to find its equivalent as $x \to 0$. In comparison with the case $\beta = 0$ treated in [23], no new idea is needed. Therefore, we just give the key argument. The reader is invited to see Section 4 in [23] for details. Consider $K_0 > 0$ to be fixed large enough later. If $x_0 \neq 0$ is small enough, we introduce for all $(\xi,\tau) \in \mathbb{R} \times [-\frac{t_0(x_0)}{T-t_0(x_0)}, 1)$,

$$\begin{array}{rcl} v(x_0,\xi,\tau) & = & (T-t_0(x_0))^{\frac{1+i\delta}{p-1}}|\log(T-t_0(x_0))|^{-i\mu}u(x,t) \\ \text{where} \qquad x & = & x_0 + \xi\sqrt{(T-t_0(x_0))},\ t = t_0(x_0) + \tau(T-t_0(x_0)), \end{array} \tag{58}$$

and $t_0(x_0)$ is uniquely determined by

$$|x_0| = K_0\sqrt{(T-t_0(x_0))|\log(T-t_0(x_0))|}. \tag{59}$$

From the invariance of equation (1) under dilations, $v(x_0,\cdot,\cdot)$ is also a solution of (1) on its domain. From (58), (59) and (5), we have

$$\sup_{|\xi|<2|\log(T-t_0(x_0))|^{1/4}} |v(x_0,\xi,0)-\varphi_0(K_0)| \le \frac{C}{|\log(T-t_0(x_0))|^{1/4}} \to 0 \text{ as } x_0\to 0$$

where φ_0 is defined in (16). Using the continuity with respect to initial data for equation (1) associated to a space-localization in the ball $B(0,|\xi|<|\log(T-t_0(x_0))|^{1/4})$, we show as in Section 4 of [23] that

$$\sup_{\{|\xi|<|\log(T-t_0(x_0))|^{1/4},\ 0\le\tau<1\}} |v(x_0,\xi,\tau)-\hat v_{K_0}(\tau)| \le \epsilon(x_0)\to 0 \text{ as } x_0\to 0,$$

where $\hat v_{K_0}(\tau) = \left((p-1)(1-\tau)+bK_0^2\right)^{-\frac{1+i\delta}{p-1}}$ is the solution of the PDE (1) with constant initial data $\varphi_0(K_0)$ (it is also a solution of the associated ODE given in (3)). Making $\tau\to 1$ and using (58), we see that

$$\begin{aligned} u^*(x_0)=\lim_{t\to T} u(x,t) &= (T-t_0(x_0))^{-\frac{1+i\delta}{p-1}}|\log(T-t_0(x_0))|^{i\mu}\lim_{\tau\to 1} v(x_0,0,\tau)\\ &\sim (T-t_0(x_0))^{-\frac{1+i\delta}{p-1}}|\log(T-t_0(x_0))|^{i\mu}\hat v_{K_0}(1)\end{aligned}$$

as $x_0\to 0$. Since we have from (59)

$$\log(T-t_0(x_0))\sim 2\log|x_0| \text{ and } T-t_0(x_0)\sim \frac{|x_0|^2}{2K_0^2|\log|x_0||}$$

as $x_0\to 0$, this yields (7) and concludes the proof of Theorem 1, assuming Proposition 4.6. ■

5 Proof of Proposition 4.6

In this section, we prove Proposition 4.6. We just have to project equations (17) and (21) to get equations satisfied by the different coordinates of the decomposition (42). More precisely, the proof will be carried out in 3 subsections,

- In the first subsection, we deal with equation (21) (which is accurate up to second order terms) to write equations satisfied by q_j and $\tilde q_j$. Then, we prove (i), (ii) and (iii) (except the two last identities) of Proposition 4.6.
- In the second subsection, we first derive from equation (21) an equation satisfied by q_- and prove the last but one identity in (iii) of Proposition 4.6.
- The third subsection is the shortest. We project equation (17) (which is simpler than (21)) to write an equation satisfied by q_e and prove the last identity in (iii) of Proposition 4.6.

5.1 The finite dimensional part: q_+

We proceed in 2 parts:

- In Part 1, we project equation (21) to get equations satisfied by q_j and $\tilde q_j$.

- In Part 2, we prove (i) and (ii) of Proposition 4.6, together with the estimates concerning q_j and $\tilde{q}_j$ in (iii).

Part 1: The projection of equation (21) on the eigenfunctions of the operator $\mathcal{L}_{\beta,\delta}$

In the following, we will find the main contribution in the projections $P_{n,M}$ and $\tilde{P}_{n,M}$ of the six terms appearing in equation (21): $\partial_s q$, $\mathcal{L}_{\beta,\delta} q$, $-i\left(\frac{\mu}{s}+\theta'(s)\right)q$, $V_1 q+V_2\bar{q}$, $B(q,y,s)$ and $R^*(\theta',y,s)$. Most of the time, we give two estimates of error terms, depending on whether we use or not the fact that $q(s)\in V_A(s)$.

First term: $\dfrac{\partial q}{\partial s}$.

From (41) and (39), its projection on h_n and $\tilde{h}_n$ is $q_n'(s)$ and $\tilde{q}_n'(s)$ respectively:

$$P_{n,M}(\frac{\partial q}{\partial s}) = q_n' \text{ and } \tilde{P}_{n,M}(\frac{\partial q}{\partial s}) = \tilde{q}_n'. \tag{60}$$

Second term: $\mathcal{L}_{\beta,\delta} q$.

We claim the following:

Lemma 5.1 (Projection of $\mathcal{L}_{\beta,\delta} q$ on h_n and $\tilde{h}_n$ for $n\le M$)
If $n\le M-2$, then

$$\left|P_{n,M}(\mathcal{L}_{\beta,\delta} q)-\left(-\frac{n}{2}q_n(s)+c_{n+2}\tilde{q}_{n+2}\right)\right|\le C\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}$$

where c_n was defined in (33).
If $M-1\le n\le M$, then

$$\left|P_{n,M}(\mathcal{L}_{\beta,\delta} q)+\frac{n}{2}q_n(s)\right|\le C\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}.$$

If $n\le M$, then the projection of $\mathcal{L}_{\beta,\delta} q$ on $\tilde{h}_n$ satisfies

$$\left|\tilde{P}_{n,M}(\mathcal{L}_{\beta,\delta} q)-\left(1-\frac{n}{2}\right)\tilde{q}_n\right|\le C\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}.$$

If in addition $q(s)\in V_A(s)$, then the error estimates can be bounded from Proposition 4.1 as follows (for (e), note that (38) implies that $M\ge 8$):

Corollary 5.2 *For all $A\ge 1$, there exists $s_9(A)\ge 1$ such that for all $s\ge s_9(A)$, if $q(s)\in V_A(s)$, then:*
a) for $n=0$, we have

$$|P_{0,M}(\mathcal{L}_{\beta,\delta} q)-c_2\tilde{q}_2|\le C\frac{A^{M+1}}{s^{\frac{M+2}{2}}}.$$

b) for $1\le n\le M-1$, we have

$$\left|P_{n,M}(\mathcal{L}_{\beta,\delta} q)+\frac{n}{2}q_n(s)\right|\le C\frac{A^{n+2}}{s^{\frac{n+3}{2}}}.$$

c) for $n = M$, we have

$$\left| P_{M,M}(\mathcal{L}_{\beta,\delta}q) + \frac{M}{2} q_M(s) \right| \le C \frac{A^{M+1}}{s^{\frac{M+2}{2}}}.$$

d) for $3 \le n \le M$, we have

$$\left| \tilde{P}_{n,M}(\mathcal{L}_{\beta,\delta}q) - \left(1 - \frac{n}{2}\right) \tilde{q}_n \right| \le C \frac{A^{n+1}}{s^{\frac{n+2}{2}}}.$$

e) for $n = 0, 1$ or 2, we have

$$\left| \tilde{P}_{n,M}(\mathcal{L}_{\beta,\delta}q) - \left(1 - \frac{n}{2}\right) \tilde{q}_n \right| \le \frac{C}{s^3}.$$

Proof of Lemma 5.1: Using (42), we write

$$\mathcal{L}_{\beta,\delta}q = \mathcal{L}_{\beta,\delta}\left(\sum_{n \le M} q_n(s)h_n(y) + \tilde{q}_n(s)\tilde{h}_n(y) \right) + \mathcal{L}_{\beta,\delta}q_-(y,s) \equiv L_1 + L_2$$

Using Lemma 3.1, we write

$$\begin{aligned} L_1 &= \sum_{n \le M} -\frac{n}{2} q_n h_n + \tilde{q}_n \left((1 - \frac{n}{2})\tilde{h}_n + c_n h_{n-2} \right) \\ &= \sum_{n \le M-2} \left(-\frac{n}{2} q_n + c_{n+2}\tilde{q}_{n+2} \right) h_n + \sum_{n=M-1}^{M} -\frac{n}{2} q_n h_n + \sum_{n \le M} \tilde{q}_n \left(1 - \frac{n}{2}\right) \tilde{h}_n. \end{aligned}$$

Using the obvious fact that when a function is of the form $w = \sum_{n=0}^{M} w_n h_n + \tilde{w}_n \tilde{h}_n$, its projections on h_n and $\tilde{h}_n$ are respectively w_n and $\tilde{w}_n$, we see that the projection of L_1 on $\tilde{h}_n$ is $\left(1 - \frac{n}{2}\right)\tilde{q}_n$, whereas its projection on h_n is $-\frac{n}{2}q_n + c_{n+2}\tilde{q}_{n+2}$ if $n \le M-2$ and $-\frac{n}{2}q_n$ if $M-1 \le n \le M$. Now, let us deal with $L_2 = \mathcal{L}_{\beta,\delta}q_-$. By definition (22) of $\mathcal{L}_{\beta,\delta}$, we have

$$\mathcal{L}_{\beta,\delta}q_- = \mathcal{L}_{\beta}q_- + (1 + i\delta)\Re q_-. \tag{61}$$

Since $\mathcal{L}_\beta$ is "self-adjoint" with respect to ρ_β (see (27)), we use (39), (27) and (29) to write the projection of $\mathcal{L}_\beta q_-$ on any f_n for $n \le M$ (up to a multiplication factor)

$$\int \mathcal{L}_\beta q_- f_n \rho_\beta dy = \int \mathcal{L}_\beta f_n q_- \rho_\beta dy = -\frac{n}{2} \int f_n q_- \rho_\beta dy$$

and this is zero thanks to the "orthogonality" relation (30) and to the fact that q_- contains no f_j with $j \le M$ in its decomposition on the f_j (see (37)). Thus, we see from (61) that the projection of $\mathcal{L}_{\beta,\delta}q_-$ is equal to the projection of $(1 + i\delta)\Re q_-$, which is controlled by $C \left\| \frac{q_-}{1+|y|^{M+1}} \right\|_{L^\infty}$. This proves Lemma 5.1. ∎

Third term: $-i\left(\frac{\mu}{s} + \theta'(s)\right) q$.
It is enough to project iq. Since the projection of iq_- is zero, we see from (42) that it is enough to project ih_n and $i\tilde{h}_n$ for all $n \le M$. Since we know from Lemma 3.1 that the

term of highest degree in h_n is iy^n and in $\tilde{h}_n$ is $(1+i\delta)y^n$, we can expand ih_n and $i\tilde{h}_n$ as follows:

$$\begin{aligned} ih_n &= \delta h_n - \tilde{h}_n + \sum_{j=0}^{n-1} c_{j,n} h_j + \tilde{c}_{j,n}\tilde{h}_j \\ i\tilde{h}_n &= (1+\delta^2)h_n - \delta\tilde{h}_n + \sum_{j=0}^{n-1} d_{j,n} h_j + \tilde{d}_{j,n}\tilde{h}_j \end{aligned}$$

where the coefficients are real. Therefore, we get the following projections:

Lemma 5.3 (Projection of the term $-i\left(\frac{\mu}{s}+\theta'(s)\right)q$ on h_n and $\tilde{h}_n$ for $n \le M$) *Its projection on h_n is given by*

$$P_{n,M}\left(-i\left(\frac{\mu}{s}+\theta'(s)\right)q\right) = -\left(\frac{\mu}{s}+\theta'(s)\right)\left(\delta q_n + (1+\delta^2)\tilde{q}_n + \sum_{j=n+1}^{M} c_{n,j}q_j + d_{n,j}\tilde{q}_j\right).$$

Its projection on $\tilde{h}_n$ is given by

$$\tilde{P}_{n,M}\left(-i\left(\frac{\mu}{s}+\theta'(s)\right)q\right) = -\left(\frac{\mu}{s}+\theta'(s)\right)\left(-q_n - \delta\tilde{q}_n + \sum_{j=n+1}^{M} \tilde{c}_{n,j}q_j + \tilde{d}_{n,j}\tilde{q}_j\right).$$

for some real coefficients $c_{n,j}$, $\tilde{c}_{n,j}$, $d_{n,j}$ and $\tilde{d}_{n,j}$.

