Reconnection of vortex with the boundary and finite time Quenching

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Abstract: We construct a stable solution of the problem of vortex reconnection with the boundary in a superconductor under the planar approximation. That is a solution of

$$\frac{\partial h}{\partial t} = \Delta h + e^{-h} H_0 - \frac{1}{h}$$

such that $h(0,t) \to 0$ as $t \to T$. We give a precise description of the vortex near the reconnection point and time.
We generalize the result to other quenching problems.
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1 Introduction

1.1 The physical motivation and results

We consider a Type II superconductor located in the region $z > 0$ of the physical space $\mathbb{R}^3$. Under some conditions, the magnetic field develops a particular type of line singularity called vortex (see Chapman, Hunton and Ockendon [5] for more details and discussion). In general, a vortex is not situated in a plane, but under some reasonable physical conditions, the planar approximation is relevant. In this case, a vortex line at time $t \geq 0$ can be viewed as $L(t) = \{(x, y, z) = (x, 0, h(x, t)) | x \in \Omega\}$ where $\Omega = (-1, 1)$ or $\Omega = \mathbb{R}$, and $h > 0$ is a regular function. The physical derivation gives that $h(x, t)$ satisfies the following equation:

$$h_t = h_{xx} + e^{-h} H_0 - F_0(h) \qquad \text{(I)}$$

where H_0 is the applied magnetic field assumed to be constant, F_0 is a regular function satisfying

$$F_0(k) \sim \frac{1}{k} \text{ and } F_0'(k) \sim -\frac{1}{k^2} \text{ as } k \to 0. \tag{1}$$

We assume :

i) In the case where $\Omega = \mathbb{R}$

$$\left\{\begin{array}{rcll} F_0(k) & \sim & Ce^{-2k} & as \quad k \to +\infty \\ |F_0'(k)| & \leq & Ce^{-2k} & as \quad k \to +\infty \\ h(x,t) & \sim & a_1 x + b_1 & as \quad x \to +\infty \\ h(x,t) & \sim & -a_2 x + b_2 & as \quad x \to -\infty \end{array}\right. \tag{2}$$

where $a_1 > 0$ and $a_2 > 0$. For simplicity, we take $b_1 = b_2 = 0$ and $a_1 = a_2$.

ii) In the case where $\Omega = (-1, 1)$,

$$h(1,t) = h(-1,t) = 1. \tag{3}$$

One can remark that boundary conditions of the type i) are closer to the physical context. Nevertheless, boundary conditions of the type *ii*) are mostly considered in the literature in order to simplify the mathematical approach of the problem.

Similar results can be shown with other types of boundary conditions (mixed boundary conditions on bounded domains). Indeed, our analysis will be local and therefore will not depend on boundary conditions.

Classical theory gives for any initial vortex line $L(0) = \{(x, 0, h_0(x)) | x \in \Omega\}$ where h_0 is positive, regular and satisfies boundary conditions, the existence and uniqueness of a solution to (I)-(2) and (I)-(3) locally in time. Therefore, there exists a unique solution to (I) on $[0, T)$ and either $T = +\infty$ or $T < +\infty$ and in this case $\lim_{t \to T} \inf_{x \in \Omega} h(x,t) = 0$, i.e. h extinguishes in finite time, and if $x_0 \in \Omega$ is such that there exists $(x_n, t_n) \to (x_0, T)$ as $n \to +\infty$ satisfying $h(x_n, t_n) \to 0$ as $n \to +\infty$, then x_0 is an extinction point of h.

This phenomenon is called a vortex reconnection with the boundary (the plane $z = 0$). Two questions arise:

- Question 1: Are there any initial data such that $T < +\infty$?
- Question 2: What does the vortex look like at the reconnection time?

Equation (I) with a more general exponent can also appear in various physical contexts (combustion for example), and the problem of reconnection is known as the quenching problem.

Indeed, we consider

$$h_t = \Delta h - F(h), \qquad h \geq 0 \tag{II}$$

where

(H1) $F \in \mathcal{C}^\infty(\mathbb{R}_+^*)$, $\quad F(k) \sim \dfrac{1}{k^\beta}$ and $F'(k) \sim -\dfrac{\beta}{k^{\beta+1}}$ as $k \to 0$

with $\beta > 0$ and h is defined on a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary condition $h \equiv 1$ on $\partial\Omega$. The case $\Omega = \mathbb{R}^N$ can also be considered with hypothesis (H1) and (H2) where

$$\text{(H2)} \left\{ \begin{array}{rcll} |F(k)| + |F'(k)| & \leq & Ce^{-k} & as \quad k \to +\infty \\ h(x,t) & \sim & a_1|x| & as \quad |x| \to +\infty \end{array} \right.$$

Few results are known on equation (II). For $\beta > 0$, some criteria of quenching are known for solutions defined on $(-1, 1)$ with Dirichlet boundary conditions (or mixed boundary conditions) in dimension one (see Deng and Levine [6], Guo [12], Levine [18]). Even in that case, few informations are known on the solution at quenching except on the quenching rate (See also Keller and Lowengrub [17] for formal asymptotic behavior). In particular, there is no answer to questions 1 and 2 for problem (I).

To answer questions 1 and 2, we will not use the classical approach which consists in finding a general quenching criterion for initial data and in studying the quenching behavior of the solution. As in [22] and [25], the techniques we use here are the reverse: we study the quenching behavior of a solution a priori, and using this information, we prove by a priori estimates the existence of a solution which has all the properties we expect. Using this type of approach, we prove then that this behavior is stable. Let us first introduce:

$$\hat{\Phi}(z) = (\beta + 1 + \frac{(\beta+1)^2}{4\beta}|z|^2)^{1/(\beta+1)}, \tag{4}$$

and $H^*_{x_0}(x)$ defined by:
i) In the case $\Omega = \mathbb{R}^N$: $H^*_{x_0}(x) = H^*(x - x_0)$ where H^* is defined by:

$$\begin{array}{lll} H^*(x) = & \left[\frac{(\beta+1)^2|x|^2}{-8\beta\log|x|}\right]^{\frac{1}{\beta+1}} & \text{for } |x| \leq C(a_1, \beta) \\ H^*(x) = & a_1|x| & \text{for } |x| \geq 1 \\ H^*(x) > & 0, |\nabla H^*(x)| > 0 & \text{for } x \neq 0 \text{ and } H^* \in \mathcal{C}^\infty(\mathbb{R}^N). \end{array} \tag{5}$$

ii) In the case where Ω is bounded:

$$\begin{array}{lll} H^*_{x_0}(x) = & \left[\frac{(\beta+1)^2|x-x_0|^2}{-8\beta\log|x-x_0|}\right]^{\frac{1}{\beta+1}} & \text{for } |x-x_0| \leq \min\Big(C(\beta), \frac{1}{4}d(x_0,\partial\Omega)\Big) \\ H^*_{x_0}(x) = & 1 & \text{for } |x-x_0| \geq \frac{1}{2}d(x_0,\partial\Omega) \\ H^*_{x_0}(x) > & 0, |\nabla H^*(x)| > 0 & \text{for } x \neq x_0 \text{ and } H^*_{x_0} \in \mathcal{C}^\infty(\Omega\backslash\{x_0\}). \end{array}$$

We also introduce H, the set to initial data:

$$H = \{k \in \psi + H^1 \cap W^{2,\infty}(\mathbb{R}^N) \mid 1/k \in L^\infty(\mathbb{R}^N)\} \text{ if } \Omega = \mathbb{R}^N \qquad (6)$$

where $\psi \in C^\infty(\mathbb{R}^N)$, $\psi \equiv 0$ for $|x| \leq 1$, $\psi(x) = a_1|x|$ for $|x| \geq 2$ and a_1 is defined in (H2),

$$H = \{k \in H^1 \cap W^{2,\infty}(\Omega) \mid 1/k \in L^\infty(\Omega)\} \text{ if } \Omega \text{ is bounded.} \qquad (7)$$

We claim the following:

Theorem (Existence and stability of a vortex reconnection with the boundary or quenching for equation (II) with $\beta > 0$)
Assume that $\Omega = \mathbb{R}^N$ and F is satisfying (H1) and (H2), or Ω is bounded and F is satisfying (H1).
1) **(Existence)** *For all $x_0 \in \Omega$, there exists a positive $h_0 \in H$ such that for a $T_0 > 0$, equation (II) with initial data h_0 has a unique solution $h(x,t)$ on $[0,T_0)$ satisfying $\lim_{t\to T_0} h(x_0,t) = 0$.*

Furthermore,
i)

$$\lim_{t\to T_0} \|\frac{(T_0-t)^{1/\beta+1}}{h(x_0+z\sqrt{-(T_0-t)\log(T_0-t)},t)} - \frac{1}{\hat{\Phi}(z)}\|_{L^\infty} = 0,$$

ii) $h^*(x) = \lim_{t\to T_0} h(x,t)$ exists for all $x \in \Omega$ and $h^*(x) \sim H^*_{x_0}(x)$ as $x \to x_0$.

2) **(Stability)** *For every $\epsilon > 0$, there exists a neighborhood $\mathcal{V}_0$ of h_0 in H with the following property:*
for each $\tilde{h}_0 \in \mathcal{V}_0$, there exist $\tilde{T}_0 > 0$ and $\tilde{x}_0$ satisfying

$$|T_0 - \tilde{T}_0| + |x_0 - \tilde{x}_0| \leq \epsilon$$

such that equation (II) with initial data $\tilde{h}_0$ has a unique solution $\tilde{h}(x,t)$ on $[0,\tilde{T}_0)$ satisfying $\lim_{t\to\tilde{T}_0} \tilde{h}(t,\tilde{x}_0) = 0$. In addition,

$$\text{-} \lim_{t\to\tilde{T}_0} \|\frac{(\tilde{T}_0-t)^{1/\beta+1}}{\tilde{h}(\tilde{x}_0+z\sqrt{-(\tilde{T}_0-t)\log(\tilde{T}_0-t)},t)} - \frac{1}{\hat{\Phi}(z)}\|_{L^\infty} = 0,$$

- $\tilde{h}^*(x) = \lim_{t \to \tilde{T}_0} \tilde{h}(x,t)$ *exists for all* $x \in \Omega$ *and* $\tilde{h}^*(x) \sim H^*_{\tilde{x}_0}(x)$ *as* $x \to \tilde{x}_0$.

Remark: In the case $\beta = 1$ (equation (I)), this Theorem implies that the vortex connects with the boundary in finite time. Let us note that the profile we obtain is $\mathcal{C}^1$ (which is not true for $\beta > 1$). Using the precise estimate of the behavior of h at extinction, it will be interesting to check the validity of the planar approximation in the physical problem near the reconnection time for a behavior like the one described in the theorem.

Remark: We can also consider a larger class of equations:

$$\frac{\partial h}{\partial t} = \nabla.(A(x)\nabla h(x)) - b(x)F(h)$$

where F satisfies (H1) and (H2) with $\beta > 0$, $A(x)$ is a uniformly elliptic $N \times N$ matrix with bounded coefficients, $b(x)$ is bounded, and $b(x_0) > 0$.

Using the stability result and techniques similar to [21], we can construct for arbitrary given k points in Ω a quenching solution h of equation (II) which quenches at time T exactly at the given points. The local quenching behavior of h near each of these points is the same as the one given in the Theorem.

Remark: We have two types of informations on the singularity:

- Part i): it describes the singularity in some refined scale variable at x_0 where we can observe the quenching dynamics. We point out that the estimate we obtain is global (convergence takes place in L^∞).
- Part ii): it describes the singularity in the original variables and shows its influence on the regular part of the solution.

We see in the estimates that these two descriptions are related.

In order to see why such a profile is selected, see [22] and [25] for similar discussions.

Remark: Part *ii*) is valid only for some extinction solutions. We suspect this kind of extinction behavior to be generic (see [15] for a related problem). Indeed, we suspect ourselves to be able to show existence of extinction solutions of (I)-(2) such that:

$$h(x,t) \to h_k^*(x)$$

where $h_k^*(x) \sim C|x|^k$, $k \in \mathbb{N}$ and $k \geq 2$. Unfortunately, this kind of behavior is suspected to be unstable.

1.2 Mathematical setting and strategy of the proof

The case $\Omega = \mathbb{R}^N$ is different from the case Ω is a bounded domain in the way how to treat the Cauchy problem outside the singularity.

Let us consider the problem of the existence of a solution such that $i)$ and $ii)$ of the Theorem hold. We first note that once the existence result is proved, the stability result can be proved in the same way as in [22]. In order to prove the Theorem, we use the following transformation:

$$u(x,t) = \frac{\alpha^{\frac{\alpha}{\beta+1}}}{h(x,t)^\alpha} \tag{8}$$

where h is the extinction solution of (II) to be constructed, and $\alpha > 0$. On its existence interval $[0,T)$, $u(t)$ satisfies

$$\frac{\partial u}{\partial t} = \Delta u - a\frac{|\nabla u|^2}{u} + f(u) \tag{III}$$

where $a = a(\alpha,\beta) = 1 + \frac{1}{\alpha}$,

$$f(u) = \alpha^{\frac{\beta}{\beta+1}} u^{1+\frac{1}{\alpha}} F(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) = u^p + f_1(u) \text{ with } p = p(\alpha,\beta) = \frac{1+\alpha+\beta}{\alpha}, \tag{9}$$

$$\text{(H3)} \begin{cases} f_1 \in \mathcal{C}^\infty(\mathbb{R}_+),\ f_1(v) = o(v^p) \text{ and } f_1'(v) = o(v^{p-1}) \text{ as } v \to +\infty \\ 1 < a < p, \end{cases}$$

and in the case $\Omega = \mathbb{R}^N$,

$$\text{(H4)} \qquad \begin{cases} |f(v)| + |f'(v)| \le C v^{1+\frac{1}{\alpha}} \exp(-\alpha^{\frac{1}{\beta+1}} v^{-\frac{1}{\alpha}}) \text{ as } v \to 0, \\ u(x,t) \sim \frac{1}{a_1|x|} \text{ as } |x| \to +\infty \end{cases}$$

Now, with the transformation $(\alpha,\beta) \to (a(\alpha,\beta), p(\alpha,\beta))$, the problem of finding a solution h of (II) such that $\lim\limits_{t\to T} \inf\limits_{x\in\mathbb{R}} h(x,t) = 0$ is equivalent to the problem of finding a solution u of (III) such that

$$\lim_{t\to T} \|u(t)\|_{L^\infty} = +\infty,$$

(that is a solution of (III) which blows-up in finite time).

Problem (III) can be viewed as a gradient perturbation of the nonlinear heat equation ($a = 0$)

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1} u \tag{IV}$$

where $u(x,t)$ is defined for $x \in \mathbb{R}^N$, $t \ge 0$, $p > 1$ and $p < (N+2)/(N-2)$ if $N \ge 3$.

For this equation, Ball [1], Kavian [16] and Levine [20] obtained obstructions to global existence in time, using monotony properties and the maximum principle. Another method has been followed by Merle and Zaag in [22] (see also Giga and Kohn [10], [9] and [8], Bricmont and Kupiainen [4], Zaag [25]). Once an asymptotic profile (that is a function from which, after a time dependent scaling, $u(t)$ approaches as $t \to T$) is derived formally, the existence of a solution $u(t)$ which blows-up in finite time with the suggested profile is then proved rigorously, using analysis of equation (IV) near the given profile and reduction of the problem to a finite dimensional one.

In the case $a = 0$, the existence and stability of a blow-up solution $u(t)$ of (IV) such that at the blow-up point x_0:

$$\lim_{t\to T} \|(T-t)^{\frac{1}{p-1}} u(x_0 + \sqrt{(T-t)\log(T-t)}z, t) - \Phi_0(z)\|_{L^\infty} = 0$$

where

$$\Phi_0(z) = (p - 1 + \frac{(p-1)^2}{4p} z^2)^{-1/(p-1)}$$

is proved in [22]. Bricmont and Kupiainen obtained the existence result using renormalization group theory (see [4]).

In these new variables, and with the introduction of

$$\Phi(z) = (p - 1 + \frac{(p-1)^2}{4(p-a)}|z|^2)^{-\frac{1}{p-1}}. \tag{10}$$

and
$$U^*_{x_0}(x) = \alpha^{\frac{\alpha}{\beta+1}} H^*_{x_0}(x)^{-\alpha}, \tag{11}$$

$= \left[\frac{8(p-a)|\log|x||}{(p-1)^2|x|^2}\right]^{\frac{1}{p-1}}$ if $\Omega = \mathbb{R}^N$, $x_0 = 0$ and $|x| \le C(a_1, \beta)$, the Theorem is equivalent to the following Proposition:

Proposition 1 (Existence of blow-up solutions for equation (III)) *Assume that $\Omega = \mathbb{R}^N$ and f is satisfying (H3) and (H4), or Ω is bounded and f is satisfying (H3).*
For each $a \in (1, p)$, for each $x_0 \in \Omega$, there exist regular initial data u_0 such that equation (III) has a unique solution $u(x,t)$ which blows-up at a time $T_0 > 0$ only at the point x_0.

Moreover,
i) $\lim_{t\to T_0} u(x,t) = u^*(x)$ exists for all $x \in \Omega\backslash\{x_0\}$ and $u^*(x) \sim U^*_{x_0}(x)$ as $x \to x_0$.
ii)

$$\lim_{t\to T_0} \left\|(T_0 - t)^{\frac{1}{p-1}} u(x_0 + ((T_0-t)|\log(T_0-t)|)^{\frac{1}{2}} z, t) - \Phi(z)\right\|_{L^\infty} = 0.$$

Remark: This proposition provides us with a blow-up solution of (III) in the case $a \in (1, p)$. Let us remark that we already know that blow-up occurs in the case $a \leq 1$:
- If $a < 1$ and $v = (1-a)^{\frac{1-a}{p-1}} u^{1-a}$, then v satisfies:

$$\frac{\partial v}{\partial t} = \Delta v + v^{p'} \quad \text{with} \quad p' = \frac{p-a}{1-a} > 1. \tag{12}$$

- If $a = 1$ and $v = (p-1)\log u$, then v satisfies

$$\frac{\partial v}{\partial t} = \Delta v + e^{v}. \tag{13}$$

It is well-known that equations (12) and (13) (and then (III)) have blow-up solutions.

We introduce *similarity variables* (see [10], [8] and [9])):

$$y = \frac{x - x_0}{\sqrt{T-t}}, s = -\log(T-t), w_{T,x_0}(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \tag{14}$$

where x_0 is the blow-up point and T the blow-up time of $u(t)$, a blow-up solution of (III) to be constructed (we will focus on the study of solutions that blow-up at one single point). We now assume $x_0 = 0$.
The study of the profile of u as $t \to T$ is then equivalent to the study of the asymptotic behavior of w_{T,x_0} (noted w) as $s \to \infty$, and each result for u has an equivalent formulation in terms of w. From equation (III), the equation satisfied by w is the following: $\forall y \in \mathbb{R}^N$, $\forall s \geq -\log T$:

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} - a\frac{|\nabla w|^2}{w} + w^p + e^{-\frac{ps}{p-1}} f_1(e^{\frac{s}{p-1}} w) \tag{15}$$

where $f_1(v) = f(v) - v^p$ and f satisfies (H3) and (H4).

The problem is then to find w a solution of (15) such that

$$\|w(y,s) - \Phi(\frac{y}{\sqrt{s}})\|_{L^\infty} \to 0 \text{ as } s \to +\infty.$$

We introduce

$$\varphi(y,s) = \Phi(\frac{y}{\sqrt{s}}) + \frac{(p-1)^{-\frac{1}{p-1}}}{2(p-a)s} \text{ and } q(y,s) = w(y,s) - \varphi(y,s) \tag{16}$$

where Φ is introduced in (10) (the introduction of the term $\frac{(p-1)^{-\frac{1}{p-1}}}{2(p-a)s}$ is not necessary but it simplifies the calculations).

Then q satisfies: $\forall y \in \mathbb{R}^N$, $\forall s \geq -\log T$:

$$\frac{\partial q}{\partial s} = (\mathcal{L} + V(y,s))q + B(q) + T(q) + R(y,s) + e^{-\frac{ps}{p-1}} f_1(e^{\frac{s}{p-1}}(\varphi + q)) \quad (17)$$

with $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$, $V(y,s) = p\varphi(y,s)^{p-1} - \frac{p}{p-1}$,
$B(q) = (\varphi + q)^p - \varphi^p - p\varphi^{p-1}q$,
$T(q) = -a\frac{|\nabla\varphi + \nabla q|^2}{\varphi + q} + a\frac{|\nabla\varphi|^2}{\varphi}$, $R(y,s) = -\frac{\partial\varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p - a\frac{|\nabla\varphi|^2}{\varphi}$.

Therefore, the question is to find w a solution of (15) or q a solution of (17) such that

$$\lim_{s\to\infty} \|q(s)\|_{L^\infty} = 0. \quad (18)$$

The equation satisfied by q is almost the same as in [22], except the term $T(q)$. As in [22], we introduce estimates on q in the blow-up region $|z| \leq K_0$ or $|y| \leq K_0\sqrt{s}$, and in the regular region $|z| \geq K_0$ or $|y| \geq K_0\sqrt{s}$ where $z = \frac{y}{\sqrt{s}}$ is the self-similar variable for q. The estimates of $T(q)$ in the region $|y| \leq K_0\sqrt{s}$ follow from regularizing effect of the heat flow. One can remark that the Cauchy problem for an equation of the type $\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 + u^p$ is suspected not to be solved in H^1 or $W^{1,p+1}$.

In the analysis of [22], the estimates in the region $|y| \geq K_0\sqrt{s}$ imply smallness of q only, and do not allow any control of $T(q)$ in this region. In other words, the analysis based on the method of [22], that is to estimate the solution in the z variable is not sufficient and must be improved. For this, we add estimates in three regions in a different variable scale (centered in the original x variable not necessarily at the considered blow-up point) using techniques similar to those used in [25] to derive the exact profile in x variable: $u(x,t) \to u^*(x)$ as $t \to T$ where $u^*(x) \sim U^*(x)$ as $x \to 0$ (see (11) for U^*). This part makes the originality of the paper. We expect that such techniques can be useful in various supercritical problems.
We first define for $K_0 > 0$, $\epsilon_0 > 0$ and $t \in [0,T)$ given, three regions covering $\mathbb{R}^N$:

$$\begin{aligned}
P_1(t) &= \{x \mid |x| \leq K_0\sqrt{-(T-t)\log(T-t)}\} \\
&= \{x \mid |y| \leq K_0\sqrt{s}\} = \{x \mid |z| \leq K_0\}, \\
P_2(t) &= \{x \mid \frac{K_0}{4}\sqrt{-(T-t)\log(T-t)} \leq |x| \leq \epsilon_0\} \\
&= \{x \mid \frac{K_0}{4}\sqrt{s} \leq |y| \leq \epsilon_0 e^{\frac{s}{2}}\} = \{x \mid \frac{K_0}{4} \leq |z| \leq \frac{e^{\frac{s}{2}}}{\sqrt{s}}\}, \\
P_3(t) &= \{x \mid |x| \geq \epsilon_0/4\} = \{x \mid |y| \geq \frac{\epsilon_0}{4}e^{\frac{s}{2}}\} = \{x \mid |z| \geq \frac{e^{\frac{s}{2}}}{\sqrt{s}}\},
\end{aligned}$$

$$\text{for } i=1,2,3,\ \ P_i = \{(x,t)\in\mathbb{R}^N\times[0,T) | x\in P_i(t)\},$$

where $s=-\log(T-t)$, $y=\frac{x}{\sqrt{T-t}}$, $z=\frac{y}{\sqrt{s}}=\frac{x}{\sqrt{(T-t)|\log(T-t)|}}$.

In P_1, the "extinction region" of h (which is also the blow-up region of u), we make the change of variables (14) and (16) to do an asymptotic analysis around the profile $\Phi(y/\sqrt{s})$.

Outside the singularity in region P_2, we control h using classical parabolic estimates on k, a rescaled function of h defined for $x\neq 0$ by

$$k(x,\xi,\tau) = (T-t(x))^{-\frac{1}{\beta+1}} h(x+\sqrt{T-t(x)}\xi, (T-t(x))\tau + t(x))$$

where $\frac{K_0}{4}\sqrt{(T-t(x))|\log(T-t(x))|} = |x|$. From equation (II), we see that k satisfies almost the same equation as h: $\forall\xi\in\mathbb{R}^N$, $\forall\tau\in[-\frac{t(x)}{T-t(x)},1)$:

$$\frac{\partial k}{\partial\tau} = \Delta_\xi k - (T-t(x))^{\frac{\beta}{\beta+1}} F((T-t(x))^{\frac{1}{\beta+1}}k)$$

where $(T-t(x))^{\frac{\beta}{\beta+1}}F((T-t(x))^{\frac{1}{\beta+1}}k)\sim\frac{1}{k^\beta}$ as $(T-t(x))^{\frac{1}{\beta+1}}k\to 0$.

We will in fact prove that h behaves for $|\xi|\leq\alpha_0\sqrt{|\log(T-t(x))|}$ and $\tau\in[\frac{t_0-t(x)}{T-t(x)},1)$ for some $t_0<T$, like the solution of

$$\frac{\partial\hat{k}}{\partial\tau} = -\frac{1}{\hat{k}^\beta}.$$

In P_3, the regular region, we estimate directly h. This will give the desired estimate.

The proof of the existence result of the Theorem will be presented in section 2. Assuming some a priori estimates in P_1, P_2 and P_3, we show in section 2 that $h(t)$ can be controlled near the profile by a finite dimensional variable. Adjusting the finite dimensional parameters, we then conclude the proof. We present a priori estimates in P_1 in section 3, and in P_2 and P_3 in section 4.

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2 Existence of a blow-up solution for equation (16)

In this section, we give the proof of the existence result of the Theorem. The proof will be given in the case $\Omega = \mathbb{R}^N$ (we will mention the differences with the case Ω is bounded, when it is necessary, see section 4). We assume $N = 1$ in order to simplify the notations. The same calculations and proof hold in a higher dimension (see [22] and [25]). We assume $x_0 = 0$ since (II) is translation invariant. For simplicity in notations, we simplify hypothesis (H1) and assume that

$$\forall v \in (0,1],\ F(v) = \frac{1}{v^\beta}. \tag{19}$$

Same calculations holds without this simplification.

Let us first remark on the following about the Cauchy problem for equation (II).

Lemma 2.1 (Local Cauchy Problem for equation (II)) *The local in time Cauchy problem for equation (II) is well-posed in H where H is defined by (7) if Ω is bounded, and by (6) if $\Omega = \mathbb{R}$.*

Moreover, in both cases, either the solution h exists for all time $t > 0$ or only on $[0,T)$ with $T < +\infty$, and in this case $\lim_{t\to T} \inf_{x\in\Omega} h(x,t) = 0$.

