Uniform blow-up estimates for nonlinear heat equations and applications

Frank Merle
IUF and Université de Cergy-Pontoise
Hatem Zaag
CNRS École Normale Supérieure

We consider the following nonlinear heat equation

$$\left\{ \begin{array}{rcl} u_t &=& \Delta u + |u|^{p-1}u \\ u(0) &=& u_0, \end{array} \right. \quad (1)$$

where $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$,

$$1 < p,\ (N-2)p < N+2 \text{ and either } u_0 \geq 0 \text{ or } (3N-4)p < 3N+8. \quad (2)$$

More general vector-valued heat equations can be considered with similar results (see [MZ99] for more details) :

$$\left\{ \begin{array}{rcl} u_t &=& \Delta u + F(|u|)u \\ u(0) &=& u_0, \end{array} \right. \quad (3)$$

where $u : \Omega \times [0,T) \to \mathbb{R}^M$, p satisfies (2), $\Omega = \mathbb{R}^N$ or Ω is a smooth bounded convex domain of $\mathbb{R}^N$, $F(|u|) \sim |u|^{p-1}$ as $|u| \to +\infty$, and $M \in \mathbb{N}$.

We are interested in the blow-up phenomenon for (1). Many authors has been interested in this topic. Let us mention for instance Friedman [Fri65], Fujita [Fuj66], Ball [Bal77], Bricmont and Kupiainen [BKL94], Chen and Matano [CM89], Galaktionov and Vázquez [GV95], Giga and Kohn [GK89], [GK87], [GK85], Herrero and Velázquez [HV93], [HV92].

In the following, we consider $u(t)$ a blow-up solution of (1) and denote its blow-up time by T. We aim at finding sharp uniform estimates at blow-up and specifying the blow-up behavior of $u(t)$. Such a study is done considering equation (1) in its self-similar form : for all $a \in \mathbb{R}^N$, we define

$$y = \frac{x-a}{\sqrt{T-t}},\ s = -\log(T-t),\ w_{a,T}(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t). \quad (4)$$

Therefore, $w_{a,T} = w$ satisfies $\forall s \geq -\log T$, $\forall y \in \mathbb{R}^N$:

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w. \tag{5}$$

Let us introduce the following Lyapunov functional associated with (5) :

$$E(w) = \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla w|^2 + \frac{1}{2(p-1)} |w|^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho(y) dy$$

where $\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}$.
The question is to estimate $w_{a,T}(s)$ as $s \to +\infty$, uniformly with respect to a, whether a is a blow-up point or not (a is called a blow-up point if there exists $(a_n, t_n) \to (a, T)$ such that $|u(a_n, t_n)| \to +\infty$).

Giga and Kohn showed that self-similar variables are convenient for describing the blow-up rate in the following sense : there exists $\epsilon_0(u_0) > 0$ such that $\forall s \geq s_0^*(u_0)$,

$$\epsilon_0 \leq |w(s)|_{L^\infty} \leq \frac{1}{\epsilon_0}. \tag{6}$$

We first aim at sharpening this result in order to obtain compactness properties in our problem.

1 A Liouville Theorem for equation (5)

We are interested in classifying all global and bounded solutions of (5), for all subcritical p :

$$p > 1 \text{ and } (N-2)p < N+2. \tag{7}$$

We claim the following :

Theorem 1 (A Liouville Theorem for equation (5)) *Assume (7) and consider w a solution of (5) defined for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$, $|w(y,s)| \leq C$. Then, either $w \equiv 0$, or $w \equiv \kappa$ or $w(y,s) = \pm\varphi(s - s_0)$ where $\kappa = (p-1)^{-\frac{1}{p-1}}$, $s_0 \in \mathbb{R}$ and $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$.*

Remark : φ is in fact an L^∞ connection between two critical points of (5): κ and 0. Indeed,

$$\dot{\varphi} = -\frac{\varphi}{p-1} + \varphi^p, \ \varphi(-\infty) = \kappa, \ \varphi(+\infty) = 0.$$

Remark : A similar classification result can be obtained with a solution w defined only on $(-\infty, s^*)$ (see [MZ98]).