If in addition $q(s) \in V_A(s)$, then the error estimates can be bounded from Proposition 4.1 as follows:

Corollary 5.4 *For all $A \ge 1$, there exists $s_{10}(A) \ge 1$ such that for all $s \ge s_{10}(A)$, if $q \in V_A(s)$ and $|\theta'(s)| \le CA^5\frac{\log s}{s^2}$, then:*
a) for $1 \le n \le M$, we have

$$\left|P_{n,M}\left(-i\left(\frac{\mu}{s}+\theta'(s)\right)q\right)\right| \le C\frac{A^n}{s^{\frac{n+3}{2}}};$$

b) for $1 \le n \le M$, we have

$$\left|\tilde{P}_{n,M}\left(-i\left(\frac{\mu}{s}+\theta'(s)\right)q\right)\right| \le C\frac{A^n}{s^{\frac{n+3}{2}}};$$

c) for $n=0$, $\left|P_{0,M}(-i\left(\frac{\mu}{s}+\theta'(s)\right)q)\right| + \left|\tilde{P}_{0,M}(-i\left(\frac{\mu}{s}+\theta'(s)\right)q)\right| \le \frac{C}{s^2}$.

Fourth term: $V_1q + V_2\bar{q}$.
We claim the following:

Lemma 5.5 (Projection of V_1q and $V_2\bar{q}$ on h_n an $\tilde{h}_n$)
(i) It holds that

$$|V_i(y,s)| \le C\frac{(1+|y|^2)}{s} \text{ for all } y \in \mathbb{R} \text{ and } s \ge 1, \tag{62}$$

and for all $k \in \mathbb{N}^*$,

$$V_i(y,s) = \sum_{j=1}^{k} \frac{1}{s^j} W_{i,j}(y) + \tilde{W}_{i,k}(y,s) \tag{63}$$

where $W_{i,j}(y)$ is an even polynomial of degree $2j$ and $\tilde{W}_{i,k}(y,s)$ satisfies

$$\text{for all } s \geq 1 \text{ and } |y| \leq \sqrt{s}, \quad \left|\tilde{W}_{i,k}(y,s)\right| \leq C \frac{(1+|y|^{2k+2})}{s^{k+1}}.$$

(ii) The projections of $V_1 q$ and $V_2 \bar{q}$ on h_n and $\tilde{h}_n$ satisfy for all $s \geq 1$,

$$\begin{aligned} & |P_{n,M}(V_1 q)| + |\tilde{P}_{n,M}(V_1 q)| \\ \leq\ & \frac{C}{s} \sum_{j=n-2}^{M} (|q_j| + |\tilde{q}_j|) + \sum_{j=0}^{n-3} \frac{C}{s^{\frac{n-j}{2}}} (|q_j| + |\tilde{q}_j|) + \frac{C}{s} \left\| \frac{q_-}{1+|y|^{M+1}} \right\|_{L^\infty}. \end{aligned} \tag{64}$$

and the same holds for $V_2 \bar{q}$.

Remark: If $n \leq 2$, the first sum in (64) runs for $j = 0$ to M and the second sum doesn't exist.

If in addition $q(s) \in V_A(s)$, then the error estimates can be bounded from Proposition 4.1 as follows:

Corollary 5.6 *For all $A \geq 1$, there exists $s_{11}(A) \geq 1$ such that for all $s \geq s_{11}(A)$, if $q(s) \in V_A(s)$, then:*
a) for $3 \leq n \leq M$, we have

$$|P_{n,M}(V_1 q)| + |\tilde{P}_{n,M}(V_1 q)| \leq C \frac{A^{n-2}}{s^{\frac{n+1}{2}}};$$

b) for $n = 0, 1$ or 2, we have

$$|P_{n,M}(V_1 q)| + |\tilde{P}_{n,M}(V_1 q)| \leq \frac{C}{s^2}.$$

Proof of Lemma 5.5:
(i) The estimates of $V_1 q$ and $V_2 \bar{q}$ are the same, so we only deal with $V_1 q$. Let $F(u) = \frac{(p+1)}{2}(1+i\delta)\left[|u|^{p-1} - \frac{1}{p-1}\right]$, where $u \in \mathbb{C}$ and consider $z = \frac{y}{\sqrt{s}}$. Note that from (16) and (22), we have

$$V_1(y,s) = F(\varphi(y,s)) \text{ where } \varphi(y,s) = \varphi_0\left(\frac{y}{\sqrt{s}}\right) + \frac{a}{s}(1+i\delta).$$

Note that there exist positive constants c_0 and s_0 such that $|\varphi_0(z)|$ and $|\varphi(y,s)| = |\varphi_0(z) + \frac{a}{s}(1+i\delta)|$ are both larger than $1/c_0$ and smaller than c_0, uniformly in $|z| < 1$ and $s \geq s_0$. Since $F(u)$ is C^∞ for $\frac{1}{c_0} \leq |u| \leq c_0$, we expand it around $u = \varphi_0(z)$ as follows: for all $s \geq s_0$ and $|z| < 1$,

$$\begin{aligned} \left| F\left(\varphi_0(z) + \frac{a}{s}(1+i\delta)\right) - F(\varphi_0(z)) \right| &\leq \frac{C}{s}, \\ \left| F\left(\varphi_0(z) + \frac{a}{s}(1+i\delta)\right) - F(\varphi_0(z)) - \sum_{j=1}^{n} \frac{1}{s^j} F_j(\varphi_0(z)) \right| &\leq \frac{C}{s^{n+1}} \end{aligned}$$

where $F_j(u)$ are C^∞. Hence, we can expand $F(u)$ and $F_j(u)$ around $u = \varphi_0(0)$ and write for all $s \geq s_0$ and $|z| < 1$,

$$\left|F\left(\varphi_0(z) + \frac{a}{s}(1+i\delta)\right) - F(\varphi_0(0))\right| \leq Cz^2 + \frac{C}{s},$$

$$\left|F\left(\varphi_0(z) + \frac{a}{s}(1+i\delta)\right) - F(\varphi_0(0)) - \sum_{l=1}^{n} c_{0,l} z^{2l} - \sum_{j=1}^{n}\sum_{l=0}^{n-j} \frac{c_{j,l}}{s^j} z^{2l}\right|$$

$$\leq \quad C|z|^{2n+2} + \sum_{j=1}^{n} \frac{C}{s^j}|z|^{2(n-j)+2} + \frac{C}{s^{n+1}}.$$

Since $F(\varphi_0(0)) = F(\kappa) = 0$ and $z = \frac{y}{\sqrt{s}}$, this yields estimates (62) and (63) for V_1, when $s \geq s_0$ and $|y| < \sqrt{s}$. Since V_1 is bounded, (62) is also valid when $|y| \geq \sqrt{s}$, and then when $s \geq 1$.

(ii) Note first from (41) that it is enough to prove the bound (64) for the projection of $V_i q$ onto f_n to get the same bound for $P_{n,M}(V_i q)$ and $\tilde{P}_{n,M}(V_i q)$. Since in addition, the proof for $V_2 \bar{q}$ is the same as for $V_1 q$, we only prove (64) for the projection of $V_1 q$ onto f_n. Using (42) and (39), we see that this projection is given (up to a multiplication factor) by

$$\int f_n V_1 q \rho_\beta = \int f_n V_1 q_- \rho_\beta + \sum_{j=0}^{M} q_j \int f_n h_j V_1 \rho_\beta + \sum_{j=0}^{M} \tilde{q}_j \int f_n \tilde{h}_j V_1 \rho_\beta. \tag{65}$$

Using (62), the first term can be bounded by

$$\int |f_n| \left(\frac{1+|y|^2}{s}\right) |q_-||\rho_\beta| \leq \frac{C}{s} \left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}. \tag{66}$$

Now, we deal with the second term. We only focus on the terms involving h_j since the estimates are the same for the terms involving $\tilde{h}_j$.
If $j \geq n-2$, we use (62) to write $|\int f_n h_j V_1 \rho_\beta| \leq C/s$.
If $j \leq n-3$, then we claim that

$$\left|\int f_n h_j V_1 \rho_\beta\right| \leq \frac{C}{s^{\frac{n-j}{2}}} \tag{67}$$

(this actually vanishes if j and n have different parities). It is clear that (64) follows from (65), (66) and (67). Let us prove (67) then. Note that $k \equiv \left[\frac{n-j-1}{2}\right]$ (which is in $\mathbb{N}^*$ since

$j \le n-3$) is the largest integer such that $j+2k<n$. We use (63) to write

$$\begin{array}{rcl} \int f_n h_j V_1 \rho_\beta &=& \int_{|y|<\sqrt{s}} f_n h_j V_1 \rho_\beta + \int_{|y|>\sqrt{s}} f_n h_j V_1 \rho_\beta \\ &=& \sum_{l=1}^{k} \frac{1}{s^l} \int_{|y|<\sqrt{s}} f_n h_j W_{1,l} \rho_\beta + O\left(\frac{1}{s^{\left[\frac{n-j-1}{2}\right]+1}} \int (1+|y|^{n-j+1}) |f_n||h_j||\rho_\beta| dy \right) \\ &+& \int_{|y|>\sqrt{s}} f_n h_j V_1 \rho_\beta \\ &=& \sum_{l=1}^{k} \frac{1}{s^l} \int_{\mathbb{R}^N} f_n h_j W_{1,l} \rho_\beta + O\left(\frac{1}{s^{\left[\frac{n-j-1}{2}\right]+1}} \right) - \sum_{l=1}^{k} \frac{1}{s^l} \int_{|y|>\sqrt{s}} f_n h_j W_{1,l} \rho_\beta \\ &+& \int_{|y|>\sqrt{s}} f_n h_j V_1 \rho_\beta \end{array}$$

Since $\deg(h_j W_{1,l}) = j+2l \le j+2k < n = \deg f_n$, f_n is "orthogonal" to $h_j W_{1,l}$ thanks to (30) and

$$\int_{\mathbb{R}^N} f_n h_j W_{1,l} \rho_\beta = 0.$$

Since $|\rho_\beta(y)| \le Ce^{-cs}$ when $|y| > \sqrt{s}$, the integrals over the domain $\{|y|>\sqrt{s}\}$ can be bounded by

$$Ce^{-cs} \int_{|y|>\sqrt{s}} |f_n||h_j|(1+|y|^{2k}) \sqrt{|\rho_\beta(y)|} dy \le Ce^{-cs}.$$

Using that $\left[\frac{n-j-1}{2}\right]+1 \ge \frac{n-j}{2}$, we deduce that (67) holds. Hence, we have proved (64) and this concludes the proof of Lemma 5.5. ■

We need further refinements when $n=2$ for the terms $\tilde{P}_{2,M}(V_1 q)$ and $\tilde{P}_{2,M}(V_2 \bar{q})$. More precisely:

Lemma 5.7 (Projection of $V_1 q$ and $V_2 \bar{q}$ on $\tilde{h}_2$)
(i) It holds that

$$\forall s \ge 1 \text{ and } |y| < \sqrt{s}, \quad \left| V_i(y,s) - \frac{1}{s} W_{i,1}(y) \right| \le \frac{C}{s^2} \left(1+|y|^4\right), \tag{68}$$

where

$$\begin{array}{rcl} W_{1,1}(y) &=& \frac{(p+1)(1+i\delta)}{8\left(\beta\delta(p+1)+\delta^2-p\right)} (y^2 - 2 + 2\beta\delta) = \frac{(p+1)}{8\left(\beta\delta(p+1)+\delta^2-p\right)} \tilde{h}_2(y), \\ W_{2,1}(y) &=& \frac{(1+i\delta)}{8(\beta\delta(p+1)+\delta^2-p)} \left(y^2(2i\delta+p-1) - 4i\delta - 2p + 2 + 4i\beta\delta^2 + 2\beta\delta(p-1) \right). \end{array} \tag{69}$$

(ii) The projections of $V_1 q$ and $V_2\bar{q}$ on $\tilde{h}_2$ satisfy

$$\left|\tilde{P}_{2,M}(V_1 q) - \frac{\tilde{q}_2(s)}{s}\frac{(p+1)\left(2\beta\delta^3 - 14\beta\delta - 8\delta^2 + 8\right)}{8\left(\beta\delta(p+1)+\delta^2-p\right)}\right| \quad (70)$$

$$+ \quad \left|\tilde{P}_{2,M}(V_2\bar{q}) - \frac{\tilde{q}_2(s)}{s}\frac{(1+\delta^2)\left(8(p-1) - 2\delta\beta(p+5)\right)}{8\left(\beta\delta(p+1)+\delta^2-p\right)}\right| \quad (71)$$

$$\leq \frac{C}{s}\sum_{j=0}^{M}|q_j| + \frac{C}{s}\sum_{j=0,\, j\neq 2}^{M}|\tilde{q}_j| + \frac{C}{s}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty} + \frac{C}{s^2}\left\|\frac{q}{1+|y|^{M+1}}\right\|_{L^\infty}.$$

Remark: The denominators in (70) and (71) are non zero thanks to condition (4).

