Isolatedness of characteristic points at blow-up for a semilinear wave equation in one space dimension *

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We consider the one dimensional semilinear wave equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{1}$$

where $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $p > 1$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}$ with $\|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} = \sup\limits_{a\in\mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$ and $\|v\|^2_{\mathrm{H}^1_{\mathrm{loc,u}}} = \|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\nabla v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}}$.
The Cauchy problem for equation (1) in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$ follows from the finite speed of propagation and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$ (see Ginibre, Soffer and Velo [6]). If the solution is not global in time, then we call it a blow-up solution. The existence of blow-up solutions is guaranteed by ODE techniques together with the finite speed of propagation, or also by the following blow-up criterion from Levine [9]:

If $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} \left(\frac{1}{2}|u_1(x)|^2 + \frac{1}{2}|\partial_x u_0(x)|^2 - \frac{1}{p+1}|u_0(x)|^{p+1} \right) dx < 0,$$

then the solution of (1) cannot be global in time.

More blow-up results can be found in Caffarelli and Friedman [5], [4], Alinhac [1] and Kichenassamy and Littman [7], [8].

If u is a blow-up solution of (1), we define (see for example Alinhac [1]) a 1-Lipschitz curve $\Gamma = \{(x, T(x))\}$ such that u cannot be extended beyond the set called the maximal influence domain of u:

$$D = \{(x,t) \mid t < T(x)\}. \tag{2}$$

$\bar{T} = \inf_{x\in\mathbb{R}} T(x)$ and Γ are called the blow-up time and the blow-up graph of u.

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Fig. 1: The maximal influence domain of the solution

A point a is a non characteristic point (or a *regular* point) if there are $\delta_0 \in (0,1)$ and $t_0 < T(a)$ such that u is defined on $\mathcal{C}_{a,T(a),\delta_0} \cap \{t \geq t_0\}$ where

$$\mathcal{C}_{\bar{x},\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x - \bar{x}|\}.$$

Fig. 2: a is a non characteristic point

If not, then we call a a characteristic point (or a *singular* point). Naturally, we denote by $\mathcal{R}$ (resp. $\mathcal{S}$) the set of non characteristic (resp. characteristic) points. Note then that

$$\mathcal{R} \cup \mathcal{S} = \mathbb{R}.$$

In our papers [14], [16], [15], [17], [18], [19] and [20], we made several contributions to the study of blow-up solutions of (1), namely the description of its blow-up graph and blow-up behavior in selfsimilar variables.

1 The blow-up graph of equation (1)

It is clear from rather simple arguments that $\mathcal{R} \neq \emptyset$ for *any* blow-up solution $u(x,t)$ (if $T(x)$ achieves its minimum at a, then $a \in \mathcal{R}$; if the infimum of $T(x)$ is at infinity, then there exists a large a such that $a \in \mathcal{R}$ as we state in the remark following Theorem 1 in [18]). On the contrary, the situation was unclear for $\mathcal{S}$, and it was commonly conjectured before our contributions that $\mathcal{S}$ was empty. In particular, that was the case in the examples constructed by Caffarelli and Friedman in [5] and [4]. In [19], we prove that the conjecture was false. More precisely, we proved the following (see Proposition 1 in [19]):

Proposition 1 (Existence of initial data with $\mathcal{S} \neq \emptyset$) *If the initial data (u_0, u_1) is odd and $u(x,t)$ blows up in finite time, then $0 \in \mathcal{S}$.*

Fig 3: Odd initial data which makes the origin a characteristic point

As we explicitly show in Theorem 4 below, the existence of characteristic points is linked to sign changes in the solution near the singular point $(a, T(a))$. This enables us to give the following criterion for the non existence of characteristic points on some finite interval (see Theorem 4 in [19]):

Proposition 2 (Non existence of characteristic points) *Consider $u(x,t)$ a blow-up solution of (1) such that $u(x,t) \geq 0$ for all $x \in (a_0, b_0)$ and $t_0 \leq t < T(x)$ for some real a_0, b_0 and $t_0 \geq 0$. Then, $(a_0, b_0) \subset \mathcal{R}$.*

Remark: This result can be seen as a generalization of the result of Caffarelli and Friedman [5] and [4], with no restriction on initial data. Indeed, from our result, taking non-negative initial data suffices to exclude the occurrence of characteristic points.