Theorem 1 has the following corollary :

Corollary 1 (A Liouville Theorem for equation (1)) *Assume (7) and consider* u *a solution of (1) defined for all* $(x,t) \in \mathbb{R}^N \times (-\infty,0)$ *such that* $\forall (x,t) \in \mathbb{R}^N \times (-\infty,0)$, $|u(x,t)| \leq C(-t)^{-\frac{1}{p-1}}$. *Then, either* $u \equiv 0$, *or* $u(x,t) = \pm\kappa(T^* - t)^{-\frac{1}{p-1}}$ *for some* $T^* \geq 0$.

The proofs can be found in [MZ99] and [MZ98]. The key tools in the proof are the following :

i) A classification of all possible linear behaviors of $w(s)$ as $s \to -\infty$ in $L^2_\rho(\mathbb{R}^N)$ ($L^\infty_{loc}(\mathbb{R}^N)$),

ii) The following geometric transformations which keeps (5) invariant :

$$w(y,s) \to w_{a,b}(y,s) = w(y + ae^{\frac{s}{2}}, s + b),$$

where $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$,

iii) A blow-up criterion for (5) used for solutions close to the constant point κ (This criterion is also a blow-up criterion for (1) via the transformation (4)) :

If for some $s_0 \in \mathbb{R}$, $I(w(s_0)) > 0$ *where*

$$I(w) = -2E(w) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^N} |w(y)|^2 \rho(y) dy\right)^{\frac{p+1}{2}},$$

then $w(s)$ *blows up in finite time.*

Remark : This criterion is sharp for solutions near constants. Indeed, if $w(s_0) \equiv C_0$, then

w *blows-up in finite time* $\Leftrightarrow |C_0| > \kappa \Leftrightarrow I(C_0) > 0$.

Remark : The proof of the Liouville Theorem strongly relies on the existence of a Lyapunov functional for equation (5) and can not be extended to other systems where the nonlinearity is not a gradient. In [Zaa], we go beyond this restriction and introduce new tools to prove a Liouville Theorem of the same type for the following system

$$u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q.$$

2 Localization at blow-up

We assume again (2). The estimate (6) of Giga and Kohn gives compactness in the problem. Using a compactness procedure in the singular zone of $\mathbb{R}^N$ (which is, say $\{y \mid |w(y,s)| \geq \frac{\kappa}{2}\}$), we find a solution satisfying the hypotheses of Theorem 1. Therefore, Δw is small with respect to $|w|^p$ in this singular zone (or equivalently, Δu is small with respect to $|u|^p$). A subcritical localization procedure introduced by Zaag [Zaa98] (under the level of the constant κ) allows us to propagate this estimate towards the intermediate zone between the singular and the regular one. We claim the following :

Theorem 2 (Comparison with the associated ordinary differential equation) *Assume (2) and consider $T \leq T_0$ and $\|u_0\|_{C^2(\mathbb{R}^N)} \leq C_0$. Then, $\forall \epsilon > 0$, there is $C(\epsilon, C_0, T_0)$ such that $\forall (x,t) \in \mathbb{R}^N \times [0,T)$,*

$$|u_t - |u|^{p-1}u| \leq \epsilon |u|^p + C.$$

Remark : This way, we prove that the solution of the PDE (1) can be uniformly and globally in space-time compared to a solution of an ODE (localized by definition). Note that the condition $u(0) \in C^2(\mathbb{R}^N)$ is not restrictive because of the regularizing effect of the Laplacian.
Remark : Many striking corollaries can be derived from this theorem. It implies in particular that no oscillation is possible near a blow-up point a, and that $|u(x,t)| \to +\infty$ as $(x,t) \to (a,T)$. Moreover, $\forall \epsilon_0 > 0$, there exists $t_0(\epsilon_0) < T$ such that for all $b \in \mathbb{R}^N$, if $|u(b,t)| \leq (1-\epsilon_0)\kappa(T-t)^{-\frac{1}{p-1}}$ for some $t \in [t_0, T)$, then b is not a blow-up point (this specifies more precisely a former result by Giga and Kohn where $t_0 = t_0(\epsilon_0, a)$).