If in addition $q(s) \in V_A(s)$, then the error estimates can be bounded from Proposition 4.1 as follows:

Corollary 5.8 *For all $A \geq 1$, there exists $s_{12}(A) \geq 1$ such that for all $s \geq s_{12}(A)$, if $q(s) \in V_A(s)$, then*

$$\left|\tilde{P}_{2,M}(V_1 q) - \frac{\tilde{q}_2(s)}{s}\frac{(p+1)\left(2\beta\delta^3 - 14\beta\delta - 8\delta^2 + 8\right)}{8\left(\beta\delta(p+1)+\delta^2-p\right)}\right| \leq C\frac{A^4}{s^3}$$

and

$$\left|\tilde{P}_{2,M}(V_2\bar{q}) - \frac{\tilde{q}_2(s)}{s}\frac{(1+\delta^2)\left(8(p-1) - 2\delta\beta(p+5)\right)}{8\left(\beta\delta(p+1)+\delta^2-p\right)}\right| \leq C\frac{A^4}{s^3}.$$

Proof of Lemma 5.7:
(i) One has to do the computations in the proof of (63) in Lemma 5.5 explicitly in order to prove (68) and (69). This is a simple but lengthy computation that we omit.

(ii) Using (68) and (42), we see that

$$\begin{aligned} V_1 q &= \frac{1}{s}W_{1,1}q + O\left(\frac{q(1+|y|^4)}{s^2}\right) \\ &= \frac{\tilde{q}_2(s)}{s}W_{1,1}\tilde{h}_2 + \frac{1}{s}W_{1,1}\sum_{j=0}^{M}q_j h_j + \frac{1}{s}W_{1,1}\sum_{j=0,\, j\neq 2}^{M}\tilde{q}_j\tilde{h}_j + \frac{1}{s}W_{1,1}q_- + O\left(\frac{q(1+|y|^4)}{s^2}\right) \end{aligned}$$

where the O is uniform with respect to $|y| < \sqrt{s}$. When projecting this on $\tilde{h}_2$ (use (41) and (39) for the definition of that projection), we write (using the definition (69) of $W_{1,1}$)

$$\left|\tilde{P}_{2,M}(V_1 q) - \frac{\tilde{q}_2(s)}{s}\frac{(p+1)}{8\left(\beta\delta(p+1)+\delta^2-p\right)}\tilde{P}_{2,M}\left(\left(\tilde{h}_2\right)^2\right)\right|$$

$$\leq \frac{C}{s}\sum_{j=0}^{M}|q_j| + \frac{C}{s}\sum_{j=0,\, j\neq 2}^{M}|\tilde{q}_j| + \frac{C}{s}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty} + \frac{C}{s^2}\left\|\frac{q}{1+|y|^{M+1}}\right\|_{L^\infty}.$$

Therefore, the problem is reduced to projecting the polynomial $\left(\tilde{h}_2\right)^2$ on $\tilde{h}_2$. Since the polynomial is of degree 4, we can expand it on the basis $(h_n, \tilde{h}_n)_{n\leq 4}$ using Lemma 3.2 to get

$$\left(\tilde{h}_2\right)^2 = \delta(1+\delta^2)h_4 + (1-\delta^2)\tilde{h}_4 + ah_2 + \left(2\beta\delta^3 - 14\beta\delta - 8\delta^2 + 8\right)\tilde{h}_2 + bh_0 + c\tilde{h}_0$$

for some real numbers a, b and c. Thus,

$$\tilde{P}_{2,M}\left(\left(\tilde{h}_2\right)^2\right) = \left(2\beta\delta^3 - 14\beta\delta - 8\delta^2 + 8\right).$$

This controls (70).
The bound on (71) can be proved similarly. Thus, we only give a sketch. When projecting $V_2\bar{q}$ on $\tilde{h}_2$, we get

$$\begin{aligned}
&\left|\tilde{P}_{2,M}(V_s\bar{q}) - \frac{\tilde{q}_2(s)}{s}\tilde{P}_{2,M}\left(W_{2,1}\bar{\tilde{h}}_2\right)\right| \\
\leq\ & \frac{C}{s}\sum_{j=0}^{M}|q_j| + \frac{C}{s}\sum_{j=0,\ j\neq 2}^{M}|\tilde{q}_j| + \frac{C}{s}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty} + \frac{C}{s^2}\left\|\frac{q}{1+|y|^{M+1}}\right\|_{L^\infty}.
\end{aligned}$$

The polynomial $W_{2,1}\bar{\tilde{h}}_2$ can be expanded on the basis $(h_n, \tilde{h}_n)_{n\leq 4}$ using Lemma 3.2:

$$\begin{aligned}
&\frac{8\left(\beta\delta(p+1)+\delta^2-p\right)}{1+\delta^2}W_{2,1}\bar{\tilde{h}}_2 \\
=\ & \delta(3-p)h_4 + (p-1)\tilde{h}_4 + ah_2 + \left(8(p-1) - 2\delta\beta(p+5)\right)\tilde{h}_2 + bh_0 + c\tilde{h}_0
\end{aligned}$$

for some real numbers a, b and c. Thus,

$$\tilde{P}_{2,M}\left(W_{2,1}\bar{\tilde{h}}_2\right) = \frac{1+\delta^2}{8\left(\beta\delta(p+1)+\delta^2-p\right)}\left(8(p-1) - 2\delta\beta(p+5)\right).$$

This proves the bound on (71) and concludes the proof of Lemma 5.7. ■

Fifth term: $B(q,y,s)$.
We have the following lemma:

Lemma 5.9 *The function $B = B(q,y,s)$ can be decomposed for all $s \geq 1$ and $|q| \leq 1$ as*

$$\sup_{|y|<\sqrt{s}}\left|B - \sum_{l=0}^{M}\sum_{\substack{0\leq j,k\leq M+1\\ 2\leq j+k\leq M+1}}\frac{1}{s^l}\left[B_{j,k}^l\left(\frac{y}{\sqrt{s}}\right)q^j\bar{q}^k + \tilde{B}_{j,k}^l(y,s)q^j\bar{q}^k\right]\right| \leq C|q|^{M+2} + \frac{C}{s^{M+1}}, \quad (72)$$

where $B_{j,k}^l$ is an even polynomial of degree less or equal to M and the rest $\tilde{B}_{j,k}^l(y,s)$ satisfies

$$\forall s \geq 1 \text{ and } |y| < \sqrt{s},\quad \left|\tilde{B}_{j,k}^l(y,s)\right| \leq C\frac{(1+|y|^{M+1})}{s^{\frac{M+1}{2}}}.$$

Moreover,

$$\forall s \geq 1 \text{ and } |y| < \sqrt{s},\quad |B_{j,k}^l\left(\frac{y}{\sqrt{s}}\right) + \tilde{B}_{j,k}^l(y,s)| \leq C.$$

On the other hand, in the region $|y| \geq \sqrt{s}$, we have

$$|B(q,y,s)| \leq C|q|^{\bar{p}} \quad (73)$$

for some constant C where $\bar{p} = min(p,2)$.

Proof: We just give a sketch of the proof. We recall from (22) that B is given by

$$B(q,y,s) = (1+i\delta)\left\{|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi - |\varphi|^{p-1}q - \frac{(p-1)}{2}|\varphi|^{p-3}\varphi\left(\varphi\bar{q}+\bar{\varphi}q\right)\right\}.$$

We notice that in the region $|y| \leq \sqrt{s}$ and for $s \geq C$ where C is a fixed constant, φ is bounded from above and from below. Using a Taylor expansion in terms of q and $\bar{q}$, we see that B can be written as

$$\forall s \geq 1 \text{ and } |y| < \sqrt{s}, \quad \left| B - \sum_{\substack{0\leq j,k\leq M+1\\ 2\leq j+k\leq M+1}} \left[E_{j,k}(\varphi) q^j \bar{q}^k\right]\right| \leq C|q|^{M+2}. \tag{74}$$

Expanding $E_{j,k}(\varphi)$ in terms of the variable $\frac{1}{s}$, we get

$$\left| E_{j,k}(\varphi) - \sum_{l=0}^{M} \frac{1}{s^l} E^l_{j,k}(\varphi_0)\right| \leq \frac{C}{s^{M+1}}$$

then expanding in $z = \frac{y}{\sqrt{s}}$, we deduce that

$$\left| E^l_{j,k}(\varphi_0) - \sum_{i=0}^{M/2} b^{l,i}_{j,k}|z|^{2i}\right| \leq C|z|^{M+2}$$

We denote $B^l_{j,k}(z) = \sum_{i=0}^{M/2} b^{l,i}_{j,k}|z|^{2i}$ and $\tilde{B}^l_{j,k}(y,s) = E^l_{j,k}(\varphi_0) - B^l_{j,k}$. Hence (72) holds and Lemma 5.9 is proved. ■

Using lemma 5.9, we have the following estimate:

Lemma 5.10 (The quadratic term $B(q,y,s)$**)** *For all $A \geq 1$, there exists $s_{13}(A)$ such that for all $s \geq s_{13}$, if $q(s) \in V_A(s)$, then:*

a) the projection of $B(q,y,s)$ on h_n and on $\tilde{h}_n$, for $n \geq 3$ satisfies

$$|P_{n,M}(B(q,y,s))| + |\tilde{P}_{n,M}(B(q,y,s))| \leq C\frac{A^n}{s^{\frac{n}{2}+1}}. \tag{75}$$

b) for $n = 0, 1$ or 2, we have

$$|P_{n,M}(B(q,y,s))| + |\tilde{P}_{n,M}(B(q,y,s))| \leq \frac{C}{s^3}. \tag{76}$$

Proof: We will only prove estimate (75) since (76) is easier and can be proved in the same way. It is enough to prove estimate (75) for the projection on f_n since it implies the same estimate for $P_{n,M}$ and $\tilde{P}_{n,M}$ through (41). We write

$$\int f_n B(q,y,s)\rho_\beta dy = \int_{|y|<\sqrt{s}} f_n B(q,y,s)\rho_\beta dy + \int_{|y|>\sqrt{s}} f_n B(q,y,s)\rho_\beta dy.$$

Using lemma 5.9, we deduce that

$$\left| \int_{|y|<\sqrt{s}} f_n B(q,y,s)\rho_\beta - \int_{|y|<\sqrt{s}} f_n\rho_\beta \sum_{l=0}^{M} \sum_{\substack{0\le j,k\le M+1\\ 2\le j+k\le M+1}} \frac{1}{s^l}\left[B^l_{j,k}(\frac{y}{\sqrt{s}})q^j\bar{q}^k + \tilde{B}^l_{j,k}(y,s)q^j\bar{q}^k\right]\right|$$
$$\le C\int_{|y|<\sqrt{s}} |f_n||\rho_\beta|(|q|^{M+2}+\frac{1}{s^{M+1}}).$$

