One dimensional behavior of singular N dimensional solutions of semilinear heat equations

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Abstract

We consider $u(x,t)$ a solution of $u_t = \Delta u + |u|^{p-1}u$ that blows up at time T, where $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $p > 1$, $(N-2)p < N+2$ and either $u(0) \geq 0$ or $(3N-4)p < 3N+8$. We are concerned with the behavior of the solution near a non isolated blow-up point, as $T - t \to 0$. Under a non degeneracy condition and assuming that the blow-up set is locally continuous and $N-1$ dimensional, we escape logarithmic scales of the variable $T-t$ and give a sharper expansion of the solution with the much smaller error term $(T-t)^{1/2-\eta}$ for any $\eta > 0$. In particular, if in addition $p > 3$, then the solution is very close to a superposition of one dimensional solutions as functions of the distance to the blow-up set. Finally, we prove that the mere hypothesis that the blow-up set is continuous implies that it is $C^{1,1/2-\eta}$ for any $\eta > 0$.

1 Introduction

In this paper, we are mainly concerned with the blow-up behavior at *non-isolated* blow-up points of the following semilinear heat equation:

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u \\ u(.,0) &= u_0 \in L^\infty(\mathbb{R}^N), \end{aligned} \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$ and Δ stands for the Laplacian in $\mathbb{R}^N$. We assume in addition that the exponent $p > 1$ is subcritical: if $N \geq 3$ then $1 < p < (N+2)/(N-2)$. Moreover, we assume that

$$\text{either } u_0 \geq 0 \text{ or } (3N-4)p < 3N+8. \tag{2}$$

This problem has attracted a lot of attention because it captures features common to a whole range of blow-up problems arising in various physical situations; particularly it highlights the role of scaling and self-similarity. Among related equations, we mention: the motion by mean curvature, surface diffusion (Bernoff, Bertozzi and Witelski [1]) and chemotaxis (Brenner *et al.* [3], Betterton and Brenner [2]). However, equation (1) is simple enough to be tractable in rigorous mathematical terms, unlike other physical equations. In this work, we build up tools that may be useful in more physical situations. As a matter of fact, in section 5 we will mention connections with a chemotaxis problem.

The behavior near singular points is a major concern in all singularity problems. One general idea of this work is to find out how to refine the singular behavior beyond first order terms and reach significantly small error terms. Through a change of variables, singular behavior reduces to the asymptotic behavior of some PDE when a small positive parameter ϵ goes to zero. For the heat equation (1), $\epsilon = T - t \to 0$, where T is the blow-up time. In previous work, an explicit profile is found to be a good first order approximation, up to ν^α where $\nu = -1/\log\epsilon$ and $\alpha > 0$. Further refinements in this direction should give an expansion of the solution in terms of powers of ν, i.e., in logarithmic scales of ϵ (see Stewartson and Stuart [18]). Logarithmic scales also arise in some singular perturbation problems such as low Reynolds number fluids and some vibrating membranes studies (see Ward [20] and the references therein, see also Segur and Kruskal [17] for a Klein-Gordon equation). Since ν goes to zero slowly, infinite logarithmic series may be of only limited practical use in approximating the exact solution. Relevant approximations, i.e., approximations up to lower order terms such as ϵ^β for $\beta > 0$, lie beyond all logarithmic scales. In this work, our idea to capture such relevant terms is to abandon the explicit profile function obtained as a first order approximation, and take a less explicit function as a first order description of the singular behavior. Both formulations agree to the first order. Through scaling and matching, we can reach the order ϵ^β by iterating the expansion around the less explicit function.

A second general idea in this work is to see how more constraints on the singular set yield more regularity for that set. This idea is found in studies of free boundary problems, where over determined boundary conditions yield regularity of the free boundary. In this work, we focus on the case where the blow-up set of (1) is a continuum. The mere hypothesis that the blow-up set is continuous, which is an unstable situation (see section 5), adds constraints in the problem, yielding $C^{1,\alpha}$ regularity for the blow-up set.

1.1 Blow-up behavior in logarithmic scales of $T-t$

A solution $u(t)$ to (1) blows up in finite time if its maximal existence time T is finite. In this case,

$$\lim_{t\to T}\|u(t)\|_{H^1(\mathbb{R}^N)} = \lim_{t\to T}\|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

Let us consider such a solution. T is called the blow-up time of u. A point $a\in\mathbb{R}^N$ is called a blow-up point if

$$|u(x,t)| \to +\infty \text{ as } (x,t)\to(a,T)$$

(this definition is equivalent to the usual local unboundedness definition, because of Corollary 2 in Merle and Zaag [15]). S denotes the blow-up set, i.e., the set of all blow-up points. From [15], we know that there exists *a blow-up profile* $u^*\in C^2_{\rm loc}(\mathbb{R}^N\backslash S)$ such that

$$u(x,t)\to u^*(x) \text{ in } C^2_{\rm loc}(\mathbb{R}^N\backslash S) \text{ as } t\to T. \tag{3}$$

Given $\hat a\in S$, we know from Velázquez [19] that up to some scalings, u approaches a particular explicit function near the singularity $(\hat a, T)$. We consider the case where for all $K_0>0$,

$$\sup_{|z|\le K_0}\left|(T-t)^{\frac{1}{p-1}}u\left(\hat a + Q_{\hat a}z\sqrt{(T-t)|\log(T-t)|},t\right) - f_{l_{\hat a}}(z)\right| \to 0 \tag{4}$$

as $t\to T$, where $Q_{\hat a}$ is an orthonormal $N\times N$ matrix, $l_{\hat a}=1,...,N$, and

$$f_l(z) = \left(p-1+\frac{(p-1)^2}{4p}\sum_{i=1}^{l} z_i^2\right)^{-\frac{1}{p-1}}. \tag{5}$$

Other behaviors with the scaling $(T-t)^{-\frac{1}{2k}}(x-\hat a)$ where $k=2,3,..$ may occur (see [19]). We suspect them to be unstable.

If $l_{\hat a}=N$, then $\hat a$ is an isolated blow-up point. An extensive literature is devoted to this case (Weissler [21], Bricmont and Kupiainen [5], Herrero and Velázquez [12] and [19],...). We have proved the stability of such a behavior with Fermanian and Merle in [8]. The key argument in our proof was the following Liouville Theorem proved by Merle and Zaag in [13] and [15]:

Consider U a solution of (1) defined for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ such that for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$, $|U(x,t)| \leq C(T-t)^{-\frac{1}{p-1}}$. Then, either $U \equiv 0$ or $U(x,t) = [(p-1)(T^*-t)]^{-\frac{1}{p-1}}$ for some $T^* \geq T$.

When $l_{\hat{a}} = N$, the blow-up behavior of $u(x,t)$ near the isolated blow-up point $\hat{a}$ is already contained in (4) which shows that the profile of $u(x,t)$ is a function of a one dimensional variable:

$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} f_1\left(\frac{d(x,S)}{\sqrt{(T-t)|\log(T-t)|}}\right), \tag{6}$$

since $S = \{\hat{a}\}$ and $d(x,S) = |x-\hat{a}|$ when x is close to $\hat{a}$. This description remains valid even when $\hat{a}$ is not isolated, as we will show later.

The case $l_{\hat{a}} < N$ is known to occur, namely when u is invariant with respect to some coordinates. However, when $l_{\hat{a}} < N$, we cannot even tell whether $\hat{a}$ is isolated or not. The first singularity description was obtained in [23]. For simplicity, we assume that locally near $\hat{a}$, S is a $(N-l_{\hat{a}})$-dimensional C^1 manifold. We have shown in Theorems 3 and 4 in [23] that for some $t_0 < T$ and $\delta > 0$, for all $K_0 > 0$, $t \in [t_0, T)$ and $x \in B(\hat{a}, \delta)$ such that $d(x,S) \leq K_0\sqrt{(T-t)|\log(T-t)|}$, we have

$$\left|(T-t)^{\frac{1}{p-1}} u(x,t) - f_1\left(\frac{d(x,S)}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| \leq C_0'(K_0)\frac{\log|\log(T-t)|}{|\log(T-t)|} \tag{7}$$

where f_1 is defined in (5). Note that formally, this is the same description as in the case $l_{\hat{a}} = N$, where $\hat{a}$ was isolated (see (6)). The variable $d(x,S)$, normal to S, appears as the blow-up variable that determines the size of u. The major step in [23] is the proof of the stability of the behavior (4) in a neighborhood of $\hat{a}$ in S. The key argument in getting this stability is the Liouville Theorem of [15], stated on page 3.

The error term in (7) shows that we fall in logarithmic scales of the small parameter $\epsilon = T - t$. In this paper, we do better, and get to error terms of order $(T-t)^\alpha$ with $\alpha > 0$. Following the ideas of page 2, we will replace the explicit profile f_1 by a less explicit function, and then go beyond all logarithmic scales, through scaling and matching.

1.2 Blow-up behavior beyond all logarithmic scales of $T-t$

A natural candidate for this non explicit function is simply a one dimensional solution of (1) that has the same profile f_1. It is classical that there exists a one dimensional even function $\tilde{u}(x_1,t)$, solution of (1), which decays on $(0,\infty)$ and blows up at time T only at the origin, with the profile f_1, in the sense that for all $K_0>0$ and $t\in[t_0,T)$, if $|x_1|\le K_0\sqrt{(T-t)|\log(T-t)|}$, then

$$\left|(T-t)^{\frac{1}{p-1}}\tilde{u}(x_1,t)-f_1\left(\frac{x_1}{\sqrt{(T-t)|\log(T-t)|}}\right)\right|\le C_0'(K_0)\frac{\log|\log(T-t)|}{|\log(T-t)|} \tag{8}$$

(see Appendix A for a proof of this fact). Hence, it follows from (7) that for all $K_0>0$, $t\in[t_0,T)$ and $x\in B(\hat{a},\delta)$ such that $d(x,S)\le K_0\sqrt{(T-t)|\log(T-t)|}$, we have

$$(T-t)^{\frac{1}{p-1}}\left|u(x,t)-\tilde{u}(d(x,S),t)\right|\le C(K_0)\frac{\log|\log(T-t)|}{|\log(T-t)|}. \tag{9}$$

This estimate remains valid even if we replace $\tilde{u}(d(x,S),t)$ by any $\tilde{u}_{\sigma(x,t)}(d(x,S),t)$ where $\tilde{u}_\sigma$ is defined by

$$\tilde{u}_\sigma(x_1,t)=e^{-\frac{\sigma}{p-1}}\tilde{u}(e^{-\frac{\sigma}{2}}x_1,T-e^{-\sigma}(T-t)), \tag{10}$$

provided that $|\sigma(x,t)|\le C(K_0)$. Indeed, for any $\sigma\in\mathbb{R}$, $\tilde{u}_\sigma$ is still a blow-up solution of (1) with the same properties and the same profile (8) as $\tilde{u}$. Moreover, $\tilde{u}_\sigma\ne\tilde{u}$, unless $\sigma=0$, because $\tilde{u}$ is not self-similar (see Appendix A).

For each blow-up point a near $\hat{a}$, we will suitably choose this free scaling parameter $\sigma=\sigma(a)$ so that the difference $(T-t)^{\frac{1}{p-1}}\left(u(x,t)-\tilde{u}_{\sigma(a)}(d(x,S),t)\right)$ along the normal direction to S at a is minimum. Following the ideas of page 2, if we refine the expansion about this well chosen, though less explicit, function $\tilde{u}_{\sigma(a)}(d(x,S),t)$, then we escape logarithmic scales. In particular, if $p>3$, then the difference $u(x,t)-\tilde{u}_{\sigma(a)}(d(x,S),t)$ is bounded and goes to zero as $t\to T$, although both functions blow up. This can be done only when

$$l_{\hat{a}}=1$$

which corresponds to a $(N-1)$-dimensional blow-up set, according to [23]. We claim the following:

Theorem 1 (The N dimensional solution seen as a superposition of one dimensional solutions of the normal variable to the blow-up set, with a suitable dilation) *Assume $N\ge 2$ and consider u a solution* *of (1) that blows up at time T on a set S which is a $(N-1)$-dimensional C^1 manifold, locally near $\hat{a}$. If u behaves as stated in (4) near $(\hat{a}, T)$ with $l_{\hat{a}} = 1$ and if $p > 3$, then for all $t \in [t_1, T)$ and $x \in B(\hat{a}, \delta)$ such that $d(x, S) < \epsilon_0$ for some $t_1 < T$, $\delta > 0$ and $\epsilon_0 > 0$, we have*

$$\left|u(x,t) - \tilde{u}_{\sigma(P_S(x))}\left(d(x,S),t\right)\right| \le h(x,t) < M < +\infty, \tag{11}$$

where $P_S(x)$ is the projection of x over S and $h(x,t) \to 0$ as $d(x,S) \to 0$ and $t \to T$.