3 Optimal L^∞ estimates at blow-up

We still assume (2). Using estimate (6) and the Liouville Theorem, we make a compactness argument to get the following sharp estimates :

Theorem 3 (L^∞ refined estimates for $w(s)$) *Assume that (2) holds. Then, there exist positive constants C_i for $i = 1,2,3$ such that if u is a solution of (1) which blows-up at time T and satisfies $u(0) \in C^3(\mathbb{R}^N)$, then $\forall \epsilon > 0$, there exists $s_1(\epsilon) \geq -\log T$ such that $\forall s \geq s_1$, $\forall a \in \mathbb{R}^N$,*

$$\begin{array}{rclrcl} \|w_{a,T}(s)\|_{L^\infty} & \leq & \kappa + (\frac{N\kappa}{2p} + \epsilon)\frac{1}{s}, & \|\nabla w_{a,T}(s)\|_{L^\infty} & \leq & \frac{C_1}{\sqrt{s}}, \\ \|\nabla^2 w_{a,T}(s)\|_{L^\infty} & \leq & \frac{C_2}{s}, & \|\nabla^3 w_{a,T}(s)\|_{L^\infty} & \leq & \frac{C_3}{s^{3/2}}. \end{array}$$

Remark : In the case $N = 1$, Herrero and Velázquez (Filippas and Kohn also) proved some related estimates, using a Sturm property introduced in particular by Chen et Matano (the number of space oscillations is a decreasing function of time).
Remark : The constant $\frac{N\kappa}{2p}$ is optimal (see Herrero and Velázquez, Bricmont and Kupiainen, Merle and Zaag).

4 Different notions of blow-up profiles and the stability question

We assume (2). We consider $a \in \mathbb{R}^N$, a blow-up point of $u(t)$, solution of (1). From translation invariance, we can assume $a = 0$. We would like to know whether $u(t)$ (or $w_{0,T}(s)$ defined in (4)) has a universal behavior or not, as $t \to T$ (or $s \to +\infty$).

Filippas, Kohn, Liu, Herrero et Velázquez prove that w behaves in two distinct ways :

- either $\forall R > 0,\ \sup_{|y|\leq R} \left| w(Qy,s) - \left[\kappa + \frac{\kappa}{2ps}\left(l - \frac{1}{2}\sum_{k=1}^{l} y_k^2\right)\right]\right| = O\left(\frac{1}{s^{1+\delta}}\right)$ as $s \to +\infty$, for some $\delta > 0$ where $l \in \{1, ..., N\}$, Q is a $N \times N$ orthogonal matrix and I_l is the $l \times l$ identity matrix.

- or $\forall R > 0,\ \sup_{|y|\leq R} |w(y,s) - \kappa| \leq C(R)e^{-\lambda_0 s}$ for some $\lambda_0 > 0$.

From a physical point of view, these results do not tell us much about the transition between the singular zone ($w \geq \alpha$ where $\alpha > 0$) and the regular one ($w \simeq 0$). In [MZ99], we specify this transition by proving the existence of a profile in the variable $z = \frac{y}{\sqrt{s}}$.

Theorem 4 (Existence of a blow-up profile for equation (1))
Assume (2) holds. There exists $l \in \{0, 1, ..., N\}$ and a $N \times N$ orthogonal matrix Q such that $w(Q(z)\sqrt{s}, s) \to f_l(z)$ uniformly on compact sets $|z| \leq C$, where $f_l(z) = (p - 1 + \frac{(p-1)^2}{4p}\sum_{i=1}^{l}|z_i|^2)^{-\frac{1}{p-1}}$ if $l \geq 1$ and $f_0(z) = \kappa = (p-1)^{-\frac{1}{p-1}}$.

