On characteristic points at blow-up for a semilinear wave equation in one space dimension

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We consider the one dimensional semilinear wave equation

\[
\begin{aligned}
\partial_{tt} u &= \partial_{xx} u + |u|^{p-1} u, \\
u(0) &= u_0 \quad \text{and} \quad u_t(0) = u_1,
\end{aligned}
\]

where \( u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}, \ p > 1, \ u_0 \in H_{loc}^1 \) and \( u_1 \in L_{loc}^2 \) with \( \|v\|_{L_{loc}^2}^2 = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 \ dx \) and \( \|v\|_{H_{loc}^1}^2 = \|v\|_{L_{loc}^2}^2 + \|\nabla v\|_{L_{loc}^2}^2 \).

The Cauchy problem for equation (1) in the space \( H_{loc}^1 \times L_{loc}^2 \) follows from the finite speed of propagation and the wellposedness in \( H^1 \times L^2 \) (see Ginibre, Soffer and Velo [5]). If the solution is not global in time, then we call it a blow-up solution. The existence of blow-up solutions is guaranteed by ODE techniques, or also by the following blow-up criterion from Levine [8]:

If \( (u_0, u_1) \in H^1 \times L^2(\mathbb{R}) \) satisfies

\[
\int_{\mathbb{R}} \left( \frac{1}{2} |u_1(x)|^2 + \frac{1}{2} |\partial_x u_0(x)|^2 - \frac{1}{p+1} |u_0(x)|^{p+1} \right) \ dx < 0,
\]

then the solution of (1) cannot be global in time.

More blow-up results can be found in Caffarelli and Friedman [4], [3], Alinhac [1] and Kichenassamy and Litman [6], [7].

If \( u \) is a blow-up solution of (1), we define (see for example Alinhac [1]) a 1-Lipschitz curve \( \Gamma = \{(x,T(x))\} \) such that \( u \) cannot be extended beyond the set called the maximal influence domain of \( u \):

\[
D = \{(x,t) \mid t < T(x)\}.
\]

\( T = \inf_{x \in \mathbb{R}} T(x) \) and \( \Gamma \) are called the blow-up time and the blow-up graph of \( u \). A point \( x_0 \) is a non characteristic point (or a regular point) if

there are \( \delta_0 \in (0,1) \) and \( t_0 < T(x_0) \) such that \( u \) is defined on \( C_{x_0,T(x_0),\delta_0} \cap \{t \geq t_0\} \) (3)

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where \( \mathcal{C}_{\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x - \bar{x}|\} \). If not, then we call \( x_0 \) a characteristic point (or a singular point). Naturally, we denote by \( \mathcal{R} \) (resp. \( \mathcal{S} \)) the set of non characteristic (resp. characteristic) points. Note then that

\[
\mathcal{R} \cup \mathcal{S} = \mathbb{R}.
\]

In our papers [11], [13], [12], [14], [15] and [16], we made several contributions to the study of blow-up solutions of (1), namely the description of its blow-up graph and blow-up behavior in selfsimilar variables.

1 The blow-up graph of equation (1)

It is clear from rather simple arguments that \( \mathcal{R} \neq \emptyset \) for any blow-up solution \( u(x,t) \) (if \( T(x) \) achieves its minimum at \( x_0 \), then \( x_0 \in \mathcal{R} \); if the infimum is at infinity, then see the remark following Theorem 1 in [15]). On the contrary, the situation was unclear for \( \mathcal{S} \), and it was commonly conjectured before our contributions that \( \mathcal{S} \) was empty. In particular, that was the case in the examples constructed by Cafarelli and Friedman in [4] and [3]. In [16], we prove that the conjecture was false. More precisely, we proved the following:

**Proposition 1 (Existence of initial data with \( \mathcal{S} \neq \emptyset \))** If the initial data \( (u_0, u_1) \) is odd and \( u(x,t) \) blows up in finite time, then \( 0 \in \mathcal{S} \).

For general blow-up solutions, we proved the following facts about \( \mathcal{R} \) and \( \mathcal{S} \) in [15] and [16] (see Theorem 1 (and the following remark) in [15], see Propositions 5 and 8 in [16]):

**Theorem 2 (Geometry of the blow-up graph)**

(i) \( \mathcal{R} \) is a non empty open set, and \( x \mapsto T(x) \) is of class \( C^1 \) on \( \mathcal{R} \);

(ii) \( \mathcal{S} \) is a closed set with empty interior, and given \( x_0 \in \mathcal{S} \), if \( 0 < |x - x_0| \leq \delta_0 \), then

\[
0 < T(x) - T(x_0) + |x - x_0| \leq \frac{C_0|x - x_0|}{\log(\delta_0)^{(k(x_0) - 1)(p - 1)}}
\]

for some \( \delta_0 > 0 \) and \( C_0 > 0 \), where \( k(x_0) \geq 2 \) is an integer. In particular, \( T(x) \) is right and left differentiable at \( x_0 \), with \( T^r_0(x_0) = 1 \) and \( T^l_0(x_0) = -1 \).