For general blow-up solutions, we proved the following facts about $\mathcal{R}$ and $\mathcal{S}$ in [18] and [19] (see Theorem 1 (and the following remark) in [18], see Theorems 1 and 2 in [20]):

Theorem 3 (Geometry of the blow-up graph)
(i) $\mathcal{R}$ is a non empty open set, and $x \mapsto T(x)$ is of class C^1 on $\mathcal{R}$;
(ii) $\mathcal{S}$ is made of isolated points, and given $a \in \mathcal{S}$, if $0 < |x-a| \le \delta_0$, then

$$\frac{1}{C_0|\log(x-a)|^{\frac{(k(a-1)(p-1)}{2}}} \le T'(x) + \frac{x-a}{|x-a|} \le \frac{C_0}{|\log(x-a)|^{\frac{(k(a)-1)(p-1)}{2}}}$$

for some $\delta_0 > 0$ and $C_0 > 0$, where $k(a) \ge 2$ is an integer. In particular, $T(x)$ is right and left differentiable at a, with $T_l'(a) = 1$ and $T_r'(a) = -1$.

Remark: In [21], Nouaili improves the regularity of the restriction of $x \mapsto T(x)$ to $\mathcal{R}$ to $C^{1,\alpha}$ for some $\alpha > 0$.
Remark: Integrating the estimate of (ii) in Theorem 3, we see that the blow-up set is corner-shaped near a in the sense that

$$\frac{|x-a|}{C_0|\log(x-a)|^{\frac{(k(a)-1)(p-1)}{2}}} \le T(x) - T(a) + |x-a| \le \frac{C_0|x-a|}{|\log(x-a)|^{\frac{(k(a)-1)(p-1)}{2}}}. \qquad (3)$$

Fig. 4: The blow-up set is corner-shaped near characteristic points

In particular, there exists no solution of the semilinear wave equation (1) with a characteristic point a such that $T(x)$ is differentiable at $x = a$.
Note from (3) that the blow-up set never touches the backward light cone with vertex $(a, T(a))$ (except of course at a), and that the distance between them is bounded from above and from below by the same rate, which is quantified in terms of the integer $k(a) \ge 2$. In particular, from the shape of the solution near $(a, T(a))$, we can recover the integer $k(a) \ge 2$, and $k(a) - 1$ is the number of sign changes of the solution near $(a, T(a))$ as we will see in (ii) of Theorem 4 below. In one word, the shape of the solution near $(a, T(a))$ gives the topology of the solution and conversely.
Remark: The fact that the elements of $\mathcal{S}$ are isolated points is not elementary. Direct arguments give no more than the fact that $\mathcal{S} \ne \mathbb{R}$ (a point a such that $T(a)$ is the blow-up time is non characteristic). The first step of the proof is done in [19] where we proved that $\mathcal{S}$ has an empty interior and that in similarity variables, the solution splits in a non trivial decoupled sum of (at least 2) solitons with alternate signs (see (ii) of Theorem 4 below for a statement). The second step is done in [20]. It consists in using this decomposition and a good understanding of the dynamics of the equation in similarity variables (see equation (5) below) near a decoupled sum of "generalized" solitons. In fact, this is the first time where flows near an unstable sum of solitons are used and where such a result is obtained.

Remark: In higher dimensions $N \geq 2$, our result would be that the $(N-1)$-dimensional Hausdorff measure of S is bounded. To prove that, we strongly need to characterize all selfsimilar solutions of equation (1) in the energy space. This is the main obstruction to extend this result to the higher dimension.
Remark: The fact that $\mathcal{S}$ is made of isolated points certainly does not hold in general for quasilinear wave equations. Indeed, in [2], Alinhac gives an explicit solution $u(x,t)$ for the following nonlinear wave equation

$$\partial_t^2 u = \partial_x^2 u + \partial_x u \partial_t u,$$

whose domain of definition is

$$D = \mathbb{R} \times [0,\infty) \backslash \{(x,t) \mid t \geq 1,\ |x| \leq t-1\}$$

(when $0 \leq t < 1$, $u(x,t) = 4 \arctan\left(\frac{x}{1-t}\right)$). In this example, we clearly see that $\mathcal{R} = \{0\}$, $\mathcal{S} = \mathbb{R}^*$, and the boundary of D is characteristic (i.e. has slope ± 1) on $\mathcal{S}$.

2 Asymptotic behavior near the blow-up graph

As one may guess from the above description, the asymptotic behavior will not be the same on $\mathcal{R}$ and on $\mathcal{S}$. In both cases, we need to use the similarity variables which we recall in the following. Let us stress the fact that the keystone of our work is the existence of a Lyapunov functional in similarity variables.