Let us write

$$B^l_{j,k}(\frac{y}{\sqrt{s}}) = \sum_{i=0}^{M/2} b^{l,i}_{j,k}(\frac{y}{\sqrt{s}})^{2i}$$

$$q^j = (\sum_{m=0}^{M} q_m h_m + \tilde{q}_m\tilde{h}_m + q_-)^j \text{ and } q^k = (\sum_{r=0}^{M} q_r h_r + \tilde{q}_r\tilde{h}_r + q_-)^k$$

where $b^{l,i}_{j,k}$ are the coefficients of the polynomial $B^l_{j,k}$. Using the fact that $\|q(s)\|_{L^\infty}\le 1$ (which holds for s large from the fact that $q(s)\in V_A(s)$ and (i) of Proposition 4.1), we deduce that

$$|q^j - q_+^j| \le C(|q_-|^j + |q_-|)$$

Using that $q(s)\in V_A(s)$ and the fact that $s\ge 2A^2$, we deduce that in the region $y\le\sqrt{s}$, we have $|q_-|\le\frac{1}{\sqrt{s}}(\frac{A}{\sqrt{s}})^{M+1}(1+|y|)^{M+1}$ and that

$$|q^j - (\sum_{m=0}^{M} q_m h_m + \tilde{q}_m\tilde{h}_m)^j| \le C\left(\frac{A}{\sqrt{s}}\right)^{M+1}\frac{1}{\sqrt{s}}(1+|y|)^{jM+j}.$$

In the same way, we have

$$|q^k - (\sum_{m=0}^{M} q_m h_m + \tilde{q}_m\tilde{h}_m)^k| \le C\left(\frac{A}{\sqrt{s}}\right)^{M+1}\frac{1}{\sqrt{s}}(1+|y|)^{kM+k},$$

hence, the contribution coming from q_- is controlled by the right-hand side of (75). Moreover for all j,k and l, we have

$$\left|\int_{|y|<\sqrt{s}} f_n\rho_\beta B^l_{j,k}(\frac{y}{\sqrt{s}})q_+^j\bar{q}_+^k - \int f_n\rho_\beta B^l_{j,k}(\frac{y}{\sqrt{s}})q_+^j\bar{q}_+^k\right| \le Ce^{-cs}. \tag{77}$$

To compute the second term on the left hand side of (77), we notice that $B^l_{j,k}(\frac{y}{\sqrt{s}})q_+^j\bar{q}_+^k$ is a polynomial in y and that the coefficient of the term of degree n is controlled by the right-hand side of (75) since $q\in V_A$.
Moreover, using that $s\ge 2A^2$, we infer that $|q|\le\frac{1}{\sqrt{s}}(1+|y|)^{M+1}$ in the region $|y|<\sqrt{s}$ and hence for all j,k and l, we have

$$\left|\int_{|y|<\sqrt{s}} f_n\rho_\beta\frac{1}{s^l}\tilde{B}^l_{j,k}(y,s)q^j\bar{q}^k\right| \le C\frac{1}{s^{l+\frac{M+1+j+k}{2}}}$$

and

$$\left|\int_{|y|<\sqrt{s}} f_n\rho_\beta(|q|^{M+2}+\frac{1}{s^{M+1}})\right|\le C\frac{1}{s^{\frac{M+2}{2}}}.$$

The terms appearing in these two inequalities are controlled by the right hand side of (75).
Using the fact that $\|q(s)\|_{L^\infty}\le 1$ and (18), we remark that $|B(q,y,s)|\le C$. Since $|\rho_\beta(y)|\le Ce^{-cs}$ for $|y|>\sqrt{s}$, it holds that

$$\left|\int_{|y|>\sqrt{s}} f_nB(q,y,s)\rho_\beta dy\right|\le Ce^{-cs}.$$

This concludes the proof of Lemma 5.10. ■

Sixth term: $R^*(\theta',y,s)$.
In the following lemma, we expand $R^*(\theta',y,s)$ as a power series of $1/s$ as $s\to\infty$, uniformly for $|y|<\sqrt{s}$:

Lemma 5.11 (Power series of R^* as $s\to\infty$) *For all $n\in\mathbb{N}$,*

$$R^*(\theta',y,s)=\Pi_n(\theta',y,s))+\tilde{\Pi}_n(\theta',y,s) \tag{78}$$

where

$$\Pi_n(\theta',y,s)=\sum_{k=0}^{n-1}\frac{1}{s^{k+1}}P_k(y)-i\theta'(s)\left(\frac{a}{s}(1+i\delta)+\sum_{k=0}^{n-1}e_k\frac{y^{2k}}{s^k}\right) \tag{79}$$

and

$$\forall|y|<\sqrt{s},\ \left|\tilde{\Pi}_n(\theta',y,s))\right|\le C(1+s|\theta'(s)|)\frac{(1+|y|^{2n})}{s^{n+1}}, \tag{80}$$

where P_k is a polynomial of degree $2k$ for all $k\ge1$.
In particular,

$$\begin{aligned}&\sup_{|y|\le\sqrt{s}}\left|R^*(\theta',y,s)-\sum_{k=0}^{1}\frac{1}{s^{k+1}}P_k(y)+i\theta'\left[\kappa+\frac{(1+i\delta)}{s}\left(a-\frac{b\kappa}{(p-1)^2}y^2\right)\right]\right|\\ \le\ & C\left(\frac{1+|y|^4}{s^3}\right)+C|\theta'|\frac{y^4}{s^2}.\end{aligned} \tag{81}$$

Proof: Using the definition of φ (16), the fact that φ_0 satisfies (13) and (18), we see that R^* is in fact a function of θ', $z=\frac{y}{\sqrt{s}}$ and s that can be written as

$$\begin{aligned}R^*(\theta',y,s)&=-\frac{z}{s}.\nabla_z\varphi_0(z)-\frac{a}{s^2}(1+i\delta)+\frac{(1+i\beta)}{s}\Delta_z\varphi_0(z)\\ &-\frac{a(1+i\delta)^2}{(p-1)s}+\left(F\left(\varphi_0(z)+\frac{a}{s}(1+i\delta)\right)-F\left(\varphi_0(z)\right)\right)\\ &-i\frac{\mu}{s}\left(\varphi_0(z)+\frac{a}{s}(1+i\delta)\right)-i\theta'(s)\left(\varphi_0(z)+\frac{a}{s}(1+i\delta)\right)\ \text{with } F(u)=(1+i\delta)|u|^{p-1}u.\end{aligned} \tag{82}$$

Since $|z|<1$, there exist positive c_0 and s_0 such that $|\varphi_0(z)|$ and $|\varphi_0(z)+\frac{a}{s}(1+i\delta)|$ are both larger that $1/c_0$ and smaller that c_0, uniformly in $|z|<1$ and $s>s_0$. Since $F(u)$ is C^∞ for $\frac{1}{c_0}\le|u|\le c_0$, we expand it around $u=\varphi_0(z)$ as follows

$$\left|F\left(\varphi_0(z)+\frac{a}{s}(1+i\delta)\right)-F\left(\varphi_0(z)\right)-\sum_{j=1}^{n}\frac{1}{s^j}F_j\left(\varphi_0(z)\right)\right|\le C\frac{1}{s^{n+1}},$$

where $F_j(u)$ are C^∞. Hence, we can expand $F_j(u)$ around $u=\varphi_0(0)$ and write

$$\left|F\left(\varphi_0(z)+\frac{a}{s}(1+i\delta)\right)-F\left(\varphi_0(z)\right)-\sum_{j=1}^{n}\sum_{l=0}^{n-j}\frac{c_{j,l}}{s^j}z^{2l}\right|\le\sum_{j=1}^{n}\frac{C}{s^j}|z|^{2(n-j)+2}+\frac{C}{s^{n+1}}.$$

Similarly, we have the following

$$\begin{aligned}\left|\frac{z}{s}.\nabla_z\varphi_0(z)-\frac{|z|^2}{s}\sum_{j=0}^{n-2}d_jz^{2j}\right| &\le \frac{C}{s}|z|^{2n},\\ \left|\frac{1}{s}\Delta_z\varphi_0(z)-\frac{1}{s}\sum_{j=0}^{n-1}b_jz^{2j}\right| &\le \frac{C}{s}|z|^{2n}\text{ and }\left|\varphi_0(z)-\sum_{j=0}^{n-1}e_jz^{2j}\right|\le C|z|^{2n}.\end{aligned}$$

Recalling that $z=\frac{y}{\sqrt{s}}$ and using (82), we get to the conclusion of (79). Estimate (81) is obtained in the same way by performing explicit computations. ■

Using the "orthogonality" relation between the f_j (30), we easily derive from Lemma 5.11 bounds on $F_j(R^*)(\theta',s)$, the projection of $R^*(\theta',y,s)$ defined in (18) on f_j:

Lemma 5.12 (Projection of R^* on the eigenfunctions of $\mathcal{L}_\beta$)
It holds that $F_j(R^*)(\theta',s)\equiv 0$ when j is odd, and $|F_j(R^*)(\theta',s)|\le C\frac{1+s|\theta'(s)|}{s^{\frac{j}{2}+1}}$ when j is even and $j\ge 4$.
If $j=0$, then

$$\begin{aligned}F_0(R^*)(\theta',s) &= -i\theta'(s)\left(\kappa+O\left(\frac{1}{s}\right)\right)+\left\{(1+i\delta)\left(a-2(1+i\beta)\frac{\kappa b}{(p-1)^2}\right)-i\mu\kappa\right\}\frac{1}{s}\\ &+ O\left(\frac{1}{s^2}\right)\\ &= -i\theta'(s)\left(\kappa+O\left(\frac{1}{s}\right)\right)+O\left(\frac{1}{s^2}\right)\end{aligned}$$

with the choice of a and μ made in (16).
If $j=2$, then

$$\begin{aligned}F_2(R^*)(\theta',s) &= \theta'(s)\left(\frac{(\delta-i)\kappa}{4(\beta\delta(p+1)+\delta^2-p)}\frac{1}{s}+O\left(\frac{1}{s^2}\right)\right)+O\left(\frac{1}{s^3}\right)\\ &+ \left\{-\frac{\kappa b}{(p-1)^2}\left[6b\delta^2\frac{(1+i\delta)}{(p-1)^2}-i\mu(1+i\delta)-\frac{6b}{(p-1)^2}(1+i\beta)\right.\right.\\ &+ \left.1+i\delta-\frac{6b}{(p-1)}(1+i\delta)(1+i\beta)-12ib\delta\frac{(1+i\beta)}{(p-1)^2}\right]\\ &- \left.\frac{ba}{(p-1)^2}\left[(p-1)(1+i\delta)+3i\delta+i\delta^3-\delta^2+1\right]\right\}\frac{1}{s^2}.\end{aligned}$$

Proof: Since R^* is even in the y variable and f_j is odd when j is odd, $F_j(R^*)(\theta', s) \equiv 0$ when j is odd. Now when j is even, we apply Lemma 5.11 with $n = [\frac{j}{2}]$ and write

$$R^*(\theta', y, s) = \Pi_{\frac{j}{2}}(\theta', y, s) + O\left(\frac{1 + s|\theta'(s)| + |y|^j}{s^{\frac{j}{2}+1}}\right)$$

where $\Pi_{\frac{j}{2}}$ is a polynomial in y of degree less than $j-1$. Using the definition (39) of $F_j(R^*)$, we write $|F_j(R^*)| \le C\left|\int_{\mathbb{R}^N} R^* f_j \rho_\beta dy\right|$ and

$$\begin{aligned}
\int_{\mathbb{R}^N} R^* f_j \rho_\beta dy &= \int_{|y|<\sqrt{s}} R^* f_j \rho_\beta dy + \int_{|y|>\sqrt{s}} R^* f_j \rho_\beta dy \\
&= \int_{|y|<\sqrt{s}} \Pi_{\frac{j}{2}} f_j \rho_\beta dy + O\left(\int_{|y|<\sqrt{s}} \frac{1 + s\theta'(s) + |y|^j}{s^{\frac{j}{2}+1}} f_j \rho_\beta dy\right) + \int_{|y|>\sqrt{s}} R^* f_j \rho_\beta dy \\
&= \int_{\mathbb{R}^N} \Pi_{\frac{j}{2}} f_j \rho_\beta dy + O\left(\frac{1 + s\theta'(s)}{s^{\frac{j}{2}+1}}\right) + \int_{|y|>\sqrt{s}} R^* f_j \rho_\beta dy - \int_{|y|<\sqrt{s}} \Pi_{\frac{j}{2}} f_j \rho_\beta dy. \quad (83)
\end{aligned}$$

First, note that $\int_{\mathbb{R}^N} \Pi_{\frac{j}{2}} f_j \rho_\beta dy = 0$ because f_j is orthogonal to all polynomials of degree less that $j-1$ (see (30)). Then, note that both integrals over the domain $\{|y| > \sqrt{s}\}$ are controlled by

$$\int_{|y|>\sqrt{s}} \left(|R^*(\theta', y, s)| + 1 + |y|^j\right)(1 + |y|^j)\rho_\beta(y) dy.$$