Thus, when $p > 3$, all the singular terms of u in a neighborhood of $(\hat{a}, T)$ are contained in the rescaled one dimensional solution $\tilde{u}_{\sigma(P_S(x))}\left(d(x,S),t\right)$, which shows that in a tubular neighborhood of the blow-up set S, the space variable splits into 2 independent variables:

- A primary variable, $d(x,S)$, normal to S. It accounts for the main singular term of u and gives the size of $u(x,t)$, as already shown in the old formulation (9), which follows directly from [23].
- A secondary variable, $P_S(x)$, whose effect is sharper. Through the optimal choice of the dilation $\sigma(P_S(x))$, it absorbs *all* next singular terms in the normal direction to S at $P_S(x)$.

Similar ideas are used by Betterton and Brenner [2] in a chemotaxis model; see section 5 for a short discussion of connections with that work. We would like to mention that we have successfully used this idea of modulation of the dilation with Fermanian in [9] to prove that for $N = 1$ and $p \ge 3$, there is only one blow-up solution of (1) with the profile (4), up to a bounded function and to the invariances of the equation (the dilation and translations in space and in time).

Theorem 1 is a direct consequence of the following result which is valid also for $1 < p \le 3$.

Theorem 2 (Blow-up behavior and profile near a blow-up point where u behaves as in (4) assuming S is locally a $(N-1)$-dimensional manifold) *Under the hypotheses of Theorem 1 and without the restriction $p > 3$, there exists $t_1 < T$ and $\epsilon_0 > 0$ such that for all $x \in B(\hat{a}, \delta)$ such that $d(x,S) \le \epsilon_0$, we have the following:*
i) For all $t \in [t_1, T)$,

$$\begin{aligned}&\left|u(x,t) - \tilde{u}_{\sigma(P_S(x))}\left(d(x,S),t\right)\right| \le \\ &C\,\mathrm{mM}\left((T-t)^{\frac{p-3}{2(p-1)}}|\log(T-t)|^{\frac{3}{2}+C_0}, d(x,S)^{\frac{p-3}{p-1}}|\log d(x,S)|^{\frac{p}{p-1}+C_0}\right),\end{aligned} \tag{12}$$

where $P_S(x)$ *is the projection of* x *over* S, $\mathrm{mM} = \min$ *if* $1 < p \le 3$ *and* $\mathrm{mM} = \max$ *if* $p > 3$.
ii) If $x \notin S$, *then* $u(x,t) \to u^*(x)$ *as* $t \to T$ *and*

$$\left| u^*(x) - e^{-\frac{\sigma(P_S(x))}{p-1}} \tilde{u}^* \left(e^{-\frac{\sigma(P_S(x))}{2}} d(x,S) \right) \right| \le C d(x,S)^{\frac{p-3}{p-1}} |\log d(x,S)|^{\frac{p}{p-1}+C_0},$$

where $\tilde{u}^*(x_1) = \lim\limits_{t \to T} \tilde{u}(x_1,t)$.

Remark: In [23], we have obtained the following explicit equivalent for u^*:

$$u^*(x) \sim \left[\frac{8p}{(p-1)^2} \frac{|\log d(x,S)|}{d(x,S)^2} \right]^{\frac{1}{p-1}} \sim \tilde{u}^*(d(x,S)) \text{ as } d(x,S) \to 0.$$

Our new estimate shows that up to a suitable dilation, all the next terms in the expansion of u^* up to the order $d(x,S)^{\frac{p-3}{p-1}} |\log d(x,S)|^{\frac{p}{p-1}+C_0}$ are the same as the particular one dimensional solution.

1.3 $C^{1,\alpha}$ regularity of the blow-up set

The splitting of the space variable x into $d(x,S)$ and $P_S(x)$, as shown in (12), induces a geometric constraint on the blow-up set S, leading to more regularity on S.

Proposition 3 ($C^{1,\frac{1}{2}-\eta}$ regularity for S and $C^{1-\eta}$ regularity for the dilation σ) *Under the hypotheses of Theorem 2, S is the graph of a function* $\varphi \subset C^{1,\frac{1}{2}-\eta}(B_{N-1}(0,\delta_1),\mathbb{R})$, *locally near* $\hat{a}$, *and* σ *is a* $C^{1-\eta}$ *function, for any* $\eta > 0$. *More precisely, there is a* $h_0 > 0$ *such that for all* $|\xi| < \delta_1$ *and* $|h| < h_0$ *such that* $|\xi + h| < \delta_1$, *we have*

$$\begin{aligned} |\varphi(\xi+h) - \varphi(\xi) - h\varphi'(\xi)| &\le C|h|^{3/2} |\log|h||^{\frac{1}{2}+C_0}, \\ |\sigma(\xi,\varphi(\xi)) - \sigma(\xi+h,\varphi(\xi+h))| &\le C|h||\log|h||^{3+C_0}. \end{aligned}$$

The regularity of the blow-up set S is our second concern in this paper. We know from Velázquez [19] that the $(N-1)$-dimensional Hausdorff measure of S is bounded on compact sets. Under a local non-degeneracy condition, we have proved in [23] that if S locally contains a continuum, then S is locally a C^1 manifold of dimension $k = 1,...,N-1$. Since Proposition 3 derives $C^{1,\frac{1}{2}-\eta}$ regularity assuming C^1 regularity, we can weaken the hypotheses of Proposition 3 and get a stronger version that derives $C^{1,\frac{1}{2}-\eta}$ regularity just assuming continuity. Stating this new version requires additional technical notation.

We consider a non isolated blow-up point $\hat{a}$ where u has the behavior (4) with $l_{\hat{a}} = 1$. We may take $Q_{\hat{a}} = \mathrm{Id}$. According to Theorem 2 in [19], for all $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that

$$S \cap B(\hat{a}, \delta) \subset \Omega_{\hat{a},\pi,\epsilon} \equiv \{x \mid |P_\pi(x-\hat{a})| \geq (1-\epsilon)|x-\hat{a}|\},$$

where P_π is the orthogonal projection over π, the subspace spanned by e_2, ..., e_N. Note that $\Omega_{\hat{a},\pi,\epsilon}$ is a cone with vertex $\hat{a}$ that shrinks to $\hat{a}+\pi$ as $\epsilon \to 0$. In fact, $\hat{a}+\pi$ is the candidate for the tangent plane to S at $\hat{a}$. We assume there is $a \in C((-1,1)^{N-1}, \mathbb{R}^N)$ such that $a(0) = \hat{a}$ and $\mathrm{Im}\, a \subset S$ where $\mathrm{Im}\, a$ is at least $(N-1)$-dimensional in the sense that

$$\forall b \in \mathrm{Im}\, a, \text{ there are } (N-1) \text{ independent vectors } v_1, ..., v_{N-1} \text{ in } \mathbb{R}^N \text{ and } a_1, .., a_{N-1} \text{ functions in } C^1([0,1], S) \text{ such that } a_i(0) = b \text{ and } a_i'(0) = v_i. \tag{13}$$

This hypothesis means that b is actually non isolated in $(N-1)$ independent directions. We also assume that $\hat{a} = 0$ is not an endpoint in $\mathrm{Im}\, a$ in the sense that

$$\forall \epsilon > 0, \text{ the projection of } a((-\epsilon,\epsilon)^{N-1}) \text{ on the plane } \hat{a}+\pi \text{ contains an open ball with center } \hat{a}. \tag{14}$$

We claim the following:

Theorem 4 (Regularity of the blow-up set near a point with the behavior (4) assuming S contains a $(N-1)$-dimensional continuum) *Take $N \geq 2$ and consider u a solution of (1) that blows up at time T on a set S and take $\hat{a} \in S$ where u behaves locally as stated in (4) with $l_{\hat{a}} = 1$. Consider $a \in C((-1,1)^{N-1}, \mathbb{R}^N)$ such that $\hat{a} = a(0) \in \mathrm{Im}\, a \subset S$ and $\mathrm{Im}\, a$ is at least $(N-1)$-dimensional in the sense (13). If $\hat{a}$ is not an endpoint (in the sense (14)), then there are $\delta > 0$, $\delta_1 > 0$ and $\varphi \in C^{1,\frac{1}{2}-\eta}(B_{N-1}(0,\delta_1), \mathbb{R})$ (for any $\eta > 0$) such that*

$$S \cap B(\hat{a}, 2\delta) = \mathrm{graph}\, \varphi \cap B(\hat{a}, 2\delta) = \mathrm{Im}\, a \cap B(\hat{a}, 2\delta). \tag{15}$$

Moreover, the conclusions of Theorem 2 and Proposition 3 hold. In particular, if $p > 3$, then the conclusion of Theorem 1 also holds.

Remark: When $N = 2$, we can replace conditions (13) and (14) just by the existence of α_0 such that for all $\epsilon > 0$, $a(-\epsilon, \epsilon)$ intersects the complimentary of any connected closed cone with vertex at $\hat{a}$ and angle $\alpha \in (0, \alpha_0]$.
Remark: In the case $l_{\hat{a}} \geq 2$ in (4), that is when the blow-up set is 2 dimensional, we are unable to suitably choose the dilation in (10) and we cannot escape the logarithmic scale in $T - t$. Hence, we cannot obtain $C^{1,\alpha}$ regularity. We can nonetheless improve estimate (9) and prove that:

For all $t \in [t_1, T)$ and $x \in B(\hat{a}, \delta)$ such that $d(x, S) \leq \epsilon_0$, we have

$$|u(x,t) - \tilde{u}(d(x,S),t)| \leq C \min\left(\frac{(T-t)^{-\frac{1}{p-1}}}{|\log(T-t)|}, \frac{d(x,S)^{-\frac{2}{p-1}}}{|\log d(x,S)|^{\frac{p-2}{p-1}}}\right).$$

Theorem 1 is a direct consequence of Theorem 2. Throughout the paper, we assume the hypotheses of Theorem 2. In section 2, we start from the conclusion given in [23] under the hypotheses of Theorem 2 and show that for any blow-up point a near $\hat{a}$, there is $\sigma(a) \in \mathbb{R}$ such that $\tilde{u}_{\sigma(a)}$ is the best profile for u along the normal direction to S at a. In section 3, we use this to get the blow-up behavior of u in a tubular neighborhood of S (Theorem 2). In section 4, we prove regularity results (Proposition 3). Theorem 4 is a direct consequence of Theorem 2 and Proposition 3 because of the results of [23]. Indeed, Theorem 4 in [23] asserts that under the hypotheses of Theorem 4, S is the graph of a C^1 function; hence Theorem 2 and Proposition 3 apply. Some connections with a chemotaxis model are presented in section 5. The results of this paper and those of [22] have been presented in the note [22].