Given some $a \in \mathbb{R}$, we introduce the following self-similar change of variables:

$$w_a(y,s) = (T(a)-t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x-a}{T(a)-t}, \quad s = -\log(T(a)-t). \tag{4}$$

The function $w = w_a$ satisfies the following equation for all $y \in B = B(0,1)$ and $s \geq -\log T(a)$:

$$\partial^2_{ss} w = \mathcal{L} w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial^2_{y,s} w \tag{5}$$

$$\text{where } \mathcal{L} w = \frac{1}{\rho} \partial_y \left(\rho(1-y^2) \partial_y w\right) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \tag{6}$$

From Antonini and Merle [3], we know the existence of the following Lyapunov functional for equation (5):

$$E(w) = \int_{-1}^{1} \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} \left(\partial_y w\right)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy, \tag{7}$$

defined for $(\partial_s w, w) \in \mathcal{H}$ where

$$\mathcal{H} = \left\{ q \ \mid \ \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + \left(q_1'\right)^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \tag{8}$$

Using this energy structure together with interpolation and the Gagliardo-Nirenberg estimate, we proved in [14], [16], [15] and [17] that $(w_a(s), \partial_s w_a(s))$ is bounded in the energy space $\mathcal{H}$. Moreover, if $a \in \mathcal{R}$, then the bound holds in $H^1 \times L^2(-1,1)$ as well by a covering technique.

From Proposition 1 in [17], we know that the only stationary solutions of (5) in the space $\mathcal{H}$ are $q \equiv 0$ or $w(y) \equiv \pm\kappa(d,y)$, where $d \in (-1,1)$ and

$$\kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}} \text{ and } |y| < 1. \tag{9}$$

We will sometimes refer to $\pm\kappa(d,y)$ as "solitons". Note that the set of stationary solutions is made of 3 connected components. One wanders whether these stationary solutions are good candidates for the convergence of $w_a(y,s)$, at least when $a \in \mathcal{R}$. In the context of Hamiltonian systems, this question is delicate, and there is no natural reason for such a convergence, since the wave equation is time reversible. This kind of difficulties is also encountered for the critical generalized Korteweg de Vries equation (see Martel and Merle [10]) and for the Nonlinear Schrödinger equation (see Merle and Raphaël [11]).

As a matter of fact, there is convergence for w_a when $a \in \mathcal{R}$ as we see from the following result (see Corollary 4 in [17] and Theorem 6 in [19]):

Theorem 4 (Asymptotic behavior near the blow-up graph)
(i) **Case where $a \in \mathcal{R}$: Existence of an asymptotic profile**. *There exist $\delta_0(a) > 0$, $|e(a)| = 1$, $s_0(a) \geq -\log T(a)$ such that for all $s \geq s_0$:*

$$\left\| \begin{pmatrix} w_a(s) \\ \partial_s w_a(s) \end{pmatrix} - e(a) \begin{pmatrix} \kappa(T'(a),.) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_0)} \tag{10}$$

for some positive μ_0 and C_0 independent from a. Moreover, $E(w_a(s)) \to E(\kappa_0)$ as $s \to \infty$.
(ii) **Case where $a \in \mathcal{S}$: Decomposition into a sum of decoupled solitons**. *It holds that*

$$\left\| \begin{pmatrix} w_a(s) \\ \partial_s w_a(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k(a)} e_i^*(a)\kappa(d_i(s),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ and } E(w_a(s)) \to k(a)E(\kappa_0) \tag{11}$$

as $s \to \infty$, for some

$$k(a) \geq 2, \quad e_i^*(a) = e_1^*(a)(-1)^{i+1} \tag{12}$$

and continuous $d_i(s) = -\tanh\zeta_i(s) \in (-1,1)$ for $i = 1,...,k(a)$. Moreover, for some $C_0 > 0$, for all $i = 1,...,k(a)$ and s large enough,

$$\left|\zeta_i(s) - \left(i - \frac{(k(a)+1)}{2}\right)\frac{(p-1)}{2}\log s\right| \leq C_0. \tag{13}$$

Remark: It happens that (i) holds uniformly for all b in some neighborhood of a. Therefore, using the Sobolev injection in one dimension and a covering technique, we can show that the convergence of (10) holds in $L^\infty(-1,1)$ in the sense that

$$\|w_a(y,s) - e(a)\kappa(T'(a),y)\|_{L^\infty(-1,1)} \to 0 \text{ as } x \to \infty.$$

Remark: To better see that the solitons in (11) are decoupled, we should use the change of variables

$$y = \tanh \xi \text{ where } \xi \in \mathbb{R}$$

and introduce

$$\bar{w}_a(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_a(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(s) = -\tanh^{-1} d_i(s).$$

In this case, estimate (11) yields the fact that

$$\|\bar{w}_a(\xi, s) - e_1^*(a) \sum_{i=1}^{k(a)} (-1)^i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

It is worth noting that the $\zeta_i(s)$ satisfy a Toda lattice system:

$$\frac{1}{c_1}\zeta_i'(s) = e^{-\frac{2}{p-1}(\zeta_i(s) - \zeta_{i-1}(s))} - e^{-\frac{2}{p-1}(\zeta_{i+1}(s) - \zeta_i(s))} + R_i(s)$$

with

$$R_i(s) = o\left(\sum_{j=1}^{k-1} e^{-\frac{2}{p-1}(\zeta_{j+1}(s) - \zeta_j(s))}\right) \text{ as } s \to \infty.$$