Using the bound (25) on R^* and the fact that $|\rho_\beta(y)| \le Ce^{-cs}$ for $|y| > \sqrt{s}$, we can bound this integral by

$$C(1 + |\theta'(s)|) \int_{\mathbb{R}^N} (1 + |y|^j)^2 ce^{-cs} \sqrt{\rho_\beta(y)} dy = C(j)(1 + |\theta'(s)|)e^{-cs}.$$

Using (83) yields the result for $j \ge 4$, j even. If $j = 0$ or $j = 2$, one has to refine Lemma 5.11 in a straightforward but long way and then do as we did for general j. We omit the details. This concludes the proof of Lemma 5.12. ■

Using the definition (41) of the coordinates $P_{j,M}(R^*)$ and $\tilde{P}_{j,M}(R^*)$ in terms of $F_j(R^*)$, we have the following estimates:

Corollary 5.13 (Projection of R^* on the eigenfunctions of $\mathcal{L}_{\beta,\delta}$)
If j is even and $j \ge 4$, then $P_{j,M}(R^*)(\theta', s)$ and $\tilde{P}_{j,M}(R^*)(\theta', s)$ are $O\left(\frac{1+s|\theta'|}{s^{\frac{j}{2}+1}}\right)$.
If j is odd, then $P_{j,M}(R^*)(\theta', s)$ and $\tilde{P}_{j,M}(R^*)(\theta', s)$ are $O\left(\frac{1+s|\theta'|}{s^{\frac{j+3}{2}}}\right)$.
If $j = 0$, then $P_{0,M}(R^*)(\theta', s) = -\theta'(s)\left(\kappa + O\left(\frac{1}{s}\right)\right) + O\left(\frac{1}{s^2}\right)$ and $\tilde{P}_{0,M}(R^*)(\theta', s) = O\left(\frac{\theta'(s)}{s}\right) + O\left(\frac{1}{s^2}\right)$ (with the choice of a and μ made in (16)).

If $j=2$, then $P_{2,M}(R^*) = O\left(\frac{1}{s^2}\right) + O\left(\frac{\theta'(s)}{s}\right)$ and

$$\tilde P_{2,M}(R^*) = \theta'(s)\left(\frac{\kappa\delta}{4(\beta\delta(p+1)+\delta^2-p)}\frac{1}{s} + O\left(\frac{1}{s^2}\right)\right) + O(\frac{1}{s^3}) + \left\{-\frac{ba(p-\delta^2)}{(p-1)^2}\right.$$

$$\left. - \quad \frac{\kappa b}{(p-1)^2}\left[\frac{6b\delta^2}{(p-1)^2} + \mu\delta - \frac{6b}{(p-1)^2} + 1 - \frac{6b}{(p-1)}(1-\delta\beta) + \frac{12b\delta\beta}{(p-1)^2}\right]\right\}\frac{1}{s^2}$$

$$= \quad \theta'(s)\left(\frac{\kappa\delta}{4(\beta\delta(p+1)+\delta^2-p)}\frac{1}{s} + O\left(\frac{1}{s^2}\right)\right) + O\left(\frac{1}{s^3}\right)$$

with the choice of b, a and μ made in (16).

Part 2: Proof of Proposition 4.6, except the last two identities

In this part, we consider $A \geq 1$ and take s large enough so that Part 1 applies.

(i) From Part 1, taking the projection of (21) on $\tilde h_0$, we see that for all $s \in [\tau, s_1]$,

$$\left|q_0' - c_2\tilde q_2 + \kappa\theta'(s)\right| \leq \frac{C}{s^2} + C\frac{|\theta'(s)|}{s} \text{ where } c_2 = 2\beta(1+\delta^2)$$

was given in (33) and computed in Lemma 3.2. Since $q(s) \in V_A(s)$ and $q_0(s) = 0$ for all $s \in [\tau, s_1]$, this yields

$$\left|\theta'(s) - 2\frac{\beta(1+\delta^2)}{\kappa}\tilde q_2(s)\right| \leq \frac{C}{s}|\tilde q_2(s)| + \frac{C}{s^2}. \tag{84}$$

Using Proposition 4.1 to bound $\tilde q_2$, we get $|\theta'(s)| \leq CA^5 s^{-2}\log s$ and conclude the proof of (i) of Proposition 4.6, provided that s_0 is large enough.

(ii) This is a direct consequence of Part 1, provided that s_0 is large enough.

(iii) *Estimate of* $\tilde q_2(s)$: We use Part 1 and the fact that $q \in V_A$ to project the different terms of (21) on $\tilde h_2$:

$\tilde P_{2,M}(\frac{\partial q}{\partial s}) = \tilde q_2'$.

$|\tilde P_{2,M}(\mathcal{L}_{\beta,\delta} q)| \leq \frac{C}{s^3}$.

$|\tilde P_{2,M}(-i\left(\frac{\mu}{s} + \theta'(s)\right)q) - \mu\delta\frac{\tilde q_2}{s}| \leq \frac{C}{s^{3/2}}|\tilde q_2(s)| + C\frac{A^3}{s^3}$ (here we have used (i) of Proposition 4.6; we recall that the value of μ was introduced in (16)).

$|\tilde P_{2,M}(V_1 q) - \frac{\tilde q_2(s)}{s}\frac{(p+1)\left(2\beta\delta^3 - 14\beta\delta - 8\delta^2 + 8\right)}{8\left(\beta\delta(p+1)+\delta^2-p\right)}| \leq C\frac{A^4}{s^3}$.

$|\tilde P_{2,M}(V_2\bar q) - \frac{\tilde q_2(s)}{s}\frac{(1+\delta^2)\left(8(p-1) - 2\delta\beta(p+5)\right)}{8\left(\beta\delta(p+1)+\delta^2-p\right)}| \leq C\frac{A^4}{s^3}$.

$|\tilde P_{2,M}(B(q,y,s))| \leq \frac{C}{s^3}$.

$|\tilde P_{2,M}(R^*(\theta' s, y, s)) - \frac{\tilde q_2(s)}{s}\frac{4\beta\delta(1+\delta^2)}{8\left(\beta\delta(p+1)+\delta^2-p\right)}| \leq C\frac{A}{s^3}$ (here, we have used Lemma 5.13 and (84)). Adding all these contributions gives -2 as the coefficient of $\frac{\tilde q_2(s)}{s}$ in the the following equation satisfied for all $s \in [\tau, s_1]$:

$$|\tilde q_2' + \frac{2}{s}\tilde q_2| \leq C\frac{A^4}{s^3} \leq C\frac{A^4}{\tau s^2}$$

for s_0 large enough. Integrating this differential inequality between τ and s gives the desired estimate on $\tilde q_2$.

Estimate of q_1, q_2, q_j and $\tilde q_j$ for $3 \le j \le M$: Using Part 1 and the fact that $q(s) \in V_A(s)$, we see that for all $s \in [\tau, s_1]$, we have

$$\begin{array}{rclrcl} |q_1' + \frac{1}{2}q_1| & \le & C\frac{A^3}{s^2}, & |q_2 + q_2'| & \le & \frac{C}{s^2}, \\ |q_j' + \frac{j}{2}q_j| & \le & C\frac{A^{j-1}}{s^{\frac{j+1}{2}}}, & |\tilde q_j' + \frac{j-2}{2}q_j| & \le & C\frac{A^{j-1}}{s^{\frac{j+1}{2}}}. \end{array}$$

Integrating these differential inequalities between τ and s_1 gives the desired estimates.

5.2 The infinite dimensional part : q_-

Here, we prove the last but one identity in (iii) of Proposition 4.6. As in the previous subsection, we proceed in two parts:
- In Part 1, we project equation (21) using the projector P_- defined in (42).
- In Part 2, we prove the estimate on q_- contained in (iii) of Proposition 4.6.

Part 1: Projection of equation (21) using the projector P_-
In the following, we will project equation (21) term by term.

First term: $\dfrac{\partial q}{\partial s}$.
From (41) and (39), its projection is

$$P_-(\frac{\partial q}{\partial s}) = \frac{\partial q_-}{\partial s}. \tag{85}$$

Second term: $\mathcal{L}_{\beta,\delta}q$.
We have the following:

Lemma 5.14 (Projection of $\mathcal{L}_{\beta,\delta}q$)

$$P_-(\mathcal{L}_{\beta,\delta}q) = \mathcal{L}_\beta(q_-(s)) + P_-[(1+i\delta)\Re q_-]$$

Third term: $-i\left(\frac{\mu}{s} + \theta'(s)\right) q$.
Since P_- commutes with the multiplication by i, we deduce that

$$P_-[-i\left(\frac{\mu}{s} + \theta'(s)\right) q] = -i\left(\frac{\mu}{s} + \theta'(s)\right) q_-.$$

Fourth term: $V_1 q + V_2 \bar q$.
We have the following:

Lemma 5.15 (Projection of $V_1 q$ and $V_2\bar q$)
The projections of $V_1 q$ and $V_2 \bar q$ satisfy for all $s \ge 1$

$$\left\|\frac{P_-(V_1 q)}{1+|y|^{M+1}}\right\|_{L^\infty} \le (||V_1||_{L^\infty} + \frac{C}{s})\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty} + \sum_{n=0}^{M}\frac{C}{s^{\frac{M+1-n}{2}}}(|q_n| + |\tilde q_n|) \tag{86}$$

and the same holds for $V_2\bar q$.

Using the fact that $q(s) \in V_A(s)$, we get the following:

Corollary 5.16 *For all $A \geq 1$, there exists $s_{14}(A)$ such that for all $s \geq s_{14}$, if $q(s) \in V_A(s)$, then*

$$\left\|\frac{P_-(V_1 q)}{1+|y|^{M+1}}\right\|_{L^\infty} \leq ||V_1||_{L^\infty}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty} + C\frac{A^M}{s^{\frac{M+2}{2}}}$$

and the same holds for $V_2\bar{q}$.

Proof of Lemma 5.15: We just give the proof for $V_1 q$ since the proof for $V_2\bar{q}$ is similar. From subsection 3.5, we write $q = q_+ + q_-$ and

$$P_-(V_1 q) = V_1 q_- - P_+(V_1 q_-) + P_-(V_1 q_+).$$

Moreover, we claim that the following estimates hold

$$\begin{aligned}\left\|\frac{V_1 q_-}{1+|y|^{M+1}}\right\|_{L^\infty} &\leq ||V_1||_{L^\infty}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}\\ \left\|\frac{P_+(V_1 q_-)}{1+|y|^{M+1}}\right\|_{L^\infty} &\leq \frac{C}{s}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}\end{aligned}$$

Indeed, the first one is obvious. To prove the second one, we use (62) to show that

$$|P_{n,M}(V_1 q_-)| + |\tilde{P}_{n,M}(V_1 q_-)| \leq \frac{C}{s}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}.$$

To control $P_-(V_1 q_+) = \sum_{n\leq M} P_-\left(V_1(q_n h_n + \tilde{q}_n \tilde{h}_n)\right)$ (use (40)), we argue as follows.
If $M-n$ is odd, we take $k = \frac{M-1-n}{2}$ in (63), hence

$$\begin{aligned}P_-\left(V_1(q_n h_n + \tilde{q}_n\tilde{h}_n)\right) &= \sum_{j=1}^{k}\frac{1}{s^j}P_-\left(W_{1,j}(q_n h_n + \tilde{q}_n\tilde{h}_n)\right)\\ &\quad + P_-\left((q_n h_n + \tilde{q}_n\tilde{h}_n)\tilde{W}_{1,k}\right)\end{aligned}$$

Since $2k+n \leq M$, we deduce that $P_-(W_{1,j}(q_n h_n + \tilde{q}_n\tilde{h}_n) = 0$ for all $0 \leq j \leq k$. Moreover, using that

$$\left|\tilde{W}_{1,k}\right| \leq C\frac{(1+|y|^{2k+2})}{s^{k+1}}$$

and applying Lemma A.3, we deduce that

$$\left\|\frac{P_-\left(V_1(q_n h_n + \tilde{q}_n\tilde{h}_n)\right)}{1+|y|^{M+1}}\right\|_{L^\infty} \leq C\frac{(|q_n| + |\tilde{q}_n|)}{s^{\frac{M+1-n}{2}}} \tag{87}$$

If $M-n$ is even, we take $k = \frac{M-n}{2}$ in (63) and use that

$$\left|\tilde{W}_{1,k}\right| \leq C\frac{(1+|y|^{2k+1})}{s^{k+\frac{1}{2}}}$$

to deduce that (87) holds. This ends the proof of Lemma 5.15. ■

Fifth term: $B(q,y,s)$.
Using (23), we have the following estimate from Lemmas 5.9 and A.3:

Lemma 5.17 *For all $K \geq 1$ and $A \geq 1$, there exists $s_{15}(K,A)$ such that for all $s \geq s_{15}$, if $q(s) \in V_A(s)$, then*

$$\left\| \frac{P_-(B(q,y,s))}{1+|y|^{M+1}} \right\|_{L^\infty} \leq C(M) \left[\left(\frac{A^{M+2}}{\sqrt{s}} \right)^{\bar{p}} + \frac{A^{[5+(M+1)^2]}}{s} \right] \frac{1}{s^{\frac{M+1}{2}}} \tag{88}$$

where $\bar{p} = \min(p,2)$.