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2 Modulation of the dilation, uniformly with respect to the blow-up point

This is a major step in our paper. Under the hypotheses of Theorem 2, there is a C^1 function φ such that

$$S_\delta \equiv S \cap B(\hat{a}, 2\delta) = \operatorname{graph} \varphi \cap B(\hat{a}, 2\delta) \tag{16}$$

for some $\delta > 0$ and $\varphi \in C^1(B_{N-1}(0,\delta_1),\mathbb{R})$, where $\delta_1 > 0$ and $B_{N-1}(0,\delta_1)$ is a ball in $\mathbb{R}^{N-1}$. If $a \in S_\delta$ and w_a is defined by

$$w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \;\; y = \frac{x-a}{\sqrt{T-t}}, \;\; s = -\log(T-t), \tag{17}$$

then we see from (1) that for all $(y,s) \in \mathbb{R}^N \times [-\log T, \infty)$,

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w. \tag{18}$$

We have proved in Propositions 3.1, 4.4 and 4.4' of [23] that for all $a \in S_\delta$ and $s \geq -\log T$,

$$\left\| w_a(Q_a y, s) - \left\{ \kappa + \frac{\kappa}{2ps}(1 - \frac{y_1^2}{2}) \right\} \right\|_{L^2_\rho} \leq C \frac{\log s}{s^2}, \tag{19}$$

where Q_a is a $N \times N$ orthogonal matrix continuous in terms of a, such that $\{Q_a e_i \mid i = 2, ..., N\}$ span the tangent plane T_a to S at a,

$$\begin{aligned} &Q_a e_1 \text{ is the normal direction to } S \text{ at } a, \\ &\kappa = (p-1)^{-\frac{1}{p-1}} \text{ and } \rho(y) = e^{-\frac{|y|^2}{4}}/(4\pi)^{N/2}. \end{aligned} \tag{20}$$

To show this, we first start from (4) and use the paper by Filippas and Kohn [10] to establish (19) at $a = \hat{a}$. Then, we use dynamical system methods to show the stability of the behavior (19) for solutions of (18). The Liouville Theorem stated on page 3 is a central argument.
The particular one dimensional solution $\tilde{u}(x_1,t)$ of page 4 can also be thought as a N dimensional solution blowing up on the hyperplane $\{x_1 = 0\}$ in $\mathbb{R}^N$. Therefore, the results of [23] apply to $\tilde{u}$ and (19) holds for $\tilde{u}$ too. Since $\tilde{u}$ is invariant in the direction of the blow-up set, we have for all $a \in \{x_1 = 0\}$, $Q_a \equiv \text{Id}$ and $\tilde{w}_a = \tilde{w}$ defined by

$$\tilde{w}(y_1,s) = (T-t)^{\frac{1}{p-1}} \tilde{u}(x_1,t), \;\; y_1 = \frac{x_1}{\sqrt{T-t}}, \;\; s = -\log(T-t). \tag{21}$$

$\tilde{w}$ is a solution of (18) and (19) yields for all $s \geq -\log T$,

$$\left\| \tilde{w}(y_1, s) - \left\{ \kappa + \frac{\kappa}{2ps}(1 - \frac{y_1^2}{2}) \right\} \right\|_{L^2_\rho} \leq C \frac{\log s}{s^2}. \tag{22}$$

Using (19) and (22), we get for all $\sigma_0 > 0$, $a \in S_\delta$, $|\sigma| \leq \sigma_0$ and $s \geq -\log T + \sigma_0$,

$$\| w_a(Q_a y, s) - \tilde{w}(y_1, s+\sigma) \|_{L^2_\rho} \leq C(\sigma_0) \frac{\log s}{s^2}. \tag{23}$$

We aim in this section at choosing a particular $\sigma = \sigma(a)$ so that this difference becomes less than $Ce^{-\frac{s}{2}}s^{C_0}$ for some $C_0 \geq 0$. This is equivalent to choosing an appropriate dilation $\lambda(a) = e^{-\sigma(a)}$ in (10) for the original function $\tilde{u}(x_1, t)$. The following proposition is the goal of this section.

Proposition 2.1 (Modulation of the dilation in the one dimensional solution) *There exist $s_0 > 0$ and $C_0 > 0$ and a continuous function $\sigma : S_\delta \to \mathbb{R}$ such that for all $a \in S_\delta$ and $s \geq s_0$,*

$$\|w_a(Q_a y, s) - \tilde{w}(y_1, s + \sigma(a))\|_{L^2_\rho} \leq C_0 e^{-\frac{s}{2}} s^{C_0}.$$

Let us first recall from [15] some consequences of the Liouville Theorem of page 3, namely some L^∞ estimates and a localization property for blow-up solutions of (1). We also need some elementary estimates of the one dimensional solution $\tilde{u}$.

2.1 Uniform L^∞ estimates

The following propositions are consequences of the Liouville Theorem of page 3.

Proposition 2.2 (L^∞ estimates for solutions to (1) at blow-up) *There exists $C > 0$ such that if u is a solution to (1) which blows up at time $T > 0$, then, there exists $\hat{s}$ such that for all $s \geq \hat{s}$ and $a \in \mathbb{R}^N$,*

$$\| w_a(s) \|_{L^\infty} \leq \kappa + \frac{C}{s} \text{ and } \| \nabla^i w_a(s) \|_{L^\infty} \leq \frac{C}{s^{i/2}} \tag{24}$$

for all $i \in \{1, 2, 3\}$, where w_a is defined in (17).

Proposition 2.3 (A uniform localization of the PDE (1) by means of the associated ODE) *Let u be a solution to (1) which blows up at time T. Then, $\forall \epsilon > 0$, $\exists C_\epsilon > 0$,*

$$\forall t \in [\frac{T}{2}, T), \;\; \forall x \in \mathbb{R}^N, \;\; \left| \frac{\partial u}{\partial t} - |u|^{p-1}u \right| \leq \epsilon |u|^p + C_\epsilon.$$

The reader will find a proof of these propositions in [15] and [14] respectively.

In the following lemma, we give some elementary estimates for the particular one dimensional solution $\tilde{u}$:

Lemma 2.4 (Elementary estimates for $\tilde{u}$)
i) There exists $C > 0$ and $\hat{s} > 0$ such that for all $s \geq \hat{s}$ and $|y_1| \leq \sqrt{s}$, we have

$$\tilde{w}(y_1, s) \leq \tilde{w}(0, s) - C\frac{y_1^2}{s}.$$

ii)
$$\int_{\mathbb{R}} \frac{\partial \tilde{w}}{\partial s}(y_1, s)\frac{(y_1^2 - 2)}{8}\frac{e^{-\frac{y_1^2}{4}}}{\sqrt{4\pi}}dy_1 \sim \frac{\kappa}{4ps^2} \text{ as } s \to \infty.$$

iii) $\frac{\partial \tilde{w}}{\partial s}(0, s) \sim -\frac{\kappa}{2ps^2}$ *as* $s \to \infty$.

Proof: See Appendix A. ■

2.2 A dynamical system formulation for the modulation problem

Our approach is identical to what we did with Fermanian in [9] for the difference of two solutions with the *radial* profile ($l_{\hat{a}} = N$) in (4), instead of the non symmetric profile ($1 = l_{\hat{a}} < N$) we handle here. Therefore, we follow in extent the strategy of [9] and emphasize the novelties. However, some technical details -most of them are straightforward and long- are omitted. The reader can find them in [9]. Consider an arbitrary $\sigma_0 \geq 0$ and fix $a \in S_\delta$ and $|\sigma| \leq \sigma_0$. If we define

$$g_{a,\sigma}(y, s) = w_a(Q_a y, s) - \tilde{w}(y_1, s + \sigma), \tag{25}$$

then we see from (18) that for all $(y, s) \in \mathbb{R}^N \times [-\log T + \sigma_0, \infty)$,

$$\partial_s g_{a,\sigma}(y, s) = (\mathcal{L} + \alpha_{a,\sigma})\, g_{a,\sigma}, \tag{26}$$

where $\mathcal{L} = \Delta - \frac{y}{2} \cdot \nabla + 1$ and $\forall (y, s) \in \mathbb{R}^N \times \mathbb{R}$,

$$\alpha_{a,\sigma}(y, s) = \frac{|\, w_a(Q_a y, s) \,|^{p-1}\, w_a - |\, \tilde{w}(y_1, s + \sigma) \,|^{p-1}\, \tilde{w}}{w_a - \tilde{w}} - \frac{p}{p-1} \tag{27}$$

if $w_a(Q_a y, s) \neq \tilde{w}(y_1, s + \sigma)$, and in general,

$$\alpha(y, s) = p\, |\, \bar{w}_{a,\sigma}(y, s) \,|^{p-1} - \frac{p}{p-1} \tag{28}$$

for some $\bar{w}_{a,\sigma}(y, s) \in \big(w_a(Q_a y, s), \tilde{w}(y_1, s + \sigma)\big)$. In the following, we drop down the index (a, σ) unless there is ambiguity. One should keep in mind that all quantities defined from g also depend on (a, σ).

According to (23) and (25), $g \to 0$ in L^2_ρ as $s \to \infty$. More precisely, for all $s \geq -\log T + \sigma_0$,

$$\|g(s)\|_{L^2_\rho} \leq C(\sigma_0)\frac{\log s}{s^2}. \tag{29}$$

Operator $\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N)$ where ρ is defined in (20). The spectrum of $\mathcal{L}$ consists of eigenvalues

$$\operatorname{spec}\mathcal{L} = \{1 - \frac{m}{2},\ \ m \in \mathbb{N}\}.$$

Note that except two positive eigenvalues (1 and $\frac{1}{2}$) and a null eigenvalue, all the spectrum is negative. The eigenfunctions of $\mathcal{L}$ are

$$h_\beta(y) = h_{\beta_1}(y_1)...h_{\beta_N}(y_N), \tag{30}$$

where $\beta = (\beta_1, ..., \beta_N) \in \mathbb{N}^N$ and for each $m \in \mathbb{N}$, h_m is the rescaled Hermite polynomial

$$h_m(\xi) = \sum_{j=0}^{[m/2]} \frac{m!}{j!(m-2j)!}(-1)^j \xi^{m-2j}. \ \text{ We note } \ k_m = \frac{h_m}{\|h_m\|^2_{L^2_{\rho_1}(\mathbb{R})}}, \tag{31}$$

where $L^2_{\rho_1}(\mathbb{R})$ is the L^2 space with the measure

$$\rho_1(\xi) = \frac{e^{-\frac{\xi^2}{4}}}{\sqrt{4\pi}} \text{ that satisfies } \rho(y) = \prod_{i=1}^N \rho_1(y_i). \tag{32}$$

The polynomials h_m and h_β satisfy

$$\mathcal{L}h_\beta = (1 - \frac{|\beta|}{2})h_\beta \text{ and } \int_{\mathbb{R}} h_m(\xi)k_j(\xi)\rho_1(\xi)d\xi = \delta_{m,j}. \tag{33}$$

Let us introduce the component of $g(.,s)$ on h_β

$$g_\beta(s) = \int_{\mathbb{R}^N} k_\beta(y)g(y,s)\rho(y)dy \text{ where } k_\beta(y) = \| h_\beta \|^{-2}_{L^2_\rho} h_\beta(y). \tag{34}$$

If P_n is the orthogonal projector of L^2_ρ over the eigenspace of $\mathcal{L}$ corresponding to $1 - \frac{n}{2}$, then $P_n g(y,s) = \sum_{|\beta|=n} g_\beta(s)h_\beta(y)$. Since the eigenfunctions of $\mathcal{L}$ span the whole space L^2_ρ, we can write

$$\begin{cases} g(y,s) = \sum_{n\in\mathbb{N}} P_n g = \sum_{\beta\in\mathbb{N}^N} g_\beta(s)h_\beta(y) \\ \|g(s)\|^2_{L^2_\rho} \equiv I(s)^2 = \sum_{n\in\mathbb{N}} l_n(s)^2 \text{ where } l_n(s) \equiv \|P_n g\|_{L^2_\rho}. \end{cases} \tag{35}$$

As for α, we claim the following:

Lemma 2.5 (Estimates on α) *For all $\sigma_0 \geq 0$, $a \in S_\delta$, $|\sigma| \leq \sigma_0$, $y \in \mathbb{R}^N$ and $s \geq -\log T + \sigma_0$,*

$$\alpha(y,s) \leq \frac{C(\sigma_0)}{s}, \quad |\alpha(y,s)| \leq \frac{C(\sigma_0)}{s}(1+|y|^2)$$

$$\text{and } |\alpha(y,s) + \frac{1}{4s}h_2(y_1)| \leq \frac{C(\sigma_0)}{s^{3/2}}(1+|y|^3). \tag{36}$$

Proof: See Lemma 2.5 in [9] where a similar lemma was derived from Proposition 2.2, (22) and (19) (note that both (22) and (19) hold in C^k_{loc} by parabolic regularity). ■

2.3 Modulation for the dilation in the one dimensional solution

We prove Proposition 2.1 here. Practically, since $g_{a,\sigma}$ satisfies equation (26), we consider that equation as a dynamical system and classify all possible asymptotic behaviors the equation can exhibit as $s \to \infty$, under the growth condition (29). It turns out that the effect of α in (26) can be neglected, except on the neutral mode of $\mathcal{L}$. Since the eigenvalues of $\mathcal{L}$ are 1, $\frac{1}{2}$, 0 and $-\frac{k}{2}$ for any integer $k \geq 1$, we expect the positive modes to be neglected. More precisely, unless $g_{a,\sigma}$ decreases faster than e^{-ks} for any $k \in \mathbb{N}$, either the null mode or a negative mode of $\mathcal{L}$ will dominate as $s \to \infty$. Moreover, we expect $g_{a,\sigma}$ to decrease polynomially in the former case (because of the effect of the $\frac{1}{s}$ term in α) and exponentially in the latter. We proceed in 3 steps:

- In Step 1, we project equation (26) on the different modes. We then show that the positive modes are relatively small and that either the null or a negative mode dominates (unless $g_{a,\sigma}$ decreases faster than e^{-ks} for any $k \in \mathbb{N}$).