An important step of the proof is to derive this system which gives two facts: the signs e_i^* in (12) are alternate and the solitons' centers $\zeta_i(s)$ are equidistant (up to a constant) as written in (13).
Remark: Note that some elements in the description in similarity variables given above have a geometrical interpretation:
- when $a \in \mathcal{R}$, the solution in similarity variables converges to the profile $e(a)\kappa(T'(a), \cdot)$ which has the slope $T'(a)$ as a parameter;
- when $a \in \mathcal{S}$, $k(a) \geq 2$ is the number of solitons in the decomposition (11) appears also in the upper bound estimate on $T(x)$ for x near a given in (3).
Remark: The proof of the convergence in (i) has two major difficulties:
- the linearized operator of equation (5) around the profile $\kappa(d, y)$ is not selfadjoint, which makes the standard tools inefficient for the control of the negative part of the spectrum. Fortunately, the Lyapunov functional structure will be useful in this matter;
- all the non zero stationary solutions of equation (5) are non isolated, which generates a null eigenvalue difficult to control in the linearization of equation (5) around $\kappa(d, y)$. A modulation technique is then used to overcome this difficulty.

Extending the definition of $k(a)$ defined for $a \in \mathcal{S}$ after (11) by setting

$$k(a) = 1 \text{ for all } a \in \mathcal{R},$$

we proved the following energy criterion in [19] and using the monotonicity of the Lyapunov functional $E(w)$, we have the following consequence from the blow-up behavior in Theorem 4:

Proposition 2.1 (An energy criterion for non characteristic points; see Corollary 7 in [19])
(i) For all $a \in \mathbb{R}$ and $s_0 \geq -\log T(a)$, we have

$$E(w_a(s_0)) \geq k(a)E(\kappa_0).$$

(ii) If for some a and $s_0 \geq -\log T(a)$, we have

$$E(w_a(s_0)) < 2E(\kappa_0),$$

then $a \in \mathcal{R}$.

3 A Liouville theorem and a trapping result near the set of stationary solutions

The following Liouville Theorem is crucial in our analysis:

Theorem 5 (A Liouville Theorem for equation (5)) *Consider $w(y,s)$ a solution to equation (5) defined for all $(y,s) \in (-\frac{1}{\delta_*}, \frac{1}{\delta_*}) \times \mathbb{R}$ such that for all $s \in \mathbb{R}$,*

$$\|w(s)\|_{H^1(-\frac{1}{\delta_*},\frac{1}{\delta_*})} + \|\partial_s w(s)\|_{L^2(-\frac{1}{\delta_*},\frac{1}{\delta_*})} \leq C^* \tag{14}$$

for some $\delta_* \in (0,1)$ and $C^* > 0$. Then, either $w \equiv 0$ or w can be extended to a function (still denoted by w) defined in

$$\{(y,s) \mid -1 - T_0 e^s < d_0 y\} \supset \left(-\frac{1}{\delta_*}, \frac{1}{\delta_*}\right) \times \mathbb{R} \text{ by } w(y,s) = \theta_0 \kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(1+T_0 e^s + d_0 y)^{\frac{2}{p-1}}}, \tag{15}$$

for some $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ and $\theta_0 = \pm 1$, where κ_0 defined in (9).

Remark: Note that deriving blow-up estimates through the proof of Liouville Theorems has been successful for different problems. For the case of the heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u \tag{16}$$

where $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $p > 1$ and $(N-2)p < N+2$, the blow-up time T is unique for equation (16). The blow-up set is the subset of $\mathbb{R}^N$ such that $u(x,t)$ does not remain bounded as (x,t) approaches (a,T). In [22], the second author proved the C^2 regularity of the blow-up set under a non degeneracy condition. A Liouville Theorem proved in [13] and [12] was crucially needed for the proof of the regularity result in the heat equation. The above Liouville Theorem is crucial for the regularity of the blow-up set for the wave equation (Theorem 4).

The second fundamental crucial result for our contributions is given by the following trapping result from [17] (See Theorem 3 in [17] and its proof):

Proposition 3.1 (A trapping result near the sheet $d \mapsto \kappa(d,y)$ of stationary solutions) *There exists $\epsilon_0 > 0$ such that if $w \in C([s^*,\infty),\mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution of equation (5) such that*

$$\forall s \ge s^*, \quad E(w(s)) \ge E(\kappa_0) \text{ and } \left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d^*,\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \le \epsilon^* \tag{17}$$

for some $d^* = -\tanh \xi^*$, $\omega^* = \pm 1$ and $\epsilon^* \in (0,\epsilon_0]$, then there exists $d_\infty = -\tanh \xi_\infty$ such that

$$|\xi_\infty - \xi^*| \le C_0 \epsilon^* \text{ and } \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d_\infty,\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0. \tag{18}$$

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