Proof: The case Ω is bounded follows from classical arguments.

For the case $\Omega = \mathbb{R}$, we define $\tilde{h}(x,t)$ by $h(x,t) = \psi(x) + \tilde{h}(x,t)$. This way, (II) is equivalent to

$$\tilde{h}_t = \tilde{h}_{xx} - F(\psi(x) + \tilde{h}) + \psi_{xx}. \tag{20}$$

Using (H1) and (H2), we see by classical arguments that this equation can be solved in H. ■

Let us consider $\beta > 0$ and $T > 0$, all fixed. The problem is to find $t_0 < T$ and h_0 such that the solution of equation (II) with data at t_0 $h(x,t_0) = h_0$ extinguishes in finite time $T > 0$ at only one extinction point $x = 0$ and:

$$\text{-}\ \lim_{t\to T} \| \frac{(T-t)^{1/\beta+1}}{h(z\sqrt{-(T-t)\log(T-t)},t)} - \frac{1}{\hat{\Phi}(z)} \|_{L^\infty(\mathbb{R})} = 0 \tag{21}$$

- $h^*(x) = \lim_{t\to T} h(x,t)$ exists for all $x \in \mathbb{R}$ and

$$h^*(x) > 0 \text{ for } x \neq 0, h^*(x) \sim H^*(x) \text{ as } x \to 0 \tag{22}$$

where $\hat{\Phi}$ and H^* are introduced in (4) and (5).

As explained in the introduction, (21) and (22) follow from the control of $h(x,t)$ for $t \in [t_0, T)$ in three different scales, depending on the three regions P_1, P_2, and P_3.

a) In P_1, the extinction region, we rescale h by means of (8), (14) and (16) in order to define for $t \in [t_0, T)$, $q(s)$ where $s = -\log(T-t)$ and

$$\begin{cases} \forall y \in \mathbb{R}, & q(y,s) = (T-t)^{\frac{1}{p-1}} u(y\sqrt{T-t}, t) - \varphi(y,s), \\ \forall x \in \mathbb{R}, & u(x,t) = \alpha^{\frac{\alpha}{\beta+1}} h(x,t)^{-\alpha} \text{ and } \alpha > 0, \\ & \varphi(y,s) = \Phi(\frac{y}{\sqrt{s}}) + \frac{(p-1)^{-\frac{1}{p-1}}}{2(p-a)s}, \\ p = \frac{\alpha+\beta+1}{\alpha}, \quad a = \frac{\alpha+1}{\alpha}, & \text{and } \Phi \text{ is given in (10).} \end{cases} \tag{23}$$

Remark: To prove the Theorem, we can take $\alpha = 1$. Nevertheless, we need to keep $\alpha > 0$ general, if we want to deduce directly Proposition 1 from the Theorem.

The equation satisfied by q is (17): $\forall y \in \mathbb{R}$, $\forall s \geq -\log(T-t_0)$:

$$\frac{\partial q}{\partial s} = (\mathcal{L} + V(y,s))q + B(q) + T(q) + R(y,s) + e^{-\frac{ps}{p-1}} f_1(e^{\frac{s}{p-1}}(\varphi + q)) \tag{24}$$

with $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$, $V(y,s) = p\varphi(y,s)^{p-1} - \frac{p}{p-1}$,
$B(q) = (\varphi+q)^p - \varphi^p - p\varphi^{p-1}q$,
$T(q) = -a\frac{|\nabla\varphi+\nabla q|^2}{\varphi+q} + a\frac{|\nabla\varphi|^2}{\varphi}$, $R(y,s) = -\frac{\partial\varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p - a\frac{|\nabla\varphi|^2}{\varphi}$,
$f_1(u) = \alpha^{\frac{\beta}{\beta+1}} u^{1+\frac{1}{\alpha}} F(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) - u^p$.

We note that $\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2(\mathbb{R}, d\mu)$ with

$$d\mu(y) = \frac{e^{-\frac{|y|^2}{4}}}{\sqrt{4\pi}} \tag{25}$$

and that its eigenvalues are $\{1 - \frac{m}{2} | m \in \mathbb{N}\}$.

In one dimension, $h_m(y) = \displaystyle\sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}$ is the eigenfunction corresponding to $1 - \frac{m}{2}$. We introduce also $k_m = h_m / \|h_m\|^2_{L^2(\mathbb{R},d\mu)}$ and note that Vect $\{h_m \mid m \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R}, d\mu)$.

We are interested in obtaining $L^\infty(\mathbb{R})$ estimates for q. Since $L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}, d\mu)$, we will expand q (actually, a cut-off of q) with respect to the eigenvalues of $\mathcal{L}$. Nevertheless, the estimates we will obtain will be L^∞ and not $L^2(\mathbb{R}, d\mu)$.

The control of $h(t)$ for $t \in [t_0, T)$ in this region P_1 is equivalent to the control of $q(s)$ for $s \in [-\log(T-t_0), +\infty)$ in a set $V_{K_0,A}(s)$ so that $\lim_{s\to\infty} \|q(s)\|_{L^\infty} = 0$. The definition of $V_{K_0,A}(s)$ requires the introduction of a cut-off function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K_0\sqrt{s}}\right) \tag{26}$$

where

$$\chi_0 \in \mathcal{C}^\infty(\mathbb{R}^+, [0,1]),\ \ \chi_0 \equiv 1 \text{ on } [0,1],\ \ \chi_0 \equiv 0 \text{ on } [2,+\infty). \tag{27}$$

b) In P_2, we control a rescaled function of h defined for $x \neq 0$ by $\forall \xi \in \mathbb{R}$, $\forall \tau \in [\frac{t_0 - t(x)}{T - t(x)}, 1)$:

$$k(x,\xi,\tau) = (T-t(x))^{-\frac{1}{\beta+1}} h(x + \sqrt{T-t(x)}\xi, (T-t(x))\tau + t(x)), \tag{28}$$

where $t(x)$ is defined by

$$|x| = \frac{K_0}{4}\sqrt{(T-t(x))|\log(T-t(x))|} = \frac{K_0}{4}\sqrt{\theta(x)|\log\theta(x)|} \tag{29}$$

$$\text{with} \qquad \theta(x) = T - t(x).$$

Let us note that $\theta(x)$ is related to the asymptotic profile $H^*(x)$.

Lemma 2.2 *For fixed K_0, we have:*
i) $H^*(x) \sim \hat{k}(1)\theta(x)^{\frac{1}{\beta+1}}$ as $x \to 0$,
ii) $|\nabla H^*(x)| \sim \frac{8}{(\beta+1)K_0} \frac{\hat{k}(1)}{\sqrt{|\log\theta(x)|}} \theta(x)^{\frac{1}{\beta+1}-\frac{1}{2}}$ as $x \to 0$ where

$$\hat{k}(\tau) = \left((\beta+1)(1-\tau) + \frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16}\right)^{\frac{1}{\beta+1}}. \tag{30}$$

Proof: From (29), we write:
$\log|x| = \log\frac{K_0}{4} + \frac{1}{2}\log\theta(x) + \frac{1}{2}\log|\log\theta(x)|$ and
$\frac{|x|^2}{-\log|x|} = \frac{2K_0^2}{16}\theta(x)\frac{\log\theta(x)}{\log\theta(x) + \log|\log\theta(x)| + 2\log\frac{K_0}{4}}$. Therefore,

$$\log\theta(x) \sim 2\log|x| \text{ and } \theta(x) \sim \frac{8}{K_0^2}\frac{|x|^2}{|\log|x||} \text{ as } x \to 0. \tag{31}$$

Since $H^*(x) = \hat{k}(1)\left[\frac{8|x|^2}{K_0^2|\log|x||}\right]^{\frac{1}{\beta+1}}$ and
$|\nabla H^*(x)| \sim \frac{4\sqrt{2}}{(\beta+1)K_0}\frac{\hat{k}(1)}{\sqrt{|\log|x||}}\left[\frac{8|x|^2}{K_0^2|\log|x||}\right]^{\frac{1}{\beta+1}-\frac{1}{2}}$ when x is small (see (5)), we get the conclusion. ■

k satisfies almost the same equation as h: $\forall \tau \in [\frac{t_0-t(x)}{\theta(x)}, 1)$, $\forall \xi \in \mathbb{R}$,

$$\frac{\partial k}{\partial \tau} = \Delta_\xi k - \theta(x)^{\frac{\beta}{\beta+1}} F(\theta(x)^{\frac{1}{\beta+1}} k). \tag{32}$$

We will see that the estimates on k allow us to write $\theta(x)^{\frac{\beta}{\beta+1}} F(\theta(x)^{\frac{1}{\beta+1}} k) = \frac{1}{k^\beta}$ for suitable ξ. If we show that $k(\tau)$ behaves like $\hat{k}$ (see (30)) which is a solution of the ODE

$$\frac{d\hat{k}}{d\tau} = -\frac{1}{\hat{k}^\beta}$$

defined for $\tau \in [0, \hat{T})$ with $\hat{T} = 1 + \frac{(\beta+1)K_0^2}{64\beta} > 1$, and that $|\nabla_\xi k(\tau)| \leq \frac{C(K_0,A)}{\sqrt{|\log\theta(x)|}}$, then according to lemma 2.2, this yields that $h(x,t)$ behaves in P_2 like $H^*(x)$ and $|\nabla h(x,t)| \leq C(K_0, A)|\nabla H^*(x)|$ if x and $T - t_0$ are small, which is almost the estimate *ii)* of the Theorem.

c) In P_3, we estimate directly h using the local in time well posedness of the Cauchy problem for equation (III).

More formally, we define for each $t \in [t_0, T)$ a set $S^*(t)$ depending on some parameters so that $h(t) \in S^*(t)$ means that h is controlled in the three regions as described before. We show then that if $\forall t \in [t_0, T)$, $h(t) \in S^*(t)$, then (21) and (22) hold and the Theorem follows.

Let us define $S^*(t)$:

Definition of $S^*(t)$ and S^*

I) For all $t_0 < T$, $K_0 > 0$, $\epsilon_0 > 0$, $\alpha_0 > 0$, $A > 0$, $\delta_0 > 0$, $C_0' > 0$, $C_0 > 0$ and $\eta_0 > 0$, for all $t \in [t_0, T)$, we define $S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ as being the set of all functions $h \in H$ satisfying:

i) **Estimates in P_1:** *$q(s) \in V_{K_0,A}(s)$ where $s = -\log(T-t)$, $q(s)$ is defined in (23) and $V_{K_0,A}(s)$ is the set of all functions r in $W^{1,\infty}(\mathbb{R})$ such that*

$$\left\{\begin{array}{rclrcl} |r_m(s)| & \leq & As^{-2}\ (m=0,1), & |r_2(s)| & \leq & A^2 s^{-2}\log s, \\ |r_-(y,s)| & \leq & As^{-2}(1+|y|^3), & |r_e(y,s)| & \leq & A^2 s^{-1/2} \\ |(\frac{\partial r}{\partial s})_\perp(y,s)| & \leq & As^{-2}(1+|y|^3), & & & \end{array}\right. \tag{33}$$

where

$$\left\{\begin{array}{rclrcl} r_e(y,s) & = & (1-\chi(y,s))r(y), & r_-(s) & = & P_-(\chi(s)r), \\ \text{for } m\in\mathbb{N},\ r_m(s) & = & \int d\mu k_m(y)\chi(y,s)r(y), & r_\perp(s) & = & P_\perp(\chi(s)r), \end{array}\right. \tag{34}$$

χ is defined in (26), P_- and $P_\perp$ are the $L^2(\mathbb{R},d\mu)$ projectors respectively on Vect $\{h_m|m\geq 3\}$ and Vect $\{h_m|m\geq 2\}$, $d\mu$, h_m and k_m are introduced in (25).

ii) **Estimates in** P_2*: For all* $|x|\in[\frac{K_0}{4}\sqrt{(T-t)|\log(T-t)|},\epsilon_0]$, $\tau=\tau(x,t)=\frac{t-t(x)}{\theta(x)}$, *and* $|\xi|\leq\alpha_0\sqrt{|\log\theta(x)|}$,
$|k(x,\xi,\tau)-\hat{k}(\tau)|\leq\delta_0$, $|\nabla_\xi k(x,\xi,\tau)|\leq\frac{C_0'}{\sqrt{|\log\theta(x)|}}$, *and* $|\nabla_\xi^2 k(x,\xi,\tau)|\leq C_0$ *where* k, $\hat{k}$, $t(x)$ *and* $\theta(x)$ *are defined in (28), (30) and (29).*

iii) **Estimates in** P_3*: For all* $|x|\geq\frac{\epsilon_0}{4}$, $|h(x,t)-h(x,t_0)|\leq\eta_0$ *and* $|\nabla h(x,t)-\nabla h(x,t_0)|\leq\eta_0$.

II) For all $t_0<T$ *we define* $S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0)=$
$\{k\in C([t_0,T),H)\mid\forall t\in[t_0,T),k(t)\in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)\}$.

Remark: Note that according to (25) and (34), we have for all $r\in L^\infty(\mathbb{R})$,

$$r(y) = \sum_{m=0}^{2} r_m(s)h_m(y)+r_-(y,s)+r_e(y,s), \tag{35}$$

$$r(y) = \sum_{m=0}^{1} r_m(s)h_m(y)+r_\perp(y,s)+r_e(y,s). \tag{36}$$

Therefore, $i)$ yields an estimate on $\|q(s)\|_{L^\infty}$ and $\|\left(\frac{\partial q}{\partial y}\right)_\perp(s)\|_{L^\infty}$.
Remark: The estimates on h are in $W^{1,\infty}(\mathbb{R})$. In particular, they are global. The estimates on $\frac{\partial q}{\partial y}$ in P_1, $\nabla_\xi k$ in P_2 and on ∇h in P_3 allow us to control the term $T(q)$ appearing in the equation satisfied by q (see (24)). We remark that the estimate $q(s)\in V_{K_0,A}(s)$ describes h mainly in P_1. The estimate on q_e involved in definition (33) is useful only in the frontier between P_1 and P_2.

Now we show that if we find suitable parameters and initial data such that $h\in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0)$, then the Theorem holds.

Proposition 2.1 (Reduction of the proof) *For given* $t_0 < T$, K_0, ϵ_0, α_0, A, δ_0, C_0', C_0 *and* η_0 *such that* $\delta_0 \le \frac{1}{2}\hat{k}(1)$ *and* $\eta_0 \le \frac{1}{2} \inf_{|x|\ge\epsilon_0/4} h(x,t_0)$, *assume that* $h \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$. *Then* $h(t)$ *extinguishes in finite time* T *only at the point* $x_0 = 0$, *that is* $\lim_{t\to T} h(0,t) = 0$ *and* $\forall x \neq 0$, *there exists* $\eta(x) > 0$ *such that*

$$\liminf_{t\to T} \min_{|x'-x|\le\eta(x)} h(x',t) > 0. \tag{37}$$

Moreover, with $\hat{\Phi}$ *and* H^* *defined by (4) and (5),*

$$\lim_{t\to T} \| \frac{(T-t)^{\frac{1}{\beta+1}}}{h(z\sqrt{-(T-t)\log(T-t)},t)} - \frac{1}{\hat{\Phi}(z)} \|_{L^\infty(\mathbb{R})} = 0, \tag{38}$$

$h^*(x) = \lim_{t\to T} h(x,t)$ *exists for all* $x \in \mathbb{R}$ *and*

$$h^*(x) > 0 \text{ for } x \neq 0 \text{ and } h^*(x) \sim H^*(x) \text{ as } x \to 0. \tag{39}$$

Proof: We assume that $h \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$. One can remark that once (38), (37) and (39) are proved, it follows that
i) $\lim_{t\to T} h(0,t) = 0$: $h(t)$ extinguishes at time T at the point $x = 0$,
ii) $x = 0$ is the only extinction point of h.
It remains then to prove (37), (38) and (39).

Proof of (37):
From iii) of Definition of $S^*(t)$, we know that if $|x| \ge \frac{\epsilon_0}{4}$, then $\forall t \in [t_0, T)$, $h(x,t) \ge h(x,t_0) - \eta_0 \ge \inf_{|x|\ge\frac{\epsilon_0}{4}} h(x,t_0) - \eta_0 \ge \frac{1}{2}\inf_{|x|\ge\frac{\epsilon_0}{4}} h(x,t_0)) > 0$. This yields (37) for $|x| \ge \epsilon_0$.

From ii) of Definition of $S^*(t)$, we have $\forall |x| \in (0, \epsilon_0]$, for t close enough to T, $|k(x,0,\tau(x,t)) - \hat{h}(\tau(x,t))| \le \delta_0$ where $\tau(x,t) = \frac{t-t(x)}{\theta(x)}$. Therefore, $k(x,0,\tau(x,t)) \ge \hat{k}(\tau(x,t)) - \delta_0 \ge \hat{k}(1) - \delta_0 \ge \frac{1}{2}\hat{k}(1)$ (from (30) and $\delta_0 \le \frac{1}{2}\hat{k}(1)$). From (28), it follows: $h(x,t) \ge \frac{1}{2}\hat{k}(1)\theta(x)^{\frac{1}{\beta+1}} > 0$. This yields (37) for $0 < |x| < \epsilon_0$.

Proof of (38):
We consider $q(s)$, the function introduced in (23). Let us show that

$$\|q(s)\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } s \to +\infty. \tag{40}$$

From i) of the definition of $S^*(t)$ and (35), we have $\forall s \in [-\log(T-t_0), +\infty)$, $q(s) \in V_{K_0,A}(s)$ and

$|q(y,s)| = |1_{\{|y|\le 2K_0\sqrt{s}\}}\left(\sum_{m=0}^{2} q_m(s)h_m(y) + q_-(y,s)\right) + q_e(y,s)|$
$\le 1_{\{|y|\le 2K_0\sqrt{s}\}}(As^{-2}(1+|y|) + A^2 s^{-2}\log s(|y|^2+2) + As^{-2}(1+|y|^3)) + A^2 s^{-1/2} \le C(K_0,A)s^{-1/2}$ and (40) follows.

Let $z\in\mathbb{R}$ and $g(z) = |(T-t)^{1/\beta+1}/h(z\sqrt{-(T-t)\log(T-t)},t) - \frac{1}{\hat\Phi(z)}|$. We have

$g(z)$
$\le C|(T-t)^{\frac{\alpha}{\beta+1}}\alpha^{\frac{\alpha}{\beta+1}}h(z\sqrt{-(T-t)\log(T-t)},t)^{-\alpha} - \alpha^{\frac{\alpha}{\beta+1}}\hat\Phi(z)^{-\alpha}|^{\frac{1}{\bar\alpha}}$
where $\bar\alpha = \max(\alpha,1)$.

Using (4) and (23), we have $\alpha = 1/(p-a)$ and $\beta = (p-a)/(a-1)$, therefore $\frac{\alpha}{\beta+1} = \frac{1}{p-1}$,

$$\alpha^{\frac{\alpha}{\beta+1}}\hat\Phi(z)^{-\alpha} = (\frac{\beta+1}{\alpha} + \frac{(\beta+1)^2}{4\beta\alpha}|z|^2)^{-\frac{1}{p-1}} = \varphi(z\sqrt{s},s) - \frac{(p-1)^{-1/(p-1)}}{2(p-a)s},$$

and $(T-t)^{\frac{\alpha}{\beta+1}}\alpha^{\frac{\alpha}{\beta+1}}h(z\sqrt{-(T-t)\log(T-t)},t)^{-\alpha}$
$= (T-t)^{\frac{1}{p-1}}u(z\sqrt{-(T-t)\log(T-t)},t)$ with $s = -\log(T-t)$.

Combining this with (23) again, we get
$g(z) \le C(\alpha,\beta)\left(|q(z\sqrt{-\log(T-t)}, -\log(T-t))| + 1/|\log(T-t)|\right)^{\frac{1}{\bar\alpha}}$
$\le C\left(\|q(s)\|_{L^\infty(\mathbb{R})} + 1/|\log(T-t)|\right)^{\frac{1}{\bar\alpha}} \to 0$ as $t\to T$ by (40). This yields (38).

Proof of (39): From the proof of (37) and classical theory (see Merle [21] for a similar problem), there exists a profile function $h^*(x)$ such that $\forall x\ne 0$, $\lim_{t\to T} h(x,t) = h^*(x) > 0$. To show that $h^*(x)\sim H^*(x)$ as $x\to 0$, we give the following localization estimate:

Proposition 2.2 (Localization in P_2) *Assume that k is a solution of equation*

$$k_\tau = \Delta k - \frac{1}{k^\beta} \tag{41}$$

for $\tau\in[0,\tau_0)$ with $\tau_0 \le 1 (< \hat T)$. Assume in addition: $\forall\tau\in[0,\tau_0]$,
i) For $|\xi|\le 2\xi_0$, $|k(\xi,0)-\hat k(0)|\le\delta$ and $|\nabla k(\xi,0)|\le\delta$,
ii) For $|\xi|\le\frac{7\xi_0}{4}$, $k(\xi,\tau)\ge\frac{1}{2}\hat k(\tau)$.
iii) For $|\xi|\le\frac{7\xi_0}{4}$, $|\nabla^2 k(\xi,\tau)|\le C_0$,
where $\hat k$ is introduced in (30). Then there exists $\epsilon=\epsilon(\delta,\xi_0)$ such that $\forall\tau\in[0,\tau_0]$, for $|\xi|\le\xi_0$,
$|k(\xi,\tau)-\hat k(\tau)|\le\epsilon$ and $|\nabla k(\xi,\tau)|\le\epsilon$, where $\epsilon\to 0$ as $\delta\to 0$ and $\xi_0\to+\infty$.

Proof: We prove in section 4 a more accurate version of this Proposition (Proposition 4.1). One can adapt without difficulties the proof to the present context. ■

Let us apply this Proposition to $k(x,\xi,\tau)$ when x is near zero with $\tau_0 = 1$ and $\xi_0 = |\log\theta(x)|^{1/4}$. We first check all the hypothesizes of the Proposition:

Lemma 2.3 *If x is small enough, then $k(x,\xi,\tau)$ satisfies (41) for $|\xi| \leq |\log\theta(x)|^{1/4}$ and $\tau \in [0,1)$. Moreover,*

$$i) \sup_{|\xi|\leq|\log\theta(x)|^{1/4}} |k(x,\xi,0) - \hat{k}(0)| + |\nabla_\xi k(x,\xi,0)| \leq \delta(x) \to 0 \text{ as } x \to 0, \tag{42}$$

ii) for $|\xi| \leq |\log\theta(x)|^{\frac{1}{4}}$, $\forall\tau \in [0,1)$, $k(x,\xi,\tau) \geq \frac{1}{2}\hat{k}(\tau)$,
iii) for $|\xi| \leq |\log\theta(x)|^{\frac{1}{4}}$, $\forall\tau \in [0,1)$, $|\nabla_\xi^2 k(x,\xi,\tau)| \leq C_0$.

Combining this lemma and Proposition 2.2, we get $\forall\tau \in [0,1)$, $|k(x,\xi,\tau) - \hat{k}(\tau)| \leq \epsilon(x) \to 0$ as $x \to 0$. Using (28), (30) and letting $\tau \to 1$, we obtain

$$\theta(x)^{-\frac{1}{\beta+1}} h^*(x) \sim \hat{k}(1) = \left(\frac{(\beta+1)^2 K_0^2}{64\beta}\right)^{\frac{1}{\beta+1}}. \tag{43}$$

By lemma 2.2, we obtain $h^*(x) \sim H^*(x)$ as $x \to 0$, which concludes the proof of Proposition 2.1.

Proof of lemma 2.3:

i) and *iii*): Since (29) implies that $\theta(x) \to 0$ as $x \to 0$, we have by combining (38) and (28):

$$\sup_{|\xi|\leq|\log\theta(x)|^{1/4}} |1/k(x,\xi,0) - 1/\hat{\Phi}(\frac{x+\xi\sqrt{\theta(x)}}{\sqrt{\theta(x)|\log\theta(x)|}})| \to 0 \text{ as } x \to 0.$$ Hence, from (4), the first part of (42) follows.

From *ii*) of the Definition of $S^*(t)$, we have $|\nabla_\xi k(x,\xi,0)| \leq \frac{C_0'}{\sqrt{|\log\theta(x)|}}$ and $|\nabla_\xi^2 k(x,\xi,0)| \leq C_0$ for $|\xi| \leq |\log\theta(x)|^{1/4}$, if x is small. This yields the second part of *i*) and *iii*).

ii): From *ii*) of the Definition of $S^*(t)$, it follows that for x small enough, we have $|k(x,\xi,\tau) - \hat{k}(\tau)| \leq \delta_0$ for $|\xi| \leq |\log\theta(x)|^{1/4}$ and $\tau \in [0,1)$. Hence, *ii*) follows from (30) since $\delta_0 \leq \frac{1}{2}\hat{k}(1)$. By the way, this implies that $|\theta(x)^{\frac{1}{\beta+1}} k(x,\xi,\tau)| \leq 1$ for $|\xi| \leq |\log\theta(x)|^{1/4}$ and $\tau \in [0,1)$. Therefore, it follows from (32) and (19) that k satisfies (41). ■

From this Proposition, the proof of the Theorem reduces to find suitable parameters $t_0 < T$, K_0, ϵ_0, α_0, A, δ_0, C_0', C_0, η_0 and $h_0 \in H$ so that the

solution h of equation (II) with data $h(t_0) = h_0$ belongs to
$S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$.

Unfortunately, the spectrum of $\mathcal{L}$ which greatly determines the dynamic of q (and then the dynamic of h too) contains two expanding eigenvalues: 1 and 1/2. Therefore, we expect that for most choices of initial data h_0, the corresponding $q_0(s)$ and $q_1(s)$ with $s = -\log(T-t)$ will force $h(t)$ to exit $S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$.

As a matter of fact, we will show through a priori estimates that for suitably chosen $t_0 < T$, K_0, ϵ_0, α_0, A, δ_0, C_0', C_0 and η_0, the control of $h(t)$ in $S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ for $t \in [t_0, T)$ reduces to the control of $(q_0(s), q_1(s))$ in

$$\hat{V}_A(s) \equiv [-As^{-2}, As^{-2}]^2 \tag{44}$$

for $s \geq -\log(T-t_0)$ ($q_0(s)$ and $q_1(s)$ correspond to expanding eigenvalues in the q variable). Hence, we will consider initial data h_0 depending on two parameters $(d_0, d_1) \in \mathbb{R}^2$, and then, we will fix (d_0, d_0) using a topological argument so that $(q_0(s), q_1(s)) \in \hat{V}_A(s)$ for all $s \geq -\log(T-t)$, which yields $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$, thanks to the finite dimensional reduction.

