On characteristic points at blow-up for a semilinear wave equation in one space dimension*

Frank Merle

Université de Cergy Pontoise, IHES and CNRS

Hatem Zaag

CNRS LAGA Université Paris 13

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We consider the one dimensional semilinear wave equation

$$\begin{cases} \partial^2_{tt} u = \partial^2_{xx} u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{1}$$

where $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $p > 1$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}$ with $\|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} = \sup\limits_{a\in\mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$ and $\|v\|^2_{\mathrm{H}^1_{\mathrm{loc,u}}} = \|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\nabla v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}}$.

The Cauchy problem for equation (1) in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$ follows from the finite speed of propagation and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$ (see Ginibre, Soffer and Velo [5]). If the solution is not global in time, then we call it a blow-up solution. The existence of blow-up solutions is guaranteed by ODE techniques, or also by the following blow-up criterion from Levine [8]:

If $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} \left(\frac{1}{2}|u_1(x)|^2 + \frac{1}{2}|\partial_x u_0(x)|^2 - \frac{1}{p+1}|u_0(x)|^{p+1} \right) dx < 0,$$

then the solution of (1) cannot be global in time.

More blow-up results can be found in Caffarelli and Friedman [4], [3], Alinhac [1] and Kichenassamy and Litman [6], [7].

If u is a blow-up solution of (1), we define (see for example Alinhac [1]) a 1-Lipschitz curve $\Gamma = \{(x, T(x))\}$ such that u cannot be extended beyond the set called the maximal influence domain of u:

$$D = \{(x,t) \mid t < T(x)\}. \tag{2}$$

$\bar{T} = \inf_{x\in\mathbb{R}} T(x)$ and Γ are called the blow-up time and the blow-up graph of u. A point x_0 is a non characteristic point (or a *regular* point) if

$$\text{there are } \delta_0 \in (0,1) \text{ and } t_0 < T(x_0) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0,T(x_0),\delta_0} \cap \{t \geq t_0\} \tag{3}$$

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where $\mathcal{C}_{\bar{x},\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x-\bar{x}|\}$. If not, then we call x_0 a characteristic point (or a *singular* point). Naturally, we denote by $\mathcal{R}$ (resp. $\mathcal{S}$) the set of non characteristic (resp. characteristic) points. Note then that

$$\mathcal{R} \cup \mathcal{S} = \mathbb{R}.$$

In our papers [11], [13], [12], [14], [15] and [16], we made several contributions to the study of blow-up solutions of (1), namely the description of its blow-up graph and blow-up behavior in selfsimilar variables.

1 The blow-up graph of equation (1)

It is clear from rather simple arguments that $\mathcal{R} \neq \emptyset$ for *any* blow-up solution $u(x,t)$ (if $T(x)$ achieves its minimum at x_0, then $x_0 \in \mathcal{R}$; if the infiimum is at infinity, then see the remark following Theorem 1 in [15]). On the contrary, the situation was unclear for $\mathcal{S}$, and it was commonly conjectured before our contributions that $\mathcal{S}$ was empty. In particular, that was the case in the examples constructed by Cafarelli and Friedman in [4] and [3]. In [16], we prove that the conjecture was false. More precisely, we proved the following:

Proposition 1 (Existence of initial data with $\mathcal{S} \neq \emptyset$) *If the initial data (u_0, u_1) is odd and $u(x,t)$ blows up in finite time, then $0 \in \mathcal{S}$.*

For general blow-up solutions, we proved the following facts about $\mathcal{R}$ and $\mathcal{S}$ in [15] and [16] (see Theorem 1 (and the following remark) in [15], see Propositions 5 and 8 in [16]):

Theorem 2 (Geometry of the blow-up graph)
(i) $\mathcal{R}$ is a non empty open set, and $x \mapsto T(x)$ is of class C^1 on $\mathcal{R}$;
(ii) $\mathcal{S}$ is a closed set with empty interior, and given $x_0 \in \mathcal{S}$, if $0 < |x - x_0| \le \delta_0$, then

$$0 < T(x) - T(x_0) + |x - x_0| \le \frac{C_0 |x - x_0|}{|\log(x - x_0)|^{\frac{(k(x_0)-1)(p-1)}{2}}} \tag{4}$$

for some $\delta_0 > 0$ and $C_0 > 0$, where $k(x_0) \ge 2$ is an integer. In particular, $T(x)$ is right and left differentiable at x_0, with $T_l'(x_0) = 1$ and $T_r'(x_0) = -1$.