This result has been proved by Velázquez in [Vel92]. However, the convergence speed depends on the considered blow-up point in [Vel92], whereas they are uniform in [MZ99]. This uniformity allows us to derive the stability of the profile f_N in [FKMZ].

Using renormalization theory, Bricmont and Kupiainen prove in [BK94] the existence of a solution of (5) such that

$$\forall s \geq s_0,\ \forall y \in \mathbb{R}^N, |w(y,s) - f_N(\frac{y}{\sqrt{s}})| \leq \frac{C}{\sqrt{s}}.$$

Merle and Zaag prove the same result in [MZ97], thanks to a technique of finite-dimension reduction. They also prove the stability of such a behavior with respect to initial data, in a neighborhood of the constructed solution.

In [Zaa98] and [Vel92], it is proved that in this case, $u(x,t) \to u^*(x)$ as $t \to T$ uniformly on $\mathbb{R}^N \backslash \{0\}$ and that $u^*(x) \sim \left[\frac{8p|\log|x||}{(p-1)^2|x|^2}\right]^{\frac{1}{p-1}}$ as $x \to 0$.

One interesting problem is to relate all known blow-up profiles' notions : profiles for $|y|$ bounded, $\frac{|y|}{\sqrt{s}}$ bounded or $x \simeq 0$. We prove in the following that all these descriptions are equivalent, in the case of single point blow-up with a non degenerate profile (generic case). This answers many questions which arose in former works.

Theorem 5 (Equivalence of blow-up behaviors at a blow-up point) *Assume (2) and consider a be an isolated blow-up point of $u(t)$ solution of (1). The following behaviors of $u(t)$ and $w_{a,T}(s)$ (defined in (4)) are equivalent :*

i) $\forall R > 0,\ \sup_{|y|\le R}\left|w(y,s) - \left[\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)\right]\right| = o\left(\frac{1}{s}\right)$ *as* $s \to +\infty$,

ii) $\forall R > 0,\ \sup_{|z|\le R}|w(z\sqrt{s},s) - f_0(z)| \to 0$ *as* $s \to +\infty$ *with* $f_0(z) = (p-1+\frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$,

iii) $\exists \epsilon_0 > 0$ *such that for all* $|x-a| \le \epsilon_0$, $u(x,t) \to u^*(x)$ *as* $t \to T$ *and* $u^*(x) \sim \left[\frac{8p|\log|x-a||}{(p-1)^2|x-a|^2}\right]^{\frac{1}{p-1}}$ *as* $x \to a$.

A further application of the Liouville Theorem is the stability of the behavior described in Theorem 5, with respect to perturbations in initial data. Using a dynamical system approach, we prove in [FKMZ], with Fermanian the following :

Theorem 6 (Stability of the blow-up profile) *Assume (7) and consider $\tilde{u}(t)$ a blow-up solution of (1) with initial data $\tilde{u}_0$ which blows-up at $t = \tilde{T}$ at only one point $\tilde{a} = 0$ and satisfies (6). Assume that*

$$\text{for all } |x| \ge R \text{ and } t \in [0,\tilde{T}),\ \ |\tilde{u}(x,t)| \le M.$$

and that the function $\tilde{w}_{0,\tilde{T}}(y,s)$ defined in (4) satisfies uniformly on compact sets of $\mathbb{R}^N$

$$\tilde{w}_{0,\tilde{T}}(y,s) - \kappa \underset{s\to+\infty}{\sim} \frac{\kappa}{2ps}(N - \frac{|\ y\ |^2}{2}). \tag{8}$$

Then, there is a neighborhood $\mathcal{V}$ in L^∞ of $\tilde{u}_0$ such that for all $u_0 \in \mathcal{V}$ the solution of (1) with initial data u_0 blows-up at time $T = T(u_0)$ at a unique point $a = a(u_0)$ and the function $w_{a,T}(y,s)$ defined in (4) satisfies uniformly on compact sets of $\mathbb{R}^N$

$$w_{a,T}(y,s) - \kappa \underset{s\to+\infty}{\sim} \frac{\kappa}{2ps}(N - \frac{|\ y\ |^2}{2}).$$