Proof: The proof is very similar to the proof of the previous lemma. From Lemma 5.9, we deduce that for all s there exists a polynomial B_M of degree M in y such that for all y and s, we have

$$|B - B_M(y)| \leq C \left[\left(\frac{A^{M+2}}{\sqrt{s}} \right)^{\bar{p}} + \frac{A^{[5+(M+1)^2]}}{s} \right] \frac{(1+|y|^{M+1})}{s^{\frac{M+1}{2}}}. \tag{89}$$

Indeed, we can take $B_M(y)$ to be the polynomial

$$B_M = P_{+,M} \left[\sum_{l=0}^{M} \sum_{\substack{0 \leq j,k \leq M+1 \\ 2 \leq j+k \leq M+1}} \frac{1}{s^l} \left[B^l_{j,k}(\frac{y}{\sqrt{s}}) q_+^j \bar{q}_+^k \right] \right].$$

The fact that $B - B_M(y)$ is controlled by the right-hand side of (89) is a consequence of the following estimates in the outer region and in the inner region.
First, in the region where $|y| \geq \sqrt{s}$, we have from Lemma 5.9,

$$|B| \leq C|q|^{\bar{p}} \leq C \left(\frac{A^{M+2}}{\sqrt{s}} \right)^{\bar{p}}$$

and from the proof of Lemma 5.10, we know that for $0 \leq n \leq M$,

$$|P_{n,M}(B_M(q,y,s))| + |\tilde{P}_{n,M}(B_M(q,y,s))| \leq C \frac{A^n}{s^{\frac{n}{2}+1}}$$

Besides, in the region $|y| \leq \sqrt{s}$, we can use the same argument as in the proof of Lemma 5.9 to deduce that the coefficient of degree $k \geq M+1$ of the polynomial

$$\sum_{l=0}^{M} \sum_{\substack{0 \leq j,k \leq M+1 \\ 2 \leq j+k \leq M+1}} \frac{1}{s^l} \left[B^l_{j,k}(\frac{y}{\sqrt{s}}) q_+^j \bar{q}_+^k \right] - B_M$$

is controlled by $C \frac{A^k}{s^{\frac{k}{2}+1}}$ and hence

$$\left| \sum_{l=0}^{M} \sum_{\substack{0 \leq j,k \leq M+1 \\ 2 \leq j+k \leq M+1}} \frac{1}{s^l} \left[B^l_{j,k}(\frac{y}{\sqrt{s}}) q^j \bar{q}^k \right] - B_M \right| \leq C \frac{A^{2M+2}}{s^{\frac{M+3}{2}}} (1+|y|^{M+1})$$

in the region $|y| \leq \sqrt{s}$.
Moreover, using that $|q| \leq C \frac{A^{M+1}}{\sqrt{s}}$ in the region $|y| \leq \sqrt{s}$, we deduce that for $s \geq 2A^2$, we have

$$\sum_{l=0}^{M} \sum_{\substack{0 \leq j,k \leq M+1 \\ 2 \leq j+k \leq M+1}} \frac{1}{s^l} \left[\tilde{B}^l_{j,k}(\frac{y}{\sqrt{s}}) q^j \bar{q}^k \right] \leq C \frac{A^{2M+2}}{s^{\frac{M+3}{2}}} (1+|y|^{M+1})$$

Finally, to control the term $|q|^{M+2}$, we use that in the region $|y| \le \sqrt{s}$, we have the following two estimates $|q| \le C\frac{A^{M+1}}{\sqrt{s}}$ and $|q| \le A^5 \frac{1}{s^{\frac{3}{2}}}(1+|y|^{M+1})$ if $s \ge 2A^2$. Hence

$$|q|^{M+2} \le C\frac{A^5}{s^{3/2}}\left(\frac{A^{M+1}}{\sqrt{s}}\right)^{M+1}(1+|y|^{M+1})$$

This ends the proof of estimate (89) and we conclude the proof of (88) by applying Lemma A.3.

Sixth term: $R^*(\theta', y, s)$.
We claim the following:

Lemma 5.18 *If* $|\theta'(s)| \le \frac{C}{s^{3/2}}$ *then the following holds*

$$\left\|\frac{P_-(R^*(\theta', y, s))}{1+|y|^{M+1}}\right\|_{L^\infty} \le C\frac{1}{s^{\frac{M+3}{2}}}$$

Proof: Taking $n = \frac{M}{2}+1$ (remember that M is even), we write from Lemma 5.11 $R^*(\theta', y, s) = \Pi_n(\theta', y, s)) + \tilde{\Pi}_n(\theta', y, s)$. Since $2n-2 = M$, we see from subsection 3.5 that

$$|\tilde{\Pi}_n(\theta', y, s))| \le C\frac{\left(1+|y|^{2n-2}\right)}{s^n} C\frac{\left(1+|y|^{M+1}\right)}{s^{\frac{M+3}{2}}} \tag{90}$$

in the region $|y| < \sqrt{s}$. It is easy to see, using (25) and the definition of Π_n that (90) holds for all $y \in \mathbb{R}$ and $s \ge 1$. Then applying Lemma A.3, we conclude easily. ■

Part 2: Proof of the last but one identity in (iii) of Proposition 4.6 (estimate on q_-)

If we apply the projection operator P_- to the equation (21) satisfied by q, we see that q_- satisfies the following equation:

$$\frac{\partial q_-}{\partial s} = \mathcal{L}_\beta q_- + P_-[(1+i\delta)\Re q_-] + P_-[-i\left(\frac{\mu}{s}+\theta'(s)\right)q + V_1 q + V_2\bar{q} + B(q, y, s) + R^*(\theta', y, s)].$$

Here, we have used the important fact that $P_-[(1+i\delta)\Re q_+] = 0$.
Unlike the estimates on q_n and $\tilde{q}_n$ where we use the properties of the operator $\mathcal{L}_{\beta,\delta}$, here we use the operator $\mathcal{L}_\beta$. The fact that M is large as fixed in (38) is crucial in the proof. Using the kernel of the semigroup generated by $\mathcal{L}_\beta$, we get for all $s \in [\tau, s_1]$,

$$\begin{aligned} q_-(s) &= e^{(s-\tau)\mathcal{L}_\beta} q_-(\tau) \\ &+ \int_\tau^s e^{(s-s')\mathcal{L}_\beta} P_-[(1+i\delta)\Re q_-] ds' \\ &+ \int_\tau^s e^{(s-s')\mathcal{L}_\beta} P_-\left[-i\left(\frac{\mu}{s}+\theta'(s')\right)q + V_1 q + V_2\bar{q} + B(q, y, s) + R^*(\theta', y, s')\right] ds'. \end{aligned}$$

Using Lemma A.2, we get

$$\left\|\frac{q_-(s)}{1+|y|^{M+1}}\right\|_{L^\infty} \le e^{-\frac{M+1}{2}(s-\tau)}\left\|\frac{q_-(\tau)}{1+|y|^{M+1}}\right\|_{L^\infty}$$

$$+\int_\tau^s e^{-\frac{M+1}{2}(s-s')}\sqrt{1+\delta^2}\left\|\frac{q_-(s')}{1+|y|^{M+1}}\right\|_{L^\infty} ds'$$

$$+\int_\tau^s e^{-\frac{M+1}{2}(s-s')}\left\|\frac{P_-\left[-i\left(\frac{\mu}{s'}+\theta'(s')\right)q+V_1q+V_2\bar{q}+B(s')+R^*(s')\right]}{1+|y|^{M+1}}\right\|_{L^\infty}$$

Assuming that $q(s')\in V_A(s')$, the results from Part 1 yield (use (i) of Proposition 4.6 to bound $|\theta'(s')|$)

$$\begin{aligned}\left\|\frac{q_-(s)}{1+|y|^{M+1}}\right\|_{L^\infty} &\le e^{-\frac{M+1}{2}(s-\tau)}\left\|\frac{q_-(\tau)}{1+|y|^{M+1}}\right\|_{L^\infty}\\ &+\int_\tau^s e^{-\frac{M+1}{2}(s-s')}\left(\sqrt{1+\delta^2}+\left\||V_1|+|V_2|\right\|_{L^\infty}\right)\left\|\frac{q_-(s')}{1+|y|^{M+1}}\right\|_{L^\infty} ds'\\ &+\int_\tau^s e^{-\frac{M+1}{2}(s-s')}\left[\frac{A^{[(M+1)^2]}}{(s')^{\frac{M+3}{2}}}+\frac{A^{(M+2)\bar{p}}}{(s')^{\frac{\bar{p}-1}{2}}}\frac{1}{(s')^{\frac{M+2}{2}}}+\frac{A^M}{(s')^{\frac{M+2}{2}}}\right]ds'\end{aligned}$$

Since we have already fixed M in (38) such that

$$M\ge 4(\sqrt{1+\delta^2}+1+2\max_{i=1,2,\ y\in\mathbb{R},\ s\ge1}|V_i(y,s)|),$$

using Gronwall's lemma we deduce that

$$\begin{aligned}e^{\frac{M+1}{2}s}\left\|\frac{q_-(s)}{1+|y|^{M+1}}\right\|_{L^\infty} &\le e^{\frac{M+1}{4}(s-\tau)}e^{\frac{M+1}{2}\tau}\left\|\frac{q_-(\tau)}{1+|y|^{M+1}}\right\|_{L^\infty}\\ &+e^{\frac{M+1}{2}s}2^{\frac{M+3}{2}}\left[\frac{A^{[(M+1)^2]}}{s^{\frac{M+3}{2}}}+\frac{A^{(M+2)\bar{p}}}{s^{\frac{\bar{p}-1}{2}}}\frac{1}{s^{\frac{M+2}{2}}}+\frac{A^M}{s^{\frac{M+2}{2}}}\right]\end{aligned}$$

which concludes the proof of the last but one identity in (iii) of Proposition 4.6.