- In Step 2, we solve the ODE satisfied by the null mode and show that it decays like $\frac{1}{s^2}$, except for a critical explicit value $\sigma(a)$ of σ, where it decays faster.

- In Step 3, we take σ equal to this critical value $\sigma(a)$ and show that the null mode can not dominate, unless $g_{a,\sigma} \equiv 0$. Thus, we drop down in the spectrum from 0 to $-\frac{1}{2}$ or less, which gives exponentially fast decay for $g_{a,\sigma}$.

Step 1: Dominance of a particular mode

Let us first project (26) on the different modes. For the null mode of $\mathcal{L}$ ($|\beta| = 2$), the main term of the equation comes from the main term of α (see (36)).

Lemma 2.6 (Projection of (26) on the different modes) *For all $\sigma_0 \geq 0$, $a \in S_\delta$, $|\sigma| \leq \sigma_0$ and $s \geq -\log T + \sigma_0$, we have the following:*
i) For all $n \in \mathbb{N}$, $|l_n' + (\frac{n}{2} - 1)l_n| \leq C(n, \sigma_0)\frac{I(s)}{s}$,
ii) For all $n \in \mathbb{N}$, $I'(s) \leq \left(1 - \frac{n+1}{2} + \frac{C_0(\sigma_0)}{s}\right) I(s) + \sum_{k=0}^{n} \frac{1}{2}(n+1-k)l_k(s)$.
iii) If $|\beta| = 2$, then $\left|g_\beta'(s) + \frac{\beta_1}{s} g_\beta(s)\right| \leq C(\sigma_0)\frac{I(s)}{s^{3/2}} + C(\sigma_0)\frac{l_0 + l_4}{s}$.

Proof: The calculation is straightforward. Parts i) and ii) follow from (26) and Lemma 2.5 exactly as in Lemma 2.7 in [9].
iii) The calculation is straightforward and similar to the proof of Proposition 2.9 in [9]. See Appendix B.1 for details. ■

Our main goal in this step is to show that one mode has to dominate all the others (unless $I(s)$ decays faster that e^{-ks} for any $k \in \mathbb{N}$). The argument would be clear if α was identically zero, because the modes would not interact in that case. In the actual proof, we rely on this simple fact and treat the term αg as a perturbation to get the result. We claim the following lemma (which was proved in [9] with no special care to uniform estimates with respect to $a \in S_\delta$):

Lemma 2.7 (Dominance of a mode) *For all $a \in S_\delta$ and $\sigma \in \mathbb{R}$, either for all $m \in \mathbb{N}$, $l_m(s) = O\left(\frac{I(s)}{s}\right)$ or there is $n \geq 2$ such that $I \sim l_n$ as $s \to \infty$. In that case, $\forall m \neq n$, $l_m = O\left(\frac{I}{s}\right)$ as $s \to \infty$.*

Proof: See Proposition 2.6 in [9]. ■

Lemma 2.7 asserts that the positive modes l_0 and l_1 are $O\left(\frac{I}{s}\right)$ as $s \to \infty$. We need to know that this holds uniformly with respect to a and σ. We claim the following:

Lemma 2.8 (Uniform smallness of the positive modes)
For all $\sigma_0 \geq 0$, there exists $s_1 > 0$ such that for all $a \in S_\delta$, $|\sigma| \leq \sigma_0$ and $s \geq s_1$, $l_0(s) + l_1(s) \leq 2C(\sigma_0)\frac{I(s)}{s}$.

Proof: It is the same as in [9], with more care about the dependence of the constants. See Appendix B.2 for the proof. ■

Step 2: Asymptotic behavior of the null mode
We first use the decay information on $I(s)$ and $l_0(s)$ contained in (29) and Lemma 2.8 to solve the ODE satisfied by the null mode and stated in iii) of Lemma 2.6. We claim the following:

Lemma 2.9 (Decay of the null mode of (26)) *For all $\sigma_0 \geq 0$, there is $s_3(\sigma_0)$ such that for all $a \in S_\delta$, $|\sigma| \leq \sigma_0$, $s \geq s_3(\sigma_0)$ and $|\beta| = 2$, we have:*

$$\begin{cases} |g_{a,\sigma,\beta}(s)| \leq C(\sigma_0)\frac{\log s}{s^{5/2}} & \text{if } \beta_1 \neq 2 \\ |g_{a,\sigma,\beta}(s) - \frac{k_{a,\sigma}}{s^2}| \leq C(\sigma_0)\frac{\log s}{s^{5/2}} & \text{if } \beta_1 = 2. \end{cases}$$

Proof: This is straightforward. See Appendix B.3. ■

Now it becomes clear that by making $k_{a,\sigma} = 0$, the decay of the null mode is faster, which suggests that the null mode may not dominate, therefore, we drop down in the spectrum to $-\frac{1}{2}$ or less, which yields exponential decay. But, can we make $k_{a,\sigma} = 0$? The answer is yes and this comes from a simple fact: the difference $k_{a,\sigma} - k_{a,0}$ does not depend on the function w or on the blow-up point $a \in S_\delta$, or even on the one dimensional solution $\tilde{w}$; it is a linear function of σ. More precisely, we have the following lemma, which is the core of our argument:

Lemma 2.10 (Modulation of the value of σ) *For all $a \in S_\delta$ and $\sigma \in \mathbb{R}$, $k_{a,\sigma} = k_{a,0} - \frac{\kappa}{4p}\sigma$.*

Proof: By definition of $k_{a,\sigma}$ (see Lemma 2.9 and (34)),

$$k_{a,\sigma} = \lim_{s\to\infty} s^2 \int_{\mathbb{R}^N} g_{a,\sigma}(y,s)k_2(y_1)\rho(y)dy. \tag{37}$$

Therefore,

$$\begin{aligned} k_{a,\sigma} - k_{a,0} &= \lim_{s\to\infty} s^2 \int_{\mathbb{R}^N} \left(g_{a,\sigma}(y,s) - g_{a,0}(y,s)\right) k_2(y_1)\rho(y)dy \\ &= \lim_{s\to\infty} s^2 \int_{\mathbb{R}} \left(\tilde{w}(y_1,s) - \tilde{w}(y_1,s+\sigma)\right) k_2(y_1)\rho_1(y_1)dy_1 \end{aligned} \tag{38}$$

according to (25) and (32). In particular, $k_{a,0} - k_{a,\sigma}$ does not depend on w or on $a \in S_\delta$. Since we know from ii) in Lemma 2.4, (31) and (32) that

$$\int_{\mathbb{R}} \frac{\partial \tilde{w}}{\partial s}(y_1,s)k_2(y_1)\rho_1(y_1)dy_1 \sim \frac{\kappa}{4ps^2} \text{ as } s \to \infty,$$

the conclusion follows by the mean value theorem. ■

In the following, we take $\sigma = \sigma(a) \equiv \frac{4p}{\kappa}k_{a,0}$, which makes $k_{a,\sigma} = 0$.

Step 3: Exponential decay of $g_{a,\sigma(a)}$ in L^2_ρ

We conclude the proof of Proposition 2.1 here. With this choice of σ, $k_{a,\sigma} = 0$, hence, iii) of Lemma 2.6 and Lemma 2.9 yield

$$l_2'(s) \geq -\frac{2}{s}l_2 - \frac{C}{s^{3/2}}I(s) \text{ and } l_2(s) = O\left(\frac{\log s}{s^{5/2}}\right) \text{ as } s \to \infty \tag{39}$$

(recall that $l_2^2 = \sum_{|\beta|=2} g_\beta^2 \|h_\beta\|_{L^2_\rho}^2$). This implies that we cannot have $I \sim l_2$, unless $I \equiv 0$. Therefore, Lemma 2.7 implies that either a negative mode dominates, or all the modes are less than $CI(s)/s$. In both cases, the differential inequality ii) in Lemma 2.6 yields exponential decay for $I(s)$, which is the desired conclusion. However, we need to make this decay uniform with respect to the blow-up point $a \in S_\delta$. We need first to fix σ_0. The uniform estimate of Lemma 2.9 along with the continuity of $g_{a,\sigma}(y,s)$ with respect to a, σ and s (see (25)) yields the continuity of $k_{a,\sigma}$ with respect to $(a,\sigma) \in S_\delta \times \mathbb{R}$ (see (37)). Hence, we can fix

$$\sigma_0 = \max_{a \in S_\delta} \frac{4p}{\kappa} |k_{a,0}| < +\infty \tag{40}$$

and define a continuous function $\sigma : S_\delta \to [-\sigma_0, \sigma_0]$ by $\sigma(a) = \frac{4p}{\kappa} k_{a,0}$. Just note that if we take $n = 2$ in i) and ii) of Lemma 2.6 and use Lemma 2.8, then we see that $x = l_2$ and $y = I$ satisfy the inequality (41) in the following ODE lemma:

Lemma 2.11 (ODE Lemma) *For all $M > 0$ and $\hat{s}$, there is $\bar{s}(M, \hat{s}) \geq \hat{s}$ such that if $0 \leq x(s) \leq y(s) \to 0$ as $s \to \infty$ and*

$$\forall s \geq \hat{s}, \ \begin{cases} x' \geq -\frac{M}{s} y \\ y' \leq -\frac{1}{2} y + \frac{M}{s} y + \frac{1}{2} x, \end{cases} \tag{41}$$

then either $\quad \forall s \geq \bar{s}, \ x(s) \leq \frac{5M}{s} y(s) \quad$ (42)

or $\quad y \geq x > 0$ *and* $y \sim x$ *as* $s \to \infty$. $\quad$ (43)

Remark: If (43) holds, then we have no uniform control with respect to M and $\hat{s}$.
Proof: See Appendix B.2. ■

We have just proved that (43) doesn't hold. Therefore, for all $a \in S_\delta$ and $s \geq s_2$ for some $s_2 > 0$, $l_2(s) \leq CI(s)/s$. Using Lemma 2.8 and ii) in Proposition 2.6 (take $n = 3$) yields for all $a \in S_\delta$, if $\sigma = \frac{4p}{\kappa} k_{a,0}$, then

$$\forall s \geq s_0, \begin{cases} l_k(s) & \leq C \frac{I(s)}{s} \text{ if } k = 0, \ 1 \text{ or } 2 \\ I'(s) & \leq \left(-\frac{1}{2} + \frac{C_0}{s}\right) I(s) + \frac{1}{2} \sum_{k=0}^{2} (n+1-k) l_k(s). \end{cases}$$

Therefore, $\forall s \geq s_0$, $I'(s) \leq (-\frac{1}{2} + \frac{C}{s}) I(s)$, hence $I(s) \leq C_0 e^{-\frac{s}{2}} s^{C_0}$ for some $C_0 > 0$. This concludes the proof of Proposition 2.1. ■

3 Blow-up behavior of u in a tubular neighborhood of S

We prove Theorem 2 here. We have proved in [23] that (7) holds. This estimate identifies for each $t \in [0, T)$ three regions in $B(\hat{a}, \delta)$:

- **The blow-up region**: It is $\{x \mid d(x, S) \leq \sqrt{(T-t)|\log(T-t)|}\}$. According to (7), it corresponds to the set $\{x \mid |u(x,t)| \geq \eta \|u(t)\|_{L^\infty}\}$ for some $0 < \eta < 1$.

- **The regular region**: It is the region far away from blow-up, where u stays bounded, say by 1. It corresponds to $\{x \mid d(x, S) \geq \epsilon_0\}$ for some $\epsilon_0 > 0$.