Moreover, $(a(u_0), T(u_0))$ *goes to* $(0, \tilde{T})$ *as* u_0 *goes to* $\tilde{u}_0$*.*

Remark : This results generalizes the stability result of [MZ97]. Note that unlike most applications of the Liouville Theorem, this result is valid for all subcritical p. In [FKZ], the same result is proved (only under the condition (2)), by a completely different approach based on the Liouville Theorem and on [MZ97].

Remark : In [FKMZ], we prove the stability with respect to initial data of the blow-up behavior with the minimal speed

$$\|u(t)\|_{L^\infty} \leq C(T-t)^{-\frac{1}{p-1}} \tag{9}$$

for all subcritical p (that is under the condition (7)). Note that this result is obvious under the weaker assumption (2), for Giga and Kohn proved in [GK87] that all blow-up solutions satisfy (9). No blow-up rate estimate is known if

$$u_0 \text{ has no sign and } (3N-4)p \geq 3N-8.$$

Therefore, our result is meaningful in this last case.

References

[Bal77] J. M. Ball. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford Ser. (2)*, 28(112):473–486, 1977.

[BK94] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994.

[BKL94] J. Bricmont, A. Kupiainen, and G. Lin. Renormalization group and asymptotics of solutions of nonlinear parabolic equations. *Comm. Pure Appl. Math.*, 47(6):893–922, 1994.

[CM89] X. Y. Chen and H. Matano. Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations. *J. Differential Equations*, 78(1):160–190, 1989.

[FKMZ] C. Fermanian Kammerer, F. Merle, and H. Zaag. Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view. preprint.

[FKZ] C. Fermanian Kammerer and H. Zaag. Boundedness till blow-up of the difference between two solutions to the semilinear heat equation. preprint.

[Fri65] A. Friedman. Remarks on nonlinear parabolic equations. In *Proc. Sympos. Appl. Math., Vol. XVII*, pages 3–23. Amer. Math. Soc., Providence, R.I., 1965.

[Fuj66] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124, 1966.

[GK85] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.*, 38(3):297–319, 1985.

[GK87] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36(1):1–40, 1987.

[GK89] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.

[GV95] V. A. Galaktionov and J. L. Vázquez. Geometrical properties of the solutions of one-dimensional nonlinear parabolic equations. *Math. Ann.*, 303:741–769, 1995.

[HV92] M. A. Herrero and J. J. L. Velázquez. Flat blow-up in one-dimensional semilinear heat equations. *Differential Integral Equations*, 5(5):973–997, 1992.

[HV93] M. A. Herrero and J. J. L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(2):131–189, 1993.

[MZ97] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.

[MZ98] F. Merle and H. Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2):139–196, 1998.

[MZ99] F. Merle and H. Zaag. A Liouville theorem for vector-valued nonlinear heat equations and applications. *Math. Annalen*, 1999. to appear.

[Vel92] J. J. L. Velázquez. Higher-dimensional blow up for semilinear parabolic equations. *Comm. Partial Differential Equations*, 17(9-10):1567–1596, 1992.

[Zaa] H. Zaag. A liouville theorem and blow-up behavior for a vector-valued nonlinear heat equation with no gradient structure. preprint.

[Zaa98] H. Zaag. Blow-up results for vector-valued nonlinear heat equations with no gradient structure. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15(5):581–622, 1998.

Address :
Département de mathématiques, Université de Cergy-Pontoise, 2 avenue Adolphe Chauvin, B.P. 222, Pontoise, 95 302 Cergy-Pontoise cedex, France.
Département de mathématiques et applications, École Normale Supérieure, 45 rue d'Ulm, 75 230 Paris cedex 05, France.

e-mail : merle@math.pst.u-cergy.fr, Hatem.Zaag@ens.fr