5.3 The outer region : q_e

Here, we finish the proof of Proposition 4.6 by proving the last inequality in (iii). Since $q(s)\in V_A(s)$ for all $s\in[\tau,s_1]$, it holds from Proposition 4.1 and (i) of Proposition 4.6 that

$$\|q(s)\|_{L^\infty(|y|<2K\sqrt{s})}\le C\frac{A^{M+1}}{\sqrt{s}} \text{ and } |\theta'(s)|\le CA^5\frac{\log s}{s^2}. \tag{91}$$

Then, we derive from (17) an equation satisfied by q_e:

$$\begin{aligned}\frac{\partial q_e}{\partial s} &= \mathcal{L}_\beta q_e-\tfrac{1}{p-1}q_e+(1-\chi)e^{\frac{i\delta}{p-1}s}\left\{L(q,\theta',y,s)+\tilde{R}(\theta',y,s)\right\}\\ &-e^{\frac{i\delta}{p-1}s}q(s)\left(\partial_s\chi+(1+i\beta)\Delta\chi+\frac{1}{2}y.\nabla\chi\right)+2e^{\frac{i\delta}{p-1}s}(1+i\beta)\,\mathrm{div}\,(q(s)\nabla\chi)\,.\end{aligned}\tag{92}$$

Writing this equation in its integral form and using the maximum principle satisfied by $e^{\tau\mathcal{L}_\beta}$ (see Lemma A.1 below), we write

$$\begin{aligned}
\|q_e(s)\|_{L^\infty} &\le e^{-\frac{s-\tau}{p-1}}\|q_e(\tau)\|_{L^\infty}\\
&\quad+\int_\tau^s e^{-\frac{s-s'}{p-1}}\left(\|(1-\chi)L(q,\theta',y,s')\|_{L^\infty}+\|(1-\chi)\tilde{R}(\theta',y,s')\|_{L^\infty}\right)ds'\\
&\quad+\int_\tau^s e^{-\frac{s-s'}{p-1}}\left\|q(s')\left(\partial_s\chi+(1+i\beta)\Delta\chi+\frac{1}{2}y.\nabla\chi\right)\right\|_{L^\infty}ds'\\
&\quad+\int_\tau^s e^{-\frac{s-s'}{p-1}}\frac{1}{\sqrt{1-e^{-(s-s')}}}\|q(s')\nabla\chi\|_{L^\infty}ds'.
\end{aligned}$$

Let us bound the norms in the three last lines of this inequality.
First of all, we have from (19) and (91)

$$\begin{aligned}
\left\|q(s')\left(\partial_s\chi+(1+i\beta)\Delta\chi+\frac{1}{2}y.\nabla\chi\right)\right\|_{L^\infty} &\le C\Big(1+\frac{1}{K^2{s'}^2}\Big)\|q(s')\|_{L^\infty(|y|<2K\sqrt{s'})}\\
&\le C\frac{A^{M+1}}{\sqrt{s'}}, \qquad (93)\\
\|q(s')\nabla\chi\|_{L^\infty}\le\frac{C}{K\sqrt{s'}}\|q(s')\|_{L^\infty(|y|<2K\sqrt{s'})} &\le C\frac{A^{M+1}}{s'} \qquad (94)
\end{aligned}$$

for s' large enough.
Second, note that the residual term $(1-\chi)R^*$ is small as well. Indeed, recalling the bound (24) on R, we write from the definition of R^* (18) and (91):

$$\left\|(1-\chi)R^*(\theta',y,s')\right\|_{L^\infty}\le\frac{C}{s'}+|\theta'(s')|\le\frac{C}{s'} \qquad (95)$$

for s' large enough.
Third, the term $(1-\chi)L(q,\theta',y,s')$ given in (18) is less than $\epsilon|q_e|$ with $\epsilon=\frac{1}{2(p-1)}$. Indeed, it holds from (91) that

$$\begin{aligned}
&\|(1-\chi)L(q,\theta',y,s')\|_{L^\infty}\\
&\le C\|q_e(s')\|_{L^\infty}\left(\|\varphi(s')\|^{p-1}_{L^\infty(|y|\ge K\sqrt{s'})}+\|q(s')\|^{p-1}_{L^\infty(|y|\ge K\sqrt{s'})}+\frac{1}{s'}+|\theta'(s')|\right) \qquad (96)\\
&\le\tfrac{1}{2(p-1)}\|q_e(s')\|_{L^\infty}
\end{aligned}$$

whenever K and s' are large (in order to ensure that $\|\varphi(s')\|_{L^\infty(|y|\ge K\sqrt{s'})}$ is small, see (16)). Notice that it is only here that we need the fact that K is big enough. Using estimates (91), (93), (94), (95) and (96), we write

$$\begin{aligned}
\|q_e(s)\|_{L^\infty} &\le e^{-\frac{s-\tau}{p-1}}\|q_e(\tau)\|_{L^\infty}\\
&\quad+\int_\tau^s e^{-\frac{s-s'}{p-1}}\left(\frac{1}{2(p-1)}\|q_e(s')\|_{L^\infty}+\frac{CA^{M+1}}{\sqrt{s'}}+\frac{A^{M+1}}{s'}\frac{1}{\sqrt{1-e^{-(s-s')}}}\right).
\end{aligned}$$

Using Gronwall's inequality, we end-up with

$$\|q_e(s)\|_{L^\infty} \le e^{-\frac{(s-\tau)}{2(p-1)}}\|q_e(\tau)\|_{L^\infty} + \frac{CA^{M+1}}{\sqrt{\tau}}\left(s-\tau+\sqrt{s-\tau}\right)$$

which concludes the proof of Proposition 4.6. ∎

6 Stability of the profile (5)

As announced in the introduction, our technique proves the stability not only of the solution constructed in Theorem 1, but of any solution trapped in some neighborhood of the profile in selfsimilar variables. More precisely, we have the following:

Theorem 2' *Consider $\hat u(x,t)$ a solution of equation (1) which blows up at some time $\hat T > 0$ at one single blow-up point $\hat a$ such that $\hat u(x-\hat a, t)$ satisfies (5) with $T = \hat T$. Assume in addition that*

$$\forall s \ge \hat s_0,\ \ \hat q_{\hat T,\hat a}(s) \in V_{\hat A}(s)$$

for some positive $\hat s_0$, $\hat K$ and $\hat A$, where $\hat q_{\hat T,\hat a}$ is defined by

$$\begin{aligned} \hat w_{\hat T,\hat a}(y,s) &= e^{i(\mu\log s+\hat\theta_{\hat T,\hat a}(s))}(\varphi(y,s)+\hat q_{\hat T,\hat a}(y,s)),\\ P_{0,M}(\hat q_{\hat T,\hat a}(s)) &= 0,\\ \hat w_{\hat T,\hat a}(y,s) &= (\hat T-t)^{\frac{1+i\delta}{p-1}}\hat u(x,t),\ \ y=\frac{x-\hat a}{\sqrt{\hat T-t}},\ \ s=-\log(\hat T-t) \end{aligned} \tag{97}$$

and φ is the profile defined in (16).
Then, there exists a neighborhood $\mathcal V_0$ of $\hat u_0 \equiv \hat u(x,0)$ in L^∞ such that for any $u_0 \in \mathcal V_0$, equation (1) has a unique solution $u(x,t)$ with initial data u_0, and $u(x,t)$ blows up in finite time $T(u_0)$ at one single blow-up point $a(u_0)$. Moreover, for all $s \ge s_0$, $q_{T(u_0),a(u_0)}(s) \in V_{\hat K,A}(s)$ for some $A \ge 3\hat A$ independent from u_0, where $q_{T(u_0),a(u_0)}(y,s)$ can be defined from $u(x,t)$ as in (97). Finally, estimate (5) is satisfied by $u(x-a(u_0),t)$ and

$$T(u_0)\to\hat T,\ \ a(u_0)\to\hat a \text{ as } u_0\to\hat u_0.$$

This section is devoted to the proof of Theorem 2' (which is a generalization of Theorem 2). The proof is the same as in the case $\beta=\delta=0$ treated in [14]. For the reader's convenience, we give a sketch of the proof here (we recommend however the reading of the stability section of [14] first, in a pedagogical approach).

The sketch of the proof is given in 3 steps:

- In Step 1, we replace the parameters (d_0,d_1) by new ones: the blow-up time and point. More precisely, using the modulation technique of the existence proof, we define $w_{T,a}$ and $q_{T,a}$ for (u_0,T,a) close to $(\hat u_0,\hat T,\hat a)$ as in (97).

- In Step 2, given an arbitrary u_0 close to $\hat u_0$, our goal is to prove that for some $(T,a)=(T(u_0),a(u_0))$, $q_{T,a}(s)$ is trapped in $V_{\hat K,A}(s)$ for all s large enough and some

$A \geq \hat{A}$. Recalling the reduction to a 2 dimensional problem from the existence problem, we show that we reduce to the control of $\tilde{P}_{j,M}(q_{T,a}(s))$ for $j = 0$ and 1.

- In Step 3, we solve the two dimensional problem by contradiction. Giving the behavior of $\tilde{P}_{j,M}(\hat{q}_{T,a}(s))$ for $j = 0$ and 1 and (T, a) close to $(\hat{T}, \hat{a})$ is crucial to find a contradiction through index theory.

Step 1: Interpretation of the 2 parameters in terms of the blow-up time and point

If in the existence proof, we had to finetune 2 parameters d_0 and d_1 in (45) in order to guarantee that $q(s)$ stays in $V_A(s)$ for any $s \geq s_0$ for some s_0, here our parameters will be (T, a) in some neighborhood of $(\hat{T}, \hat{a})$. More precisely, given initial data u_0 close to $\hat{u}_0$, we define $u(x, t)$ the local solution of equation (1). Then, given any (T, a) close to $(\hat{T}, \hat{a})$, we introduce

$$w_{T,a}(y, s) = (T - t)^{\frac{1+i\delta}{p-1}} u(x, t), \;\; y = \frac{x - a}{\sqrt{T - t}}, \;\; s = -\log(T - t). \tag{98}$$

Given any $s_1 \geq \hat{s}_0$, we see that $w_{T,a}$ is close to $\hat{w}_{\hat{T},\hat{a}}$, provided that (u_0, T, a) is close to $(\hat{u}_0, \hat{T}, \hat{a})$. More precisely, we have the following:

Proposition 6.1 (Continuity with respect to initial data on a finite time interval) *For all $s_1 \geq \hat{s}_0$ and $\delta_1 > 0$, there exist $\mathcal{V}_1$ a neighborhood of $\hat{u}_0$ and $\epsilon_1 > 0$ such that for all $u_0 \in \mathcal{V}_1$, equation (1) with initial data u_0 has a unique solution defined for all $t \in [0, \hat{T} - \frac{e^{-s_1}}{2}]$. Moreover, for all (a, T) such that*

$$|a - \hat{a}| + |T - \hat{T}| < \epsilon_1,$$

the function $w_{T,a}$ (98) is well defined for all $s \in [-\log T, s_1]$ and

$$\sup_{s \in [-\log T, s_1]} \|w_{T,a}(s) - \hat{w}_{\hat{T},\hat{a}}(\sigma)\|_{L^\infty} < \delta_1 \text{ where } \sigma = s - \log(1 - (T - \hat{T})e^s).$$

Remark: All the quantities having a hat are defined from $\hat{u}$.
Idea of the proof: This is just the continuity with respect to initial data of solutions of equation (1) when time belongs to a finite interval (here $[0, \hat{T} - \frac{e^{-s_1}}{2}]$). ■

Now, we can make modulation theory to define $q_{T,a}(y, s)$ as in the 2 first lines of (97). More precisely, we have the following:

Proposition 6.2 (Modulation theory) *There exists $\hat{s}_1 \geq \hat{s}_0$ such that for all $s_1 \geq \hat{s}_1$, there exist $\mathcal{V}_2$ and $\epsilon_2 > 0$ such that for all $u_0 \in \mathcal{V}_2$, for all $(T, a) \in B((\hat{T}, \hat{a}), \epsilon_2)$ and $s \in [\hat{s}_1, s_1]$, there exists $\theta_{T,a}(s)$ (C^1 in terms of s) such that if $q_{T,a}(y, s)$ is defined by*

$$w_{T,a}(y, s) = e^{i(\mu \log s + \theta_{T,a}(s))}(q_{T,a}(y, s) + \varphi(y, s)), \; P_{0,M}(q_{T,a}(s)) = 0 \tag{99}$$

where φ is defined in (16), then

$$\forall s \in [\hat{s}_1, s_1], \;\; q_{T,a}(s) \in V_{2\hat{A}}(s).$$

Idea of the proof: This proposition is analogous to Lemma 4.4. Using Proposition 6.1, one has to apply the implicit function theorem to the function

$$F(v,s,\theta) \equiv \Im((1-i\delta)\int (e^{-i(\mu\log s+\theta)}v(y)-\varphi(y,s))\rho_\beta(y)dy) = 0, \qquad (100)$$

near the point $(v,s,\theta) = (\hat{w}_{\hat{T},\hat{a}}(\sigma),\sigma,\hat{\theta}_{\hat{T},\hat{a}}(\sigma))$ for s large enough. ■

Step 2: Reduction to a finite dimensional problem

The two parameters T and a in $q_{T,a}$ replace d_0 and d_1 in (45) in the existence proof. Then, given $u_0 \in \mathcal{V}_0 \cap \mathcal{V}_1$, if we prove that for some $A \geq \hat{A}$, $\tilde{s}_0$ and $(T,a) = (T(u_0), a(u_0))$, we have $q_{T,a}(s) \in V_A(s)$ for all $s \geq \tilde{s}_0$, then, as in the proof of Theorem 1, T, a will be respectively the blow-up time and the blow-up point of $u(x,t)$, and u will have the profile (5).