- **The intermediate region**, between the two others, that is $\{x \mid 1 \leq |u(x,t)| \leq \eta \|u(t)\|_{L^\infty}\}$ or $\{x \mid \sqrt{(T-t)|\log(T-t)|} \leq d(x,S) \leq \epsilon_0\}$.

We handle separately the blow-up and the intermediate regions whose union makes the tubular neighborhood. Our technique is the same as in [9]. Although we had only one blow-up point in [9], it turns out that the techniques of [9] hold uniformly with respect to the blow-up point, when they are adapted to the present case. Therefore, we follow in extent the method of [9]. However, we omit technical details; the reader can find them in [9] and in the appendix. We proceed in 3 steps:

- In Step 1, we use the transport effect of the term $-\frac{1}{2} y . \nabla g$ in equation (26) to extend the convergence of Proposition 2.1 from compact sets to larger sets $|y| \leq \sqrt{s}$, i.e., the blow-up region $d(x, S) \leq \sqrt{(T-t)|\log(T-t)|}$, after the change (17).

- In Step 2, we use the information on the edge of the blow-up region, i.e., when $d(x, S) = \sqrt{(T-t)|\log(T-t)|}$ as initial data to solve the ODE $u' = u^p$, which turns out to be a very good approximation for the PDE in the intermediate region $\sqrt{(T-t)|\log(T-t)|} \leq d(x, S) \leq \epsilon_0$, as mentioned in Proposition 2.3.

- In Step 3, we just gather the previous information to prove Theorem 2.

Step 1: The blow-up region
The L^2_ρ estimate of Proposition 2.1 also holds uniformly on compact sets. The convection term $-\frac{1}{2} y . \nabla g$ in equation (26) allows us to carry estimates from compact sets to sets $|y| \leq \sqrt{s}$ along characteristics of the type $y =$

$Re^{\frac{s-s'}{2}}$. The following lemma is a corollary of Proposition 2.1 in Velázquez [19]. It is proved in the course of the proof of Proposition 2.13 in [9].

Lemma 3.1 (Velźaquez - Extension of the convergence from compact sets to sets $|y| \le \sqrt{s}$) *Assume g is a solution of*

$$\partial_s g = \Delta g - \frac{1}{2} y.\nabla g + g + \alpha(y,s) g \text{ for } (y,s) \in \mathbb{R}^N \times [\hat{s}, \infty),$$

where $\alpha(y,s) \le \frac{M}{s}$ and $|g(y,s)| \le M$. Then, for all $s' \ge \hat{s}$ and $s \ge s'+1$ such that $e^{\frac{s-s'}{2}} = \sqrt{s}$, we have

$$\sup_{|y| \le \sqrt{s}} |g(y,s)| \le C(M) e^{s-s'} \|g(s')\|_{L^2_\rho}.$$

This lemma along with Proposition 2.1 yields for all $a \in S_\delta$ and $s \ge s_0 + 1$,

$$\sup_{|y| \le \sqrt{s}} |g_{a,\sigma(a)}(y,s)| \le C e^{s-s'} C_0 e^{-\frac{s'}{2}} s'^{C_0},$$

where $e^{\frac{s-s'}{2}} = \sqrt{s}$. Since $s' = s - \log s$, we have just proved part i) of the following proposition:

Proposition 3.2 (Uniform estimates for w_a in larger sets $|y| \le \sqrt{s}$) *For all $a \in S_\delta$, $s \ge s_0 + 1$ and $|y| \le \sqrt{s}$,*

$$i) \qquad |g_{a,\sigma(a)}(y,s)| \le C e^{-\frac{s}{2}} s^{\frac{3}{2}+C_0},$$

$$ii) \qquad |w_a(y,s) - \tilde{w}(y.Q_a e_1, s + \sigma(a))| \le C e^{-\frac{s}{2}} s^{\frac{3}{2}+C_0},$$

where s_0 and C_0 are defined in Proposition 2.1.

Proof of ii): Just change $Q_0 y$ into y in part i) and use the definition of g given in (25). ■

Now, we just rewrite part ii) of the previous proposition in the original variables $u(x,t)$ through the transformation (17) to get the following corollary:

Corollary 3.3 (Uniform estimates for $u(x,t)$ in the larger sets $|x - a| \le \sqrt{(T-t)|\log(T-t)|}$) *For all $a \in S_\delta$, $t \ge T - e^{-s_0-1}$ and $|x-a| \le \sqrt{(T-t)|\log(T-t)|}$,*

$$\left| u(x,t) - (T-t)^{-\frac{1}{p-1}} \tilde{w} \left(\frac{d(x,T_a)}{\sqrt{T-t}}, -\log(T-t) + \sigma(a) \right) \right| =$$
$$\left| u(x,t) - \tilde{u}_{\sigma(a)}(d(x,T_a), t) \right| \le C(T-t)^{\frac{1}{2}-\frac{1}{p-1}} |\log(T-t)|^{3/2+C_0},$$

where T_a is the tangent plane to S at a and $\tilde{u}_{\sigma(a)}$ is defined in (10).

End of the proof: The only delicate point in this transformation is the computation of $y.Q_a e_1$ in terms of x, a and t. Using (17), we have $|y.Q_a e_1| = |(x-a).Q_a e_1|(T-t)^{-1/2} = d(x,T_a)(T-t)^{-1/2}$, because $Q_a e_1$ is the normal direction to the blow-up set S at the blow-up point a (see (20)). The relation between $\tilde{w}$ and $\tilde{u}_\sigma$ follows directly from the definition of $\tilde{w}$ (21) and the definition of $\tilde{u}_\sigma$ (10). ■

Now, if we choose a to be the closest blow-up point to x, that is $a = P_S(x)$, the projection of x on the blow-up set S, then we get $d(x,T_a) = d(x,S)$, which yields the following corollary:

Corollary 3.4 (Uniform estimates for $u(x,t)$ in the blow-up region $d(x,S) \le \sqrt{(T-t)|\log(T-t)|}$) *For all $t \ge T - e^{-s_0-1}$ and $x \in B(\hat{a},\delta)$ such that $d(x,S) \le \sqrt{(T-t)|\log(T-t)|}$,*

$$\left|u(x,t) - \tilde{u}_{\sigma(P_S(x))}\left(d(x,S),t\right)\right| \le C(T-t)^{\frac{1}{2}-\frac{1}{p-1}}|\log(T-t)|^{3/2+C_0},$$

where $P_S(x)$ is the projection of x over S.

Remark : We need the restriction $|x-\hat{a}| < \delta$ to guarantee the fact that $P_S(x)$ is in $S_\delta \equiv S \cap B(\hat{a},2\delta)$, defined in (16), so that Corollary 3.3 applies. Indeed, if $|x-\hat{a}| < \delta$, then $|P_S(x)-\hat{a}| \le |P_S(x)-x|+|x-\hat{a}| \le 2|x-\hat{a}| < 2\delta$, because $\hat{a} \in S$. Hence $P_S(x) \in S_\delta$.

Step 2: Estimates in the intermediate region

We consider a point (x,t) in the intermediate region, i.e. such that $d(x,S) \ge \sqrt{(T-t)|\log(T-t)|}$. We remark that the point $(x,\tilde{t}(d(x,S)))$ where $\tilde{t}(d)$ is defined by

$$d = \sqrt{(T-\tilde{t})|\log(T-\tilde{t})|} \qquad (44)$$

is on the frontier of the two regions (note that $\tilde{t} \le t$). Therefore, we have an estimate on u and on $u-\tilde{u}_{\sigma(P_S(x))}$ at $(x,\tilde{t}(d(x,S)))$, respectively from (7) and from Corollary 3.4. Moreover, the PDE (1) can be uniformly localized by the ODE $u' = u^p$, according to Proposition 2.3. The one dimensional solution $\tilde{u}$ too. Our idea is simple: we use the ODE to propagate the information on $u-\tilde{u}_{\sigma(P_S(x))}$ from time $\tilde{t}$ to t. Thus, the error term on $u-\tilde{u}_{\sigma(P_S(x))}$ in the intermediate region will be the same as the one on the edge. More precisely:

Proposition 3.5 (Estimates in the intermediate region $\sqrt{(T-t)|\log(T-t)|} \le d(x,S) \le \epsilon_0$) *There exists $\epsilon_0 > 0$ such that for all*

$x \in B(\hat{a},\delta)$ and $t \in [0,T)$, if $\sqrt{(T-t)|\log(T-t)|} \le d(x,S) \le \epsilon_0$, then

$$\begin{aligned}|u(x,t) - \tilde{u}_{\sigma(P_S(x))}(d(x,S),t)| &\le C(T-\tilde{t})^{\frac{1}{2}-\frac{1}{p-1}}|\log(T-\tilde{t})|^{3/2+C_0}\\ &\le Cd(x,S)^{1-\frac{2}{p-1}}|\log d(x,S)|^{\frac{p}{p-1}+C_0},\end{aligned}$$

where $P_S(x)$ is the orthogonal projection of x on S and $\tilde{t} = \tilde{t}(d(x,S))$ is defined by (44).

Proof : The main argument of the proof has just been given. The reader can find the "technical" proof in Appendix C. ■

Step 3: Estimates in a tubular neighborhood of S

We prove Theorem 2 here. Let $t_1 = \max(T - e^{-s_0-1}, \tilde{t}(\epsilon_0))$ where ϵ_0 and $\tilde{t}(\epsilon_0)$ are given in Proposition 3.5, and consider some $x \in B(\hat{a},\delta)$ such that $d(x,S) \le \epsilon_0$.

i) Let $t \in [t_1, T)$.

If $t \le \tilde{t}(d(x,S))$ defined in (44), then $d(x,S) \le \sqrt{(T-t)|\log(T-t)|}$. Use Corollary 3.4.

If $t \ge \tilde{t}(d(x,S))$, then $d(x,S) \ge \sqrt{(T-t)|\log(T-t)|}$. Use Proposition 3.5.

ii) Just make $t \to T$ in i) and use (10). ■

4 Regularity of the blow-up set

We prove Theorems 4 and Proposition 3 here. To keep up with the notation of [23], we assume that $\hat{a} = 0$ and $Q_{\hat{a}} = \mathrm{Id}$, and consider that S_δ, the intersection of S with $B(\hat{a}, 2\delta)$ (see (16)), is the graph of a function $\varphi \in C^1(B_{N-1}(0,\delta_1),\mathbb{R})$ of the variable $\tilde{x} = (x_2, ..., x_N)$. If we introduce

$$A(\tilde{x}) = (\varphi(\tilde{x}), \tilde{x}),$$

then $\mathrm{Im}\, A \cap B(\hat{a},2\delta) = \mathrm{graph}\, \varphi \cap B(\hat{a},2\delta) = S_\delta$. Given x near S_δ, Corollary 3.3 gives many different asymptotic behaviors for $u(x,t)$, depending on the choice of the point $a \in \mathrm{Im}\, A \cap B(x, \sqrt{(T-t)|\log(T-t)|})$. All these possible behaviors have to agree, up to the error term in Corollary 3.3. This implies a geometric constraint on S_δ, which gives some more regularity on A (and φ).