Let us define

$$h_0(d_0, d_1, x) = (T-t_0)^{\frac{1}{\beta+1}} \alpha^{\frac{1}{\beta+1}} \left\{ \Phi(z) + (d_0 + d_1 z)\chi_0\left(\frac{|z|}{K_0/16}\right) \right\}^{-\frac{1}{\alpha}} \chi_1(x, t_0) + H^*(x)(1 - \chi_1(x, t_0)) \tag{45}$$

where $z = x/\sqrt{(T-t_0)|\log(T-t_0)|}$,

$$\chi_1(x, t_0) = \chi_0\left(\frac{x}{(T-t_0)^{\frac{1}{2}}|\log(T-t_0)|^{\frac{p}{2}}}\right), \tag{46}$$

Φ, χ_0 and H^* are defined in (10), (27) and (5). The problem now reduces to find (d_0, d_1) in some $\mathcal{D} \subset \mathbb{R}^2$ such that
$h(d_0, d_1) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$.

The proof is divided in two parts:

i) Finite dimensional reduction:

From the technique of a priori estimates, we find suitable parameters $t_0 < T$, K_0, ϵ_0,α_0, A, δ_0, C_0', C_0 and η_0 so that the following property is true: Assume that for $t_* \in [t_0, T)$, we have $\forall t \in [t_0, t_*]$,
$h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ and
$h(t_*) \in \partial S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_*)$, then

$(q_0(s_*), q_1(s_*)) \in \partial\hat{V}_A(s_*)$ where $s_* = -\log(T-t_*)$, q_0 and q_1 follow from q by (34), q and $\hat{V}_A(s)$ are defined in (23) and (44).

ii) Solution of the finite dimensional problem:

We use a topological argument to find a parameter $(d_0, d_1) \in \mathbb{R}^2$ such that $(q_0(s), q_1(s)) \in \hat{V}_A(s)$ for all $s \geq -\log(T-t_0)$, and therefore, $h \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$. This yields the Theorem.

Part I: A priori estimates of $h(t)$, solution of equation (II) and finite dimensional reduction

Step 0: Initialization of the problem

We claim the following lemma:

Lemma 2.4 (Initialization of the problem) *There exists $K_{01} > 0$ such that for each $K_0 \geq K_{01}$ and $\delta_1 > 0$, $\exists \alpha_1(K_0, \delta_1) > 0$ and $C^*(K_0) > 0$ such that $\forall \alpha_0 \leq \alpha_1(K_0, \delta_1)$, $\exists \epsilon_1(K_0, \delta_1, \alpha_0) > 0$, such that $\forall \epsilon_0 \leq \epsilon_1(K_0, \delta_1, \alpha_0)$, $\forall C_1 > 0$, $\forall A \geq 1$, $\exists t_1(K_0, \delta_1, \epsilon_0, A, C_1) < T$ such that $\forall t_0 \in [t_1, T)$, there exists a rectangle $\mathcal{D}(t_0, K_0, A) \subset \mathbb{R}^2$ with the following properties:*
If $h(x, t_0)$ is defined by (45), then:

i) $\forall (d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$, $h(t_0) \in H$ defined in (6), $(q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0)$ defined in (44) and $h(t_0) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_1, C^*(K_0), C_1, 0, t_0)$, with $s_0 = -\log(T-t_0)$. More precisely:

$$\begin{array}{llll} |q_0(s_0)| & \leq A s_0^{-2} & |q_1(s_0)| & \leq A s_0^{-2} \\ |q_2(s_0)| & \leq s_0^{-2}\log s_0 & |q_-(y, s_0)| & \leq C s_0^{-2}(1+|y|^3) \\ |q_e(y, s_0)| & \leq s_0^{-1/2} & \left|\left(\frac{\partial q}{\partial y}\right)_\perp(y, s_0)\right| & \leq s_0^{-2}(1+|y|^3), \\ \left|\frac{\partial q}{\partial y}(y, s_0)\right| & \leq s_0^{-\frac{1}{2}} & \text{for } |y| \geq K_0\sqrt{s_0}, & \end{array}$$

for all $|x| \in [\frac{K_0}{4}\sqrt{(T-t)|\log(T-t)|}, \epsilon_0]$, $\tau_0 = \frac{t_0 - t(x)}{\theta(x)}$, and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$, $|k(x, \xi, \tau_0) - \hat{k}(\tau_0)| \leq \delta_1$, $|\nabla_\xi k(x, \xi, \tau_0)| \leq \frac{C^*(K_0)}{\sqrt{|\log\theta(x)|}}$ and $|\nabla_\xi^2 k(x, \xi, \tau_0)| \leq C_1$ where k, $\hat{k}$, $t(x)$ and $\theta(x)$ are defined in (28), (30) and (29).

ii) $$(d_0, d_1) \in \mathcal{D}(t_0, K_0, A) \Leftrightarrow (q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0),$$

$$(d_0, d_1) \in \partial\mathcal{D}(t_0, K_0, A) \Leftrightarrow (q_0(s_0), q_1(s_0)) \in \partial\hat{V}_A(s_0),$$

$(q_0(s_0), q_1(s_0))$ is an affine function of (d_0, d_1) when $(d_0, d_1) \in \partial\mathcal{D}(t_0, K_0, A)$.

Proof: See Appendix A.

Step 1: A priori estimates

We now claim the following estimates:

Proposition 2.3 (A priori estimates in P_1) *There exists $K_{02}>0$ such that for each $K_0 \geq K_{02}$, there exists $A_2(K_0)>0$ such that for each $A \geq A_2(K_0)$, $\epsilon_0>0$ and $C_0' \leq A^3$, there exist $\eta_2(\epsilon_0)>0$ and $t_2(K_0,\epsilon_0,A,C_0')<T$ such that for each $t_0 \in [t_2(K_0,\epsilon_0,A,C_0'),T)$, $\delta_0 \leq \frac{1}{2}\hat{k}(1)$, $\alpha_0>0$, $C_0>0$ and $\eta_0 \leq \eta_2(\epsilon_0)$, we have the following property:*

- if $h(x,t_0)$ is given by (45) and if (d_0,d_1) is chosen so that $(q_0(s_0),q_1(s_0)) \in \hat{V}_A(s_0)$ defined in (44) with $s_0=-\log(T-t_0)$,

- if for some $t_* \in [t_0,T)$, we have
$\forall t \in [t_0,t_*]$, $h(t) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$ then

$$\begin{array}{rclrcl} |q_2(s_*)| & \leq & A^2 s_*^{-2}\log s_* - s_*^{-3}, & |q_-(y,s_*)| & \leq & \frac{A}{2}s_*^{-2}(1+|y|^3) \\ |q_e(y,s_*)| & \leq & \frac{A^2}{2}s_*^{-1/2}, & |(\frac{\partial q}{\partial y})_\perp(y,s_*)| & \leq & \frac{A}{2}s_*^{-2}(1+|y|^3), \end{array}$$

where $s_*=-\log(T-t_*)$, q is defined in (23) and the notation is given in (34).

Proof: See section 3. ■

Proposition 2.4 (A priori estimates in P_2) *There exists $K_{03}>0$ such that for all $K_0 \geq K_{03}$, $\delta_1 \leq 1$, $\xi_0 \geq 1$, $C_0^*>0$, $C_0'^*>0$ and $C_0''^*>0$ we have the following property:*
Assume that k is a solution of equation

$$\frac{\partial k}{\partial \tau} = \Delta k - \frac{1}{k^\beta} \tag{47}$$

for $\tau \in [\tau_1,\tau_2)$ with $0 \leq \tau_1 \leq \tau_2 \leq 1$ $(<\hat{T})$.
Assume in addition: $\forall \tau \in [\tau_1,\tau_2]$,
i) $\forall \xi \in [-2\xi_0,2\xi_0]$, $|k(\xi,\tau_1)-\hat{k}(\tau_1)| \leq \delta_1$ and $|\nabla k(\xi,\tau_1)| \leq \frac{C_0''^*}{\xi_0}$,
ii) $\forall \xi \in [-\frac{7\xi_0}{4},\frac{7\xi_0}{4}]$, $|\nabla k(\xi,\tau)| \leq \frac{C_0'^*}{\xi_0}$ and $|\nabla^2 k(\xi,\tau)| \leq C_0^*$,
iii) $\forall \xi \in [-\frac{7\xi_0}{4},\frac{7\xi_0}{4}]$, $k(\xi,\tau) \geq \frac{1}{2}\hat{k}(\tau)$,
where $\hat{k}$ is given by (30). Then, for $\xi_0 \geq \xi_{03}(C_0'^*,C_0^*,C_0''^*)$ there exists $\epsilon=\epsilon(K_0,C_0'^*,\delta_1,\xi_0)$ such that $\forall \xi \in [-\xi_0,\xi_0]$, $\forall \tau \in [\tau_1,\tau_2]$, $|k(\xi,\tau)-\hat{k}(\tau)| \leq \epsilon$ and $|\nabla k(\xi,\tau)| \leq \frac{2C_0''^*}{\xi_0}$, where $\epsilon \to 0$ as $(\delta_1,\xi_0) \to (0,+\infty)$.

Proof: See section 4. ■

Proposition 2.5 (A priori estimates in P_3) *For all $\epsilon>0$, $\epsilon_0>0$, $\sigma_0>0$, and $\sigma_1>0$, there exists $t_4(\epsilon,\epsilon_0,\sigma_0,\sigma_1)<T$ such that $\forall t \in [t_4,T)$, if h is*

a solution of (II) on $[t_0, t_*]$ *for some* $t_* \in [t_0, T)$ *satisfying*
i) for $|x| \in [\frac{\epsilon_0}{6}, \frac{\epsilon_0}{4}]$, $\forall t \in [t_0, t_*]$,

$$\sigma_0 \le h(x,t) \le \sigma_1, \ |\nabla h(x,t)| \le \sigma_1 \text{ and } |\nabla^2 h(x,t)| \le \sigma_1, \tag{48}$$

ii) $h(x, t_0) = H^*(x)$ *for* $|x| \ge \frac{\epsilon_0}{6}$ *where* H^* *is defined by (5),*
then for $|x| \in [\frac{\epsilon_0}{4}, +\infty)$, $\forall t \in [t_0, t_*]$,

$$|h(x,t) - h(x,t_0)| + |\nabla h(x,t) - \nabla h(x,t_0)| \le \epsilon.$$

Proof: See section 4. ■

Step 2: Finite dimensional reduction

From Propositions 2.3, 2.4 and 2.5, we have the following:

Proposition 2.6 (Finite dimensional reduction) *We can choose parameters* $t_0 < T$, K_0, ϵ_0, α_0, A, δ_0, C_0' *and* C_0 *and* η_0 *such that the following properties hold: Assume that* $h(x, t_0)$ *is given by (45) and* $(d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$. *Then,*
i) $h(t_0) \in H \cap S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_0)$.
Assume in addition that for some $t_* \in [t_0, T)$, *we have* $\forall t \in [t_0, t_*]$,
$h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ *and*
$h(t_*) \in \partial S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_*)$ *then*
ii) $(q_0(s_*), q_1(s_*)) \in \partial \hat{V}_A(s_*)$ *where* q *is defined in (23) and* $s_* = -\log(T - t_*)$.
iii) **(Transversality)** *there exists* $\nu_0 > 0$ *such that* $\forall \nu \in (0, \nu_0)$,
$(q_0(s_* + \nu), q_1(s_* + \nu)) \notin \hat{V}_A(s_* + \nu)$ *(hence*
$h(t_* + \nu) \notin S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_* + \nu)$*).*

Proof: We proceed in two steps: we first show that we can fix K_0, δ_0 and C_0 independently from A, take $A \ge A_7$ and choose ϵ_0, α_0, C_0', η_0 and t_0 in terms of A, so that *i*) and *ii*) hold. In the second step, we fix A and t_0 so that *iii*) holds too.

Proof of i) and ii)

It follows from the following lemma:

Lemma 2.5 *There exist constants* K_0, δ_0, C_0, *and* $A_7 > 0$ *such that for all* $A \ge A_7$, *there exist* $\epsilon_0(A) > 0$, $\alpha_0(A)$, $C_0'(A)$, $\eta_7(A)$ *and* $t_7(A) < T$ *such that for all* $\eta_0 \le \eta_7$ *and* $t_0 \in [t_7, T)$, *and under the hypotheses of Proposition 2.6, i) and ii) hold.*

Proof

Let us first choose suitably the constants, and then show that *i*) and *ii*) of Proposition 2.6 follow for this choice.

All the constants we are referring to below appear either in lemma 2.4 or Propositions 2.3, 2.4 or 2.5.

We proceed in ten steps:

i) Fix $K_0 = 4\max(K_{01}, K_{02}, K_{03})$.

ii) Fix $\delta_0 = \frac{1}{4}\min(\hat{k}(1), 1)$ (note that $\hat{k}(1)$ depends only on K_0). Fix $C_0 = 1$. Let $A_7(K_0)$ be large enough so that $A_7 \geq \max(1, A_2(K_0))$ and for all $A \geq A_7(K_0)$, $A^3 \geq C_0'(A)$ where we introduce

$C_0'(A) = 4\max\left(C_3 A^2 K_0^3 + \|\nabla\hat{\Phi}\|_{L^\infty(B(0,2K_0))}, \frac{20\hat{k}(1)}{(\beta+1)K_0}, C^*(K_0)\right)$ with $C^*(K_0)$ defined in lemma 2.4 and C_3 a constant which is independent of all the parameters and appears in lemma 2.6.

Consider A any number larger than $A_7(K_0)$, and consider $C_0'(A)$.

iii) Applying Proposition 2.4 with K_0, $C_0^* = 2$, $C_0'^*(A) = 2C_0'(A)$ and $C_0''^*(A) = \frac{1}{4}C_0'(A)$, we get $\xi_0^*(A) \geq 1$ and $\delta_1^*(A) \leq 1$ such that for all $\xi_0 \geq \xi_0^*$ and $\delta_1 \leq \delta_1^*$, the conclusion of the Proposition holds with $\epsilon = \frac{\delta_0}{2}$.

iv) Let $\delta_1(A) = \min(\frac{1}{2}\delta_1^*(A), \delta_0)$ and $C_1 = \frac{1}{2}$.

v) We claim the following lemma:

Lemma 2.6 *$\forall A \geq A_7$, there exist $\alpha_5(K_0, \delta_1(A)) > 0$ such that for all $\alpha_0 \leq \alpha_5$, there exists $\epsilon_5(\alpha_0, A) > 0$ such that for all $\epsilon_0 \leq \epsilon_5(\alpha_0, A)$, there are $t_5(\epsilon_0, A) < T$ and $\eta_5(\epsilon_0, A) > 0$ such that for all $\eta_0 \leq \eta_5(\epsilon_0, A)$ and $t_0 \in [t_5(\epsilon_0, A), T)$,*

if for all $t \in [t_0, t_*]$, $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ for some $t_* \in [t_0, T)$, then we have for $|x| \in [\frac{K_0}{4}\sqrt{(T-t_*)|\log(T-t_*)|}, \epsilon_0]$:

i) For $|\xi| \leq \frac{7}{4}\alpha_0\sqrt{|\log\theta(x)|}$ and for all $\tau \in [\max(0, \frac{t_0 - t(x)}{\theta(x)}), \frac{t_* - t(x)}{\theta(x)}]$:

$k(x,.,.)$ satisfies (47) and, $|\nabla_\xi k(x,\xi,\tau)| \leq \frac{2C_0'(A)}{\sqrt{|\log\theta(x)|}}$, $|\nabla_\xi^2 k(x,\xi,\tau)| \leq 2C_0$ and $k(x,\xi,\tau) \geq \frac{1}{2}\hat{k}(\tau)$.

ii) For $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$ and $\tau = \max(\frac{t_0 - t(x)}{\theta(x)}, 0)$: $|k(x,\xi,\tau) - \hat{k}(\tau)| \leq \delta_1$ and $|\nabla_\xi k(x,\xi,\tau)| \leq \frac{C_0'(A)}{4\sqrt{|\log\theta(x)|}}$.

Proof: We focus on the proof of the fact that for $|x| \in (0, \epsilon_0]$, for $|\xi| \leq \frac{7}{4}\alpha_0\sqrt{|\log\theta(x)|}$, for $t \in [\max(0, t(x)), T)$, we have

$$|\nabla_\xi k(x,\xi,\tau)| \leq \frac{2C_0'(A)}{\sqrt{|\log\theta(x)|}} \tag{49}$$

where $\tau = \frac{t-t(x)}{\theta(x)}$, and : for $|x| \in (0, \epsilon_0]$, for $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$,

$$|k(x,\xi,\tau_0(x)) - \hat{k}(\tau_0(x))| \leq \delta_1 \tag{50}$$

$$\text{and } |\nabla_\xi k(x,\xi,\tau_0(x))| \leq \frac{\frac{1}{4}C_0'(A)}{\sqrt{|\log\theta(x)|}} \tag{51}$$

where $\tau_0(x) = \max(\frac{t_0-t(x)}{\theta(x)}, 0)$.
The other estimates follow by similar techniques.

Let $\delta > 0$ to be fixed later. If $\alpha_0 \leq \alpha_7(K_0,\delta)$ for some $\alpha_7(K_0,\delta) > 0$, then we have from (29): for $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$,

$$(1-\delta)|x| \leq |x + \xi\sqrt{\theta(x)}| \leq (1+\delta)|x|. \tag{52}$$

Proof of (49):
From (28), we have

$$\nabla_\xi k(x,\xi,\tau) = \theta(x)^{-\frac{1}{\beta+1}+\frac{1}{2}}\nabla h(x+\xi\sqrt{\theta(x)}, t). \tag{53}$$

Let us denote $x + \xi\sqrt{\theta(x)}$ by X and distinguish three cases:

- Case where $|X| \leq \frac{K_0}{4}\sqrt{(T-t)|\log(T-t)|}$:
From (8), we write $\nabla h(X,t) = C\frac{\nabla u}{u^{1+\frac{1}{\alpha}}}(X,t)$.
From $i)$ of the Definition of $S^*(t)$, we get
$|(T-t)^{\frac{1}{p-1}}u(X,t) - \Phi(\frac{X}{\sqrt{(T-t)|\log(T-t)|}})|$
$= |q(\frac{X}{\sqrt{T-t}}, -\log(T-t)) + \frac{\kappa}{2(p-a)|\log(T-t)|}| \leq \frac{CA^2K_0^3}{\sqrt{|\log(T-t)|}}$ by lemma B.1.
Moreover,
$|\nabla u(X,t) - (T-t)^{-\frac{1}{p-1}-\frac{1}{2}}|\log(T-t)|^{-\frac{1}{2}}\nabla\Phi(\frac{X}{\sqrt{(T-t)|\log(T-t)|}})| =$
$(T-t)^{-\frac{1}{p-1}-\frac{1}{2}}|\nabla q(\frac{X}{\sqrt{T-t}}, -\log(T-t))|$
$\leq (T-t)^{-\frac{1}{p-1}-\frac{1}{2}}|\log(T-t)|^{-\frac{1}{2}}CA^2K_0^3$ (see the proof of lemma B.1)
Hence, by (9), we obtain:
$|(T-t)^{-\frac{1}{\beta+1}+\frac{1}{2}}\nabla h(X,t) - |\log(T-t)|^{-\frac{1}{2}}\nabla\hat{\Phi}(\frac{X}{\sqrt{(T-t)|\log(T-t)|}})|$
$\leq \frac{C_3A^2K_0^3}{\sqrt{|\log(T-t)|}}$ and
$|\nabla h(X,t)| \leq \left(C_3A^2K_0^3 + \|\nabla\hat{\Phi}\|_{L^\infty(B(0,K_0))}\right)(T-t)^{\frac{1}{\beta-1}-\frac{1}{2}}|\log(T-t)|^{-\frac{1}{2}}$.
This gives by (53):
$|\nabla_\xi k(x,\xi,\tau)| \leq \left(\frac{T-t}{\theta(x)}\right)^{\frac{1}{\beta+1}-\frac{1}{2}}|\log(T-t)|^{-\frac{1}{2}}C_0'(A)$.

Since $(1-\delta)|x| \le |X|$ (see (52)) and $|X| \le K_0\sqrt{(T-t)|\log(T-t)|}$, we have $|x| \le \frac{K_0}{4(1-\delta)}\sqrt{(T-t)|\log(T-t)|}$.
From (29), we have $|x| \to \theta(x)$ is an increasing function. Therefore, $\theta(x) \le \theta(\frac{K_0}{4(1-\delta)}\sqrt{(T-t)|\log(T-t)|}) \sim \frac{8}{K_0^2}\frac{K_0^2(T-t)|\log(T-t)|}{16(1-\delta)^2\frac{1}{2}|\log(T-t)|} = \frac{(T-t)}{(1-\delta)^2}$ by (31). Moreover, we have $t \ge t(x)$, therefore, $T-t \le \theta(x)$. Hence, $|\nabla_\xi k(x,\xi,\tau)| \le 2C_0'(A)|\log\theta(x)|^{-\frac{1}{2}}$ if δ is small enough.

- Case where $|X| \in [\frac{K_0}{4}\sqrt{(T-t)|\log(T-t)},\epsilon_0]$:
We write $\nabla h(X,t) = \theta(X)^{\frac{1}{\beta+1}-\frac{1}{2}}\nabla_\xi k(X,0,\frac{t-t(X)}{\theta(X)})$. This gives by (53): $\nabla_\xi k(x,\xi,t) = \left(\frac{\theta(X)}{\theta(x)}\right)^{\frac{1}{\beta+1}-\frac{1}{2}}\nabla_\xi k(X,0,\frac{t-t(X)}{\theta(X)})$.
From $ii)$ of the Definition of $S^*(t)$, we obtain:
$|\nabla_\xi k(x,\xi,\tau)| \le C_0'(A)|\log\theta(x)|^{-\frac{1}{2}} \times \frac{\theta(X)^{\frac{1}{\beta+1}-\frac{1}{2}}|\log\theta(X)|^{-\frac{1}{2}}}{\theta(x)^{\frac{1}{\beta+1}-\frac{1}{2}}|\log\theta(x)|^{-\frac{1}{2}}}$.
Using (52) and taking δ small enough, this yields
$|\nabla_\xi k(x,\xi,\tau)| \le 2C_0'(A)|\log\theta(x)|^{-\frac{1}{2}}$.

- Case $|X| \ge \epsilon_0$: If $\eta_0 \le \delta \min\limits_{|x'|\ge\epsilon_0}|\nabla h(x',t_0)|$, then we have from $iii)$ of the Definition of $S^*(t)$:
$|\nabla h(X,t)| \le (1+\delta)|\nabla h(X,t_0)| \le (1+\delta)|\nabla h(\gamma x,t_0)|$ where $\gamma = 1-\delta$ if $\beta > 1$ and $\gamma = 1+\delta$ if $\beta \le 1$ (see (52)).
From lemma 2.2, we get:
$|\nabla h(X,t)| \le (1+\delta)\frac{10\hat{k}(1)}{(\beta+1)K_0}\theta(\gamma x)^{\frac{1}{\beta+1}-\frac{1}{2}}|\log\theta(\gamma x)|^{-\frac{1}{2}}$.
Arguing as before, we obtain from (53):
$|\nabla_\xi k(x,\xi,\tau)| \le \frac{20\hat{k}(1)}{(\beta+1)K_0}|\log\theta(x)|^{-\frac{1}{2}} \le 2C_0'(A)|\log\theta(x)|^{-\frac{1}{2}}$ if δ is small enough. This concludes the proof of (49).

Proof of (50):
If $|x| \ge \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$, then (29) yields $t(x) \le t_0$ and $\tau_0(x) = \frac{t_0-t(x)}{\theta(x)}$. Hence, (50) follows from lemma 2.4.
If $|x| \le \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$, then $t(x) \ge t_0$ and $\tau_0(x) = 0$. From (28) and (30), we let $X = x+\xi\sqrt{\theta(x)}$ and write:
$|k(x,\xi,0)-\hat{k}(0)| = |\theta(x)^{-\frac{1}{\beta+1}}h(X,t(x)) - \left((\beta+1)+\frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16}\right)^{\frac{1}{\beta+1}}| \le I+II$
where $I = |\theta(x)^{-\frac{1}{\beta+1}}h(X,t(x)) - \left((\beta+1)+\frac{(\beta+1)^2}{4\beta}\frac{|X|^2}{\theta(X)|\log\theta(x)|}\right)^{\frac{1}{\beta+1}}|$
and $II = |\left((\beta+1)+\frac{(\beta+1)^2}{4\beta}\frac{|X|^2}{\theta(X)|\log\theta(x)|}\right)^{\frac{1}{\beta+1}} - \left((\beta+1)+\frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16}\right)^{\frac{1}{\beta+1}}|$.
From $i)$ of the Definition of $S^*(t)$, (23) and the fact that
$|X| \le (1+\delta)|x| \le \frac{(1+\delta)K_0}{4}\sqrt{\theta(x)|\log\theta(x)|} \le K_0\sqrt{\theta(x)|\log\theta(x)|}$, we get

$I \le CA^2K_0^3|\log\theta(x)|^{-\frac12} \le CA^2K_0^3|\log(T-t_0)|^{-\frac12}$, since $|x| \le \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$. Now, if $T-t_0$ is small enough, then $I \le \frac{\delta_1}{2}$. From (52) and (29), we have $(1-\delta)^2\frac{K_0^2}{16} \le \frac{|X|^2}{\theta(X)|\log\theta(X)|} \le (1+\delta)^2\frac{K_0^2}{16}$. Hence, if δ is small enough, we obtain $II \le \frac{\delta_1}{2}$.

This concludes the proof of (50).

Proof of (51):
If $|x| \ge \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$, then (29) yields $t(x) \le t_0$ and $\tau_0(x) = \frac{t_0-t(x)}{\theta(x)}$. Hence, lemma 2.4 yields: for $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$,
$|\nabla_\xi k(x,\xi,\tau_0(x))| \le C^*(K_0)|\log\theta(x)|^{-\frac12} \le \frac14 C_0'(A)$.
If $|x| \le \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$, then $t(x) \ge t_0$ and $\tau_0(x) = 0$. With $X = x+\xi\sqrt{\theta(x)}$, we write: $\nabla_\xi k(x,\xi,0) = \theta(x)^{-\frac{1}{\beta+1}+\frac12}\nabla h(X,t(x))$. Arguing as for the first case in the proof of (49), we get: $|\nabla_\xi k(x,\xi,0)| \le \left[C_3A^2K_0^3 + \|\nabla\hat\Phi\|_{L^\infty(B(0,K_0))}\right]|\log\theta(x)|^{-\frac12} \le \frac14 C_0'(A)|\log\theta(x)|^{-\frac12}$.