2 Asymptotic behavior near the blow-up graph

As one may guess from the above description, the asymptotic behavior will not be the same on $\mathcal{R}$ and on $\mathcal{S}$. In both cases, we need to use the similarity variables which we recall in the following. Let us stress the fact that the keystone of our work is the existence of a Lyapunov functional in similarity variables.

Given some (x_0, T_0) such that $0 < T_0 \le T(x_0)$, we introduce the following self-similar change of variables:

$$w_{x_0,T_0}(y,s) = (T_0 - t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t). \tag{5}$$

If $T_0 = T(x_0)$, then we simply write w_{x_0} instead of $w_{x_0,T(x_0)}$. The function $w = w_{x_0,T_0}$ satisfies the following equation for all $y \in B = B(0,1)$ and $s \geq -\log T_0$:

$$\partial^2_{ss} w = \mathcal{L} w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial^2_{y,s} w \tag{6}$$

$$\text{where } \mathcal{L} w = \frac{1}{\rho} \partial_y \left(\rho (1-y^2) \partial_y w \right) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \tag{7}$$

From Antonini and Merle [2], we know the existence of the following Lyapunov functional for equation (6):

$$E(w) = \int_{-1}^{1} \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy, \tag{8}$$

defined for $(\partial_s w, w) \in \mathcal{H}$ where

$$\mathcal{H} = \left\{ q \mid \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \tag{9}$$

From Proposition 1 in [14], we know that the only stationary solutions of (6) in the space $\mathcal{H}$ are $q \equiv 0$ or $w(y) \equiv \pm \kappa(d,y)$, where $d \in (-1,1)$ and

$$\kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} \text{ and } |y| < 1. \tag{10}$$

Note that the set of stationary solutions is made of 3 connected components. One wanders whether these stationary solutions are good candidates for the convergence of $w_{x_0}(y,s)$. As a matter of fact, that is the case when $x_0 \in \mathcal{R}$ as we see from the following result (see Corollary 4 in [14] and Theorem 6 in [16]):

Theorem 3 (Asymptotic behavior near the blow-up graph)
(i) **Case where $x_0 \in \mathcal{R}$: Existence of an asymptotic profile**. *There exist $\delta_0(x_0) > 0$, $d(x_0) \in (-1,1)$, $|\theta(x_0)| = 1$, $s_0(x_0) \geq -\log T(x_0)$ such that for all $s \geq s_0$:*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0), .) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0 (s-s_0)} \tag{11}$$

for some positive μ_0 and C_0 independent from x_0. Moreover, $E(w_{x_0}(s)) \to E(\kappa_0)$ as $s \to \infty$.
(ii) **Case where $x_0 \in \mathcal{S}$: decomposition into a sum of decoupled solitons**. *It holds that*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k(x_0)} e_i^* \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ and } E(w_{x_0}(s)) \to k(x_0) E(\kappa_0) \tag{12}$$

as $s \to \infty$, for some

$$k(x_0) \geq 2, \ \ e_i^* = e_1^* (-1)^{i+1}$$

and continuous $d_i(s) = -\tanh \zeta_i(s) \in (-1,1)$ for $i = 1, ..., k(x_0)$. Moreover, for some $C_0 > 0$, for all $i = 1, ..., k(x_0)$ and s large enough,

$$\left|\zeta_i(s) - \left(i - \frac{(k(x_0)+1)}{2}\right)\frac{(p-1)}{2}\log s\right| \le C_0. \tag{13}$$

Remark: Note that some elements in the description in similarity variables given above have a geometrical interpretation:
- when $x_0 \in \mathcal{R}$, the paramter $d(x_0)$ appearing in (11) is equal to the slope $T'(x_0)$ of the blow-up curve (hence $d(x_0) = T'(x_0)$);
- when $x_0 \in \mathcal{S}$, $k(x_0) \ge 2$ is the number of solitons in the decomposition (12) appears also in the upper bound estimate on $T(x)$ for x near x_0 given in (4).
Remark: The proof of the convergence in (i) has two major difficulties:
- the linearized operator of equation (6) around the profile $\kappa(d,y)$ is not selfadjoint, which makes the standard tools unefficient for the control of the negative part of the spectrum. Fortunately, the Lyapunov functional structure will be useful in this matter;
- all the non zero stationary solutions of equation (6) are non isolated, which generates a null eigenvalue difficult to control in the linearization of equation (6) around $\kappa(d,y)$. A modulation technique is then used to overcome this difficulty.