We consider some $|\tilde{x}| < \delta_1$ and some $\tilde{h} \in \mathbb{R}^{N-1}$ such that $|\tilde{x} + \tilde{h}| < \delta_1$ and $A(\tilde{x})$ as well as $A(\tilde{x}+\tilde{h})$ are in S_δ. Since A is C^1 and σ is continuous (see Proposition 2.1), there is C^* such that

$$|\varphi'(\tilde{x})| \le C^*, \ \ |A(\tilde{x}+\tilde{h}) - A(\tilde{x})| \le C^*|\tilde{h}| \text{ and } |\sigma(A(\tilde{x}))| \le C^*. \qquad (45)$$

For any time $t \geq T - e^{-s_0-1}$ such that
$|A(\tilde{x}) - A(\tilde{x} + \tilde{h})| \leq \sqrt{(T-t)|\log(T-t)|}$, we can estimate $u(A(\tilde{x}+\tilde{h}), t)$ from Corollary 3.3 in two ways:

- First by taking $x = a = A(\tilde{x} + \tilde{h})$ and $s = -\log(T-t)$, which gives

$$\left|(T-t)^{\frac{1}{p-1}} u(A(\tilde{x}+h), t) - \tilde{w}(0, s + \sigma(A(\tilde{x}+\tilde{h})))\right| \leq Ce^{-\frac{s}{2}} s^{\frac{3}{2}+C_0}. \quad (46)$$

- Second, by taking $a = A(\tilde{x})$, $x = A(\tilde{x} + \tilde{h})$ and $s = -\log(T-t)$, which gives

$$\left|(T-t)^{\frac{1}{p-1}} u(A(\tilde{x}+\tilde{h}), t) - \tilde{w}\left(d\left(A(\tilde{x}+\tilde{h}), T_{A(\tilde{x})}\right) e^{\frac{s}{2}}, s + \sigma(A(\tilde{x}))\right)\right| \\ \leq Ce^{-\frac{s}{2}} s^{\frac{3}{2}+C_0}. \quad (47)$$

Now, if we fix $t = \tilde{t}(\tilde{x}, \tilde{h})$ such that

$$\left|A(\tilde{x}+\tilde{h}) - A(\tilde{x})\right| = \sqrt{(T - \tilde{t}(\tilde{x},\tilde{h}))|\log(T - \tilde{t}(\tilde{x},\tilde{h}))|} \quad (48)$$

and take $|\tilde{h}| < h_1(s_0)$ for some $h_1(s_0) > 0$, then we see from (45) that $\tilde{t}(\tilde{x},\tilde{h}) \geq T - e^{-s_0-1}$, hence (46) and (47) hold. Therefore, if $\tilde{s} = -\log(T - \tilde{t})$, then

$$\left|\tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}+\tilde{h}))) - \tilde{w}\left(d\left(A(\tilde{x}+\tilde{h}), T_{A(\tilde{x})}\right) e^{\frac{\tilde{s}}{2}}, \tilde{s} + \sigma(A(\tilde{x}))\right)\right| \\ \leq Ce^{-\frac{\tilde{s}}{2}} \tilde{s}^{\frac{3}{2}+C_0}. \quad (49)$$

By changing the roles of $\tilde{x}$ and $\tilde{x} + \tilde{h}$, we don't change $\tilde{t}(\tilde{x},\tilde{h})$ and obtain similarly

$$\left|\tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}))) - \tilde{w}\left(d\left(A(\tilde{x}), T_{A(\tilde{x}+\tilde{h})}\right) e^{\frac{\tilde{s}}{2}}, \tilde{s} + \sigma(A(\tilde{x}+\tilde{h}))\right)\right| \\ \leq Ce^{-\frac{\tilde{s}}{2}} \tilde{s}^{\frac{3}{2}+C_0}. \quad (50)$$

Since $\tilde{u}$, hence $\tilde{w}$ are radially decreasing (see page 4), this yields

$$\left|\tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}))) - \tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}+\tilde{h})))\right| \leq Ce^{-\frac{\tilde{s}}{2}} \tilde{s}^{\frac{3}{2}+C_0}. \quad (51)$$

Indeed, if $\tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}))) - \tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}+\tilde{h}))) \geq 0$, then

$$\begin{aligned} 0 &\leq \tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}))) - \tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}+\tilde{h}))) \\ &\leq \tilde{w}\left(0, \tilde{s} + \sigma(A(\tilde{x}))) - \tilde{w}(d\left(A(\tilde{x}), T_{A(\tilde{x}+\tilde{h})}\right) e^{\frac{\tilde{s}}{2}}, \tilde{s} + \sigma(A(\tilde{x}+\tilde{h}))\right) \end{aligned}$$

because $\tilde{w}$ is radially decreasing. Hence, (51) follows from (50). Do the same and use (49) in the other case.
Therefore, with a triangular identity, we get from (51) and (49)

$$\begin{aligned} &0 \le \tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}))) - \tilde{w}(d\left(A(\tilde{x} + \tilde{h}), T_{A(\tilde{x})}\right) e^{\frac{\tilde{s}}{2}}, \tilde{s} + \sigma(A(\tilde{x}))) \\ &\le Ce^{-\frac{\tilde{s}}{2}} \tilde{s}^{\frac{3}{2}+C_0}. \end{aligned} \tag{52}$$

Note that since $A(\tilde{x}) \in T_{A(\tilde{x})}$, we have $\frac{d(A(\tilde{x}+\tilde{h}), T_{A(\tilde{x})})}{|A(\tilde{x}+\tilde{h}) - A(\tilde{x})|} \le 1$. Therefore, i) of Lemma 2.4 implies that there is $C > 0$ and $h_2 > 0$ such that if $|\tilde{h}| < h_2$ then $\tilde{s} + \sigma(A(\tilde{x})) \ge \hat{s}$ by (48) and (45) and

$$\begin{aligned} &\tfrac{C}{\tilde{s}+\sigma(A(\tilde{x}))} \left(d\left(A(\tilde{x} + \tilde{h}), T_{A(\tilde{x})}\right) e^{\frac{\tilde{s}}{2}} \right)^2 \\ &\le \tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}))) - \tilde{w}\left(d\left(A(\tilde{x} + \tilde{h}), T_{A(\tilde{x})}\right) e^{\frac{\tilde{s}}{2}}, \tilde{s} + \sigma(A(\tilde{x})) \right). \end{aligned} \tag{53}$$

Since $\operatorname{Im} A$ is the graph of φ, we have

$$d\left(A(\tilde{x} + \tilde{h}), T_{A(\tilde{x})}\right) = \frac{\left|\varphi(\tilde{x} + \tilde{h}) - \varphi(\tilde{x}) - \tilde{h}.\nabla\varphi(\tilde{x})\right|}{\sqrt{1 + |\nabla\varphi(\tilde{x})|^2}}. \tag{54}$$

Using iii) in Lemma 2.4, we get $h_3 > 0$ such that if $|\tilde{h}| < h_3$, then $\tilde{s}$ is large enough by (48) and (45) and

$$\begin{aligned} &\tfrac{C}{\tilde{s}^2} |\sigma(A(\tilde{x})) - \sigma(A(\tilde{x} + \tilde{h}))| \\ &\le \left| \tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x}))) - \tilde{w}(0, \tilde{s} + \sigma(A(\tilde{x} + \tilde{h}))) \right|. \end{aligned} \tag{55}$$

If $\tau(d)$ is given by $d = \sqrt{\tau |\log \tau|}$, then

$$\log \tau \sim 2 \log d \text{ and } \tau \sim \frac{d^2}{2|\log d|} \text{ as } d \to 0.$$

Therefore, $\frac{\log|\log \tau|}{|\log \tau|} \le \frac{\log|\log d|}{|\log d|}$ if $|d| \le d_0$ for some $d_0 > 0$. Combining this with (48) and (45), we have for all $|\tilde{h}| < h_4$ for some $h_4 > 0$,

$$\begin{aligned} &e^{-\frac{\tilde{s}}{2}} \sqrt{(\tilde{s} + \sigma(A(\tilde{x})))\, e^{-\frac{\tilde{s}}{2}} \tilde{s}^{\frac{3}{2}+C_0}} \le Cd^{\frac{3}{2}} |\log d|^{\frac{1}{2}+\frac{C_0}{2}} \le C|\tilde{h}|^{\frac{3}{2}} |\log|\tilde{h}||^{\frac{1}{2}+\frac{C_0}{2}} \\ &\tilde{s}^2 e^{-\frac{\tilde{s}}{2}} \tilde{s}^{\frac{3}{2}+C_0} \le Cd|\log d|^{3+C_0} \le C|\tilde{h}||\log|\tilde{h}||^{3+C_0}, \end{aligned} \tag{56}$$

where $d = |A(\tilde{x}) - A(\tilde{x} + h)|$. Take $h_0 = \min(h_1, h_2, h_3, h_4)$. Combining (53), (54), (52), (56) and (45) gives the regularity estimate for φ. Combining (55), (51) and (56) gives the regularity estimate for σ and closes the proof of Proposition 3. ■

5 Connection with a chemotaxis problem

We would like to mention connections between the ideas of this paper and the chemotaxis problem of Betterton and Brenner [2]. Chemotaxis refers to the movement of bacteria under a gradient of some chemical substance. Under special conditions, bacteria excrete a substance to attract neighboring individuals. This way, bacteria aggregate and their density blows up in finite time $T > 0$. For simplicity, we assume that the cellular division is much slower than the dynamics of chemotaxis, and that the diffusion of bacteria is much slower than the diffusion of the attractant. Therefore, we have from [2] the equations satisfied by ρ, the bacterial density and c, the chemical attractant concentration:

$$\begin{array}{ll} \frac{\partial \rho}{\partial t} & = \Delta\rho - \nabla.(\rho\nabla c) = \Delta\rho + \rho^2 - \nabla\rho.\nabla c \\ 0 & = \Delta c + \rho. \end{array} \tag{57}$$

Many blow-up regimes are possible, depending on the relative importance of the three terms in the right hand side of (57). A global picture is presented by Brenner *et. al* in [3], in the case of radial solutions. One of those regimes has the same scaling $\sqrt{(T-t)|\log(T-t)|}$ as equation (1) with $p = 2$ (see section 4.3 in [3]).

In an experiment conducted by Budrene and Berg [6, 7], (see also Brenner, Levitov and Budrene [4]), it appears clearly that the dynamics are 3 dimensional and not radial. The authors observe two regimes in this finite time blow-up:

- **The transient regime**, for $t \leq t_1$ for some $t_1 < T$. The bacteria aggregate along cylindrical structures that shrink towards their common axis, as time grows. This suggests that the axis of the cylinder would be the singular set.

- **The asymptotic regime**: The cylinder is destabilized at time $t = t_1$ and breaks up into spherical aggregates. Then, the three dimensions of the sphere shrink simultaneously, leading to isolated blow-up points.

Although the chemotaxis equation is non local, it has the same one dimensional scaling as the heat equation (1). Both equations deal with blow-up on a continuum (say on a line) and share the idea of the instability of such a behavior (only single point blow-up is thought to be generic for equation (1)). However, the goals of the two papers are different. Indeed, while [2] proves the instability of the blow-up on a line, we prove here that if this occurs, which is exceptional, then we have more constraints, hence more regularity on that line. Although the goals are different, the same idea is used in both works: how to connect all local singular behavior near singular points (or candidates for singular points in the case of [2]) to get a global picture of the situation?

In [2], the destabilization of the cylinder at time t_1 breaks the symmetry and induces a variation of a "local blow-up time", or *phase*. The variation of the phase along the line is governed by a phase equation. The minimum of the phase determines the actual blow-up point. In our case, the connection between local behaviors is done through the dilation $\sigma(a)$, $a \in S$, analogous to the phase of [2]. The Liouville theorem of [15] cited on page 3 is the key tool to connect local descriptions. We are unable to find a non trivial phase equation for σ, analogous to that of chemotaxis. However, since σ is linked to the one dimensional scaling of (1), which is also present for chemotaxis, we believe that if one adopts our point of view in chemotaxis, σ would satisfy a non trivial equation, related to the phase equation of [2]

A Properties of the particular single point blow-up solution in one dimension

A.1 Existence of the one dimensional solution

We prove here the existence of the particular one dimensional solution announced on page 4. Take g a symmetric positive continuous function, decreasing on $(0, \infty)$ and going to zero at infinity. The solution $\tilde{u}(x_1, t)$ of (1) with initial data kg is symmetric and decreasing on $(0, \infty)$ as well. If k is large enough, then $\tilde{u}(x_1, t)$ blows up in finite-time $\tilde{T}$, only at the origin (see Theorems 1 and 2 in Mueller and Weissler [16]). We can assume $\tilde{T} = T$ by changing $\tilde{u}$ into some

$$\tilde{u}_\lambda(x_1, t) = \lambda^{\frac{2}{p-1}} \tilde{u}(\lambda x_1, \lambda^2 t).$$

Theorem 1 in Herrero and Velázquez [12] then asserts that $\tilde{u}$ has the profile f_1 defined in (5). $\tilde{u}$ is not self-similar, because the only self-similar solutions of (1) are independent of space, hence trivial (see Theorem 1' in Giga and Kohn [11]). ■

A.2 Elementary estimates for the one dimensional solution

We prove Lemma 2.4 here.