This concludes the proof of (51) and the proof of lemma 2.6. ■

vi) We now fix $\alpha_0(A) = \min(\frac12\alpha_1(K_0,\delta_1(A)),\alpha_5(K_0,\delta_1(A)),1)$. We also fix $\epsilon_0(A) \le \min(\epsilon_1(K_0,\delta_1(A),\alpha_0(A)),\epsilon_5(\alpha_0(A),A))$ such that
$\alpha_0(A)\sqrt{|\log\theta(\epsilon_0)|} \ge \xi_0^*(A)$.
vii) Then, we take $\eta_7(A) = \frac12\min(\eta_2(\epsilon_0(A)),\eta_5(\epsilon_0(A),A))$ and consider $\eta_0 \le \eta_7$.
viii) By direct parabolic estimates, it is easy to see that there exists $t_6(A) < T$ such that for all $t_0 \in [t_6,T)$, if
$h(t_0) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_1,\eta_0,t_0)$ and $\forall t \in [t_0,t']$,
$h(t) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$, then
$h(t') \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',\frac34,\eta_0,t')$.
ix) Let $\sigma_0(A) = \frac12\hat k(1)\theta(\frac{\epsilon_0}{6})^{\frac{1}{\beta+1}}$ and $\sigma_1(A) = \max(\frac32\hat k(0)\theta(\frac{\epsilon_0}{4})^{\frac{1}{\beta+1}}$,
$C_0'\frac{\theta(\frac{\epsilon_0}{6})^{\frac{1}{\beta+1}-\frac12}}{\sqrt{|\log\theta(\frac{\epsilon_0}{6})|}}, C_0'\frac{\theta(\frac{\epsilon_0}{4})^{\frac{1}{\beta+1}-\frac12}}{\sqrt{|\log\theta(\frac{\epsilon_0}{4})|}}, C_0\theta(\frac{\epsilon_0}{6})^{\frac{1}{\beta+1}-1})$.
x) Let $t_7(A) = \max(t_1(K_0,\delta_1(A),\epsilon_0(A),A,C_1),t_2(K_0,\epsilon_0(A),A,C_0'(A))$, $t_4(\frac{\eta_0}{2},\epsilon_0,\sigma_0,\sigma_1),t_5(\epsilon_0(A),A),t_6(A))$, and consider t_0 an arbitrary number in $[t_7(A),T)$.

Now, we show that *i)* and *ii)* of Proposition 2.6 hold for this choice. Let us assume that $h(t_0)$ is given by (45) and $(d_0,d_1) \in \mathcal{D}(t_0,K_0,A)$. Then, lemma 2.4 applies and $h(t_0) \in H \cap S^*(t_0.K_0,\epsilon_0.\alpha_0,A,\delta_1,C^*(K_0),0,t_0)$. Since $\delta_1 \le \delta_0$, $C^*(K_0) \le C_0'$ and $0 < \eta_0$, *i)* follows.
We now assume that in addition, we have $\forall t \in [t_0,t_*]$,
$h(t) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$ and

$h(t_*) \in \partial S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_*)$ for some $t_* \in [t_0, T)$. According to the Definition of $S^*(t)$, three cases may occur:

Case 1: $q(s_*) \in \partial V_{K_0,A}(s_*)$. From *ii*) of lemma 2.4, Proposition 2.3 and *i*) of the Definition of $S^*(t)$, we have $(q_0(s_*), q_1(s_*)) \in \partial \hat{V}_A(s_*)$ which is *i*) of Proposition 2.6.

Case 2: There exist x and ξ such that
$|x| \in [\frac{K_0}{4}\sqrt{(T-t_*)|\log(T-t_*)|}, \epsilon_0]$ and $|\xi| \le \alpha_0\sqrt{|\log\theta(x)|}$, and either $|k(x,\xi,\tau_1) - \hat{k}(\tau_1)| = \delta_0$ or $|\nabla_\xi k(x,\xi,\tau_1)| = \frac{C_0'}{\sqrt{|\log\theta(x)|}}$ or $|\nabla_\xi^2 k(x,\xi,\tau_1)| = C_0 = 1$, where $\tau_1 = \frac{t_*-t(x)}{\theta(x)} < 1$.

According to viii) and lemma 2.4, we have $|\nabla_\xi^2 k(x,\xi,\tau_*)| \le \frac{3}{4}$. Let $\tau_0 = \max(\frac{t_0-t(x)}{\theta(x)}, 0)$ and $\xi_0 = \alpha_0\sqrt{|\log\theta(x)|}$. Note that $\xi_0 \ge \alpha_0\sqrt{|\log\theta(\epsilon_0)|} \ge \xi_0^*$.
Since $\alpha_0 \le 1$, it follows from lemma 2.6:
- For $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, $|k(x,\xi,\tau_0) - \hat{k}(\tau_0)| \le \delta_1$ and $|\nabla_\xi k(x,\xi,\tau_0)| \le \frac{C_0'(A)}{4\sqrt{|\log\theta(x)|}} \le \frac{C_0'(A)}{4\xi_0}$.
- For $|\xi| \le \frac{7}{4}\alpha_0\sqrt{|\log\theta(x)|}$ and for all $\tau \in [\tau_0, \tau_1]$: $k(x,.,.)$ satisfies (87) and $|\nabla_\xi k(x,\xi,\tau)| \le \frac{2C_0'(A)}{\xi_0}$, $|\nabla_\xi^2 k(x,\xi,\tau)| \le 2C_0$ and $k(x,\xi,\tau) \ge \frac{1}{2}\hat{k}(\tau)$.
Applying Proposition 2.4 yields:
For $|\xi| \le \alpha_0\sqrt{|\log\theta(x)|}$, $|k(x,\xi,\tau_1) - \hat{k}(\tau_1)| \le \frac{\delta_0}{2}$ and $|\nabla_\xi k(x,\xi,\tau_1)| \le \frac{2\frac{1}{4}C_0'(A)}{\sqrt{|\log\theta(x)|}} < \frac{C_0'(A)}{\sqrt{|\log\theta(x)|}}$, which contradicts the hypotheses of Case 2.

Case 3: There exists $x \in \mathbb{R}$ such that $|x| \ge \frac{\epsilon_0}{4}$ and $|h(x,t_*) - h(x,t_0)| = \eta_0$ or $|\nabla h(x,t_*) - \nabla h(x,t_0)| = \eta_0$. From *ii*) of the Definition of $S(t)$, we have: $\forall t \in [t_0, t_*]$, for $|x| \in [\frac{\epsilon_0}{6}, \frac{\epsilon_0}{4}]$: $|k(x,0,\tau) - \hat{k}(\tau)| \le \delta_0$, $|\nabla_\xi k(x,0,\tau)| \le \frac{C_0'}{\sqrt{|\log\theta(x)|}}$ and $|\nabla_\xi^2 k(x,0,\tau)| \le C_0$, where $\tau = \frac{t-t(x)}{\theta(x)}$. Using (28) and the fact that $\delta_0 \le \frac{1}{2}\hat{k}(1) \le \frac{1}{2}\hat{k}(0)$, we obtain:
$\frac{1}{2}\hat{k}(1)\theta(x)^{\frac{1}{\beta+1}} \le h(x,t) \le \frac{3}{2}\hat{k}(0)\theta(x)^{\frac{1}{\beta+1}}$, $|\nabla h(x,t)| \le C_0' \frac{\theta(x)^{\frac{1}{\beta+1}-\frac{1}{2}}}{\sqrt{|\log\theta(x)|}}$ and
$|\nabla^2 h(x,t)| \le C_0\theta(x)^{\frac{1}{\beta+1}-1}$. Therefore, $\sigma_0(A) \le h(x,t) \le \sigma_1(A)$,
$|\nabla h(x,t)| \le \sigma_1$ and $|\nabla^2 h(x,t)| \le \sigma_1$. From (45), we have $h(x,t_0) = H^*(x)$ for $|x| \ge \frac{\epsilon_0}{6}$. Hence, Proposition 2.5 applies and we get: $|h(x,t) - h(x,t_0)| + |\nabla h(x,t) - \nabla h(x,t_0)| \le \frac{\eta_0}{2} < \eta_0$, which contradicts the hypotheses of Case 3.

This concludes the proof of *i*) and *ii*) of Proposition 2.6.

Proof of iii):
Let us recall that K_0, δ_0 and C_0 are fixed independently of A, where A

is taken larger than some $A_7 > 0$, ϵ_0, α_0 and C_0' are fixed in terms of A, and $t_0 \in [t_7(A), T)$, $\eta_0 \leq \eta_7(A)$, for some $t_7(A) < T$. and $\eta_7(A) > 0$. Let us prove this lemma:

Lemma 2.7 *There exists $A_8 > 0$ such that for all $A \geq A_8$, there exist $t_8(A) < T$ and $\eta_8(A)$ such that for all $t_0 \in [t_8, T)$ and $\eta_0 \leq \eta_8(A)$, and under the hypotheses of Proposition 2.6, the conclusion iii) holds.*

Proof: From lemma 2.5, we have: $\forall t \in [t_0, t_*]$,
$h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ and $(q_0(s_*), q_1(s_*)) \in \partial \hat{V}_A(s_*)$, which means that $q_m(s_*) = \epsilon A s_*^{-2}$ for some $m \in \{0,1\}$ and $\epsilon \in \{-1,1\}$. From (44), the conclusion follows if we show that $\epsilon \frac{dq_m}{ds}(s_*) > 0$.

From (24) and (34), we have: $\int \chi(s_*) \frac{\partial q}{\partial s}(s_*) k_m d\mu = \int \chi(s_*) \mathcal{L} q(s_*) k_m d\mu +$ $\int \chi(s_*) \left[V(s_*) q(s_*) + B(q) + T(q) + R(s_*) + e^{-\frac{ps_*}{p-1}} f_1(e^{\frac{s_*}{p-1}}(\varphi + q)) \right] k_m d\mu.$
If we take $t_0 \in [t_{11}(K_0, \epsilon_0(A), A, 0, C_0'), T)$ and $\eta_0 \leq \eta_{11}(\epsilon_0(A))$, then we get from lemma 3.2 (see section 3):

$$|\frac{dq_m}{ds}(s_*) - (1 - \frac{m}{2}) q_m(s_*)| \leq \frac{C_6}{s_*^2}$$

for some C_6 independent from all the other constants. Since $q_m(s_*) = \epsilon A s_*^{-2}$, we have $\epsilon \frac{dq_m}{ds}(s_*) > 0$ for $A \geq 4C_6$. ■

Conclusion of the proof: If we take $A = \max(A_7, A_8)$ and
$\eta_0 = \min(\eta_7(A), \eta_8(A), \frac{1}{2} \min\limits_{|x| \geq \frac{\epsilon_0}{4}} h(x, t_0))$ ($\min\limits_{|x| \geq \frac{\epsilon_0}{4}} h(x, t_0) > 0$ according to (45) and (5)), and $t_0 = \max(t_7(A), t_8(A))$, then both i) and ii) of Proposition 2.6 hold. This concludes the proof of Proposition 2.6. Let us note that with this choice, the reduction of the proof of Proposition 2.1 holds. ■

Part II: Topological argument

From Proposition 2.6, we claim that there exist $(d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$ such that $h(d_0, d_1) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$. The proof is similar to the analogous one in [22], let us give its main ideas.

We proceed by contradiction: From i) of Proposition 2.6, we have
$\forall (d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$,
$h(d_0, d_1, t_0) \in H \cap S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_0)$. Therefore, we define for each $(d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$ a time $t_*(d_0, d_1)$ as being the infinitum of all $t \in [t_0, T)$ such that
$h(d_0, d_1, t) \notin S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$. By ii) of Proposition 2.6, we have
$(q_0, q_1)(d_0, d_1, s_*(d_0, d_1)) \in \partial \hat{V}_A(s_*(d_0, d_1))$ where $s_* = -\log(T - t_*)$.

Hence, we can define from (44) the following function:

$$\begin{array}{rlll} \Psi: & \mathcal{D}(t_0,K_0,A) & \to & \partial\mathcal{C} \\ & (d_0,d_1) & \to & \frac{s_*(d_0,d_1)^2}{A}(q_0,q_1)(d_0,d_1,s_*(d_0,d_1)) \end{array}$$

where $\mathcal{C}$ is the unit square of $\mathbb{R}^2$.
Now we claim

Proposition 2.7 *i) Ψ is a continuous mapping from $\mathcal{D}(t_0,K_0,A)$ to $\partial\mathcal{C}$.*
ii) There exists a non trivial affine function $T:\mathcal{D}(t_0,K_0,A)\to\mathcal{C}$ such that $\Psi\circ T^{-1}_{|\partial\mathcal{C}} = Id_{|\partial\mathcal{C}}$.

Proof: The proof is very similar to the proof of Proposition 3.3 in [22], that is the reason why we give only the important arguments.

i) follows from the continuity in H of the solution $h(t)$ at a fixed time t with respect to initial data, and the transversality property *iii*) of Proposition 2.6.

From *ii*) of lemma 2.4, we have $\forall(d_0,d_1)\in\partial\mathcal{D}(t_0,K_0,A)$, $s_*(d_0,d_1)=s_0$ and *ii*) follows. ■

From Proposition 2.7, a contradiction follows (Index Theory). Therefore, there exist $(d_0,d_1)\in\mathcal{D}(t_0,K_0,A)$ such that
$h(d_0,d_1)\in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0)$. By Proposition 2.1 and the *Conclusion of the proof* of Proposition 2.6, the main Theorem follows.

3 A priori estimates of $u(t)$ in the blow-up zone

This section is devoted to the proof of Proposition 2.3. Let us consider $t_0<T$, K_0, ϵ_0, α_0, A, δ_0, C_0', C_0 and η_0. We assume that (d_0,d_1) is chosen so that $(q_0(s_0),q_1(s_0))\in\hat{V}_A(s_0)$ where $s_0=-\log(T-t_0)$, and that $\forall t\in[t_0,t_*]$, $h(t)\in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$ for some $t_*\in[t_0,T)$.

Then we improve some of the bounds given in *i*) of the Definition of $S^*(t)$ for $h(t_*)$. More precisely, we improve the bounds of $q_2(s_*)$, $q_-(y,s_*)$, $q_e(y,s_*)$ and $\left(\frac{\partial q}{\partial y}\right)_\perp(y,s_*)$ with $s_*=-\log(T-t_*)$.

For this purpose, we consider the equation (24) satisfied by $q(s)$ and the one satisfied by $\frac{\partial q}{\partial y}(s)$ as well as their integral formulations:

$$0=-\frac{\partial q}{\partial s}+(\mathcal{L}+V(y,s))q+B(q)+T(q)+R(y,s)+e^{-\frac{ps}{p-1}}f_1(e^{\frac{s}{p-1}}(\varphi+q)) \quad (54)$$

with $\mathcal{L}=\Delta-\frac{1}{2}y.\nabla+1$, $V(y,s)=p\varphi(y,s)^{p-1}-\frac{p}{p-1}$,
$B(q)=(\varphi+q)^p-\varphi^p-p\varphi^{p-1}q$,
$T(q)=-a\frac{|\nabla\varphi+\nabla q|^2}{\varphi+q}+a\frac{|\nabla\varphi|^2}{\varphi}$, $R(y,s)=-\frac{\partial\varphi}{\partial s}+\Delta\varphi-\frac{1}{2}y.\nabla\varphi-\frac{\varphi}{p-1}+\varphi^p-a\frac{|\nabla\varphi|^2}{\varphi}$,
$f_1(u)=\alpha^{\frac{\beta}{\beta+1}}u^{1+\frac{1}{\alpha}}F(\alpha^{\frac{1}{\beta+1}}u^{-\frac{1}{\alpha}})-u^p$,
if $r(y,s)=\frac{\partial q}{\partial y}(y,s)$ then

$$\frac{\partial r}{\partial s}=(\mathcal{L}-\frac{1}{2}+V)r+\frac{\partial}{\partial y}(B(q)+T(q))(y,s)+R_1(y,s)$$

$$+e^{-s}(\frac{\partial\varphi}{\partial y}+r)f_1'(e^{\frac{s}{p-1}}(\varphi+q))$$

with $R_1(y,s)=\frac{\partial R}{\partial y}(y,s)+\frac{\partial V}{\partial y}q(y,s)$,

if $K(s,\sigma)$ and $K_1(s,\sigma)$ are respectively the fundamental solution of $\mathcal{L}+V$ and $\mathcal{L}-\frac{1}{2}+V$ (note that $K_1(s,\sigma)=e^{-\frac{s-\sigma}{2}}K(s,\sigma)$), then for $s\geq\sigma\geq s_0$:

$$q(s)=K(s,\sigma)q(\sigma)+\int_\sigma^s d\tau K(s,\tau)(B(q(\tau))+T(q(\tau)))+\int_\sigma^s d\tau K(s,\tau)R(\tau)$$
$$+\int_\sigma^s d\tau K(s,\tau)e^{-\frac{p\tau}{p-1}}f_1(e^{\frac{\tau}{p-1}}(\varphi(\tau)+q(\tau))), \tag{55}$$

and

$$r(s)=K_1(s,\sigma)r(\sigma)+\int_\sigma^s d\tau K_1(s,\tau)(\frac{\partial}{\partial y}(B(q)+T(q))(\tau)+R_1(\tau)) \tag{56}$$
$$+\int_\sigma^s d\tau K_1(s,\tau)e^{-\tau}(\frac{\partial\varphi}{\partial y}(\tau)+r(\tau))f_1'(e^{\frac{\tau}{p-1}}(\varphi(\tau)+q(\tau))).$$

We proceed in two steps: in Step 1, using the fact that $h(t)\in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$ for $t\in[T-e^{-\sigma},T-e^{-(\sigma+\rho)}]$ for some $\sigma\geq s_0$ and $\rho\geq 0$, we derive bounds on terms in the right hand side of equation (54) truncated by χ and projected on h_2, and on terms in the right hand sides of equations (55) and (56), expanded respectively as in (35) and (36).

In Step 2, we use these bounds and these equations to find new bounds on q_-, q_e and $r_\perp$ on one hand, and a bound on $\frac{dq_2}{ds}(s)$ on the other hand. This latter bound yields a better estimate on $q_2(s)$ (this estimate is obtained differently from the analogous term in [22] and [25]) .

Step 1: A priori estimates of $q(s)$

We first show that if (d_0,d_1) is chosen so that $(q_0(s_0),q_1(s_0))\in\hat{V}_A(s_0)$, then $q(s_0)$ is strictly included in $V_{K_0,A}(s_0)$. In other words, at initial time

s_0, the finite dimensional variable $(q_0(s_0), q_1(s_0))$ determines the size of the hole function $q(s_0)$. In fact we have an estimate more precise than the one in lemma 2.4:

Lemma 3.1 *For each* $A > 0$*, there exists* $s_2(A) > 0$ *such that for each* $s_0 \geq s_2(A)$ *and* $K_0 > 20$*, if* $h(x, t_0)$ *is given by (45) and* (d_0, d_1) *is chosen so that* $(q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0)$*, then*

$$\begin{array}{rlrl} |q_2(s_0)| & \leq s_0^{-2} \log s_0, & |q_-(y, s_0)| & \leq C s_0^{-2}(1 + |y|^3), \\ |q_e(y, s_0)| & \leq s_0^{-1/2}, & |r_\perp(y, s_0)| & \leq s_0^{-2}(1 + |y|^3), \end{array}$$

and $|r(y, s_0)| \leq s_0^{-1/2}$ *for* $|y| \geq K_0\sqrt{s_0}$.

Proof: The proof is included in the proof of lemma 2.4: See the end of its Step 2. ■

Now we consider $\sigma \geq s_0$ and $\rho \in [0, \rho^*]$. We suppose that $\forall t \in [T - e^{-\sigma}, T - e^{-(\sigma+\rho)}]$ $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$. Then we give bounds on terms in right hand sides of equations (54), (55) and (56), expanded as in (34).

Remark: In fact, we give in lemma 3.2 estimates on equation (54) projected on h_m with $m = 0, 1$ or 2. Only $m = 2$ is useful for the proof of Proposition 2.3. The estimates for $m = 0$ or 1 are in a large part the same, they are useful for the proof of Proposition 2.6.

Lemma 3.2 *There exists* $K_{11} > 0$ *and* $A_{11} > 0$ *such that for each* $K_0 \geq K_{11}$*,* $\epsilon_0 > 0$*,* $A \geq A_{11}$*,* $\rho^* > 0$*,* $C_0' > 0$*, there exists* $t_{11}(K_0, \epsilon_0, A, \rho^*, C_0')$ *with the following property:*

$\forall t_0 \in [t_{11}(K_0, \epsilon_0, A, \rho^*, C_0'), T)$*,* $\forall \rho \in [0, \rho^*]$*, for all* $\delta_0 \leq \frac{1}{2}\hat{k}(1)$*,* $\alpha_0 > 0$*,* $C_0 > 0$ *and* $\eta_0 \leq \eta_{11}(\epsilon_0)$ *for some* $\eta_{11}(\epsilon_0) > 0$*, assume that*

- $h(x, t_0)$ *is given by (45) and* (d_0, d_1) *is chosen so that* $(q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0)$

- for some $\sigma \geq -\log(T - t_0)$*, we have* $\forall t \in [T - e^{-\sigma}, T - e^{-(\sigma+\rho)}]$ $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$*. Then,* $\forall s \in [\sigma, \sigma + \rho]$*,*

I) Equation (54): If $m = 0, 1$ *or* 2*,*

$$|\int \chi(y, s) k_m(y) \frac{\partial q}{\partial s}(y, s) d\mu - q_m'(s)| \leq e^{-s} \quad (57)$$

$$|\int \chi(y, s) k_m(y) \mathcal{L} q(y, s) d\mu - (1 - \frac{m}{2}) q_m(s)| \leq e^{-s} \quad (58)$$

$$
\begin{aligned}
|\int \chi(y,s)k_m(y)V(y,s)q(y,s)d\mu| &\leq s^{-5/2} && (59)\\
|\int \chi(y,s)k_m(y)B(q)(y,s)d\mu| &\leq Cs^{-3} && (60)\\
|\int \chi(y,s)k_m(y)T(q)(y,s)d\mu| &\leq s^{-2-1/4} && (61)\\
|\int \chi(y,s)k_m(y)R(y,s)d\mu| &\leq Cs^{-2} && (62)\\
|\int \chi(y,s)k_m(y)e^{-\frac{ps}{p-1}}f_1(e^{\frac{s}{p-1}}(\varphi+q))d\mu| &\leq e^{-s}. && (63)
\end{aligned}
$$

If $m=2$, then we have more precisely:

$$
\begin{aligned}
|\int \chi(y,s)k_2(y)V(y,s)q(y,s)d\mu + \frac{2p}{s(p-a)}q_2(s)| &\leq CAs^{-3} && (64)\\
|\int \chi(y,s)k_2(y)T(q)(y,s)d\mu - \frac{2a}{s(p-a)}q_2(s)| &\leq CAs^{-3} && (65)\\
|\int \chi(y,s)k_2(y)R(y,s)d\mu| &\leq Cs^{-3} && (66)
\end{aligned}
$$

II) Equation (55):
Case $\sigma \geq s_0$:

$$
\begin{aligned}
|\alpha_-(y,s)| &\leq C(Ae^{-(s-\sigma)/2} + A^2e^{-(s-\sigma)^2})s^{-2}(1+|y|^3) && (67)\\
|\alpha_e(y,s)| &\leq C(A^2e^{-(s-\sigma)/p} + AK_0^3e^{s-\sigma})s^{-1/2} && (68)
\end{aligned}
$$

where $\alpha(s) = K(s,\sigma)q(\sigma)$ is expanded as in (35),

$$
\begin{aligned}
|\beta_-(y,s)| &\leq C(s-\sigma)s^{-2}(1+|y|^3) && (69)\\
|\beta_e(y,s)| &\leq (s-\sigma)s^{-1/2} && (70)
\end{aligned}
$$

where $\beta(s) = \int_\sigma^s d\tau K(s,\tau)\left(B(q(\tau)) + T(q(\tau))\right)$,

$$
\begin{aligned}
|\gamma_-(y,s)| &\leq C(s-\sigma)s^{-2}(1+|y|^3) && (71)\\
|\gamma_e(y,s)| &\leq CK_0^3(s-\sigma)e^{s-\sigma}s^{-1/2} && (72)\\
& && (73)
\end{aligned}
$$

where $\gamma(s) = \int_\sigma^s d\tau K(s,\tau)R(\tau)$ is expanded as in (35),

$$
\begin{aligned}
|\delta_-(y,s)| &\leq C(s-\sigma)s^{-2}(1+|y|^3) && (74)\\
|\delta_e(y,s)| &\leq C(s-\sigma)s^{-1/2} && (75)
\end{aligned}
$$

where $\delta(s) = \int_\sigma^s d\tau K(s,\tau)e^{-\frac{ps}{p-1}}f_1(e^{\frac{\tau}{p-1}}(\varphi+q))$ is expanded as in (35).

Case $\sigma = s_0$: More precisely,

$$\begin{aligned} |\alpha_-(y,s)| &\leq Cs^{-2}(1+|y|^3) & (76)\\ |\alpha_e(y,s)| &\leq CK_0^3 e^{s-\sigma} s^{-1/2}. & (77) \end{aligned}$$

III) Equation (56):
Case $\sigma \geq s_0$:

$$|P_\perp(\chi(s)K_1(s,\sigma)r(\sigma))| \leq C(Ae^{-(s-\sigma)/2} + C(K_0)C_0' e^{-(s-\sigma)^2})\frac{1+|y|^3}{s^2} \quad (78)$$

$$|P_\perp(\chi(s)\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial}{\partial y}(B(q)+T(q))(\tau))| \leq C(s-\sigma)^{1/2}\frac{1+|y|^3}{s^2} \quad (79)$$

$$|P_\perp(\chi(s)\int_\sigma^s d\tau K_1(s,\tau)R_1(\tau))| \leq C(s-\sigma)\frac{1+|y|^3}{s^2} \quad (80)$$

$$|P_\perp(\chi\int_\sigma^s d\tau K_1(s,\tau)e^{-\tau}(\frac{\partial\varphi}{\partial y}+r)f_1'(e^{\frac{\tau}{p-1}}(\varphi+q))| \leq C(s-\sigma)\frac{1+|y|^3}{s^2} \quad (81)$$

where $P_\perp$ is defined in (34).
Case $\sigma = s_0$: More precisely,

$$|P_\perp(\chi(s)K_1(s,\sigma)r(\sigma))| \leq Cs^{-2}(1+|y|^3). \quad (82)$$

Proof: See Appendix B. ■

Step 2: Lemma 3.2 implies Proposition 2.3

Let $K_0 \geq K_{02} > 0$, $\epsilon_0 > 0$, $A \geq A_2(K_0) > 0$ where $A_2(K_0)$ will be fixed later, and $C_0' \leq A^3$. Let $t_0 > 0$ to be fixed in $[t_2(K_0,\epsilon_0,A,C_0'),T)$ (where $t_2(K_0,\epsilon_0,A,C_0')$ will be defined later). Consider $\delta_0 \leq \frac{1}{2}\hat{k}(1)$, $\alpha_0 > 0$, $C_0 > 0$ and $\eta_0 \leq \eta_2(\epsilon_0)$. Let $h(d_0,d_1)$ be a solution of equation (II) with initial data (45) defined on $[t_0,t_*]$ with $t_* \in [t_0,T)$, such that
- (d_0,d_1) is chosen so that $(q_0(s_0),q_1(s_0)) \in \hat{V}_A(s_0)$ ($s_0 = -\log(T-t_0)$ and q is defined by (23)),
- $\forall t \in [t_0,t_*]$, $h(t) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$ and $q(s_*) \in \partial V_{K_0,A}(s_*)$.