Extending the definition of $k(x_0)$ defined for $x_0 \in \mathcal{S}$ after (12) by setting

$$k(x_0) = 1 \text{ for all } x_0 \in \mathcal{R},$$

we proved the following energy criterion in [16] and using the monotonicity of the Lyapunov functional $E(w)$, we have the following consequence from the blow-up behavior in Theorem 3:

Proposition 2.1 (An energy criterion for non characteristic points; see Corollary 7 in [16])
(i) For all $x_0 \in \mathbb{R}$ and $s_0 \ge -\log T(x_0)$, we have

$$E(w_{x_0}(s_0)) \ge k(x_0)E(\kappa_0).$$

(ii) If for some x_0 and $s_0 \ge -\log T(x_0)$, we have

$$E(w_{x_0}(s_0)) < 2E(\kappa_0),$$

then $x_0 \in \mathcal{R}$.

3 A Liouville theorem and a trapping result near the set of stationary solutions

The following Liouville thoerem is crucial in our analysis:

Theorem 4 (A Liouville Theorem for equation (6)) *Consider $w(y,s)$ a solution to equation (6) defined for all $(y,s) \in (-\frac{1}{\delta_*}, \frac{1}{\delta_*}) \times \mathbb{R}$ such that for all $s \in \mathbb{R}$,*

$$\|w(s)\|_{H^1(-\frac{1}{\delta_*},\frac{1}{\delta_*})} + \|\partial_s w(s)\|_{L^2(-\frac{1}{\delta_*},\frac{1}{\delta_*})} \le C^* \tag{14}$$

for some $\delta_* \in (0,1)$ and $C^* > 0$. Then, either $w \equiv 0$ or w can be extended to a function (still denoted by w) defined in

$$\{(y,s) \mid -1 - T_0 e^s < d_0 y\} \supset \left(-\frac{1}{\delta_*}, \frac{1}{\delta_*}\right) \times \mathbb{R} \text{ by } w(y,s) = \theta_0 \kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(1+T_0 e^s + d_0 y)^{\frac{2}{p-1}}}, \quad (15)$$

for some $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ and $\theta_0 = \pm 1$, where κ_0 defined in (10).

Remark: Note that deriving blow-up estimates through the proof of Liouville Theorems has been successful for different problems. For the case of the heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u \qquad (16)$$

where $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $p > 1$ and $(N-2)p < N+2$, the blow-up time T is unique for equation (16). The blow-up set is the subset of $\mathbb{R}^N$ such that $u(x,t)$ does not remain bounded as (x,t) approaches (x_0,T). In [17], the second author proved the C^2 regularity of the blow-up set under a non degeneracy condition. A Liouville Theorem proved in [10] and [9] was crucially needed for the proof of the regularity result in the heat equation. The above Liouville Theorem is crucial for the regularity of the blow-up set for the wave equation (Theorem 3).

The second fundamental crucial result for our contributions is given by the following trapping result from [14] (See Theorem 3 in [14] and its proof):

Proposition 3.1 (A trapping result near the sheet $d \mapsto \kappa(d,y)$ of stationnary solutions) *There exists $\epsilon_0 > 0$ such that if $w \in C([s^*,\infty),\mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution of equation (6) such that*

$$\forall s \geq s^*, \quad E(w(s)) \geq E(\kappa_0) \text{ and } \left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d^*,\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon^* \qquad (17)$$

for some $d^* = -\tanh \xi^*$, $\omega^* = \pm 1$ and $\epsilon^* \in (0,\epsilon_0]$, then there exists $d_\infty = -\tanh \xi_\infty$ such that

$$|\xi_\infty - \xi^*| \leq C_0 \epsilon^* \text{ and } \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d_\infty,\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0. \qquad (18)$$

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Address:
Université de Cergy Pontoise, Département de mathématiques, 2 avenue Adolphe Chauvin, BP 222, 95302 Cergy Pontoise cedex, France.

`e-mail: merle@math.u-cergy.fr`

Université Paris 13, Institut Galilée, Laboratoire Analyse, Géométrie et Applications, CNRS UMR 7539, 99 avenue J.B. Clément, 93430 Villetaneuse, France.

`e-mail: Hatem.Zaag@univ-paris13.fr`