i) Using a Taylor expansion, we write

$$\tilde{w}(y_1,s) = \tilde{w}(0,s) + y_1\frac{\partial \tilde{w}}{\partial y_1}(0,s) + \frac{1}{2}y_1^2\frac{\partial^2 \tilde{w}}{\partial y_1^2}(0,s) + \frac{1}{6}y_1^3\frac{\partial^3 \tilde{w}}{\partial y_1^3}(z_1,s)$$

for some $z_1 \in (0, y_1)$. Since $\tilde{w}$ is even, we have $\frac{\partial \tilde{w}}{\partial y_1}(0,s) \equiv 0$. Since (22) also holds in C^k_{loc}, we have $\frac{\partial^2 \tilde{w}}{\partial y_1^2}(0,s) \le -\frac{4\tilde{C}}{s}$ for some $\tilde{C} > 0$. Since Proposition 2.2 implies that $|\frac{\partial^3 \tilde{w}}{\partial y_1^3}(z_1,s)| \le \frac{C_3}{s^{3/2}}$, we combine all the previous estimates with the Taylor expansion to get

$$\forall |y_1| \le \frac{6\tilde{C}}{C_3}\sqrt{s} \equiv \tilde{\delta}\sqrt{s}, \;\; \tilde{w}(y_1,s) \le \tilde{w}(0,s) - \frac{\tilde{C}}{s}y_1^2. \tag{58}$$

If $\tilde{\delta} \ge 1$, then the proof is complete.
If $\tilde{\delta} < 1$, then recall that

$$\sup_{|y_1| \le \sqrt{s}} \left| \tilde{w}(y_1,s) - f_1\left(\frac{y_1}{\sqrt{s}}\right)\right| \to 0 \text{ as } s \to \infty, \tag{59}$$

since $\tilde{u}$ has the profile f_1 defined in (5). Therefore, there is $\hat{s} > 0$ such that if $s \ge \hat{s}$ and $\tilde{\delta}\sqrt{s} \le |y_1| \le \sqrt{s}$, then

$$|\tilde{w}(0,s) - \tilde{w}(y_1,s)| \ge \frac{1}{2}\left|f_1(0) - f_1(\tilde{\delta})\right| \ge \frac{1}{2}\left|f_1(0) - f_1(\tilde{\delta})\right|\frac{|y_1|^2}{s}. \tag{60}$$

The conclusion then follows from (58) and (60).

ii) See identity (5.34) on page 854 in Filippas and Kohn.

iii) We know from (59) that

$$\tilde{w}(y_1,s) \to f_1(0) = (p-1)^{-\frac{1}{p-1}} \text{ as } s \to \infty$$

uniformly on compact sets. Since $\|\Delta\tilde{w}(s)\|_{L^\infty}$ and $\|\nabla\tilde{w}(s)\|_{L^\infty}$ go to 0 as $s \to \infty$ (see Proposition 2.2), we use equation (18) to get

$$\frac{\partial \tilde{w}}{\partial s}(y_1,s) \to 0 \text{ as } s \to \infty$$

uniformly on compact sets. By the Lebesgue Theorem, we obtain

$$\left\|\frac{\partial \tilde{w}}{\partial s}(s)\right\|_{L^2_{\rho_1}(\mathbb{R})} \to 0 \text{ as } s \to \infty.$$

Let us introduce $q(y_1,s)=\frac{\partial\tilde w}{\partial s}(y_1,s)$. From (18), we see that q satisfies an equation of the same type as (26):

$$\frac{\partial q}{\partial s}=(\mathcal{L}+\alpha(y_1,s))\,q, \tag{61}$$

where $\mathcal{L}q=\frac{\partial^2 q}{\partial y_1^2}-\frac{1}{2}y_1\frac{\partial q}{\partial y_1}+q$ and $\alpha(y_1,s)=p\tilde w(y_1,s)^{p-1}$. In particular, we have the same dynamical system techniques as for equation (26). Therefore, we just sketch our argument and borrow techniques from section 2 and from [9] where the same equation was considered. Since $\tilde w$ satisfies Proposition 2.2 and (22), α satisfies the estimates of Lemma 2.5. If we borrow the notations we used for g in section 2 and write

$$q(y_1,s)=\sum_{n\in\mathbb{N}}q_n(s)h_n(y_1),\quad I(s)=\|q(s)\|_{L^2_{\rho_1}},\quad l_n(s)=|q_n(s)|\|h_n(y_1)\|_{L^2_{\rho_1}}, \tag{62}$$

then we have $I(s)^2=\sum_{n\in\mathbb{N}}q_n(s)^2\|h_n\|^2_{L^2_{\rho_1}}$ and equations i) and ii) in Lemma 2.6 hold. Let us remark that

$$I(s)\geq\frac{C}{s^2}\text{ for }s\text{ large, where }C>0. \tag{63}$$

Indeed, $I(s)\geq|q_2(s)|\|h_2\|_{L^2_{\rho_1}}$ and by definition (see (31) and (32)),

$$q_2(s)=\int\frac{\partial\tilde w}{\partial s}(y_1,s)k_2(y_1)\rho_1(y_1)dy_1=w_2'(s)\sim\frac{\kappa}{4ps^2} \tag{64}$$

as mentioned in ii) of the lemma we are proving.
Like for equation (26), Lemma 2.7 holds and either no mode dominates in $I(s)$ or $I(s)\sim l_n(s)$ as $s\to\infty$ for some $n\geq 2$. We claim that

$$I(s)\sim l_2(s)\text{ as }s\to\infty.$$

Indeed, if no mode dominates or if $I(s)\sim l_n(s)$ with $n\geq 3$, then Lemmas 2.6 and 2.7 imply that $I(s)$ has to decay exponentially fast. Contradiction with (63).
Using (64), we see that

$$I(s)\sim l_2(s)=2\sqrt{2}|q_2(s)|\sim\frac{\kappa\sqrt{2}}{2ps^2}\text{ as }s\to\infty. \tag{65}$$

Our conclusion follows if we prove that

$$\|q(y_1,s)-q_2(s)h_2(y_1)\|_{L^2_{\rho_1}}=O\left(\frac{1}{s^3}\right). \qquad (66)$$

Indeed, parabolic regularity implies that (66) also holds in L^∞_{loc}, in particular, at $y_1=0$:

$$\frac{\partial\tilde{w}}{\partial s}(0,s)=q(0,s)\sim q_2(s)h_2(0)\sim-\frac{\kappa}{2ps^2}\text{ as } s\to\infty,$$

which is the desired conclusion (note that $h_2(0)=-2$, by (31)). Let us prove (66).

Proof of (66): From (62), we see that

$$\|q-q_2(s)h_2(y_1)\|^2_{L^2_{\rho_1}}=c_0q_0(s)^2+c_1q_1(s)^2+l_3(s)^2, \qquad (67)$$

where $l_3=\|\pi_3q\|_{L^2_{\rho_1}}$ and $\pi_3q=\sum_{n=3}^{\infty}q_n(s)h_n(y_1)$. Using i) of Lemma 2.6 with $n=0$ or 1, along with (65), we see that $\left|l_n'(s)+\left(\frac{n}{2}-1\right)l_n(s)\right|\le\frac{C}{s^3}$ which yields

$$l_n(s)=O\left(\frac{1}{s^3}\right)\text{ as } s\to 0 \text{ for } n=0 \text{ or } 1. \qquad (68)$$

If we project (61) using π_3, we see that

$$\partial_s\pi_3q=\mathcal{L}\pi_3q+\pi_3(\alpha q).$$

Multiplying this equation by $\pi_3q\rho_1(y_1)$ and integrating over $\mathbb{R}$, we see that

$$\frac{1}{2}\frac{d}{ds}l_3^2=\int\mathcal{L}\pi_3q.\pi_3q\rho_1dy_1+\int\pi_3(\alpha q)\pi_3q\rho_1dy_1\le-\frac{1}{2}l_3^2+\int\pi_3(\alpha q)\pi_3q\rho_1dy_1$$

because π_3 is the projector over the negative part of the spectrum. Using Cauchy-Schwartz's inequality twice, we write

$$\begin{aligned}\left|\int\pi_3(\alpha q)\pi_3q\rho_1dy_1\right| &\le\|\pi_3(\alpha q)\|_{L^2_{\rho_1}}\|\pi_3q\|_{L^2_{\rho_1}}\\ &\le\|\alpha q\|_{L^2_{\rho_1}}l_3 \text{ (because } \pi_3 \text{ is a projector)}\\ &\le\|\alpha\|_{L^4_{\rho_1}}\|q\|_{L^4_{\rho_1}}l_3.\end{aligned}$$

Therefore,

$$l_3'\le-\frac{1}{2}l_3+\|\alpha\|_{L^4_{\rho_1}}\|q\|_{L^4_{\rho_1}}. \qquad (69)$$

Using Proposition 2.5, we see that $\|\alpha\|_{L^4_{\rho_1}} \leq \frac{C}{s}\|(1+y_1^2)\|_{L^4_{\rho_1}} \equiv \frac{C}{s}$. Equation (61) has a nice property of control of the $L^4_{\rho_1}$ norm by the $L^2_{\rho_1}$ norm up to some delay in time (see Lemma 2.3 in [12]):

$$\left(\int q(y_1,s)^4\rho_1 dy_1\right)^{1/4} \leq C\left(\int q(y_1,s-s_*)^2\rho_1 dy_1\right)^{1/2}$$

for some $s_* > 0$. Using (65), we end-up with $\|q\|_{L^4_{\rho_1}} \leq \frac{C}{s^2}$. Therefore, (69) becomes

$$l_3' \leq -\frac{1}{2}l_3 + \frac{C}{s^3}$$

which yields

$$l_3(s) = O\left(\frac{1}{s^3}\right) \text{ as } s \to \infty. \tag{70}$$

Thus, (66) follows from (67), (68) and (70). This concludes the proof of Lemma 2.4. ■

B Projection of equation (26) on the different modes

We prove in this appendix various technical lemmas from Section 2. In subsection B.1, we prove part iii) of Lemma 2.6. We prove Lemma 2.8 and Lemma 2.11 in subsection B.2. Subsection B.3 is devoted to the proof of Lemma 2.9.

B.1 Equation on the null mode

We prove iii) of Lemma 2.6 here. Take $\beta \in \mathbb{N}^N$ such that $|\beta| = 2$. If we multiply (26) by $k_\beta(y)\rho(y)$ and integrate over $\mathbb{R}^N$, then we get from (34) and (33)

$$g_\beta'(s) = \int \alpha g k_\beta \rho dy.$$

Using (36) and Cauchy-Schwartz's inequality, we write for all $a \in S_\delta$, $|\sigma| < \sigma_0$ and $s \geq -\log T + \sigma_0$,

$$\begin{aligned} |g_\beta' + \tfrac{1}{4s}\textstyle\int h_2(y_1) g k_\beta \rho dy| &\leq \tfrac{C}{s^{3/2}}\textstyle\int (1+|y|^3)|g||k_\beta|\rho dy \\ &\leq \tfrac{C}{s^{3/2}}\|(1+|y|^3)k_\beta\|_{L^2_\rho}\|g\|_{L^2_\rho} \equiv \tfrac{C(\beta)}{s^{3/2}} I(s). \end{aligned}$$

Using (35), (30), (32) and (33), we write

$$\begin{aligned}
& \int h_2(y_1) g(y,s) k_\beta(y) \rho(y) dy \\
= & \sum_{\gamma \in \mathbb{N}^N} g_\gamma(s) \int h_2(y_1) h_\gamma(y) k_\beta(y) \rho(y) dy \\
= & \sum_{\gamma \in \mathbb{N}^N} g_\gamma \int h_2(y_1) h_{\gamma_1}(y_1) k_{\beta_1}(y_1) \rho_1(y_1) dy_1 \prod_{i=2}^N \int h_{\gamma_i} k_{\beta_i} \rho_1(y_i) dy_i \\
= & \sum_{\gamma \in \mathbb{N}^N} g_\gamma(s) \int h_2(y_1) h_{\gamma_1}(y_1) k_{\beta_1}(y_1) \rho_1(y_1) dy_1 \prod_{i=2}^N \delta_{\gamma_i, \beta_i}
\end{aligned}$$

Because of the orthogonality relation (33) and symmetry, the above term is zero except when for all $i = 2, ..., N$, $\gamma_i = \beta_i$ and $|\gamma_1 - \beta_1| = 0$ or 2.
If $\gamma = \beta$, then the term is $g_\beta(s) \int h_2(y_1) h_{\beta_1}(y_1) k_{\beta_1}(y_1) \rho_1(y_1) dy_1 = 4\beta_1 g_\beta(s)$ after straightforward calculations based on (31) and (33), performed for $\beta_1 = 0$, 1 or 2.
If $\gamma = \beta \pm (2, 0..., 0)$, then the term is $\left| g_\gamma(s) \int h_2 h_{\beta_1 \pm 2} k_{\beta_1} \rho_1 dy_1 \right| \equiv C |g_\gamma(s)| \leq C\left(l_0 + l_4\right)$ by (35). This concludes the proof of iii) in Lemma 2.6.