We want to show that

$$\begin{array}{llll} |q_2(s_*)| \leq A^2 s_*^{-2}\log s_* - s_*^{-3}, & |q_-(y,s_*)| \leq \frac{A}{2}s_*^{-2}(1+|y|^3) \\ |q_e(y,s_*)| \leq \frac{A^2}{2}s_*^{-1/2}, & |r_\perp(y,s_*)| \leq \frac{A}{2}s_*^{-2}(1+|y|^3) \end{array} \quad (83)$$

where

$$r(y,s) = \frac{\partial q}{\partial y}(y,s).$$

We consider $\rho_1(K_0, A) \geq \rho_2(K_0, A)$ two positive numbers (which will be fixed later in terms of K_0 and A). The conclusion follows if we treat Case 1 where $s_* - s_0 \leq \rho_1$ and then Case 2 where $s_* - s_0 \geq \rho_2$. The proof relies strongly on estimates of lemma 3.2. Therefore, we suppose $K_0 \geq K_{11}$, $A \geq A_{11}$, $C_0' \leq A^3$, $t_0 \geq \max(t_{11}(K_0, \epsilon_0, A, \rho_1, C_0'), t_{11}(K_0, \epsilon_0, A, \rho_2, C_0'))$, $s_0 = -\log(T - t_0) \geq \max(\rho_1, \rho_2)$, $\epsilon_0 > 0$, $\delta_0 \leq \frac{1}{2}\hat{k}(1)$, $C_0 > 0$ and $\eta_0 \leq \eta_{11}(\epsilon_0)$.

Case 1: $s_* - s_0 \leq \rho_1(K_0, A)$

We apply lemma 3.2 with A, $\rho^* = \rho_1$, $\rho = s_* - s_0$ and $\sigma = s_0$.

From equation (54) with $m = 2$, we obtain: $\forall s \in [s_0, s_*]$, $|q_2'(s) + 2s^{-1}q_2(s)| \leq CAs^{-3} + 2e^{-s} \leq CAs^{-3}$. Therefore, $\forall s \in [s_0, s_*]$, $|\frac{d}{ds}(s^2 q_2(s))| \leq CAs^{-1}$, and then, using $s_* \leq 2s_0$ (indeed, $s_* = s_0 + \rho \leq s_0 + \rho_1 \leq 2s_0$), we obtain $|q_2(s_*)| \leq s_*^{-2} s_0^2 |q_2(s_0)| + 2A(s_* - s_0)s_*^{-3}$. Using $|q_2(s_0)| \leq s_0^{-2}\log s_0$ which follows from lemma 3.1, we get $|q_2(s_*)| \leq s_*^{-2}\log s_* + CA(s_* - s_0)s_*^{-3}$. Together with estimates concerning equations (55) and (56) in lemma 3.2, we obtain:

$$\begin{aligned}
|q_2(s_*)| &\leq s_*^{-2}\log s_* + 2C_1 A s_*^{-2} \\
|q_-(y, s_*)| &\leq C_1(1 + s_* - s_0)s_*^{-2}(1 + |y|^3) \\
|q_e(y, s_*)| &\leq C_1 K_0^3 e^{s_* - s_0}(1 + s_* - s_0)s_*^{-1/2} \\
|r_\perp(y, s_*)| &\leq C_1(1 + (s_* - s_0)^{1/2} + (s_* - s_0))s_*^{-2}(1 + |y|^3) \\
&\leq 2C_1(1 + s_* - s_0)s_*^{-2}(1 + |y|^3).
\end{aligned}$$

To have (83), it is enough to have

$$1 \leq \frac{A^2}{2},\ 2C_1(1 + s_* - s_0) \leq \frac{A}{2},\ \text{and } C_1 K_0^3 e^{s_* - s_0}(1 + s_* - s_0) \leq \frac{A^2}{2} \quad (84)$$

on one hand and

$$2C_1 A s_*^{-2} \leq \frac{A^2}{2}\frac{\log s_*}{s_*^2} - s_*^{-3} \quad (85)$$

on the other hand.

If we restrict ρ_1 to satisfy $2C_1(1+\rho_1) \leq A/2$ and $C_1 K_0^3 e^{\rho_1}(1+\rho_1) \leq A^2/2$ (which is possible with $\rho_1 = 3/2\log A$ for $A \geq A_6(K_0)$ large enough), then (84) is satisfied, since $s_* - s_0 \leq \rho_1$. Now if $s_0 \geq s_6(A)$, then (85) is satisfied. Thus (83) is satisfied also. This concludes Case 1.

Case 2: $s_* - s_0 \geq \rho_2(K_0, A)$

We apply lemma 3.2 with A, $\rho = \rho^* = \rho_2$ and $\sigma = s_* - \rho_2$. From equation (54) with m=2, we obtain $\forall s \in [\sigma, s_*]$, $|q_2'(s) + 2s^{-1}q_2(s)| \leq CAs^{-3}$. Using the same argument as Case 1 and $|q_2(\sigma)| \le A^2\sigma^{-2}\log\sigma$, and then estimates on equation (55) and (56), we obtain:

$$\begin{array}{rcl}
|q_2(s_*)| & \le & A^2 s_*^{-2}\log(s_*-\rho_2) + 2C_2A\rho_2 s_*^{-3} \\
|q_-(y,s_*)| & \le & C_2(Ae^{-\rho_2/2} + A^2e^{-\rho_2^2} + \rho_2)s_*^{-2}(1+|y|^3) \\
|q_e(y,s_*)| & \le & C_2(A^2e^{-\rho_2/p} + AK_0^3e^{\rho_2} + K_0^3\rho_2e^{\rho_2})s_*^{-1/2} \\
|r_\perp(y,s_*)| & \le & C_2(Ae^{-\rho_2/2} + C(K_0)C_0'e^{-\rho_2^2} + {\rho_2}^{1/2} + \rho_2)s_*^{-2}(1+|y|^3).
\end{array}$$

Since $C_0' \le A^3$, in order to obtain (83), it is enough to have

$$\begin{array}{rcl}
f_{A,\rho_2}(s_*) & \ge & 0 \\
C_2(Ae^{-\rho_2/2} + A^2e^{-\rho_2^2} + \rho_2) & \le & \frac{A}{2} \\
C_2(A^2e^{-\rho_2/p} + AK_0^3e^{\rho_2} + K_0^3\rho_2e^{\rho_2}) & \le & \frac{A^2}{2} \\
C_2(Ae^{-\rho_2/2} + C(K_0)A^3e^{-\rho_2^2} + {\rho_2}^{1/2} + \rho_2) & \le & \frac{A}{2}
\end{array} \qquad (86)$$

with $f_{A,\rho_2}(s_*) = A^2s_*^{-2}\log s_* - s_*^{-3} - \left[A^2s_*^{-2}\log(s_*-\rho_2) + 2C_2A\rho_2s_*^{-3}\right]$.

We now fix ρ_2 so that $C_2K_0^3Ae^{\rho_2} = A^2/8$, i.e. $\rho_2 = \log\left(A/(8C_2K_0^3)\right)$. Then, the conclusion follows if A is large enough. Indeed, for all $A>1$, we write

$$|f_{A,\log\frac{A}{8C_2K_0^3}}(s_*) - s_*^{-3}\left(A^2\log\frac{A}{8C_2K_0^3} - 2C_2A\log\frac{A}{8C_2K_0^3} - 1\right)|$$

$$\le \frac{A^2(\log\frac{A}{8C_2K_0^3})^2}{s^2(s-\log\frac{A}{8C_2K_0^3})^2}.$$

Then we take $A \ge A_7(K_0, C_0')$ for some $A_7(K_0)$ such that

$$\begin{array}{rcl}
A^2\log\frac{A}{8C_2K_0^3} - 2C_2A\log\frac{A}{8C_2K_0^3} - 1 & \ge & 1 \\
C_2(A(\frac{A}{8C_2K_0^3})^{-1/2} + A^2e^{-(\log\frac{A}{8C_2K_0^3})^2} + \log\frac{A}{8C_2K_0^3}) & \le & \frac{A}{2} \\
C_2(A^2(\frac{A}{8C_2K_0^3})^{-1/p} + AK_0^3\frac{A}{8C_2K_0^3} + K_0^3\log\frac{A}{8C_2K_0^3}\frac{A}{8C_2K_0^3}) & \le & \frac{A^2}{2}
\end{array}$$

$$C_2(A(\frac{A}{8C_2K_0^3})^{-1/2} + C(K_0)A^3e^{-(\log\frac{A}{8C_2K_0^3})^2} + (\log\frac{A}{8C_2K_0^3})^{1/2}$$

$+\log\frac{A}{8C_2K_0^3}) \le \frac{A}{2}$.

Afterwards, we take $s_0 \geq s_7(K_0, A)$ so that $\forall s \geq s_0$,
$A^2(\log \frac{A}{8C_2K_0^3})^2 s^{-2}(s - \log \frac{A}{8C_2K_0^3})^{-2} \leq s^{-3}/2$.

This way, (86) is satisfied for $A \geq A_7(K_0)$ and $s_0 \geq s_7(K_0, A)$. This concludes Case 2.

We remark that for $A \geq A_8(K_0)$, we have $\rho_1 = \frac{3}{2}\log A \geq \log \frac{A}{8C_2K_0^3} = \rho_2$.

If we take now $K_{02} = K_{11}$, $A_2(K_0) = \max(A_{11}, A_6(K_0), A_7(K_0), A_8(K_0))$ and $t_2 = \max(t_{11}(K_0, \epsilon_0, A, \rho_1(A), C_0'), T - e^{-\rho_1(A)}$, $t_{11}(K_0, \epsilon_0, A, \rho_2(K_0, A), C_0')$, $T - e^{-\rho_2(K_0,A)}, T - e^{-s_6(A)}, T - e^{-s_7(K_0,A)})$, $\eta_2(\epsilon_0) = \eta_{11}(\epsilon_0)$, then we conclude the proof of Proposition 2.3.

4 A priori estimates in P_2 and P_3

In this section, we estimate directly the solutions of equation (II).

4.1 Estimates in P_2

Let us recall that $\hat{k}(\tau) = \left((\beta+1)(1-\tau) + \frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16}\right)^{\frac{1}{\beta+1}}$ and that it is defined for $\tau \in [0, \hat{T}]$ with $\hat{T} > 1$.

Proposition 4.1 *There exists $K_{03} > 0$ such that for all $K_0 \geq K_{03}$, $\delta_1 \leq 1$, $\xi_0 \geq 1$ and $C_0^* > 0$, $C_0'^* > 0$, $C_0''^* > 0$ we have the following property:*
Assume that k is a solution of equation

$$\frac{\partial k}{\partial \tau} = \Delta k - \frac{1}{k^\beta} \tag{87}$$

for $\tau \in [\tau_1, \tau_2)$ with $0 \leq \tau_1 \leq \tau_2 \leq 1$ $(< \hat{T})$. Assume in addition: $\forall \tau \in [\tau_1, \tau_2]$,
i) $\forall \xi \in [-2\xi_0, 2\xi_0]$, $|k(\xi, \tau_1) - \hat{k}(\tau_1)| \leq \delta_1$ and $|\nabla k(\xi, \tau_1)| \leq \frac{C_0''^*}{\xi_0}$,
ii) $\forall \xi \in [-\frac{7\xi_0}{4}, \frac{7\xi_0}{4}]$, $|\nabla k(\xi, \tau)| \leq \frac{C_0'^*}{\xi_0}$ and $|\nabla^2 k(\xi, \tau)| \leq C_0^*$,
iii) $\forall \xi \in [-\frac{7\xi_0}{4}, \frac{7\xi_0}{4}]$, $k(\xi, \tau) \geq \frac{1}{2}\hat{k}(\tau)$. Then, for $\xi_0 \geq \xi_{03}(C_0'^*, C_0^*, C_0''^*)$ there exists $\epsilon = \epsilon(K_0, C_0'^*, \delta_1, \xi_0)$ such that $\forall \xi \in [-\xi_0, \xi_0]$, $\forall \tau \in [\tau_1, \tau_2]$,
$|k(\xi, \tau) - \hat{k}(\tau)| \leq \epsilon$ and $|\nabla k(\xi, \tau)| \leq \frac{2C_0''^*}{\xi_0}$, where $\epsilon \to 0$ as $(\delta_1, \xi_0) \to (0, +\infty)$.

Proof: We can assume $\tau_1 = 0$ and $\tau_2 = \tau_0 \leq 1$.
Step 1: Gradient estimate

Lemma 4.1 *Under the assumptions of Proposition 4.1, we have*
$\forall \xi \in [-\frac{5\xi_0}{4}, \frac{5\xi_0}{4}]$, $\forall \tau \in [0,\tau_0]$ $|\nabla k(\xi,\tau)| \le \frac{2C_0''^*}{\xi_0}$, *if* $\xi_0 \ge \xi_{03}(C_0'^*, C_0^*, C_0''^*)$.

Proof: We have $\forall \xi \in [-2\xi_0, 2\xi_0]$, $\forall \tau \in [0,\tau_0]$,

$$\frac{\partial}{\partial \tau}\nabla k = \Delta(\nabla k) + \beta \frac{\nabla k}{k^{\beta+1}}.$$

From *iii*), we have for $|\xi| \le \frac{7\xi_0}{4}$, $|\frac{1}{k^{\beta+1}}| \le 1$ for K_0 large. If $\theta = |\nabla h|^2$, then, by a direct calculation, $2\frac{\partial k}{\partial \xi}\Delta\left(\frac{\partial k}{\partial \xi}\right) \le \Delta\theta$ and $\theta_\tau \le \Delta\theta + C\theta$ for $|x| \le \frac{7\xi_0}{4}$. Let us consider $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $\chi_1(x) = 1$ for $|x| \le \frac{3\xi_0}{2}$, $\chi_1(x) = 0$ for $|x| \ge \frac{7\xi_0}{4}$, $0 \le \chi_1 \le 1$, $|\nabla\chi_1| \le \frac{C}{\xi_0}$ and $|\Delta\chi_1| \le \frac{C}{\xi_0^2}$. Then, $\theta_1 = \chi_1\theta$ satisfies the following inequality:
$\theta_{1\tau} \le \Delta\theta_1 - 2\nabla\chi_1.\nabla\theta - \Delta\chi_1\theta + C\theta_1$
$\le \Delta\theta_1 + C(C_0'^*, C_0^*)\xi_0^{-2}1_{\{\frac{3\xi_0}{2} \le |x| \le 2\xi_0\}} + C\theta_1$. With $\theta_2 = e^{-C\tau}\theta_1$, we have

$$\theta_{2\tau} \le \Delta\theta_2 + C(C_0'^*, C_0^*)\xi_0^{-2}1_{\{\frac{3\xi_0}{2} \le |x| \le 2\xi_0\}} \text{ and } 0 \le \theta_2(0) \le \frac{C_0''^{*2}}{\xi_0^2}.$$

Therefore, by the maximum principle, $\forall \xi \in [-\frac{5\xi_0}{4}, \frac{5\xi_0}{4}]$,$\forall \tau \in [0,\tau_0]$, $\theta(\xi,\tau) \le \frac{C_0''^{*2}}{\xi_0^2} + C(C_0'^*, C_0^*)^2\xi_0^{-2}e^{-C'\xi_0^2}$. Hence, for $|\xi| \le \frac{5\xi_0}{4}$, $\forall \tau \in [0,1]$, $|\nabla k(\xi,\tau)| \le \frac{C_0''^*}{\xi_0} + \frac{C(C_0'^*, C_0^*)}{\xi_0}e^{-C'\xi_0^2} \le \frac{2C_0''^*}{\xi_0}$, if $\xi_0 \ge \xi_{03}(C_0'^*, C_0^*, C_0''^*)$, which yields the conclusion. ■

Step 2: Estimates on k

We are now able to conclude the proof of Proposition 4.1.

Lemma 4.2 *For* $|\xi| \le \xi_0$, $\forall \tau \in [0,\tau_0]$, *we have* $|k(\xi,\tau) - \hat{k}(\tau)| \le \epsilon$, *where* $\epsilon \to 0$ *as* $\xi_0 \to +\infty$ *and* $\delta_1 \to 0$.

Proof: Let us consider k_1 a solution of equation (87) such that $\forall \xi \in [-2,2]$, $\forall \tau \in [0,\tau_0]$, $|k_1(\xi,0) - \hat{k}(0)| \le \delta_1$, $|\nabla k_1(\xi,\tau)| \le \epsilon$. Let us show that for $|\xi| \le 2$, $\forall \tau \in [0,\tau_0]$, $|k_1(0,\tau) - \hat{k}(\tau)| \le C(K_0)\epsilon + \delta_1$ where $C(K_0)$ is independent from ϵ.

We have $\forall \tau \in [0,\tau_0]$, $k_1(0,\tau) = \frac{1}{|B_2(0)|}\int_{|\xi|\le 2} k_1(\xi,\tau)dx + k_2(\tau)$, and $\frac{1}{k_1(0,\tau)^\beta} = \frac{1}{|B_2(0)|}\int_{|\xi|\le 2}\frac{1}{k_1(\xi,\tau)^\beta}d\xi + k_3(\tau)$, where $|B_2(0)|$ is the volume of the sphere of radius 2 in $\mathbb{R}^N$, $\|k_2\|_{L^\infty} \le 2\epsilon$ and $\|k_3\|_{L^\infty} \le C\epsilon$.

In the distribution sense, for ϵ small enough, considering $\tilde{k}(\tau) = \frac{1}{|B_2(0)|}\int_{|\xi|\le 2} k_1(\xi,\tau)d\xi$, we have

$$-\frac{1}{\tilde{k}^\beta} - C\epsilon \le \frac{d\tilde{k}}{d\tau} \le -\frac{1}{\tilde{k}^\beta} + C\epsilon$$

and $|\tilde{k}(0) - \hat{k}(0)| \leq C\epsilon + \delta_1$.

Together with (87), we obtain by classical a priori estimates that $\forall \tau \in [0, \tau_0]$, $|\tilde{k}(\tau) - \hat{k}(\tau)| \leq C(K_0)\epsilon + \delta_1$ (since $C_1 \leq |\hat{k}(\tau)| \leq C_1'(K_0)$) and therefore $\forall |\xi| \leq 2$, $\forall \tau \in [0, \tau_0]$, $|h_1(0, \tau) - \hat{h}(\tau)| \leq C(K_0)\epsilon + \delta_1$. Applying this result to $h_1(\xi, \tau) = h(\tau, \xi - x_0)$ for $x_0 \in [-\xi_0 + 2, \xi_0 - 2]$, from the assumption and step 1 we obtain lemma 4.2.

Lemmas 4.1 and 4.2 yield Proposition 4.1.

4.2 Estimates in P_3

We claim the following

Proposition 4.2 *For all $\epsilon > 0$, $\epsilon_0 > 0$, $\sigma_0 > 0$, and $\sigma_1 > 0$, there exists $t_4(\epsilon, \epsilon_0, \sigma_0, \sigma_1) < T$ such that $\forall t \in [t_4, T)$, if h is a solution of (II) on $[t_0, t_*]$ for some $t_* \in [t_0, T)$ satisfying*
i) for $|x| \in [\frac{\epsilon_0}{6}, \frac{\epsilon_0}{4}]$, $\forall t \in [t_0, t_*]$,

$$\sigma_0 \leq h(x,t) \leq \sigma_1,\ |\nabla h(x,t)| \leq \sigma_1 \text{ and } |\nabla^2 h(x,t)| \leq \sigma_1, \tag{88}$$

ii) $h(x, t_0) = H^*(x)$ for $|x| \geq \frac{\epsilon_0}{6}$ where H^* is defined by (5), then for $|x| \in [\frac{\epsilon_0}{4}, +\infty)$, $\forall t \in [t_0, t_*]$,

$$|h(x,t) - h(x,t_0)| + |\nabla h(x,t) - \nabla h(x,t_0)| \leq \epsilon.$$

Proof:

Let us obtain the estimates on h for $|x| \geq \frac{\epsilon_0}{4}$. The estimates on ∇h can be obtained similarly. We argue by contradiction. Let us consider $t_\epsilon \in (t_0, t_*)$ such that $\forall t \in [t_0, t_\epsilon)$,

$$\|h(x,t) - h(x,t_0)\|_{L^\infty(|x| \geq \frac{\epsilon_0}{4})} \leq \epsilon \text{ and } \|h(x,t_\epsilon) - h(x,t_0)\|_{L^\infty(|x| \geq \frac{\epsilon_0}{4})} = \epsilon. \tag{89}$$

We can assume $\epsilon \leq \frac{1}{4} \min_{|x| \geq \frac{\epsilon_0}{6}} H^*(x)$. We can remark that (5) implies that $|h(x,t_0)| = H^*(x) > C_0(\epsilon_0) > 0$ for $|x| \geq \frac{\epsilon_0}{6}$, therefore, we have $|F(h(x,t))| \leq C(\epsilon_0)$ for $|x| \geq \frac{\epsilon_0}{6}$ and $t \in [t_0, t_\epsilon)$.
From assumption i), we have in fact $\forall t \in [t_0, t_\epsilon]$, for $\frac{\epsilon_0}{6} \leq |x| \leq \frac{\epsilon_0}{4}$, $h(x,t) \geq \sigma_0 > 0$ and $|F(h(x,t))| \leq C(\sigma_0)$. We then consider $h_1(x,t) = \chi_1(x)h(x,t)$ where $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}^N, [0,1])$, $\chi_1 \equiv 1$ for $|x| \geq \frac{\epsilon_0}{5}$, $\chi_1 \equiv 0$ for $|x| \leq \frac{\epsilon_0}{6}$, $|\nabla \chi_1| \leq \frac{C}{\epsilon_0}$ and $|\Delta \chi_1| \leq \frac{C}{\epsilon_0^2}$. We then have:

$$\frac{\partial h_1}{\partial t} = \Delta h_1 - 2\nabla \chi_1 . \nabla h - \Delta \chi_1 h - \chi_1 F(h).$$

Since $\forall t \in [t_0, t_*]$, $|2\nabla\chi_1.\nabla h| + |\Delta\chi_1 h| \le C(\epsilon_0, \sigma_1) 1_{\{\frac{\epsilon_0}{6} \le |x| \le \frac{\epsilon_0}{5}\}}(x)$, we write

$$\frac{\partial h_1}{\partial t} = \Delta h_1 + \tilde{f}_1(x,t) - \chi_1 F(h)$$

with $|\tilde{f}_1(x,t)| \le C(\epsilon_0, \sigma_1) 1_{\{\frac{\epsilon_0}{6} \le |x| \le \frac{\epsilon_0}{5}\}}(x)$.

Let us now consider the case of a bounded domain Ω and the case $\Omega = \mathbb{R}^N$, since there is a small difference in the proof.

i) Ω is a bounded domain:

In this case,
$\forall t \in [t_0, t_\epsilon)$, $h_1(t) - S(t-t_0)h_1(t_0) = \int_{t_0}^t ds S(t-s)[\tilde{f}_1(x,t) - \chi_1 F(h)]$ where $S(.)$ is the linear heat flow. Hence,
$|h_1(t) - h_1(t_0)|_{L^\infty} \le |h_1(t) - S(t-t_0)h_1(t_0)|_{L^\infty} + |S(t-t_0)h_1(t_0) - h_1(t_0)|_{L^\infty} \le \int_{t_0}^t ds[|S(t-s)\tilde{f}_1(s)|_{L^\infty} + |S(t-s)C(\epsilon_0,\sigma_0)\chi_1 F(h)|_{L^\infty}]$
$+ |S(t-t_0)h_1(t_0) - h_1(t_0)|_{L^\infty}$
$\le \int_{t_0}^t ds[\frac{ds}{\sqrt{t-s}}|\tilde{f}_1(s)|_{L^N} + |S(t-s)C(\epsilon_0,\sigma_0)1_{\{\Omega\}}|_{L^\infty}]$
$+ |S(t-t_0)h_1(t_0) - h_1(t_0)|_{L^\infty}$
$\le C(\epsilon_0, \sigma_0, \sigma_1)\sqrt{t-t_0} + |S(t-t_0)\chi_1 H^* - \chi_1 H^*|_{L^\infty}$.
Now, if $t_0 \in [t_5(\epsilon, \epsilon_0, \sigma_0, \sigma_1), T)$, then we have $|h_1(t_\epsilon) - h_1(t_0)|_{L^\infty} \le \frac{\epsilon}{2}$, which is a contradiction with (89).

Therefore, $\forall t \in [t_0, t_*]$ $|h(x,t) - h(x,t_0)|_{L^\infty(|x| \ge \frac{\epsilon_0}{4})} \le \epsilon$.

ii) Case $\Omega = \mathbb{R}^N$: we define $h_2(x,t) = \psi(x) + h_1(x,t)$ where $\psi(x)$ is introduced in the introduction (such that $\psi \in C^\infty(\mathbb{R}^N)$, $\psi \equiv 0$ on $[-1,1]$, $\psi(x) = a_1|x|$ for $|x| \ge 2$). From the fact that $\frac{\partial h_2}{\partial t} = \Delta h_2 + F(h_2(x) + \psi(x)) + \Delta\psi$ and that for $|v| \ge 1$, $|F(v)| + |F'(v)| \le Ce^{-v}$, we obtain using similar techniques:
$\forall t \in [t_0, t_*)$, $|h_2(x,t) - h_2(x,t_0)|_{L^\infty} \le \epsilon$ or equivalently: $\forall t \in [t_0, t_*)$, $|h_1(x,t) - h_1(x,t_0)|_{L^\infty} \le \epsilon$. This concludes the proof of Proposition 4.2.