2 Asymptotic behavior near the blow-up graph

As one may guess from the above description, the asymptotic behavior will not be the same on \( \mathcal{R} \) and on \( \mathcal{S} \). In both cases, we need to use the similarity variables which we recall in the following. Let us stress the fact that the keystone of our work is the existence of a Lyapunov functional in similarity variables.

Given some \( (x_0, T_0) \) such that \( 0 < T_0 \leq T(x_0) \), we introduce the following self-similar change of variables:

\[
w_{x_0,T_0}(y,s) = (T_0 - t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t).
\]
If \( T_0 = T(x_0) \), then we simply write \( w_{x_0} \) instead of \( w_{x_0,T(x_0)} \). The function \( w = w_{x_0,T_0} \) satisfies the following equation for all \( y \in B = B(0,1) \) and \( s \geq -\log T_0 \):

\[
\partial_s^2 w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y\partial_{y,s}^2 w
\]

(6)

where \( \mathcal{L}w = \frac{1}{\rho} \partial_y \left( \rho (1 - y^2) \partial_y w \right) \) and \( \rho(y) = (1 - y^2)^{\frac{2}{p-1}} \).

(7)

From Antonini and Merle [2], we know the existence of the following Lyapunov functional for equation (6):

\[
E(w) = \int_{-1}^{1} \left( \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1 - y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy,
\]

(8)

defined for \( (\partial_s w, w) \in \mathcal{H} \) where

\[
\mathcal{H} = \left\{ q \mid \|q\|_{\mathcal{H}}^2 = \int_{-1}^{1} \left( q_1^2 + (q_1')^2 (1 - y^2) + q_2^2 \right) \rho dy < +\infty \right\}.
\]

(9)

From Proposition 1 in [14], we know that the only stationary solutions of (6) in the space \( \mathcal{H} \) are \( q \equiv 0 \) or \( w(y) \equiv \pm \kappa(d,y) \), where \( d \in (-1,1) \) and

\[
\kappa(d,y) = \frac{\kappa_0 (1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^2} \quad \text{where} \quad \kappa_0 = \left( \frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} \quad \text{and} \quad |y| < 1.
\]

(10)

Note that the set of stationary solutions is made of 3 connected components. One wonders whether these stationary solutions are good candidates for the convergence of \( w_{x_0}(y,s) \).

As a matter of fact, that is the case when \( x_0 \in \mathcal{R} \) as we see from the following result (see Corollary 4 in [14] and Theorem 6 in [16]):

**Theorem 3** (Asymptotic behavior near the blow-up graph)

(i) Case where \( x_0 \in \mathcal{R} \): Existence of an asymptotic profile. There exist \( \delta_0(x_0) > 0 \), \( d(x_0) \in (-1,1) \), \( |\theta(x_0)| = 1 \), \( s_0(x_0) \geq -\log T(x_0) \) such that for all \( s \geq s_0 \):

\[
\left\| \left( \begin{array}{c}
\frac{w_{x_0}(s)}{\partial_s w_{x_0}(s)} \\
\kappa(d(x_0) ) \end{array} \right) - \theta(x_0) \begin{pmatrix} \kappa(d(x_0)) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_0)}
\]

(11)

for some positive \( \mu_0 \) and \( C_0 \) independent from \( x_0 \). Moreover, \( E(w_{x_0}(s)) \to E(\kappa_0) \) as \( s \to \infty \).

(ii) Case where \( x_0 \in \mathcal{S} \): decomposition into a sum of decoupled solitons. It holds that

\[
\left\| \left( \begin{array}{c}
\frac{w_{x_0}(s)}{\partial_s w_{x_0}(s)} \\
k(x_0)
\end{array} \right) - \left( \sum_{i=1}^{k(x_0)} e_i^* \kappa(d_i(s),\cdot) \right) \right\|_{\mathcal{H}} \to 0 \quad \text{and} \quad E(w_{x_0}(s)) \to k(x_0)E(\kappa_0)
\]

(12)

as \( s \to \infty \), for some \( k(x_0) \geq 2 \), \( e_i^* = e_i^* \frac{(-1)^{i+1}}{i} \).
and continuous \( d_i(s) = -\tanh \zeta_i(s) \in (-1, 1) \) for \( i = 1, \ldots, k(x_0) \). Moreover, for some \( C_0 > 0 \), for all \( i = 1, \ldots, k(x_0) \) and \( s \) large enough,

\[
|\zeta_i(s) - \left( i - \frac{(k(x_0) + 1)}{2}\right) \frac{(p-1)}{2} \log s | \leq C_0.
\]  

(13)

Remark: Note that some elements in the description in similarity variables given above have a geometrical interpretation:
- when \( x_0 \in \mathbb{R} \), the parameter \( d(x_0) \) appearing in (11) is equal to the slope \( T'(x_0) \) of the blow-up curve (hence \( d(x_0) = T'(x_0) \));
- when \( x_0 \in \mathcal{S} \), \( k(x_0) \geq 2 \) is the number of solitons in the decomposition (12) appears also in the upper bound estimate on \( T(x) \) for \( x \) near \( x_0 \) given in (4).