B.2 Uniform smallness of the positive modes

We prove Lemmas 2.8 and Lemma 2.11 here.

Proof of Lemma 2.8: If we take $n = 0$ in i) and ii) in Lemma 2.6, then we see that $x = e^{-s} l_0(s)$ and $y = e^{-s} I(s)$ satisfy inequality (41) in the ODE lemma 2.11. Therefore, either (42) or (43) holds. Let us assume by contradiction that (43) holds. Then, we see that $I(s) \sim l_0 > 0$ as $s \to \infty$. Using i) of Lemma 2.6 with $n = 0$, we see that l_0 and I go to infinity. Contradiction. Thus (42) holds and we get the estimate for l_0. We do the same for l_1 and I, using Lemma 2.11 with $x = e^{-\frac{s}{2}} l_1$ and $y = e^{-\frac{s}{2}} I$. This closes the proof of Lemma 2.8. Remains to prove Lemma 2.11. ∎

Proof of Lemma 2.11: This lemma was proved in [9] with no attention to the dependence of the conclusion on the data. We have proved there that

$$\text{either } x = O\left(\frac{y}{s}\right) \text{ or } x \sim y > 0 \text{ as } s \to \infty, \tag{71}$$

with no uniform estimates. Let us prove the uniform version.
Define $\bar{s}(M, \hat{s}) \geq \hat{s}$ such that

$$\forall s \geq \bar{s}, \frac{3M}{2s} + \frac{M}{s^2}\left(5 - \frac{35}{2}M\right) \geq 0. \tag{72}$$

If (42) doesn't hold, then there is $\tilde{s} \geq \bar{s}$ such that $\gamma(\tilde{s}) > 0$ where $\gamma(s) = x(s) - \frac{5M}{s} y(s)$ ($\tilde{s}$ may depend on x and y). Using (41) and (72), we get

$$\forall s \geq \tilde{s}, \quad \gamma' \geq y \left(\frac{3M}{2s} + \frac{M}{s^2} \left(5 - \frac{35}{2} M \right) \right) - \frac{5M}{2s} \gamma \geq -\frac{5M}{2s} \gamma.$$

Therefore, $\gamma(s) \geq \gamma(\tilde{s}) \left(\frac{\tilde{s}}{s} \right)^{\frac{5M}{2}} > 0$ and

$$\forall s \geq \tilde{s}, \quad x(s) > \frac{5M}{s} y(s). \tag{73}$$

In particular, $y \geq x > 0$ and we can write from (41) the following equation for all $s \geq \tilde{s}$,

$$\forall s \geq \tilde{s}, \left(\frac{x}{y} \right)' \geq -\frac{M}{s}(1 + \frac{x}{y}) + \frac{1}{2} \frac{x}{y}(1 - \frac{x}{y}) \geq -\frac{2M}{s} + \frac{1}{2} \frac{x}{y}(1 - \frac{x}{y}). \tag{74}$$

The proof will be completed if we rule out the first possibility in (71). We proceed by contradiction. If $x = O\left(\frac{y}{s}\right)$, then we have from (73) and (74),

$$\left(\frac{x}{y} \right)' \geq -\frac{2M}{s} + \frac{5M}{2s}(1 - \frac{5M}{s}) = \frac{M}{2s} - \frac{25M^2}{4s^2}$$

for s large. This implies that $\frac{x}{y} \to \infty$ as $s \to \infty$. Contradiction with $x \leq y$. Thus, only the second case in (71) holds and Lemma 2.11 as well as Lemma 2.8 are proved. ∎

B.3 Decay of the null mode

We prove Lemma 2.9 here. We use equation iii) in Lemma 2.6. We need to estimate the error terms there. Let $s_3(\sigma_0) = \max\left(-\log T + \sigma_0, s_1(\sigma_0)\right)$ where $s_1(\sigma_0)$ is defined in Lemma 2.8. Consider some $a \in S_\delta$ and $|\sigma| \leq \sigma_0$. According to Lemma 2.8 and (29), we have for all $s \geq s_3(\sigma_0)$,

$$l_4(s) \leq I(s) \leq C(\sigma_0) \frac{\log s}{s^2} \text{ and } l_0 \leq C \frac{I(s)}{s} \leq C(\sigma_0) \frac{\log s}{s^3}. \tag{75}$$

As for the size of l_4, we integrate the equation in i) of Lemma 2.6 with $n = 4$ to get $\forall s \geq s_3$,

$$l_4(s) \leq e^{-(s-s_3)} l_4(s_3) + e^{-s} \int_{s_3}^{s} e^t \frac{I(t)}{t} dt.$$

Using (75), we see that

$$\int_{s_3}^{s} e^t \frac{I(t)}{t} dt \le C(\sigma_0) \int_{s_3}^{s} e^t \frac{\log t}{t^3} dt \le C(\sigma_0) e^s \frac{\log s}{s^3}.$$

Therefore,

$$\forall s \ge s_3,\ l_4(s) \le C(\sigma_0) \frac{\log s}{s^3}. \tag{76}$$

Using iii) of Lemma 2.6 along with (75) and (76) yields

$$\forall s \ge s_3, \forall |\beta| = 2, \quad \left| g_\beta'(s) + \frac{\beta_1}{s} g_\beta(s) \right| \le C(\sigma_0) \frac{\log s}{s^{7/2}}.$$

Since $\beta_1 = 0$, 1 or 2 and $|g_\beta(s)| \le C l_2(s) \le C I(s) \le C(\sigma_0) \frac{\log s}{s^2}$ by (75), this yields the conclusion. ■

C Estimates in the intermediate region

We prove Proposition 3.5 here. From (7) and Corollary 3.4, we have information on u and $u - \tilde{u}_{\sigma(P_S(x))}$ at $(x, \tilde{t}(d(x,S)))$, a point on the edge of the blow-up region. We use this as initial data, and solve the 2 ODEs of Proposition 2.3 between $\tilde{t}$ and t to get an estimate on u and $u - \tilde{u}_{\sigma(P_S(x))}$ at (x,t), when $t \in [\tilde{t}, T)$. For clearness, we work with rescaled versions of u and $\tilde{u}$, defined for all $(\xi, \tau) \in \mathbb{R}^2 \times [-\frac{\tilde{t}}{T-\tilde{t}}, 1)$ by:

$$\left\{ \begin{array}{l} v(x,\xi,\tau) = (T-\tilde{t})^{\frac{1}{p-1}} u(x + \xi\sqrt{T-\tilde{t}}, \tilde{t} + \tau(T-\tilde{t})) \\ \tilde{v}(x,\xi,\tau) = (T-\tilde{t})^{\frac{1}{p-1}} \tilde{u}_{\sigma(P_S(x))}(d(x,S) + \xi_1\sqrt{T-\tilde{t}}, \tilde{t} + \tau(T-\tilde{t})) \\ h(x,\xi,\tau) = v - \tilde{v}, \end{array} \right. \tag{77}$$

where $\tilde{t} = \tilde{t}(d(x,S))$ is defined in (44) and goes to T as $d(x,S) \to 0$.
We start with initial data at $\tau = 0$ for v, $\tilde{v}$ and h (which corresponds to information on u at time $\tilde{t}$, i.e. at the frontier between the blow-up and the intermediate regions). We see from Corollary 3.4 and (7) that there is $\epsilon_1 > 0$ such that if $|x - \hat{a}| < \delta$ and $d(x,S) < \epsilon_1$, then

$$\left\{ \begin{array}{rl} |v(x,0,0) - f_1(1)| & \le C \dfrac{\log|\log(T-\tilde{t})|}{|\log(T-\tilde{t})|} \\ |h(x,0,0)| & \le C(T-\tilde{t})^{\frac{1}{2}} |\log(T-\tilde{t})|^{3/2+C_0} \end{array} \right. \tag{78}$$

As rescaled versions, v and $\tilde{v}$ are still solutions of the PDE (1). However, it is easier to work with the localizing ODE given in Proposition 2.3: for all $\epsilon > 0$ and $(x,t) \in \mathbb{R}^N \times [\frac{T}{2}, T)$,

$$|\partial_t u - |u|^{p-1}u| \le \epsilon |u|^p + C_\epsilon, \quad |\partial_t \tilde{u} - |\tilde{u}|^{p-1}\tilde{u}| \le \epsilon |\tilde{u}|^p + C_\epsilon,$$

where C_ϵ denotes hereafter a constant depending only on ϵ. Since $\sigma(a)$ is continuous in terms of a (see Proposition 2.1), we see from the definition of $\tilde{u}_\sigma$ (10) that for all $a \in S_\delta$ and $(x,t) \in \mathbb{R}^N \times [T - e^{-\sigma_0}\frac{T}{2}, T)$

$$|\partial_t \tilde{u}_{\sigma(a)} - |\tilde{u}_{\sigma(a)}|^{p-1}\tilde{u}_{\sigma(a)}| \le \epsilon |\tilde{u}_{\sigma(a)}|^p + C_\epsilon.$$

Using (77), we get for all $\epsilon > 0$, $x \in B(\hat{a}, \delta)$ and $\tau \in [0,1)$,

$$\begin{array}{l} |\partial_\tau v(x,0,\tau) - |v|^{p-1}v| \le \epsilon |v|^p + C_\epsilon (T - \tilde{t})^{\frac{p}{p-1}} \\ |\partial_\tau \tilde{v}(x,0,\tau) - |\tilde{v}|^{p-1}\tilde{v}| \le \epsilon |\tilde{v}|^p + C_\epsilon (T - \tilde{t})^{\frac{p}{p-1}} \\ |\partial_\tau h(x,0,\tau) - p|\bar{v}|^{p-1}h| \le \epsilon \left(|v|^p + |\tilde{v}|^p\right) + C_\epsilon (T - \tilde{t})^{\frac{p}{p-1}} \end{array} \tag{79}$$

for some $\bar{v} \in [v, \tilde{v}]$. Since the solution of

$$v_0' = v_0^p, \quad v_0(0) = f(1)$$

is $v_0(\tau) = \left(\frac{(p-1)^2}{4p} + (p-1)(1-\tau)\right)^{-\frac{1}{p-1}}$, a bounded function for all $\tau \in [0,1]$, we use the continuity of ODE solutions with respect to initial data to get

$$\sup_{\tau \in [0,1)} |v(x,0,\tau) - v_0(\tau)| + |\tilde{v}(x,0,\tau) - v_0(\tau)| \to 0 \text{ as } d(x,S) \to 0$$

and

$$\sup_{\tau \in [0,1)} |h(x,0,\tau)| \le C |h(x,0,0)|$$

whenever $d(x,S) \le \epsilon_0$ for some $\epsilon_0 > 0$. Therefore, we get from (77) and (78):

$$\sup_{\tilde{t} \le t < T} \left| u(x,t) - \tilde{u}_{\sigma(P_S(x))}(d(x,S),t) \right| \le C (T - \tilde{t})^{\frac{1}{2} - \frac{1}{p-1}} |\log(T - \tilde{t})|^{\frac{3}{2} + C_0}.$$

Since $d(x,S) \ge \sqrt{(T-t)|\log(T-t)|}$ whenever $t \ge \tilde{t}$ (see (44)) and

$$(T - \tilde{t})^{\frac{1}{2} - \frac{1}{p-1}} |\log(T - \tilde{t})|^{\frac{3}{2} + C_0} \sim C d(x,S)^{1 - \frac{2}{p-1}} |\log d(x,S)|^{\frac{p}{p-1} + C_0}$$

as $d(x,S) \to 0$ (see (44)), this concludes the proof of Proposition 3.5. ■

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