A Proof of lemma 2.4

We must show that for suitable $(d_0, d_1) \in \mathbb{R}^2$, the estimates of the Definition of $S^*(t)$ hold for $t = t_0$. Since estimate *iii)* holds obviously, we show in a first step that $h(t_0) \in H$ and estimate *ii)* holds, for all choices of (d_0, d_1), provided that t_0 is near T. Then, in step 2, we find $\mathcal{D}(t_0, K_0, A)$ such that $\forall (d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$, $q(s_0) \in V_{K_0,A}(s_0)$, where q is the function introduced in (23).

Step 1: Estimate *ii)* of the Definition of $S^*(t)$

Let us first remark that from (45), (5) and (6), we have $h(t_0) \in \psi + H^1 \cap W^{2,\infty}(\mathbb{R})$. Moreover, one can see from (45), (10), (27) and (5) that $\forall x \in \mathbb{R}$, $h(x,t_0) \geq C(t_0, d_0, d_1, \epsilon_0) > 0$. Therefore, $h(t_0) \in H$.

Let us consider $t_0 < T$, K_0, ϵ_0, α_0, δ_1, and C_1, and show that if these constants are suitably chosen, then for $|x| \in [\frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)}, \epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$, we have

$$|k(x,\xi,\frac{t_0 - t(x)}{\theta(x)}) - \hat{k}\left(\frac{t_0 - t(x)}{\theta(x)}\right)| \leq \delta_1, \;\; |\frac{\partial k}{\partial \xi}| \leq \frac{C^*(K_0)}{\sqrt{|\log\theta(x)|}}, \tag{90}$$

and $|\nabla_\xi^2 k| \leq C_1$ where k, $\hat{k}$, $t(x)$ and $\theta(x)$ are defined in (28), (30) and (29).

Let us first introduce some useful notations:

$$\theta_0 = T - t_0, \;\; r(t_0) = \frac{K_0}{4}\sqrt{\theta_0|\log\theta_0|} \text{ and } R(t_0) = \theta_0^{\frac{1}{2}}|\log\theta_0|^{\frac{p}{2}}, \tag{91}$$

and remark that thanks to (31), we have for fixed K_0:

$$\theta(r(t_0)) \sim \theta_0, \;\; \theta(R(t_0)) \sim \frac{16}{K_0^2}\theta_0|\log\theta_0|^{p-1}, \quad \theta(2R(t_0)) \sim \frac{64}{K_0^2}\theta_0|\log\theta_0|^{p-1},$$

$$\log\theta(r(t_0)) \sim \log\theta(R(t_0)) \sim \log\theta(2R(t_0)) \sim \log\theta_0 \text{ as } t_0 \to T. \tag{92}$$

If $\alpha_0 \leq \frac{K_0}{16}$ and $\epsilon_0 \leq \frac{2}{3}C(a_1,\beta)$, then it follows from (29) that for $|x| \in [r(t_0), \epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$, we have $|\xi\sqrt{\theta(x)}| \leq \frac{|x|}{2}$ and

$$\frac{r(t_0)}{2} \leq \frac{|x|}{2} = |x| - \frac{|x|}{2} \leq |x + \xi\sqrt{\theta(x)}| \leq \frac{3}{2}|x| \leq \frac{3}{2}\epsilon_0 \leq C(a_1,\beta). \tag{93}$$

Therefore, we get from (28), (45), and (27): for $|x| \in [r(t_0), \epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$,

$$k(x,\xi,\frac{t_0 - t(x)}{\theta(x)}) = (I)\chi_1(x + \xi\sqrt{\theta(x)}, t_0) + (II)(1 - \chi_1(x + \xi\sqrt{\theta(x)}, t_0)) \tag{94}$$

with $(I) = \left(\frac{\theta_0}{\theta(x)}\right)^{\frac{1}{\beta+1}} \hat{\Phi}(\frac{x+\xi\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}})$ and $(II) = \theta(x)^{-\frac{1}{\beta+1}}H^*(x + \xi\sqrt{\theta(x)})$.

Estimate on k:

By linearity and (46), it is enough to prove that for $|x| \in [r(t_0), 2R(t_0)]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$,

$$\left|(I) - \hat{k}\left(\frac{t_0 - t(x)}{\theta(x)}\right)\right| \leq \frac{\delta_1}{2} \tag{95}$$

and for $|x| \in [R(t_0), \epsilon_0]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$,

$$\left|(II) - \hat{k}\left(\frac{t_0 - t(x)}{\theta(x)}\right)\right| \le \frac{\delta_1}{2}. \tag{96}$$

We begin with (95). From (4) and (30), we have:

$\left|(I) - \hat{k}\left(\frac{t_0-t(x)}{\theta(x)}\right)\right| = |\left((\beta+1)(\frac{\theta_0}{\theta(x)}) + \frac{(\beta+1)^2}{4\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{\theta(x)|\log\theta_0|}\right)^{\frac{1}{\beta+1}}$

$-\left((\beta+1)(\frac{\theta_0}{\theta(x)}) + \frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16}\right)^{\frac{1}{\beta+1}} | \le C|\frac{|x+\xi\sqrt{\theta(x)}|^2}{\theta(x)|\log\theta_0|} - \frac{K_0^2}{16}|^{\frac{1}{\beta+1}}$

$\le CK_0^{\frac{2}{\beta+1}}\left||\sqrt{\frac{\log\theta(x)}{\log\theta_0}} + \frac{4\xi}{K_0\sqrt{|\log\theta_0|}}|^2 - 1\right|^{\frac{1}{\beta+1}}.$

Since $|x| \in [r(t_0), R(t_0)]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, we have

$\left(\sqrt{\frac{\log(\theta(R(t_0)))}{\log\theta_0}}(1-4\frac{\alpha_0}{K_0})\right)^2 - 1 \le \left(\sqrt{\frac{\log\theta(x)}{\log\theta_0}} + \frac{4\xi}{K_0\sqrt{\log\theta_0}}\right)^2 - 1$

$$\le \left(\sqrt{\frac{\log(\theta(r(t_0)))}{\log\theta_0}}(1+4\frac{\alpha_0}{K_0})\right)^2 - 1. \tag{97}$$

From (97) and (92), we find $\alpha_5(K_0,\delta_1)$ and $t_5(K_0,\delta_1) < T$ such that $\forall\alpha_0 \le \alpha_5$, $\forall t_0 \in [t_5, T)$,

$|(I) - \hat{k}\left(\frac{t_0-t(x)}{\theta(x)}\right)| \le CK_0^{\frac{2}{\beta+1}}||\sqrt{\frac{\log\theta(x)}{\log(T-t_0)}} + \frac{4\xi}{K_0\sqrt{|\log(T-t_0)|}}|^2 - 1|^{\frac{1}{\beta+1}} \le \frac{\delta_1}{2}.$

Now, we treat (96). Let $|x| \in [R(t_0), \epsilon_0]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$. We have from (94), (5), (29) and (30),

$(II) = \left[\frac{(\beta+1)^2|x+\xi\sqrt{\theta(x)}|^2}{8\beta\theta(x)|\log|x+\xi\sqrt{\theta(x)}||}\right]^{\frac{1}{\beta+1}} = \left[\frac{(\beta+1)^2|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\xi|^2}{8\beta|\log|x+\xi\sqrt{\theta(x)}||}\right]^{\frac{1}{\beta+1}}$ and

$\left|(II) - \hat{k}\left(\frac{t_0-t(x)}{\theta(x)}\right)\right|$

$= \left|\left[\frac{(\beta+1)^2|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\xi|^2}{8\beta|\log|x+\xi\sqrt{\theta(x)}||}\right]^{\frac{1}{\beta+1}} - \left[(\beta+1)\left(\frac{\theta_0}{\theta(x)}\right) + \frac{(\beta+1)^2K_0^2}{64\beta}\right]^{\frac{1}{\beta+1}}\right|$

$\le \left|\frac{(\beta+1)^2}{8\beta}\left(\frac{\left|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\xi\right|^2}{|\log|x+\xi\sqrt{\theta(x)}||} - \frac{K_0^2}{8}\right) - (\beta+1)\left(\frac{\theta_0}{\theta(x)}\right)\right|^{\frac{1}{\beta+1}}$

$\le C\left[(I_1) + (I_2)\right]$

with $(I_1) = \left|\frac{\left|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\xi\right|^2}{|\log|x+\xi\sqrt{\theta(x)}||} - \frac{K_0^2}{8}\right|^{\frac{1}{\beta+1}}$ and $(I_2) = \left|\frac{\theta_0}{\theta(x)}\right|^{\frac{1}{\beta+1}}$.

Let us bound (I_1). Since $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, we have from (29),

$|(I_1)| \le \left|\frac{\left|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\alpha_0\sqrt{|\log\theta(x)|}\right|^2}{|\log|x+\alpha_0\sqrt{\theta(x)|\log\theta(x)|}||} - \frac{K_0^2}{8}\right|^{\frac{1}{\beta+1}}$

$= \left| \frac{\log\theta(x)}{\log x + \log(1+\frac{4\alpha_0}{K_0})} \left(\frac{K_0}{4} + \alpha_0\right)^2 - \frac{K_0^2}{8} \right|^{\frac{1}{\beta+1}}.$

Since $|x| \leq \epsilon_0$ and $\log\theta(x) \sim 2\log|x|$ as $x \to 0$ (see (31)), we find $\alpha_6(K_0, \delta_1)$ such that for each $\alpha_0 \leq \alpha_6(K_0, \delta_1)$, there is $\epsilon_6(K_0, \delta_1, \alpha_0)$ such that for all $\epsilon_0 \leq \epsilon_6(K_0, \delta_1, \alpha_0)$, for $|x| \in [R(t_0), \epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$, we have

$$|(I_1)| \leq \frac{\delta_1}{2}. \tag{98}$$

Let us bound (I_2). Since $|x| \geq R(t_0)$, we have from (92), $|(I_2)| \leq \left|\frac{\theta_0}{\theta(R(t_0))}\right|^{\frac{1}{\beta+1}} \leq C(K_0)|\log\theta_0|^{-\frac{(p-1)}{\beta+1}}$. Therefore, if $t_0 \geq t_6(K_0, \delta_1)$, then

$$|(I_2)| \leq \frac{\delta_1}{2}. \tag{99}$$

Combining (98) and (99), we get: If $\alpha_0 \leq \alpha_6(K_0, \delta_1)$, $\epsilon_0 \leq \epsilon_6(K_0, \delta_1, \alpha_0)$ and $t_0 \geq t_6(K_0, \delta_1)$, then for $|x| \in [R(t_0), \epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$, (96) holds.

Estimate on $\frac{\partial k}{\partial \xi}$:

From (94), we have for $|x| \in [r(t_0), \epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$, $\frac{\partial k}{\partial \xi}(x, \xi, \frac{t_0 - t(x)}{\theta(x)}) = E_1 + E_2 + E_3$ where

$$E_1 = \left(\frac{\theta_0}{\theta(x)}\right)^{\frac{1}{\beta+1}} \frac{\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}} \nabla\hat{\Phi}\left(\frac{x + \xi\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}}\right) \chi_1(x + \xi\sqrt{\theta(x)}, t_0), \tag{100}$$

$$E_2 = \theta(x)^{\frac{1}{2} - \frac{1}{\beta+1}} \nabla H^*(x + \xi\sqrt{\theta(x)})(1 - \chi_1(x + \xi\sqrt{\theta(x)}, t_0)), \tag{101}$$

$$E_3 = E_4\theta(x)^{\frac{1}{2} - \frac{1}{\beta+1}} \frac{\partial \chi_1}{\partial x}(x + \xi\sqrt{\theta(x)}, t_0) \text{ with} \tag{102}$$

$$E_4 = \theta_0^{\frac{1}{\beta+1}} \hat{\Phi}\left(\frac{x + \xi\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}}\right) - H^*(x + \xi\sqrt{\theta(x)}).$$

In order to get the estimate on $\frac{\partial k}{\partial \xi}$, it is enough to show that

$$\text{for } |x| \in [r(t_0), 2R(t_0)] \text{ and } |\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|},\ |E_1| \leq \frac{C(K_0)}{\sqrt{|\log\theta(x)|}}, \tag{103}$$

$$\text{for } |x| \in [R(t_0), \epsilon_0] \text{ and } |\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|},\ |E_2| \leq \frac{C(K_0)}{\sqrt{|\log\theta(x)|}}, \tag{104}$$

for $|x| \in [R(t_0), 2R(t_0)]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, $|E_3| \le \dfrac{C(K_0)}{\sqrt{|\log\theta(x)|}}$. (105)

We begin with E_1. Let $|x| \in [r(t_0), 2R(t_0)]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$. From (4), it follows that $|\nabla\hat{\Phi}(z)| \le C|z|^{\frac{1-\beta}{\beta+1}}$. Therefore, by (100),

$|E_1| \le \left(\frac{\theta_0}{\theta(x)}\right)^{\frac{1}{\beta+1}} \frac{\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}} \frac{|x+\xi\sqrt{\theta(x)}|^{\frac{1-\beta}{\beta+1}}}{(\theta_0|\log\theta_0|)^{\frac{1-\beta}{2(\beta+1)}}}$

$\le |\log\theta_0|^{-\frac{1}{\beta+1}}\theta(x)^{\frac12-\frac{1}{\beta+1}}C(\beta)|x|^{\frac{1-\beta}{\beta+1}}$ (by (93))

$\le C(K_0)|\log\theta(x)|^{-\frac12}|\log\theta_0|^{-\frac{1}{\beta+1}}|\log\theta(x)|^{\frac{1}{\beta+1}}$ (by (29))

$\le C(K_0)|\log\theta(x)|^{-\frac12}|\log\theta_0|^{-\frac{1}{\beta+1}}|\log\theta\left(r(t_0)\right)|^{\frac{1}{\beta+1}}$ (since $|x| \ge r(t_0)$)

$\le C(K_0)|\log\theta(x)|^{-\frac12}$ for $t_0 \ge t_7(K_0)$ (use (92)), which implies (103).

Now we treat E_2. Let $|x| \in [R(t_0), \epsilon_0]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$. From (101), we have $|E_2| \le \theta(x)^{\frac12-\frac{1}{\beta+1}}|\nabla H^*(x+\xi\sqrt{\theta(x)})|$
$\le \theta(x)^{\frac12-\frac{1}{\beta+1}}|\nabla H^*(\gamma x)|$ with $\gamma = \frac32$ is $\beta \le 1$ and $\gamma = \frac12$ if $\beta > 1$ (use (93) and (5)). According to lemma 2.2,

$|\nabla H^*(\gamma x)| \sim C(K_0)\frac{\theta(\gamma x)^{\frac{1}{\beta+1}-\frac12}}{\sqrt{|\log\theta(\gamma x)|}} \sim C'(K_0)\frac{\theta(x)^{\frac{1}{\beta+1}-\frac12}}{\sqrt{|\log\theta(x)|}}$ as $x \to 0$. This implies (104) for $\epsilon \le \epsilon_7(K_0)$.

Now we show the bound on E_3. We consider $|x| \in [R(t_0), 2R(t_0)]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, and find a bound on E_4. From (102),

$E_4 = \left[(\beta+1)\theta_0 + \frac{(\beta+1)^2}{4\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log\theta_0|}\right]^{\frac{1}{\beta+1}} - \left[\frac{(\beta+1)^2}{8\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log|x+\xi\sqrt{\theta(x)}||}\right]^{\frac{1}{\beta+1}}$. From (93) and (91), we have

$$\alpha(t_0) \le (\beta+1)\theta_0 + \frac{(\beta+1)^2}{4\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log\theta_0|} \le C\alpha(t_0)$$

$$\text{and } \alpha(t_0) \le \frac{(\beta+1)^2}{8\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log|x+\xi\sqrt{\theta(x)}||} \le C\alpha(t_0)$$

with $\alpha(t_0) \sim C\theta_0|\log\theta_0|^{p-1}$. Therefore, $|E_4| \le C\left[\theta_0|\log\theta_0|^{p-1}\right]^{-\frac{\beta}{\beta+1}}$
$\times\left|(\beta+1)\theta_0 + \frac{(\beta+1)^2}{8\beta}|x+\xi\sqrt{\theta(x)}|^2\left(\frac{2}{|\log\theta_0|} - \frac{1}{|\log|x+\xi\sqrt{\theta(x)}||}\right)\right|$
$\le C\left[\theta_0|\log\theta_0|^{p-1}\right]^{-\frac{\beta}{\beta+1}}\left|\theta_0 + \frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log\theta_0||\log|x+\xi\sqrt{\theta(x)}||}\left|\log\frac{|x+\xi\sqrt{\theta(x)}|^2}{\theta_0}\right|\right|$
$\le C\left[\theta_0|\log\theta_0|^{p-1}\right]^{-\frac{\beta}{\beta+1}}\left|\theta_0 + \theta_0|\log\theta_0|^{p-2}\log\log\theta_0\right|$ (use (93), (91) and $|x| \in [R(t_0), 2R(t_0)]$). Hence

$$|E_4| \le C\theta_0^{\frac{1}{\beta+1}}|\log\theta_0|^{-\frac{(p-1)\beta}{\beta+1}}\left[1 + |\log\theta_0|^{p-2}\log\log\theta_0\right]. \qquad (106)$$

Using (46) and (27), we have

$$\left|\frac{\partial \chi_1}{\partial x}\right| \le C\theta_0^{-\frac{1}{2}} |\log \theta_0|^{-\frac{p}{2}}. \tag{107}$$

From (92) and the fact that $|x| \in [R(t_0), 2R(t_0)]$, we have:
$\theta(x)^{\frac{1}{2}-\frac{1}{\beta+1}} \le \theta\left(\delta R(t_0)\right)^{\frac{1}{2}-\frac{1}{\beta+1}} \le C(K_0)\left[\theta_0 |\log \theta_0|^{p-1}\right]^{\frac{1}{2}-\frac{1}{\beta+1}}$ if $t_0 \ge t_8(K_0)$, with $\delta = 2$ if $\beta \ge 1$ and $\delta = 1$ if $\beta < 1$.

Combining this with (102), (106) and (107), we get
$|E_3| \le C(K_0)|\log \theta_0|^{-p+\frac{1}{2}}\left[1 + |\log \theta_0|^{p-2} \log\log \theta_0\right] \le |\log \theta_0|^{-\frac{1}{2}}$ if $t_0 \ge t_8(K_0)$.

Since $\log \theta_0 \sim \log \theta(R(t_0))$ as $t_0 \to T$ (see (92)) and $R(t_0) \le |x|$, this yields (105) for $t_0 \ge t_9(K_0)$.

The expected bound (90) on $\frac{\partial k}{\partial \xi}$ follows from (103), (104) and (105).

Estimate on Δk:

In the same way, we show that if $t_0 \ge t_{10}(K_0, \epsilon_0, C_1)$, then
for $|x| \in [r(t_0), \epsilon_0]$ and $|\xi| \le 2\alpha_0\sqrt{|\log \theta(x)|}$, we have $|\nabla^2_\xi k(x, \xi, \frac{t-t(x)}{\theta(x)}| \le C_1$.

Step 2: Estimate $i)$ of the Definition of $S^*(t)$

From (23) and (45), we have:

$$\chi(y, s_0)q(y, s_0) = (d_0 + d_1\frac{y}{\sqrt{s_0}})\chi_0(\frac{|y|}{\sqrt{s_0}K_0/16}) - \frac{\kappa}{2(p-a)s_0}\chi_0(\frac{|y|}{\sqrt{s_0}K_0}). \tag{108}$$

Using (34), (26), (25) and simple calculations, and taking $K_0 \ge 20$, we have: if $t_0 \ge t_{11}$, then

$$\begin{array}{lll} q_0(s_0) & = d_0 & \int \chi_0(\frac{|y|}{\sqrt{s_0}K_0/16})d\mu - \frac{\kappa}{2(p-a)s_0}\int \chi_0(\frac{|y|}{\sqrt{s_0}K_0})d\mu, \\ q_1(s_0) & = \frac{d_1}{\sqrt{s_0}} & \int \frac{y^2}{2}\chi_0(\frac{|y|}{\sqrt{s_0}K_0/16})d\mu, \end{array} \tag{109}$$

and

$$q_0(s_0) = d_0(1 + O(e^{-s_0})) - \frac{\kappa}{2(p-a)s_0} + O(e^{-s_0}) \tag{110}$$

$$q_1(s_0) = \frac{d_1}{\sqrt{s_0}}(1 + O(e^{-s_0})) \tag{111}$$

$$q_2(s_0) = d_0 O(e^{-s_0}) + O(e^{-s_0}), \tag{112}$$

$|q_-(y, s_0)| \le Ce^{-s_0}(1 + |d_0|)(1 + |y|^2) + C|d_1|s_0^{-\frac{1}{2}}e^{-s_0}|y|$
$+\frac{\kappa}{2(p-a)s_0}(1 - \chi_0(\frac{|y|}{K_0\sqrt{s_0}})) + (|d_0| + |d_1\frac{y}{\sqrt{s_0}}|)(1 - \chi_0(\frac{|y|}{\sqrt{s_0}K_0/16}))$.

Since $\forall n \in \mathbb{N}$, $|\chi_0(z) - 1| \leq C_n |z|^n$, and $K_0 \geq 20$, we get

$$|q_-(y, s_0)| \leq (|d_0| + |d_1| + \frac{C}{s_0}) \frac{(1 + |y|^3)}{s_0^{3/2}}. \tag{113}$$

Let us show that

$$|q_e(y, s_0)| \leq \frac{C}{s_0}. \tag{114}$$

From (23), we have $q_e(y, s_0) = Q_1 + Q_2$ where $Q_2 = \frac{\kappa}{2(p-a)s_0}(1 - \chi(y, s)) \leq Cs_0^{-1}$ and $Q_1 = (1 - \chi(y, s)) \left[\frac{e^{-\frac{s_0}{p-1}} \alpha^{\frac{\alpha}{\beta+1}}}{h(x,t_0)^\alpha} - \Phi(\frac{y}{\sqrt{s_0}}) \right]$ with $x = ye^{-s_0/2}$ and $t_0 = T - e^{-s_0}$.

If $|x| \leq R(t_0)$ (see (91) for $R(t_0)$), then we have from (45), (46) and (27) $Q_1 = 0$.

If $|x| \geq R(t_0)$, then we have from (10), (45), (91), (9) and easy calculations: $\Phi(\frac{y}{\sqrt{s_0}}) \leq \Phi(\frac{e^{\frac{s_0}{2}} R(t_0)}{\sqrt{s_0}}) \leq Cs_0^{-1}$ and

$h(x, t_0) \geq \chi_1(x, t_0)(T - t_0)^{\frac{1}{\beta+1}} C \left[\Phi(\frac{e^{\frac{s_0}{2}} R(t_0)}{\sqrt{s_0}}) \right]^{-\frac{1}{\alpha}} + (1 - \chi_1(x, t_0)) H^*(R(t_0))$

$\geq C(T - t_0)^{\frac{1}{\beta+1}} s_0^{\frac{1}{\alpha}}$.

Therefore, by (9), $|Q_1| \leq Cs_0^{-1}$, which yields (114).

By analogous calculations, one can easily obtain:

$$|\left(\frac{\partial q}{\partial y}\right)_\perp (y, s_0)| \leq C \frac{(|d_0| + |d_1| + 1/s_0)}{\sqrt{s_0}} \frac{(1 + |y|^3)}{s_0^{3/2}} \tag{115}$$

and $|\frac{\partial q}{\partial y}(y, s_0)| \leq s_0^{-1}$ for $|y| \geq K_0 \sqrt{s_0}$.

From (109), one sees that $g : (d_0, d_1) \to (q_0(s_0), q_1(s_0))$ is an affine function. Let us introduce $\mathcal{D}(t_0, K_0, A) = g^{-1} \left([-\frac{A}{s_0^2}, \frac{A}{s_0^2}]^2 \right)$. $\mathcal{D}(t_0, K_0, A)$ is obviously a rectangle.

If $(d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$, or equivalently $|q_m(s_0)| \leq \frac{A}{s_0^2}$ for $m = 0, 1$, then, from (110) and (111), we obtain $|d_0| \leq Cs_0^{-1}$ and $|d_1| \leq CAs_0^{-3/2}$. Combining this with (112), (113), (114) and (115), we obtain $\forall A > 0$, there exists $t_{12}(A) < T$ such that for each $t_0 \in [t_{12}, T)$:

$$\begin{array}{llll} |q_2(s_0)| & \leq s_0^{-2} \log s_0, & |q_-(y, s_0)| & \leq Cs_0^{-2}(1 + |y|^3), \\ |q_e(y, s_0)| & \leq s_0^{-1/2}, & |\left(\frac{\partial q}{\partial y}\right)_\perp (y, s_0)| & \leq s_0^{-2}(1 + |y|^3), \\ |\frac{\partial q}{\partial y}(y, s_0)| & \leq s_0^{-1} & \text{for } |y| \geq K_0 \sqrt{s_0} & \end{array}$$

and $q(s_0) \in V_{K_0, A}(s_0)$.

Now, putting the conclusions of Steps 1 and 2 together and taking $K_{01} = 20$, $\alpha_1(K_0,\delta_1) = \min\left(\frac{K_0}{4}, \alpha_5(K_0,\delta_1), \alpha_6(K_0,\delta_1)\right)$, $\epsilon_1(K_0,\delta_1,\alpha_0) = \min\left(\frac{2}{3}C(a_1,\beta), \epsilon_6(K_0,\delta_1,\alpha_0), \epsilon_7(K_0)\right)$, $t_1(K_0,\delta_1,\epsilon_0,A,C_1) =$ $\max(t_5(K_0,\delta_1), t_6(K_0,\delta_1), t_7(K_0), t_8(K_0), t_9(K_0), t_{10}(K_0,\epsilon_0,C_1), t_{11}, t_{12}(A))$, we reach the conclusion of lemma 2.4 *i*). *ii*) is obviously true by construction and by (109). ■

B Proof of lemma 3.2

We start with some technical results on equations (54), (55) and (56) (Step 1). In Step 2, we conclude the proof of lemma 3.2.