Remark: The proof of the convergence in (i) has two major difficulties:
- the linearized operator of equation (6) around the profile \( \kappa(d, y) \) is not selfadjoint, which makes the standard tools inefficient for the control of the negative part of the spectrum. Fortunately, the Lyapunov functional structure will be useful in this matter;
- all the non zero stationary solutions of equation (6) are non isolated, which generates a null eigenvalue difficult to control in the linearization of equation (6) around \( \kappa(d, y) \). A modulation technique is then used to overcome this difficulty.

Extending the definition of \( k(x_0) \) defined for \( x_0 \in \mathcal{S} \) after (12) by setting

\[
k(x_0) = 1 \text{ for all } x_0 \in \mathbb{R},
\]

we proved the following energy criterion in [16] and using the monotonicity of the Lyapunov functional \( E(w) \), we have the following consequence from the blow-up behavior in Theorem 3:

Proposition 2.1 (An energy criterion for non characteristic points; see Corollary 7 in [16])

(i) For all \( x_0 \in \mathbb{R} \) and \( s_0 \geq -\log T(x_0) \), we have

\[
E(w_{x_0}(s_0)) \geq k(x_0)E(\kappa_0).
\]

(ii) If for some \( x_0 \) and \( s_0 \geq -\log T(x_0) \), we have

\[
E(w_{x_0}(s_0)) < 2E(\kappa_0),
\]

then \( x_0 \in \mathcal{R} \).

3 A Liouville theorem and a trapping result near the set of stationary solutions

The following Liouville theorem is crucial in our analysis:

Theorem 4 (A Liouville Theorem for equation (6)) Consider \( w(y, s) \) a solution to equation (6) defined for all \( (y, s) \in (-\frac{1}{\delta^*}, \frac{1}{\delta^*}) \times \mathbb{R} \) such that for all \( s \in \mathbb{R} \),

\[
\| w(s) \|_{H^1(-\frac{1}{\delta^*}, \frac{1}{\delta^*})} + \| \partial_y w(s) \|_{L^2(-\frac{1}{\delta^*}, \frac{1}{\delta^*})} \leq C^*
\]

(14)
for some $\delta_* \in (0,1)$ and $C^* > 0$. Then, either $w \equiv 0$ or $w$ can be extended to a function (still denoted by $w$) defined in
\[
\{(y,s) \mid -1 - T_0 e^s < d_0 y\} \supset \left( -\frac{1}{\delta_*}, \frac{1}{\delta_*} \right) \times \mathbb{R} \text{ by } w(y,s) = \theta_0 \kappa_0 \frac{(1 - d_0^2)^{\frac{1}{p-1}}}{(1 + T_0 e^s + d_0 y)^{\frac{2}{p-1}}},
\]
(15)
for some $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ and $\theta_0 = \pm 1$, where $\kappa_0$ defined in (10).

**Remark:** Note that deriving blow-up estimates through the proof of Liouville Theorems has been successful for different problems. For the case of the heat equation
\[
\partial_t u = \Delta u + |u|^{p-1} u
\]
(16)
where $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $p > 1$ and $(N - 2)p < N + 2$, the blow-up time $T$ is unique for equation (16). The blow-up set is the subset of $\mathbb{R}^N$ such that $u(x,t)$ does not remain bounded as $(x,t)$ approaches $(x_0,T)$. In [17], the second author proved the $C^2$ regularity of the blow-up set under a non degeneracy condition. A Liouville Theorem proved in [10] and [9] was crucially needed for the proof of the regularity result in the heat equation. The above Liouville Theorem is crucial for the regularity of the blow-up set for the wave equation (Theorem 3).

The second fundamental crucial result for our contributions is given by the following trapping result from [14] (See Theorem 3 in [14] and its proof):

**Proposition 3.1 (A trapping result near the sheet $d \mapsto \kappa(d,y)$ of stationary solutions)** There exists $\epsilon_0 > 0$ such that if $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution of equation (6) such that
\[
\forall s \geq s^*, \ E(w(s)) \geq E(\kappa_0) \text{ and } \left\| \left( \begin{array}{c} w(s) \\ \partial_s w(s) \end{array} \right) - \omega^* \left( \begin{array}{c} \kappa(d^*, \cdot) \\ 0 \end{array} \right) \right\|_{\mathcal{H}} \leq \epsilon^*
\]
(17)
for some $d^* = -\tanh \xi^*$, $\omega^* = \pm 1$ and $\epsilon^* \in (0,\epsilon_0]$, then there exists $d_\infty = -\tanh \xi_\infty$ such that
\[
|\xi_\infty - \xi^*| \leq C_0 \epsilon^* \text{ and } \left\| \left( \begin{array}{c} w(s) \\ \partial_s w(s) \end{array} \right) - \omega^* \left( \begin{array}{c} \kappa(d_\infty, \cdot) \\ 0 \end{array} \right) \right\|_{\mathcal{H}} \to 0.
\]
(18)

**References**


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