As in Proposition 4.5, we reduce the problem to a finite dimensional one:

Proposition 6.3 (Control of $q(s)$ in $V_A(s)$ by $(\tilde{q}_0(s),\tilde{q}_1(s))$) *There exists A_3 such that for all $A \geq A_3$, there exist $\hat{s}_3$, $\mathcal{V}_3$ a neighborhood of u_0 and $\epsilon_3 > 0$ such that for all $T \leq T_3$, the following holds:*
If $u_0 \in \mathcal{V}_3$ and $|T-\hat{T}| + |a-\hat{a}| \leq \epsilon_3$ and $q_{T,a}(s) \in V_A(s)$ for all $s \in [s_0,s_1]$ with $q_{T,a}(s_1) \in \partial V_A(s_1)$ for some $s_1 \geq s_0$, then:

(i) **(Smallness of the modulation parameter)** *For all $s \in [s_0,s_1]$, $|\theta'_{T,a}(s)| \leq C\frac{A^5\log s}{s^2}$.*

(ii) $(\tilde{q}_0(s_1),\tilde{q}_1(s_1))_{T,a} \in \partial[-\frac{A}{s_1^2},\frac{A}{s_1^2}]^2$.

(iii) **(Transverse crossing)** *There exists $m \in \{0,1\}$ and $\omega \in \{-1,1\}$ such that*

$$\omega\tilde{q}_m(s_1) = \frac{A}{s_1^2} \text{ and } \omega\frac{d\tilde{q}_m}{ds}(s_1) > 0.$$

Proof: Propositions 6.3 and 4.5 are essentially the same. They both follow from Proposition 4.6. The only difference is in the data at $s = s_0$. Therefore, in the case $s \geq s_0+\sigma$ where $\sigma = \log A$, we don't use the data at $s = s_0$ and the proof is the same. On the contrary, in the case $s \leq s_0+\sigma$, the proof is different. Indeed, in Proposition 4.5, we take $(d_0,d_1) \in \mathcal{D}_T$, so that data at time $s = s_0$ is very small and stays small up to $s = s_0+\sigma$, whereas in Proposition 6.3, the fact that $q(s)$ does not touch the boundary of $V_A(s)$ for $s \leq s_0+\sigma$ follows directly from the continuity result of Proposition 6.2 by taking $A \geq 3\hat{A}$.

Step 3: Solution of the 2 dimensional problem

As in the existence proof, we derive from Steps 1 and 2 the existence of some large $A \geq 3\hat{A}$, $\hat{s}_4 \geq \hat{s}_0$, a neighborhood $\mathcal{V}_4$ of $\hat{u}_0$ and a rectangle $\hat{\mathcal{D}} \subset \mathbb{R}^2$ containing $(\hat{T},\hat{a})$ such that for all $u_0 \in \mathcal{V}_0$, we will be able to find $(T,a) \in \hat{\mathcal{D}}$ such that $q_{T,a}(s) \in V_A(s)$ for all $s \geq \hat{s}_1$. We proceed by contradiction, and assume that for all $(T,a) \in \hat{\mathcal{D}}$, there exists $s_*(T,a) \geq \hat{s}_4$ such that $q_{T,a}(s) \in V_A(s)$ for all $s \in [\hat{s}_4,s_*]$ and $q_{T,a}(s_*) \in \partial V_A(s_*)$. Using Proposition 6.3, we see that the following function is well defined and continuous:

$$\begin{aligned}\Phi_{u_0} \quad &: \hat{\mathcal{D}}_T \to \partial[-1,1]^2\\ &(T,a) \to \tfrac{s_*^2}{A}\,(\tilde{q}_0,\tilde{q}_1)_{T,a}\,(s_*).\end{aligned}$$

If one proves that the degree of Φ_{u_0} on the boundary of $\hat{\mathcal{D}}$ is not zero, then a contradiction follows from index theory and the proof is terminated. From the continuity of the degree and Step 1, it is enough to show this for $u_0 = \hat{u}_0$. This comes from the following Proposition which is analogous to Lemma B.2 page 186 in [14] concerning the case $\beta = \delta = 0$. Here, we only give the expansion of $\tilde{P}_{j,M}(\hat{q}_{T,a})(s)$. Other estimates (in particular concerning the derivatives with respect to T and a) are omitted here. They are completely analogous to [14]. The following lemma allows us to conclude

Lemma 6.4 (Expansion of $\tilde{P}_{j,M}(\hat{q}_{T,a})(s)$ for (T,a) close to $(\hat{T},\hat{a})$ and $j=0$ and 1) *There exist $c_0 = c_0(p,\beta,\delta) \in \mathbb{R}$ and $\hat{s}_5 \geq \hat{s}_1$ such that for all $s \geq \hat{s}_5$, for all $(T,a) \in \mathbb{R}^2$ satisfying $|\tau| \leq \frac{1}{2}$ and $|\alpha| \leq \frac{1}{2}$, where $\tau = (T-\hat{T})e^s$ and $\alpha = (a-\hat{a})e^{\frac{s}{2}}$, we have*

$$\begin{aligned}
|\tilde{P}_{0,M}(\hat{q}_{T,a}(s)) - \frac{c_0}{s^2} - \tau\frac{\kappa}{p-1}| &\leq C\left(\frac{\log s}{s^{5/2}} + \frac{|\tau|}{s} + \tau^2 + \frac{\alpha^2}{s} + |\alpha|\frac{\log s}{s^2} + |\alpha|^3\right),\\
|\tilde{P}_{1,M}(\hat{q}_{T,a}(s)) + \frac{2b\kappa}{(p-1)^2}\frac{\alpha}{s}| &\leq C\left(\frac{\log s}{s^3} + \frac{|\tau|}{s} + \frac{\alpha^4}{s} + \tau^2 + |\alpha|\frac{\log s}{s^2}\right).
\end{aligned}$$

Indeed, as in [14], taking τ and $\frac{\alpha}{s}$ of the size of $\frac{1}{s^2}$, we see that $\Phi_{\hat{u}_0}$ is a linear function (up to some perturbation) whose degree on the boundary is -1. By continuity, Φ_{u_0} has the same degree, which yields a contradiction and proves the stability result. It remains to prove Lemma 6.4.

Proof of Lemma 6.4: Using the definition of $\hat{w}_{T,a}$, (98), we write

$$\hat{w}_{T,a}(y,s) = (1-\tau)^{-\frac{1+i\delta}{p-1}}\hat{w}_{\hat{T},\hat{a}}(z,\sigma)$$

where

$$z = \frac{y+\alpha}{\sqrt{1-\tau}},\ \sigma = s - \log(1-\tau),\ \tau = (T-\hat{T})e^s \text{ and } \alpha = (a-\hat{a})e^{\frac{s}{2}}.$$

Using the definition of $\hat{q}_{T,a}$ (99), this gives

$$\hat{q}_{T,a}(y,s) = e^{i\psi}I$$

where

$$\begin{aligned}
I &= (1-\tau)^{-\frac{1+i\delta}{p-1}}\hat{q}_{\hat{T},\hat{a}}(z,\sigma) + (1-\tau)^{-\frac{1+i\delta}{p-1}}\varphi(z,\sigma) - \varphi(y,s),\\
\psi &= \mu\log\sigma - \mu\log s + \hat{\theta}_{\hat{T},\hat{a}}(\sigma) - \hat{\theta}_{T,a}(s).
\end{aligned}$$

The application of the implicit function theorem to $F(v,s,\theta)$ (100) gives an expansion of $\hat{\theta}_{T,a}(s) - \hat{\theta}_{\hat{T},\hat{a}}(\sigma)$ in terms of $s-\sigma$ and $\hat{w}_{T,a}(y,s) - \hat{w}_{\hat{T},\hat{a}}(z,\sigma)$, which gives after straightforward computations analogous to [14]

$$|e^{i\psi} - 1| \leq C\tau^2 + C\frac{|\tau|}{s} + C|\alpha|\frac{\log s}{s^2} + C\frac{\alpha^2}{s} + C|\alpha|^3. \tag{101}$$

Proceeding as in [14], we make the expansion of I and obtain its projections on $\tilde{h}_0$ and $\tilde{h}_1$. Gathering the information on ψ and I concludes the proof of Lemma 6.4, Theorems 2' and 2. ■

A Spectral properties and the semigroup generated by the operator $\mathcal{L}_\beta$

We aim at defining a semi-group for the operator $\mathcal{L}_\beta$ and showing some of its properties. For that, we introduce the more general operator defined for all $z \in \mathbb{C}$, $\Re z > 0$,

$$\tilde{\mathcal{L}}_z w = z\Delta w - \frac{1}{2} y.\nabla w = \frac{1}{\rho_z} \operatorname{div}(\rho_z \nabla w)$$

where $\rho_z(y) = e^{-\frac{|y|^2}{4z}}/(4\pi z)^{N/2}$. Let us remark that this problem in analytical in terms of the z variable. Therefore, it is enough to solve it for $z \in \mathbb{R}_+^*$ and to deduce its properties for general z by holomorphic extension. As a matter of fact, when $z \in \mathbb{R}_+^*$, we have

$$\int_{\mathbb{R}^N} \rho_\beta(y) dy = 1,$$

moreover, the operator $\tilde{\mathcal{L}}_z$ is well known. It is self-adjoint with respect to the weight ρ_z in the sense that

$$\int_{\mathbb{R}^N} u(y)\tilde{\mathcal{L}}_z w(y)\rho_z(y)dy = \int_{\mathbb{R}^N} w(y)\tilde{\mathcal{L}}_z u(y)\rho_z(y)dy. \tag{102}$$

In one space dimension ($N = 1$), the eigenfunctions f_n of $\tilde{\mathcal{L}}_z$ are dilations of the standard Hermite polynomials $H_n(y)$:

$$f_n(y) = H_n(\frac{y}{2\sqrt{z}}) \text{ where } \tilde{\mathcal{L}}_z H_n = -\frac{n}{2} H_n.$$

If $N \geq 2$, its eigenfunctions $f_\alpha(y_1, ..., y_N)$ where $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$ is a multi-index are given by

$$f_\alpha(y) = \Pi_{i=1}^N f_{\alpha_i}(y_i) = \Pi_{i=1}^N H_{\alpha_i}(\frac{y_i}{2\sqrt{z}}).$$

The family f_α is orthogonal in the sense that for all α and ζ in $\mathbb{N}^N$,

$$\int f_\alpha f_\zeta \rho_z dy = \delta_{\alpha,\zeta} \int f_\alpha^2 \rho_z dy. \tag{103}$$

The semigroup generated by $\tilde{\mathcal{L}}_z$ is well defined and has the following kernel:

$$e^{s\tilde{\mathcal{L}}_z}(y, x) = \frac{1}{[4\pi z(1 - e^{-s})]^{N/2}} \exp\left[-\frac{\left| x - ye^{-\frac{s}{2}} \right|^2}{4z(1 - e^{-s})} \right].$$

Using the holomorphic extension, it is clear that all the above properties hold for all $z \in \mathbb{C}$ such that $\Re z > 0$. The following two lemmas will be used to prove some decay

Lemma A.1 *a) The semigroup satisfies the maximum principle:*

$$\|e^{s\mathcal{L}_\beta}\varphi\|_{L^\infty} \leq \|\varphi\|_{L^\infty}.$$

b) Moreover, we have

$$\|e^{s\mathcal{L}_\beta} div(\varphi)\|_{L^\infty} \leq \frac{C}{\sqrt{1 - e^{-s}}} \|\varphi\|_{L^\infty}$$

where C only depends on β.

Lemma A.2 *There exists a constant C such that if ϕ satisfies*

$$\forall x \in \mathbb{R} \quad |\phi(x)| \leq (1+|x|^{M+1})$$

then for all $y \in \mathbb{R}$, we have

$$|e^{s\mathcal{L}_\beta} P_-(\phi(y))| \leq Ce^{-\frac{M+1}{2}s}(1+|y|^{M+1})$$

Moreover, we have the following useful lemma about P_-

Lemma A.3 *For all $k \geq 0$, we have*

$$\left\| \frac{P_-(\phi)}{1+|y|^{M+k}} \right\|_{L^\infty} \leq C \left\| \frac{\phi}{1+|y|^{M+k}} \right\|_{L^\infty}$$

Remark: Even though (102) and (103) hold for all z, one should bear in mind that $\tilde{\mathcal{L}}_z$ and $\mathcal{L}_\beta$ are neither self-adjoint nor Hermitian with respect to the weight ρ_z. Moreover, we can't say that family $(f_\alpha)_{\alpha\in\mathbb{N}}$ is orthogonal, because the symmetric bilinear form

$$(u,v) \to \int uv\rho_z$$

is not even positive.

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