Step 1: Estimates on equations (54), (55) and (56)

i) Sizes of q and ∇q:

Lemma B.1 *For all $K_0 \geq 1$ and $\epsilon_0 > 0$, there exists $t_1(K_0,\epsilon_0)$ such that $\forall t_0 \in [t_1, T)$, for all $A \geq 1$, $\alpha_0 > 0$, $C_0 > 0$, $C_0' > 0$, $\delta_0 \leq \frac{1}{2}\hat{k}(1)$ and $\eta_0 \leq \eta_1(\epsilon_0)$ for some $\eta_1(\epsilon_0) > 0$, we have the following property: Assume that $h(x,t_0)$ is given by (45) and that for some $t \in [t_0,T)$, we have $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, t)$, then:*
i) $|q(y,s)| \leq CA^2K_0^3 s^{-1/2}$ and $|q(y,s)| \leq CA^2 s^{-2}\log s(1+|y|^3)$,
ii) $|\nabla q(y,s)| \leq C(K_0,C_0')A^2s^{-1/2}$, $|\nabla q(y,s)| \leq C(K_0,C_0')A^2s^{-2}\log s(1+|y|^3)$, $|(1-\chi(y,s))\nabla q(y,s)| \leq C(K_0)C_0's^{-\frac{1}{2}}$, where $s = -\log(T-t)$ and q is defined in (23).

Proof:
i): From *i*) of the definition of $S^*(t)$, we have $q(s) \in V_{K_0,A}(s)$. Therefore, the proof of lemma 3.8 in [22] holds.
ii): Arguing similarly as for *i*), we obtain from *i*) of the definition of $S^*(t)$ and (26):

$$|\chi(y,s)\nabla q(y,s)| \leq CA^2\frac{\log s}{s^2}(1+|y|^3) \text{ and } |\chi(y,s)\nabla q(y,s)| \leq C\frac{A^2K_0^3}{\sqrt{s}}.$$

Since $|\nabla\varphi(y,s)| \leq Cs^{-1/2}$ and $s^{-1/2} \leq s^{-2}|y|^3$ for $|y| \geq K_0\sqrt{s}$ and $K_0 \geq 1$, we have to prove that $|(1-\chi(y,s))\nabla(q+\varphi)(y,s)| \leq C(K_0)C_0's^{-1/2}$ in order to conclude the proof.

From (23), this reduces to show that $\forall t \geq t_0$, for $|x| \geq r(t)$,

$$|\nabla u|(x,t) = C(\alpha)\frac{|\nabla h|}{h^{\alpha+1}}(x,t) \leq C(K_0, C_0')\frac{(T-t)^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(T-t)|}} \tag{116}$$

$$\text{where} \qquad r(t) = K_0\sqrt{(T-t)|\log(T-t)|}. \tag{117}$$

Let us consider two cases:

Case 1: $|x| \in [r(t), \epsilon_0]$. We use the information contained in *ii*) of the definition of $S^*(t)$. From (28), we have

$$h(x,t) = \theta(x)^{\frac{1}{\beta+1}} k(x, 0, \tau(x,t)) \tag{118}$$

$$\text{and} \qquad \nabla_x h(x,t) = \theta(x)^{\frac{1}{\beta+1}-\frac{1}{2}} \nabla_\xi k(x, 0, \tau(x,t)) \tag{119}$$

with $\tau(x,t) = \frac{t-t(x)}{\theta(x)}$. Therefore, since $p = \frac{\alpha+\beta+1}{\alpha}$,

$$\frac{|\nabla h|}{h^{\alpha+1}}(x,t) = \theta(x)^{-(\frac{1}{p-1}+\frac{1}{2})}\frac{|\nabla_\xi k|}{k^{\alpha+1}}(x, 0, \tau(x,t)). \tag{120}$$

Using the definition of $S^*(t)$, we have for $|x| \in [r(t), \epsilon_0]$

$$|k(x, 0, \tau(x,t)) - \hat{k}(\tau)| \leq \delta_0 \text{ and } |\nabla_\xi k(x, 0, \tau(x,t))| \leq \frac{C_0'}{\sqrt{|\log\theta(x)|}}. \tag{121}$$

Since $\delta_0 \leq \frac{1}{2}\hat{k}(1)$, (120) and (29) yield for $|x| \in [r(t), \epsilon_0]$:

$$\frac{|\nabla h|}{h^{\alpha+1}}(x,t) \leq C(K_0)C_0'\frac{\theta(x)^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log\theta(x)|}} \leq C(K_0)C_0'\frac{\theta\left(r(t)\right)^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log\theta\left(r(t)\right)|}} \tag{122}$$

with $C(K_0) = \frac{C}{\hat{k}(0)^{\alpha+1}}$. Since $r(t) \to 0$ as $t \to T$ (see (117)), we have from (31)

$$\theta(r(t)) \sim \frac{2}{K_0^2}\frac{r(t)^2}{|\log r(t)|} \text{ and } \log\theta(r(t)) \sim \log r(t) \text{ as } t \to T. \tag{123}$$

Using (117), we get

$$\frac{(\theta(r(t)))^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(\theta(r(t)))|}} \sim C_4\frac{(T-t)^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(T-t)|}} \text{ as } t \to T$$

for some constant C_4. Therefore, if $t_0 \in [t_2(K_0), T)$ for some $t_2(K_0) < T$, then we have for $t \geq t_0$

$$\frac{(\theta(r(t)))^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(\theta(r(t)))|}} \leq 2C_4 \frac{(T-t)^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(T-t)|}}. \tag{124}$$

Using (122) and (124), we find (116) for $|x| \in [r(t), \epsilon_0]$, provided that $t_0 \in [t_2(K_0), T)$.

Case 2: $|x| \geq \epsilon_0$. We use here the information contained in *iii)* of the definition of $S^*(t)$, which asserts that

$$|h(x,t) - h(x,t_0)| \leq \eta_0 \text{ and } |\nabla h(x,t) - \nabla h(x,t_0)| \leq \eta_0$$

for $|x| \geq \epsilon_0$. Let $\eta_1(\epsilon_0) = \frac{1}{2}\min\{\min_{|x|\geq\epsilon_0} |h(x,t_0)|, \min_{|x|\geq\epsilon_0} |\nabla h(x,t_0)|\}$. According to (45) and (5), we have $\eta_1(\epsilon_0) > 0$. If $\eta_0 \leq \eta_1(\epsilon_0)$, we get for $|x| \geq \epsilon_0$:

$$\frac{|\nabla h|}{h^{\alpha+1}}(x,t) \leq C\frac{|\nabla h|}{h^{\alpha+1}}(x,t_0) = C\frac{|\nabla H^*|}{H^{*(\alpha+1)}}(x)$$

from (45). Therefore, proving (116) for all $t \geq t_0$ reduces to prove it for $t = t_0$. From (5), one easily remarks that $\frac{|\nabla H^*|}{H^{*(\alpha+1)}}(x) \leq C(\epsilon_0)$ for $|x| \geq \epsilon_0$. Therefore, if $t_0 \in [t_4(\epsilon_0), T)$ for some $t_4(\epsilon_0) < T$, then we get (116) for $t = t_0$.

This concludes the proof of (116) for $t = t_0$ and $|x| \geq \epsilon_0$, hence for $t \geq t_0$ and $|x| \geq \epsilon_0$. Thus, with $t_1(K_0, \epsilon_0) = \max(t_2(K_0), t_4(\epsilon_0))$, this concludes the proof of (116) and the proof of lemma B.1. ■

ii) Estimates on K and K_1:

As we remarked before, $K_1(s,\sigma) = e^{-(s-\sigma)/2}K(s,\sigma)$. Hence, any estimate on K holds for K_1 with the adequate changes.

Since K_1 is the fundamental solution of $\mathcal{L} - 1/2 + V$ and $\mathcal{L} - 1/2$ is conjugated to the harmonic oscillator $e^{-x^2/8}(\mathcal{L} - 1/2)e^{x^2/8} = \partial^2 - x^2/16 + 1/4 + 1/2$, we give a Feynman-Kac representation for K_1:

$$K_1(s,\sigma,y,x) = e^{(s-\sigma)(\mathcal{L}-1/2)}(y,x)E(s,\sigma,y,x) \tag{125}$$

where

$$E(s,\sigma,y,x) = \int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{s-\sigma} V(\omega(\tau),\sigma+\tau)d\tau} \tag{126}$$

and $d\mu_{yx}^{s-\sigma}$ is the oscillator measure on the continuous paths $\omega : [0, s-\sigma] \to \mathbb{R}$ with $\omega(0) = x$, $\omega(s-\sigma) = y$, i.e. the Gaussian probability measure with covariance kernel $\Gamma(\tau,\tau')$

$$= \omega_0(\tau)\omega_0(\tau') + 2(e^{-\frac{1}{2}|\tau-\tau'|} - e^{-\frac{1}{2}|\tau+\tau'|} + e^{-\frac{1}{2}|2(s-\sigma)-\tau'+\tau|} - e^{-\frac{1}{2}|2(s-\sigma)-\tau'-\tau|}, \tag{127}$$

which yields $\int d\mu_{yx}^{s-\sigma}\omega(\tau)=\omega_0(\tau)$ with
$\omega_0(\tau)=(\sinh\frac{s-\sigma}{2})^{-1}(y\sinh\frac{\tau}{2}+x\sinh\frac{s-\sigma-\tau}{2})$.

We have in addition

$$e^{\theta(\mathcal{L}-1/2)}(y,x)=\frac{e^{\theta/2}}{\sqrt{4\pi(1-e^{-\theta})}}\exp[-\frac{(ye^{-\theta/2}-x)^2}{4(1-e^{-\theta})}]. \tag{128}$$

Using this formulation for K_1, we give estimates on the dynamics of K and K_1 in the following lemma:

Lemma B.2 *i) $\forall s\geq\tau\geq 1$ with $s\leq 2\tau$, $\int|K(s,\tau,y,x)|(1+|x|^m)dx\leq e^{s-\tau}(1+|y|^m)$.*

ii) There exists $K_2>0$ such that for each $K_0\geq K_2$, $A'>0$, $A''>0$, $A'''>0$, $\rho^*>0$, there exists $s_2(K_0,A',A'',A''',\rho^*)$ with the following property: $\forall s_0\geq s_2$, assume that for $\sigma\geq s_0$, $q(\sigma)$ is expanded as in (35) and satisfies

$$\begin{array}{llll}|q_m(\sigma)| & \leq A'\sigma^{-2}, m=0,1, & |q_2(\sigma)| & \leq A''(\log\sigma)\sigma^{-2},\\ |q_-(y,\sigma)| & \leq A'''(1+|y|^3)\sigma^{-2}, & |q_e(y,\sigma)| & \leq A''\sigma^{-\frac{1}{2}},\end{array}$$

then, $\forall s\in[\sigma,\sigma+\rho^*]$

$$\begin{array}{rl}|\alpha_-(y,s)| & \leq C(e^{-\frac{1}{2}(s-\sigma)}A'''+A''e^{-(s-\sigma)^2})(1+|y|^3)s^{-2},\\ |\alpha_e(y,s)| & \leq C(A''e^{-\frac{(s-\sigma)}{p}}+A'''K_0^3e^{(s-\sigma)})s^{-\frac{1}{2}},\end{array}$$

where $\alpha(y,s)=K(s,\sigma)q(\sigma)$ is expanded as in (35).

iii) There exists $K_3>0$ such that for each $K_0\geq K_3$, $A'>0$, $A''>0$, $A'''>0$, $A''''>0$, $\rho^*>0$, there exists $s_3(K_0,A',A'',A''',A'''',\rho^*)$ with the following property: $\forall s_0\geq s_3$, assume that for $\sigma\geq s_0$, $r(\sigma)$ is expanded as in (36) and satisfies

$$\begin{array}{llll}|r_0(\sigma)| & \leq A'\sigma^{-2}, & |r_1(\sigma)| & \leq A''(\log\sigma)\sigma^{-2},\\ |r_-(y,\sigma)| & \leq A'''(1+|y|^3)\sigma^{-2}, & |r_e(y,\sigma)| & \leq A''''\sigma^{-\frac{1}{2}},\end{array}$$

then, $\forall s\in[\sigma,\sigma+\rho^*]$

$$|P_\perp(\chi(s)K_1(s,\sigma)r(\sigma))|\leq C(e^{-\frac{1}{2}(s-\sigma)}A'''+A''''e^{-(s-\sigma)^2})(1+|y|^3)s^{-2}.$$

Proof: See corollary 3.1 in [22] for i). See Lemma 3.5 in [22] for ii).

Since $K_1(s,\sigma)=e^{-(s-\sigma)/2}K(s,\sigma)$, and ii) and iii) have similar structure, one can adapt without difficulty the proof of ii) (given in [22]) to get iii).■

iii) Estimates on $B(q)$:

Lemma B.3 *$\forall K_0 \geq 1$, $\forall A \geq 1$, $\exists s_5(K_0, A)$ such that $\forall s \geq s_5(A, K_0)$, $q(s) \in V_{K_0,A}(s)$ implies $|\chi(y,s)B(q(y,s))| \leq C(K_0)|q|^2$ and $|B(q)| \leq C|q|^{\bar{p}}$ with $\bar{p} = \min(p, 2)$.*

Proof: See Lemma 3.6 in [22]. ■

iv) Estimates on $T(q)$:

Lemma B.4 *For all $K_0 \geq 1$, $A \geq 1$ and $\epsilon_0 > 0$, there exists $t_6(K_0, \epsilon_0, A) < T$ and $\eta_6(\epsilon_0)$ such that for each $t_0 \in [t_6(K_0, \epsilon_0, A), T)$, $\alpha_0 > 0$, $C_0' > 0$, $\delta_0 \leq \frac{1}{2}\hat{k}(1)$, $C_0 > 0$ and $\eta_0 \leq \eta_6(\epsilon_0)$:*
if $h(x, t_0)$ is given by (45) and $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ for some $t \in [t_0, T)$, then

$$|\chi(y,s)(T(q) + 2a\frac{\nabla\varphi}{\varphi}.\nabla q)| \leq C(K_0, A)\chi(y,s)(\frac{|y|^2}{s^2}|q| + x^{-1}|q|^2 + |\nabla q|^2), \tag{129}$$

$$|\chi(y,s)T(q)| \leq C(K_0, A)\chi(y,s)\left(s^{-1}|q| + s^{-1/2}|\nabla q|\right) \tag{130}$$

$$|(1-\chi(y,s))T(q)| \leq C(K_0, C_0')\min(s^{-1}, s^{-5/2}|y|^3) \tag{131}$$

where $s = -\log(T-t)$ and q is defined in (23).

Proof:

Proof of (129) and (130): They both follow from the Taylor expansion of $F(\theta) = -\frac{|\nabla\varphi+\theta\nabla q|^2}{\varphi+\theta q} + \frac{|\nabla\varphi|^2}{\varphi}$ for $\theta \in [0,1]$. Let us compute
$F'(\theta) = q\frac{|\nabla\varphi+\theta\nabla q|^2}{(\varphi+\theta q)^2} - 2\frac{\nabla q.(\nabla\varphi+\theta\nabla q)}{\varphi+\theta q}$,
$F''(\theta) = -2q^2\frac{|\nabla\varphi+\theta\nabla q|^2}{(\varphi+\theta q)^3} + 4q\frac{\nabla q.(\nabla\varphi+\theta\nabla q)}{(\varphi+\theta q)^2} - 2\frac{|\nabla q|^2}{\varphi+\theta q}$.
From $F(1) = F(0) + F'(0) + \int_0^1 (1-\theta)F'(\theta)d\theta$, we write

$$\chi(y,s)T(q) = a\chi(y,s)(q\frac{|\nabla\varphi|^2}{\varphi^2} - 2\nabla q.\frac{\nabla\varphi}{\varphi}) + a\int_0^1 (1-\theta)\chi(y,s)F''(\theta)d\theta.$$

Using (23), lemma B.1 and (26), we claim that for $s_0 \geq s_7(A, K_0)$, $\forall s \geq s_0$, $\forall\theta \in [0,1]$, $|\nabla\varphi| \leq Cs^{-\frac{1}{2}}$, $\frac{|\nabla\varphi|^2}{\varphi^2} \leq C\frac{|y|^2}{s^2}$ and
$|\chi(y,s)F''(\theta)| \leq C(K_0, A)\chi(y,s)(s^{-1}|q|^2 + |\nabla q|^2)$
$\leq C(K_0, A)(s^{-1}|q| + s^{-\frac{1}{2}}|\nabla q|)$. Therefore, (129) and (130) follow.

Proof of (131): From (23), we have $\frac{|\nabla\varphi|^2}{\varphi}(y,s) \leq Cs^{-1}$. Therefore, if $K_0 \geq 1$, (26) implies that $(1-\chi(y,s))\frac{|\nabla\varphi|^2}{\varphi}(y,s) \leq \min(Cs^{-1}, Cs^{-5/2}|y|^3)$.

In order to prove (131), it then remains to prove that $(1-\chi(y,s))\frac{|\nabla\varphi+\nabla q|^2}{\varphi+q}(y,s) \le \min(Cs^{-1}, Cs^{-5/2}|y|^3)$, or simply, for $|y| \ge K_0\sqrt{s}$, $\frac{|\nabla\varphi+\nabla q|^2}{\varphi+q}(y,s) \le Cs^{-1}$, since $Cs^{-1} \le Cs^{-5/2}|y|^3$ for $|y| \ge K_0\sqrt{s}$, if $K_0 \ge 1$.

From (23), this reduces to show that $\forall t \ge t_0$, for $|x| \ge r(t)$,

$$\frac{|\nabla u|^2}{u}(x,t) = C(\alpha)\frac{|\nabla h|^2}{h^{\alpha+2}}(x,t) \le C(K_0, C_0')\frac{(T-t)^{-\frac{p}{p-1}}}{|\log(T-t)|} \quad (132)$$

where $r(t)$ is introduced in (117). The proof of (132) is in all its steps completely analogous to the proof of (116) given during the course of the proof of lemma B.1, that is the reason why we escape it here. ■

v) Estimates on $R(y,\tau)$:

Lemma B.5 *$\forall y \in \mathbb{R}$, $\forall s \ge 1$,$|R(y,s)| \le Cs^{-1}$, $|R(y,s) - C_1(p,a)s^{-2}| \le Cs^{-3}(1+|y|^4)$ for some $C_1(p,a) \in \mathbb{R}$, and $|\frac{\partial R}{\partial y}(y,s)| \le Cs^{-1-\bar{p}}(|y|+|y|^3)$ where $\bar{p} = \min(p,2)$.*

Proof: From (54), we have

$$R(y,s) = -\frac{\partial\varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p - a\frac{|\nabla\varphi|^2}{\varphi} \text{ where}$$

$$\varphi(y,s) = \Phi^p + \frac{\alpha}{s},\ \Phi = (p-1+bz^2)^{-\frac{1}{p-1}},\ b = \frac{(p-1)^2}{4(p-a)},\ z = \frac{y}{\sqrt{s}}, \quad (133)$$

$\alpha = \frac{\kappa}{2(p-a)}$ and $\kappa = (p-1)^{-\frac{1}{p-1}}$. Therefore,

$$\begin{aligned} R(y,s) &= -\tfrac{bz^2}{(p-1)s}\Phi^p + \tfrac{\alpha}{s^2} - \tfrac{2b}{(p-1)s}\Phi^p + \tfrac{4pb^2z^2}{(p-1)^2 s}\Phi^{2p-1} \\ &- \Phi^p - \tfrac{\alpha}{(p-1)s} + \varphi^p - \tfrac{4ab^2z^2}{(p-1)^2 s}\tfrac{\Phi^{2p}}{\varphi}. \end{aligned} \quad (134)$$

Proof of $|R(y,s)| \le Cs^{-1}$: It follows form (134), and the fact that $|z|^2\Phi^{p-1} + \Phi \le C$, $\varphi^{-1} \le \Phi^{-1}$ and $|\Phi^p - \varphi^p| \le C\alpha s^{-1}$.

Proof of $|R(y,s) - C_1(p,a)s^{-2}| \le Cs^{-3}(1+|y|^4)$: If $|z| \ge 1$, then $1 \le s^{-1}|y|^2$ and $|R(y,s) - C_1(p,a)s^{-2}| \le Cs^{-1} \times \left(s^{-1}|y|^2\right)^2 \le Cs^{-3}(1+|y|^4)$. Let us focus on the case $|z| \le 1$. The method we use consists in expanding each term of (134) in terms of powers of s^{-1} and z^2. From (133), one can easily obtain the following bounds: for $|z| \le 1$, $\forall s \ge 1$,
$|\Phi^p - \kappa^p + \frac{pb\kappa}{(p-1)^3}z^2| \le Cz^4$, $|\Phi^{2p-1} - \kappa^{2p-1}| \le Cz^2$,

$|\varphi^p - \Phi^p - \frac{p\alpha}{s}\Phi^{p-1} - \frac{p(p-1)\alpha^2}{2s^2}\Phi^{p-2}| \leq Cs^{-2}$, $|\Phi^{p-1} - \frac{1}{p-1} + \frac{b}{(p-1)^2}z^2| \leq Cz^4$, $|\Phi^{p-2} - \kappa^{p-2}| \leq Cz^2$ and $|\frac{\Phi^{2p}}{\varphi} - \Phi^{2p-1}| \leq Cs^{-1}$.
Combining all these bounds with (134) and (133), and using $|z| \leq 1$, we get the result.

Proof of $|\frac{\partial R}{\partial y}(y,s)| \leq Cs^{-1-\bar{p}}(|y|+|y|^3)$: The proof is completely similar to the above estimates. We just give its main steps. First, use (134) to compute $\frac{\partial R}{\partial y}$. Then, show that $\forall y \in \mathbb{R}$, $\forall s \geq 1$, $|\frac{\partial R}{\partial y}(y,s)| \leq Cs^{\frac{1}{2}-\bar{p}}$, in the same way as for $|R(y,s)| \leq Cs^{-1}$. Therefore, if $|z| \geq 1$, this gives the expected bound. If $|z| \leq 1$, expand all the terms with respect to s and z^2 to conclude. ■

vi) Estimates on f_1:

Lemma B.6 *$\forall u \geq 0$, $|f_1(u)| + |f_1'(u)| \leq C$.*

Proof: According to (24), (H2) and (19), we have:
$f_1(u) = \alpha^{\frac{\beta}{\beta+1}} u^{1+\frac{1}{\alpha}} F(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) - u^p$, $f_1'(u) = -F'(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) - pu^{p-1}$,
$\forall v \in (0,1]$, $F(v) = v^{-\beta}$ $\forall v \geq 1$, $|F(v)| \leq Ce^{-v} \leq C$. Therefore,
- if $\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}} \leq 1$, then $f_1(u) = f_1'(u) = 0$,
- if $\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}} \geq 1$, then $u \leq \alpha^{\frac{\alpha}{\beta+1}}$ and $|f_1(u)| + |f_1'(u)| \leq C(\alpha)$. ■

Step 2: Conclusion of the proof

Here, we use the lemmas of Step 1 in order to conclude the proof. Therefore, we assume that $K_0 \geq \max(1, K_2, K_3)$, $\epsilon_0 > 0$, $A \geq 1$,
$t_0 \geq \max\left(t_1(K_0,\epsilon_0), T - \exp(-s_2(K_0, A, A^2, A, \rho^*)),\right.$
$T - \exp(-s_2(K_0, C, C, C, \rho^*)), T - \exp(-s_2(K_0, A, 1, C, \rho^*)),$
$T - \exp(-s_3(K_0, CA, CA^2, CA, C(K_0)C_0', \rho^*)),$
$\left. T - \exp(-s_3(K_0, CA, C, 1, 1, \rho^*)), T - \exp(-s_5(K_0, A)), t_6(K_0, \epsilon_0, A)\right)$,
$\delta_0 \leq \frac{1}{2}\hat{k}(1)$, $\alpha_0 > 0$, $C_0 > 0$, $C_0' \geq 0$, $\eta_0 \leq \min(\eta_1(\epsilon_0), \eta_6(\epsilon_0))$.

We consider $\sigma \geq -\log(T-t_0)$ and $\rho \leq \rho^*$, and suppose that $\forall t \in [T - e^{-\sigma}, T - e^{-(\sigma+\rho)}]$, $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, t)$. Using the definition of $S^*(t)$, and the lemmas of Step 1, we start the proof of the estimates of lemma 3.2.

Below, $O(f)$ stands for a function bounded by f and not by Cf. We use the notations introduced in (34).

I) Equation (54)

Since $q_m'(s) = \frac{d}{ds}\int \chi(y,s)k_m(y)q(y,s)d\mu = \int \frac{\partial}{\partial s}(\chi q)k_m d\mu$, we obtain:
$|\int \chi(y,s)k_m(y)\frac{\partial q}{\partial s}(y,s)d\mu - q_m'(s)| = |\int \frac{\partial \chi}{\partial s}(y,s)k_m(y)q(y,s)d\mu|$

$\leq C\frac{A^2K_0^3}{s^{1/2}}e^{-2s}$ by lemma B.1, (25) and (26). If $s_0 \geq s_{12}(K_0, A)$, then (57) follows.

Since $\mathcal{L}$ is self adjoint and $\mathcal{L}k_m = (1-\frac{m}{2})k_m$, there exist two polynomials P_m and Q_m such that $|\int \chi(y,s)k_m(y)\mathcal{L}q(y,s)d\mu - (1-\frac{m}{2})q_m(s)| =$
$|\int [\mathcal{L}(\chi(s)k_m) - \chi(s)k_m]q(s)d\mu| = |\int (\frac{\partial\chi}{\partial y}P_m(y) + \frac{\partial^2\chi}{\partial y^2}Q_m(y))q(s)d\mu|$
$\leq CA^2K_0^3 s^{-1/2}e^{-2s}$ by lemma B.1, (25) and (26). Therefore,
$|\int \chi(y,s)k_m(y)\mathcal{L}q(y,s)d\mu| \leq e^{-s}$ if $s_0 \geq s_{13}(K_0, A)$, which yields (58).

From (54), $|V(y,s)| \leq Cs^{-1}(1+|y|^2)$. Therefore,
$|\int \chi(y,s)k_m(y)V(y,s)d\mu| \leq CA^2s^{-3}\log s \leq s^{-5/2}$ for $s \geq s_{34}(A)$, by lemma B.1 and (25). This yields (59).

From lemmas B.3 and B.1, and (25), we have
$|\int \chi(y,s)k_m(y)B(q)(y,s)d\mu| \leq C(K_0)A^4s^{-4}(\log s)^2$.
Now, if $s_0 \geq s_{15}(K_0, A)$, then (60) follows.

By lemmas B.4 and B.1, and (25), we write:
$|\int \chi(y,s)k_2(y)T(q)(y,s)d\mu| \leq s^{-2-1/4}$ for $s_0 \geq s_{36}(K_0, A)$, which is (61).

From (54), $|V(y,s) + 2p/(s(p-a))k_2| \leq Cs^{-2}(1+|y|^4)$. Since $|q_0(s)| + |q_4(s)| \leq CAs^{-2}$ follows from $q(s) \in V_{K_0,A}(s)$, and since $\int \chi(s)k_2^2q(s)d\mu = q_2(s) + c_0q_0(s) + c_4q_4(s)$, we get (64) for $s_0 \geq s_7(A)$.

From lemma B.5, we have $|R(y,s)| \leq C(s^{-2} + s^{-3}|y|^4)$. Using (25), we get (62).

From lemma B.6, we have $|e^{-\frac{ps}{p-1}}f_1(e^{\frac{s}{p-1}}(\varphi+q))| \leq Ce^{-\frac{ps}{p-1}}$. Therefore, as before, $|\int \chi(y,s)k_m(y)e^{-\frac{ps}{p-1}}f_1(e^{\frac{s}{p-1}}(\varphi+q))d\mu| \leq Ce^{-\frac{ps}{p-1}} \leq e^{-s}$ for s large and (63) follows.

From (54), $|V(y,s) + 2p/(s(p-a))k_2| \leq Cs^{-2}(1+|y|^4)$. Since $|q_0(s)| + |q_4(s)| \leq CAs^{-2}$ follows from $q(s) \in V_{K_0,A}(s)$, and since $\int \chi(s)k_2^2q(s)d\mu = q_2(s) + c_0q_0(s) + c_4q_4(s)$, we get (64) for $s_0 \geq s_7(A)$.

By lemmas B.4 and B.1, and (25), we write:
$|\int \chi(y,s)k_2(y)T(q)(y,s)d\mu + E| \leq s^{-3}$ for $s_0 \geq s_{16}(K_0, A, C_0')$, where
$E = a/4\int \nabla q(y,s)(\chi(y,s)\frac{\nabla\varphi}{\varphi}(y^2-2)e^{-|y|^2/4}/\sqrt{4\pi})dy$
$-a/4\int q(y,s)\nabla.(\chi(y,s)\nabla\varphi/\varphi(y^2-2)e^{-|y|^2/4}/\sqrt{4\pi})dy$
$= O(e^{-s}) - a/4\int q(y,s)\chi(y,s)\nabla.(\nabla\varphi/\varphi(y^2-2)e^{-|y|^2/4}/\sqrt{4\pi})dy.$
By simple calculation,
$|\nabla.(\nabla\varphi/\varphi(y^2-2)e^{-|y|^2/4}/\sqrt{4\pi}) - (h_2(y) + h_4(y)/4)/(s(p-a)).e^{-|y|^2/4}/\sqrt{4\pi}|$
$\leq P(|y|)e^{-|y|^2/4}/s^2$ where P is a polynomial. Hence $E = O(CA^2s^{-4}\log s) -$

$a/(4s(p-a))(8q_2(s)+c_4q_4(s)) = O(CAs^{-3}) - 2a/(s(p-a))q_2(s)$ and (65) holds.

(66) follows from lemma B.5, (26) and (25).

II) Equation (55)

(67) and (68) follow from lemma B.2 *ii*) applied with $A' = A''' = A$ and $A'' = A^2$.

Lemmas B.3 and B.1 yield
$|B(q(x,\tau))| \le C|q(x,\tau)|^{\bar p} \le CA^{2\bar p}\tau^{-2\bar p}(\log\tau)^{\bar p}(1+|x|^3)^{\bar p}$.
Lemmas B.4 and B.1 yield
$|T(q(x,\tau))| \le |\chi(x,\tau)T(q(x,\tau))| + |(1-\chi(x,\tau))T(q(x,\tau))|$
$\le C(K_0,A)\tau^{-5/2}\log\tau(1+|x|^3) + C(K_0,C_0')\tau^{-5/2}|x|^3$.
Therefore,

$$|B(q(\tau)) + T(q(\tau))| \le C(K_0,A,C_0')\left\{\frac{(\log\tau)^{\bar p}}{\tau^{2\bar p}}(1+|x|^{3\bar p}) + \frac{\log\tau}{\tau^{5/2}}(1+|x|^3)\right\}. \tag{135}$$

This way, $|\beta(y,s)| = |\int_\sigma^s d\tau K(s,\tau)\left(B(q(\tau)) + T(q(\tau))\right)|$
$\le \int_\sigma^s d\tau \int dx |K(s,\tau,y,x)|\,|B(q(x,\tau)) + T(q(\tau))|$
$\le C(K_0,A,C_0')\int_\sigma^s d\tau \left\{\tau^{-2\bar p}(\log\tau)^{\bar p}\int dx|K(s,\tau,y,x)|(1+|x|^{3\bar p})\right.$
$\left. +\tau^{-5/2}\log\tau\int dx|K(s,\tau,y,x)|(1+|x|^3)\right\}$
$\le C(K_0,A,C_0')(s-\sigma)e^{s-\sigma}\left\{s^{-2\bar p}(\log s)^{\bar p}(1+|y|^{3\bar p}) + s^{-5/2}\log s(1+|y|^3)\right\}$
if $s_0 \ge \rho^*$ (Indeed, $s \le \sigma+\rho \le \sigma+\rho^* \le \sigma+s_0 \le 2\sigma \le 2\tau$, and lemma B.2 applies). Hence,
$|\chi(y,s)\beta(y,s)| \le C(K_0,A,C_0')(s-\sigma)e^{s-\sigma}\left\{s^{-2\bar p}(\log s)^{\bar p}(1+|y|^3|y|^{3\bar p-3})\right.$
$\left. +s^{-5/2}\log s(1+|y|^3)\right\}$
$\le C(K_0,A,C_0')(s-\sigma)e^{s-\sigma}\left\{s^{-2\bar p}(\log s)^{\bar p}(1+|y|^3(K_0\sqrt{s})^{3\bar p-3})\right.$
$\left. +s^{-5/2}\log s(1+|y|^3)\right\} \le (s-\sigma)s^{-2}(1+|y|^3)$, if $s_0 \ge s_{17}(K_0,A,\rho^*,C_0')$ (use $\bar p > 1$). This yields $|\beta_m(s)| \le C(s-\sigma)s^{-2}$ for $m = 0,1,2$ and then (69).

Lemmas B.3 and B.1 yield $|B(q(x,\tau))| \le C|q(x,\tau)|^{\bar p} \le CK_0^{3\bar p}A^{2\bar p}\tau^{-\bar p/2}$.
Lemmas B.4 and B.1 yield
$|T(q(x,\tau))| \le C(K_0,A)\tau^{-1} + C(K_0,C_0')\tau^{-1}$.
Therefore, $|B(q(\tau)) + T(q(\tau))| \le C(K_0,A,C_0')\tau^{-\bar p/2}$.
This way, $|\int_\sigma^s d\tau K(s,\tau)(B(q(\tau)) + T(q(\tau)))|$
$\le \int_\sigma^s d\tau\int dx|K(s,\tau,y,x)||B(q(x,\tau)) + T(q(x.\tau))|$
$\le C(K_0,A,C_0')\int_\sigma^s d\tau\tau^{-\bar p/2}\int dx|K(s,\tau,y,x)|$
$\le C(K_0,A,C_0')s^{-\bar p/2}(s-\sigma)e^{s-\sigma}$ if $s_0 \ge \rho^*$ (Indeed, $s \le 2\tau$ and lemma B.2

applies). Hence $|\beta_e(y,s)| \le C(K_0,A,C_0')s^{-\bar p/2}(s-\sigma)e^{\rho^*} \le (s-\sigma)s^{-1/2}$ if $s_0 \ge s_{18}(A,\rho^*,C_0')$ (use $\bar p > 1$). This yields (70).

Lemma B.5 implies that $\forall \tau > 1$, $\forall x \in \mathbb{R}$, $|R_m(\tau)| \le C\tau^{-2}$, $m = 0,1$, $|R_2(\tau)| \le C\tau^{-2}\log\tau$, $|R_-(x,\tau)| \le C\tau^{-2}(1+|x|^3)$ and $|R_e(x,\tau)| \le C\tau^{-1/2}$. Applying lemma B.2 *ii)* with $A' = A'' = A''' = C$ and then integrating with respect to $\tau \in [\sigma,s]$ yields (71) and (72).

From lemma B.6, we have $|e^{-\frac{p\tau}{p-1}} f_1(e^{\frac{\tau}{p-1}}(\varphi+q))| \le Ce^{-\frac{p\tau}{p-1}}$. Therefore, $|\delta(y,s)| = |K(s,\tau)e^{-\frac{p\tau}{p-1}} f_1(e^{\frac{\tau}{p-1}}(\varphi+q))| \le Ce^{s-\tau}e^{-\frac{p\tau}{p-1}}$ according to *i)* of lemma B.2. Hence,
$|\int_\sigma^s K(s,\tau)e^{-\frac{p\tau}{p-1}} f_1(e^{\frac{\tau}{p-1}}(\varphi+q))| \le C(s-\sigma)e^{s-\sigma}e^{-\frac{p\sigma}{p-1}}$
$\le C(s-\sigma)e^{\rho^*}e^{-\frac{p}{p-1}\frac{s}{2}}$ if $s_0 \ge \rho^*$,
$\le (s-\sigma)s^{-2}$ if $s \ge s_{19}(A,\rho^*)$. As before, this implies (74) and (75).

From lemma 3.1 we have $|q_m(s_0)| \le As_0^{-2}$, $m = 0,1$,
$|q_2(s_0)| \le s_0^{-2}\log s_0$, $|q_-(y,s_0)| \le Cs_0^{-2}(1+|y|^3)$ and $|q_e(y,s_0)| \le s_0^{-1/2}$. If we apply lemma B.2 *ii)* with $A' = A$, $A'' = 1$, $A''' = C$, then (76) and (77) follow.

III) Equation (56)

From definition 34, we have for $m = 0,1$,
$r_m(\sigma) = \int \nabla q(y,\sigma)\chi(y,\sigma)k_m(y)d\mu$
$= -\int q(y,\sigma)\nabla(\chi(y,\sigma)k_m e^{-y^2/4}/\sqrt{4\pi})dy$
$= O(e^{-\sigma}) - \int q(y,\sigma)\chi(y,\sigma)\nabla(k_m e^{-y^2/4}/\sqrt{4\pi})dy$
$= O(e^{-\sigma}) + (m+1)\int q(y,\sigma)\chi(y,s)k_{m+1}(y)d\mu = O(e^{-\sigma}) + (m+1)q_{m+1}(\sigma)$.
Hence, if $\sigma \ge s_0 \ge s_{21}$, then $|r_0(\sigma)| \le CA\sigma^{-2}$ and $|r_1(\sigma)| \le CA^2\sigma^{-2}\log\sigma$. We have $|r_\perp(y,\sigma)| \le A\sigma^{-2}(1+|y|^3)$ since $q(\sigma) \in V_{K_0,A}(\sigma)$ (see the definition of $S^*(t)$), and $|r_e(y,\sigma)| \le C(K_0)C_0'\sigma^{-1/2}$ by lemma B.1. Now, we apply lemma B.2 *iii)* with $A' = A''' = CA$, $A'' = CA^2$ and $A'''' = C(K_0)C_0'$ to conclude the proof of (78)

Estimate (79) is harder than estimate (78) because it involves a parabolic estimate on the kernel K_1.

Setting $I(x,\tau) = B(q(x,\tau)) + T(q(x,\tau))$, we write
$K_1(s,\tau)\frac{\partial}{\partial y}(B(q)+T(q))(\tau) = \int dx e^{(s-\tau)(\mathcal{L}-1/2)}(y,x)E(s,\tau,y,x)\frac{\partial I}{\partial x}(x,\tau)$
$= (I) + (II)$ with $(I) = -\int dx\partial_x e^{(s-\tau)(\mathcal{L}-1/2)}(y,x)E(s,\tau,y,x)I(x,\tau)$ and
$(II) = -\int dx e^{(s-\tau)(\mathcal{L}-1/2)}(y,x)\partial_x E(s,\tau,y,x)I(x,\tau)$. Let us first bound (I).
From (128), $(I) =$
$\int dx \frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}} \frac{2(x-ye^{-(s-\tau)/2})}{4\pi(1-e^{-(s-\tau)})} \exp\left(-\frac{(ye^{-(s-\tau)/2}-x)^2}{4\pi(1-e^{-(s-\tau)})}\right) E(s,\tau,y,x)I(x,\tau)$.
If $s_0 \ge \rho^*$, then $0 \le E(s,\tau,y,x) \le C$ (use for this $V(x,\tau) \le C\tau^{-1}$ which is a

consequence of (54), (126), $d\mu_{yx}^{s-\tau}$ is a probability and $s \le \sigma+\rho \le \sigma+\rho^* \le \sigma+s_0 \le 2\sigma \le 2\tau$). Using (135), we get
$|(I)| \le C(K_0,A,C_0')\frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}}\int \frac{dx}{\sqrt{4\pi(1-e^{-(s-\tau)})}} \times \frac{2|ye^{-(s-\tau)/2}-x|}{\sqrt{4\pi(1-e^{-(s-\tau)})}}\times$
$\exp\left(-\frac{(ye^{-(s-\tau)/2}-x)^2}{4\pi(1-e^{-(s-\tau)})}\right)\left(\tau^{-2\bar p}(\log\tau)^{\bar p}(1+|x|^{3\bar p})+\tau^{-5/2}\log\tau(1+|x|^3)\right)$
where $\bar p=\min(p,2)>1$. With the change of variables $\xi = \frac{x-ye^{-(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}}$,
$|(I)| \le C(K_0,A,C_0')\frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}}\{\tau^{-2\bar p}(\log\tau)^{\bar p}\times$
$\int d\xi|\xi|e^{-\xi^2}(1+|\xi\sqrt{4\pi(1-e^{-(s-\tau)})}-ye^{-(s-\tau)/2}|^{3\bar p})+\tau^{-5/2}\log\tau\times$
$\int d\xi|\xi|e^{-\xi^2}(1+|\xi\sqrt{4\pi(1-e^{-(s-\tau)})}-ye^{-(s-\tau)/2}|^3)\Big\}$, hence $|(I)|$

$$\le C(K_0,A,C_0')\frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}}\left\{\frac{(\log\tau)^{\bar p}}{\tau^{2\bar p}}(1+|y|^{3\bar p})+\frac{\log\tau}{\tau^{5/2}}(1+|y|^3)\right\}. \tag{136}$$

Let us bound (II) now. Using the integration by parts formula for Gaussian measures (see [11]), we have $\partial_x E(s,\sigma,y,x)$:

$$= \frac{1}{2}\int_0^{s-\tau}\int_0^{s-\tau} d\tau_1 d\tau_2 \partial_x\Gamma(\tau_1,\tau_2)\int d\mu_{yx}^{s-\tau}(\omega)V'(\omega(\tau_1),\sigma+\tau_1)\times$$
$$V'(\omega(\tau_2),\sigma+\tau_2)e^{\int_0^{s-\tau}d\tau_3 V(\omega(\tau_3),\sigma+\tau_3)} \tag{137}$$
$$+\frac{1}{2}\int_0^{s-\tau} d\tau_1\partial_x\Gamma(\tau_1,\tau_1)\int d\mu_{yx}^{s-\tau}(\omega)V''(\omega(\tau_1),\sigma+\tau_1)e^{\int_0^{s-\tau}d\tau_3 V(\omega(\tau_3),\sigma+\tau_3)}.$$

By (54), we have $|\frac{\partial^n V}{\partial y^n}| \le Cs^{-n/2}$ for $n=0,1,2$. Combining this with (127) and (137), we get (for $s_0 \ge \rho^*$)
$|\partial_x E(s,\sigma,y,x)| \le Cs^{-1}(s-\tau)(1+s-\tau)(|y|+|x|)$.
Using this, (128) and (135), we obtain
$|(II)| \le e^{(s-\tau)/2}\int \frac{dx}{\sqrt{4\pi(1-e^{-(s-\tau)})}}\exp\left(-\frac{(y^{-(s-\tau)/2}-x)^2}{4\pi(1-e^{-(s-\tau)})}\right)(|y|+|x|)\times$
$Cs^{-1}(s-\tau)(1+s-\tau)C(K_0,A,C_0')\{\tau^{-2\bar p}(\log\tau)^{\bar p}(1+|x|^{3\bar p})$
$+\tau^{-5/2}\log\tau(1+|x|^3)\Big\}$.
Arguing as for (I), we get:

$$|(II)| \le C(K_0,A,C_0')e^{(s-\tau)/2}(s-\tau)(1+s-\tau)s^{-1}(1+|y|)\times$$
$$\left\{\frac{(\log\tau)^{\bar p}}{\tau^{2\bar p}}(1+|y|^{3\bar p})+\frac{\log\tau}{\tau^{5/2}}(1+|y|^3)\right\}. \tag{138}$$

Combining (136) and (138), we obtain
$|\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial I}{\partial y}(\tau)| \le C(K_0,A,C_0')\left\{s^{-2\bar p}(\log s)^{\bar p}(1+|y|^{3\bar p})\right.$
$\left.+s^{-5/2}\log s(1+|y|^3)\right\}\times$
$\int_\sigma^s \left\{\frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)/2})}} + e^{(s-\tau)/2}(s-\tau)(1+s-\tau)s^{-1}(1+|y|)\right\}d\tau$
$\le C(K_0,A,C_0')\left\{s^{-2\bar p}(\log s)^{\bar p}(1+|y|^{3\bar p}) + s^{-5/2}\log s\right\}\times$
$(e^{s-\sigma}\sqrt{s-\sigma}+e^{(s-\sigma)/2}((s-\sigma)^2+(s-\sigma)^3)s^{-1}(1+|y|))$ ($s_0 \ge \rho^*$, which implies $2\tau \ge s$). Multiplying this by $\chi(y,s)$ and replacing some $|y|$ by $2K_0\sqrt{s}$, we get: $\forall s \in [\sigma, \sigma+\rho]$,
$|\chi(y,s)\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial I}{\partial y}(\tau)| \le C(K_0,A,C_0')\left\{s^{-(\bar p+3)/2} + s^{-5/2}\right\}(1+|y|^3)\times$
$\sqrt{s-\tau}(e^{\rho^*} + e^{\rho^*/2}(\rho^{*3/2}+\rho^{*5/2})s^{-1/2})$. If $s \ge s_0 \ge s_{22}(A,\rho^*)$, then
$|\chi(y,s)\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial I}{\partial y}(\tau)| \le Cs^{-2}\sqrt{s-\tau}(1+|y|^3)$ (use $\bar p > 1$). Therefore,
$|P_\perp(\chi(y,s)\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial I}{\partial y}(\tau))| \le Cs^{-2}\sqrt{s-\tau}(1+|y|^3)$.
This concludes the proof of (79).

By definition, $R_1(x,\tau) = \frac{\partial R}{\partial y}(x,\tau) + \frac{\partial V}{\partial y}q(x,\tau)$. From (54), we have $|\frac{\partial V}{\partial y}(x,\tau)| = 2pb\varphi(x,\tau)^{p-2}(p-1+bx^2/\tau)^{-p/(p-1)}x\tau^{-1}$ with
$b=(p-1)^2/(4(p-a))$. Setting $z = x\tau^{-1/2}$, we easily see that
$|\frac{\partial V}{\partial y}(x,\tau)| \le C\tau^{-1/2}$. Using lemmas B.1 and B.5, we get
$|R_1(x,\tau)| \le C\tau^{-(1+\bar p)}(|x|+|x|^3) + CA^2\tau^{-5/2}\log\tau(1+|x|^3)$
$\le C\tau^{-(2+\epsilon_2(p))}(1+|x|^3)$ with $\epsilon_2(p) > 0$ if $s_0 \ge s_{33}(A)$. Therefore,
$|K_1(s,\tau)R_1(\tau)| = |\int K_1(s,\tau,y,x)R_1(x,\tau)dx|$
$\le C\tau^{-(2+\epsilon_2(p))}\int |K_1(s,\tau,y,x)(1+|x|^3)dx$
$\le C\tau^{-(2+\epsilon_2(p))}e^{(s-\tau)/2}(1+|y|^3)$ by lemma B.2 i). Hence,
$|\int_s^\sigma d\tau K_1(s,\tau)R_1(\tau)| \le C(1+|y|^3)\int_\sigma^s d\tau\tau^{-(2+\epsilon_2(p))}e^{(s-\tau)/2}$
$\le C(s-\sigma)e^{(s-\sigma)/2}s^{-(2+\epsilon_2(p))}(1+|y|^3)$ if $\sigma \ge s_0 \ge \rho^*$.
Now, if $\sigma \ge s_0 \ge s_{23}(\rho^*)$, then
$|\int_s^\sigma d\tau K_1(s,\tau)R_1(\tau)| \le C(s-\sigma)e^{\rho^*/2}s^{-(2+\epsilon_2(p))}(1+|y|^3)$
$\le (s-\sigma)s^{-2}(1+|y|^3)$. By classical arguments, this yields (80).

From lemmas B.2 and B.6, and the fact that $\left|\frac{\partial\varphi}{\partial y}\right| \le C\tau^{-1/2}$, we have:
$|e^{-\tau}(\frac{\partial\varphi}{\partial y}+r)f_1'(e^{\frac{\tau}{p-1}}(\varphi+q)| \le C(K_0,C_0')A^2\tau^{-1/2}e^{-\tau}$. Therefore, i) of lemma B.2 yields:
$|K_1(s,\tau)e^{-\tau}(\frac{\partial\varphi}{\partial y}+r)f_1'(e^{\frac{\tau}{p-1}}(\varphi+q))| \le C(K_0,C_0')A^2e^{\frac{s-\tau}{2}}\tau^{-1/2}e^{-\tau}$. Hence,
$|\int_\sigma^s d\tau K_1(s,\tau)e^{-\tau}(\frac{\partial\varphi}{\partial y}+r)f_1'(e^{\frac{\tau}{p-1}}(\varphi+q))| \le C(K_0,C_0')A^2(s-\sigma)e^{\frac{s-\sigma}{2}}\frac{e^{-\sigma}}{\sqrt{\sigma}}$
$\le C(K_0,C_0')A^2(s-\sigma)e^{\rho^*}s^{-1/2}e^{-\frac{s}{2}}$ if $s_0 \ge \rho^*$
$\le (s-\sigma)s^{-2}$ if $s \ge s_{24}(K_0,A,\rho^*)$. Thus, by classical arguments, (81) follows.

Since $r_m(s_0) = O(e^{-s_0}) + (m+1)q_{m+1}(s_0)$, we have from lemma 3.1 $|r_0(s_0)| \leq CAs_0^{-2}$, $|r_1(s_0)| \leq Cs_0^{-2}\log s_0$, $|r_\perp(y,s_0)| \leq s_0^{-2}(1+|y|^3)$ and $|r_e(y,s_0)| \leq s_0^{-1/2}$. Applying lemma *iii)* of B.2 with $A' = CA$, $A'' = C$, $A''' = A'''' = 1$ yields (82).

References

[1] Ball, J., *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations,* Quart. J. Math. Oxford 28, 1977, pp. 473-486.

[2] Berger, M., and Kohn, R., *A rescaling algorithm for the numerical calculation of blowing-up solutions*, Comm. Pure Appl. Math. 41, 1988, pp. 841-863.

[3] Bricmont, J., and Kupiainen, A., *Renormalization group and nonlinear PDEs,* Quantum and non-commutative analysis, past present and future perspectives, Kluwer (Boston), 1993.

[4] Bricmont, J., and Kupiainen, A., *Universality in blow-up for nonlinear heat equations,* Nonlinearity 7, 1994, pp. 539-575.

[5] Chapman, S.J., Hunton, B.J., and Ockendon, J.R., *Vortices and Boundaries*, Quart. Appl. Math. to appear.

[6] Deng, K., Levine, H.A., *On the blow-up of u_t at quenching*, Proc. Am. Math. Soc. 106, 1989, pp. 1049-1056

[7] Filippas, S., and Kohn, R., *Refined asymptotics for the blowup of $u_t - \Delta u = u^p$,* Comm. Pure Appl. Math. 45, 1992, pp. 821-869.

[8] Giga, Y., and Kohn, R., *Asymptotically self-similar blowup of semilinear heat equations,* Comm. Pure Appl. Math. 38, 1985, pp. 297-319.

[9] Giga, Y., and Kohn, R., *Characterizing blowup using similarity variables,* Indiana Univ. Math. J. 36, 1987, pp. 1-40.

[10] Giga, Y., and Kohn, R., *Nondegeneracy of blow-up for semilinear heat equations,* Comm. Pure Appl. Math. 42, 1989, pp. 845-884.

[11] Glimm, J., Quantum Physics, Springer, New York, 1981.

[12] Guo, J., *On the quenching behavior of the solution of a semilinear parabolic equation*, J. Math. Analysis Applic. 151, 1990, pp 58-79.

[13] Guo, J., *On the quenching rate estimate*, Quart. Appl. Math. 49, 1991, pp. 747-752.

[14] Guo, J., *On the semilinear elliptic equation* $\Delta w - \frac{1}{2}y.\nabla w + \lambda w - w^{-\beta} = 0$ *in* $\mathbb{R}^n$, Chinese J. Math. 19, no 4, 1991, pp. 355-377.

[15] Herrero, M.A, and Velazquez, J.J.L., *Blow-up profiles in one-dimensional semilinear parabolic problems,* Com. PDE 17, 1992 pp. 205-219.

[16] Kavian, O., *Remarks on the large time behavior of a nonlinear diffusion equation*, Ann. Inst. H. Poincaré Analyse Nonlinéaire 4, 1987, pp. 423-452.

[17] Keller, J.B., Lowengrub, J.S., *Straight forward asymptotics and numerci for blowing-up solutions to semilinear heat equations*, Proc. from NATO Sing. Conf. 1992

[18] Levine, H.A., *Advances in quenching*, Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), pp. 319-346.

[19] Levine H.A., *Quenching, nonquenching, and beyond quenching for solutions of some parabolic equations,* Annali Mat. Pura. Appl. 155, 1989, pp. 243-260.

[20] Levine, H., *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + F(u)$, Arch. Rat. Mech. Anal. 51, 1973, pp. 371-386.

[21] Merle, F., *Solution of a nonlinear heat equation with arbitrary given blow-up points,* Comm. Pure Appl. Math. 45, 1992 pp. 263-300.

[22] Merle, F., and Zaag, H., *Stability of blow-up profile for equation of the type* $u_t = \Delta u + |u|^{p-1}u$, Duke Math. J. . 86, 1997, pp. 143-195.

[23] Souplet, P., Tayachi, S., and Weissler, F., B., *Exact self-similar blow-up solutions of a semilinear parabolic equation with a nonlinear gradient term*, preprint 96-14 of Paris-Nord University.

[24] Velazquez, J.J.L., *Classification of singularities for blowing up solutions in higher dimensions* Trans. Amer. Math. Soc. 338, 1993, pp. 441-464.

[25] Zaag, H., *Blow-up results for vector-valued nonlinear heat equations with no gradient structure*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.

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